Competencies and Mathematical Learning

Ideas and inspiration for the development of mathematics teaching and learning in Denmark

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Mogens Niss & Tomas Højgaard (eds.)
English edition, October 2011

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Ideas and inspiration for the development of
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by: Mogens Niss & Tomas Højgaard (eds.)

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The present text is a translation from Danish of the main parts of a report published by the Danish Ministry of Education (Niss & Jensen; 2002). The translation, funded by the Ministry of Education, has been under way for quite some time, due to lengthy breaks caused by a variety of circumstances. The main delays were never caused by the translators, first Janet Reid, later Stine Timmermann Ottesen, both of whom have done an excellent job to make a very complicated text with lots of references to Danish peculiarities accessible to an international readership. Both translators deserve my sincere and warm thanks.
Right after the publication of the Danish original text it was decided to produce an English translation, in the first place in order to pave the way for international reactions to the report. Later, the rationale changed a little bit. The main authors of the report, Tomas Højgaard (now associate professor of mathematics education at DPU, Aarhus University) and I as the project director have given numerous talks, workshops etc. about the KOM project at international meetings and in institutions in numerous countries. This has given rise to a fair degree of international interest in the report itself, an interest which has grown over the years rather than diminished. So, even if this translation does indeed appear far too late in relation to the initial plans, it is nevertheless our judgment that it is not outdated.
The present translation covers the first six parts of the seven parts included in the original report. This corresponds to almost 60% of the original text, comprising 332 pages. Part VII, which has not been translated, deals in considerable detail with issues and topics which are very specific to Denmark and Danish circumstances. We have found that this material is not sufficiently relevant to an international audience to warrant translation.
The fact that the original report was published in 2002 implies that certain pieces of factual information about the state of affairs in Denmark are now outdated. As a matter of fact some of the changes have been caused by the report itself. It is, however, our view that these changes are insignificant to an international readership, so we have not accounted for such changes in the translation except in a very few singular cases.
I would like to finish by thanking the Ministry of Education for its support of the project in general and the translation endeavour in particular, the members of the KOM task group, above all its secretary Tomas Højgaard, and by repeating my thanks to the two translators.

Roskilde, October 2011

Mogens Niss
Professor of Mathematics and Mathematics Education
IMFUFA/NSM, Roskilde University
Director of the KOM project

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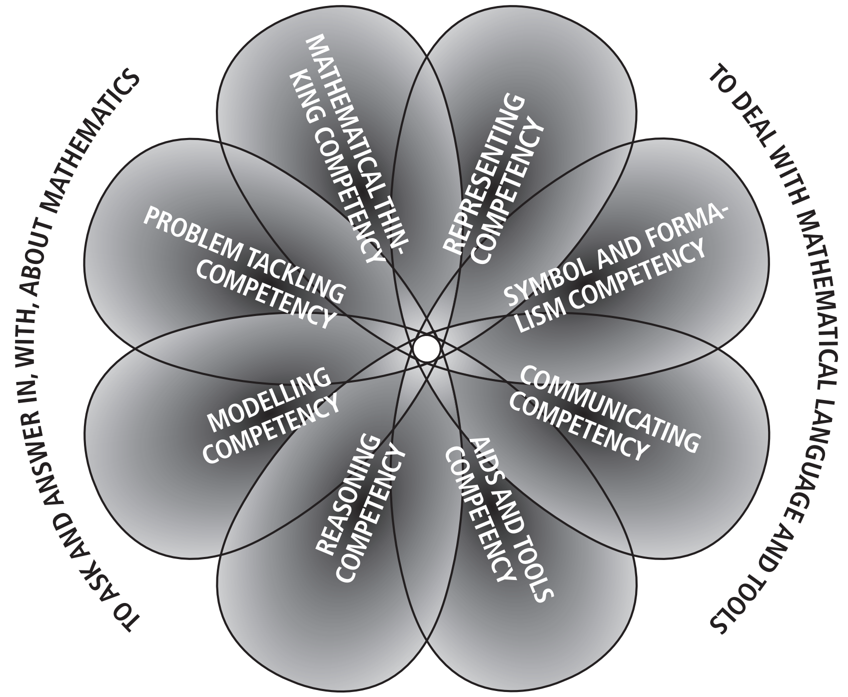

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Roskilde, October 2011

Mogens Niss
Professor of Mathematics and Mathematics Education
IMFUFA/NSM, Roskilde University
Director of the KOM project

Preface

Hereby the task group behind the project *Competencies and Mathematical Learning* (The KOM project) presents its report. The KOM task group was establiched in August 2000 jointly by the Council of Science Education and the Ministry of Education. The work was funded by a grant from the Ministry of Education, which, first and foremost, covered the salary of the academic secretary and of student assistants, in addition to the expenses of task group meetings and meetings of this group with its group of "sparring partners".

I would like to take this opportunity to thank the members of the KOM task group Tage Bai Andersen, Rune Wåhlin Andersen, Torben Christoffersen, Søren Damgaard, Therese Gustavsen (who participated in the first phase of the work), Kristine Jess, Jakob Lange, Lena Lindenskov, Malene Bonné Meyer (who participated in the first phase of the work) and Knud Nissen, and not least the secretary of the task group, Tomas Højgaard Jensen, for a huge and constructive work. Furthermore, the large group of sparring partners (who are listed in Chapter 1) and other people with an interest in the project also deserves many thanks for their contribution to the project. The collaboration with the Ministry of Education, above all with Torben Christoffersen, Jarl Damgaard and Jørgen Balling Rasmussen, but also mathematics inspectors and others, has been excellent and construtive. For this, too, I want to express my gratitude. Finally, there is good reason to thank the changing but always ready and effecient student assistants in the project, Eva Uhre, Gitte Jensen, Nesli Saglanmak and Arnold Skimminge, all students of mathematics and physics at Roskilde University, for their efforts.

Roskilde, May 21 2002

Mogens Niss, Director of the KOM project

Overview of the report

This report is divided into six parts, which in total contain eleven chapters. The point of departure for the project, the terms of reference, the structure and limitations are treated in Chapter 1, which together with the answers to the questions included in the terms of reference (provided in Chapter 2), comprice Part I of the report. So, the answers to the questions in the terms of reference are presented early in the report, even though the foundation of the answers is established only in the following chapters.

Part I: Introduction

The basic chapters of the report – that is, the chapters that present the thinking on which the project is founded – are Chapter 3, where the task for the theoretical part of the project is presented, and Chapter 4, which is devoted to an in depth description of the competency based approach to mathematical mastery. Here eight mathematical competencies and three forms of overview and judgement concerning mathematics as a discipline are presented as the common constituents of the mastery of mathematics/mathematical competence, irrespective of the educational level and the mathematical subject matter that the mastery concerns. It is the fundamental idea of the project that all mathematics teaching must aim at promoting the development of pupils' and students' mathematical competencies and (different forms of) overview and judgement. Together these two chapters constitute Part II of the report.

Part II: Competencies as a means to describe mathematics as a subject

Considering that mathematics teachers have a key role to play if the teaching of mathematics is to pursue the development of mathematical mastery as defined in this report, we find it important to give a normative description of teachers' competencies a prominent position in the report. This is done in Part III, shortly introduced in Chapter 5, which underlines the importance of a fruitful interplay between different kinds of competencies with mathematics teachers. It must be emphasised that Part III treats all mathematics teacher educational programmes under one hat, that is, primary school teachers, high school teachers, as well as teachers at institutions of higher education are being considered. Then two chapters follow, of which the first, Chapter 6, presents six forms of specific didactic and pedagogical competencies which a mathematics teacher should possess, while Chapter 7 focuses on the mathematical competencies of mathematics teachers as they are manifested in a teaching practice marked by a subject specific pedagogic agility, efficiency, and reserves of enegy and ability.

Part III: Education of mathematics teachers

Part IV: Competencies and mathematical subject matter

Since mathematical competencies are both developed and practiced in the handling of different sorts of mathematical subject matter, it is important to clarify the relationships between competencies and subject matter. This is done in Part IV, which only contains one chapter (Chapter 8). It is emphasised there that the relationship between competencies and the subject matter at a given educational level has a (two-dimensional) matrix representation. If the various educational levels are included as a variable we are led to a three dimensional structure. In the chapter, ten mathematical subject matter domains are presented which the KOM task group has identified as providing the foundation for mathematics teaching and learning throughout the school system as well as in the introductory levels of tertiary education.

Part V: Progression and assessment of competence development

One of the main points in the project is to contribute to the advancement of the progression and coherence of mathematics teaching and learning both lengthwise and crosswise in the education system, and also to create valid and reliable forms of assessment of a person's mastery of mathematical competencies. These issues are on the agenda in Part V, where the (only) chapter, Chapter 9, on the one hand discusses static and dynamic assessment of competency possession in relation to existing, respectively desirable, forms and instruments of assessment and, on the other hand characterises progression in mathematical competency development and the possibilities for (dynamic) assessment hereof.

Part VI: Looking ahead: Challenges and recommendations

This report is concluded with Part VI, which contains two chapters. In the first of these, Chapter 10, a presentation of selected main problems in Danish mathematics education is provided. It is the overall conclusion that in many respects mathematics teaching is structured, carried out, and functions quite satisfactorily, but that a number of problems and challenges exist which can and should be dealt with. It might have appeared more natural to have this chapter placed earlier in the report as a means to set the stage. But this choice might have given tise to the inadequate impression that the framework for description and assessment of mathematical competencies is derived from these specific problems and challenges, which is not the case, so, we have chosen to place this chapter in Part VI instead. This also makes it possible to mention problems that the project does not address.

The recommendation of the report

The final chapter (Chapter 11) in Part VI is devoted to those recommendations that the KOM task group wished to propose to different institutions, namely the Ministry of Education, universities and institution of higher educations, teacher training institutions/CVUs, vocational professions programmes, local school administrations and local authorities, mathematics teachers and their associatons, textbook authors and publishers and researchers of mathematics education.

The report is ended with a list of references.

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Part I

Introduction

1 Introduction

1.1 Starting point

Started in summer of 2000 at the initiative of the National Council for Science Education and the Ministry of Education

The project *Competence Development and Mathematics Learning* started during the course of the summer of 2000. The initiative came from the National Council for Science Education which wanted to pave the way for a development in mathematics teaching as a spearhead project for a possible corresponding development in other subjects. The Ministry of Education then undertook to finance the project and – provided that results were achieved that could be implemented – would be active in the question of implementing the concrete proposals which the appointed task group reached, cf. The terms of reference below. Let it be clearly stated here that from the appointment of the task group and the acceptance of the terms of reference, the task group has operated entirely without attempts at external influence on its work from either those issuing the terms of reference or anyone else. The results of the work are therefore solely the product of the task group's own activities, considerations and decisions.

The autonomy of the task group

1.1.1 Those involved

The members of the task group

Those involved in the project are, first and foremost, the twelve members of the task group in question: Rune Wåhlin Andersen (geologist; chairperson of the College of Nature and Communication initiative), Tage Bai Andersen (mathematician; study adviser, Aarhus University), Torben Christoffersen (former high school teacher in mathematics and physics; assistant secretary, the Ministry of Education; liaison officer between the task group and the Ministry), Søren Damgaard (physicist; employed by IBM; chairperson of the National Council for Science Education), Therese Gustavsen (primary school teacher, Brøndby municipality), Tomas Højgaard Jensen (project *secretary*; Ph.D. student in mathematical didactics, Roskilde University Centre), Kristine Jess (teacher training college lecturer in mathematics, the Copenhagen Day and Evening College of Teacher Training), Jakob Lange (graduate in social studies; office manager, the University of Copenhagen; head of The Coordinated Application System for University and College Entry), Lena Lindenskov (mathematical didactics expert, The Danish University of Education), Malene Bonné Meyer (human biologist, The Copenhagen School of Medical Laboratory Technologists), Mogens Niss (project *chairperson*; mathematician and mathematical didactics expert, Roskilde University Centre) and Knud Nissen (high school teacher in mathematics and computer science, Aarhus Adult Education Centre). For a number of reasons, Therese Gustavsen and Malene Bonné Meyer were unable to participate in the later phases of the work.

Sparring partner group

The task group has received substantial help from a large group of "sparring partners" who kept up a running contribution with constructive reactions to the ideas and notions of the task group as well as, in the case of some people, help with the actual writing of concrete sections of the report. With a few substitutions due to changing jobs along the way, the people concerned are: Knud Flemming Andersen, Søren Antonius, Søren Bjerregaard, Michael Caspersen, Bjørn Grøn, Anne-Marie Kristensen, Karsten Enggaard, Dan Eriksen, Erik von Essen, Bent Hirsberg, Marianne Holmer, Eva Høg, Tom Høholdt, Torben Pilegaard Jensen, Claus Jessen, Hanne Kock, Jens Helveg Larsen, Kjeld Bagger Laursen, Peter Limkilde, Nikolaj Lomholt, Marianne Nissen, Elsebeth Pedersen, Bjarne Sonberg, Hans Søndergaard, Søren Vagner, Jørn Vesterdal and Karsten Wegener.

Contact with the outside world

Besides this, the group of people involved has been constantly extended to include a large number of people connected to mathematics teaching in Denmark. This attempt to make the project "owned by many" right from the beginning has been achieved by asking a range of people from outside the task group, and to start with from outside the sparring group too, for help when it came to clarification and written work. Furthermore, the chairperson and secretary of the task group, as well as other members of the group, were invited to a large number of meetings at schools, in societies, organisations, counties, councils and ministries so as to, through meetings and conferences, etc. discuss the fundamental ideas of the project while it was still in its developmental phase. The contact to "the outside world" did not only cover mathematical education circles, but also those connected to other subject areas. For example, many meetings and pedagogical afternoons were held with the entire teaching staff at high schools and adult education centres. Contact was also established and considerations and texts were exchanged with more or less parallel task groups for the *Danish subject in the Future* under the leadership of Frans Gregersen and *Competence development in Physics/Chemistry-learning* under the leadership of Ove Poulsen, as well as with the designated leaders of planned future projects in the natural sciences and foreign languages. The intention with this form of contact was, right from the start, to ensure the project's ownership among those instances which would be obliged to carry out its implementation, not least current and future mathematics teachers at all levels. In support of this intention, working texts in draft form were made available on the project's website for orientation and reaction from interested parties.

1.1.2 Working procedure

Meetings and report writing

Work on the project took place in the interaction during meetings of the task group, 17 in total, meetings with the task group and the sparring group, 4 in total, as well as in between meetings where the chairperson and secretary of the task group, with the help of the above-mentioned people, worked to produce draft texts, and produce background material, etc. As the report stands, the principal author is the chairman, *Mogens Niss*, with the secretary, *Tomas Højgaard Jensen*, as co-author. The entire report was, in the mean time, produced by means of an iterative process comprising a great number of steps, including alternate presentations of drafts and discussions in the task group. The report is therefore the product of the entire task group.

1.2 Terms of Reference

The terms of reference of the project were, in cooperation with the National Council for Science Education, the Ministry of Education, and the task group's chairperson, formulated as follows:

Terms of reference

> "The undertaking of the task group is to throw light on the following questions (in arbitrary order of importance):
>
> a) To what extent is there a need for a renewal of existing forms of mathematics teaching?
>
> b) Which mathematical competencies need to be developed in students at the different stages of the education system?
>
> c) How does one ensure progression and coherence in mathematics teaching throughout the education system?
>
> d) How does one measure mathematical competencies?

e) What should be the content of up-to-date mathematics teaching?
f) How does one ensure the ongoing development of mathematics as a subject and mathematics teaching?
g) What does society demand of mathematics teaching?
h) What do future mathematical teaching materials look like?
i) How can one, in Denmark, make use of international experience with mathematics teaching?
j) How should mathematics teaching be organised in the future?

The group will, furthermore, put forward a number of recommendations on how the project should continue and how the answers to the above-mentioned questions should be operationalised.

The group starts its work in September 2000 and will complete its task at the end of 2001."

1.3 Structure and limitations

Two phases to the work

As implied in the terms of reference, the project mainly comprises *two closely connected phases* of an analytic and a pragmatic nature respectively; an academic *clarification phase* and a pragmatic *operationalisation phase*. Since the project has a significant educational politics element to it, right from the start there have, as intimated in the terms of reference, been plans to follow up the work in a third *implementation phase*.

1.3.1 Clarification phase

Problem clarification and competence description are the "mainstays"

This phase of the project has involved an analysis of the questions in the terms of reference and the problem formulations derived there from, as well as a report of these results. In structuring this work, the task group chose the first two questions of the terms of reference a) and b) as its "mainstays". In this way the analysis has been built up around a descriptive *problem characteristics* and a normative *competence description* of mathematics as a subject.

Problem characteristics

Point of departure: "Something" is not as it ought to be

The main point of departure for the work has been the general agreement among the initiators and members of the task group, that the answer to question a) in the terms of reference is not: "to no extent". "Something" about the connection between society's actual or desirable mathematical abilities and the mathematics teaching which ought to be a corner stone in the production of these abilities is, from our viewpoint, not as it ought to be.

Central questions

Where then does the problem lie? Are the current conceptions of who ought to possess which mathematical skills out of date and in need of critical revision? If this is the case, what changes to mathematics teaching and its framework should this give rise to? Or is the problem, on the contrary, that these conceptions are still valid but that the different situations in and around which mathematics teaching finds itself have resulted in too wide a gap between the desirable skills and those actually acquired. If this is the case, where are the weak points? What has caused them, what are they like, and who is able to do anything about them? What measures can be taken to solve the problems and on what conditions and with what consequences?

Open approach

The task group has in principle seen its job as regarding all these questions as open ended, without taking the answers to any of them for granted. Obviously it is unrealistic to expect them all to be clarified in a project like the present one. On the international plane, each of these questions is the object of comprehensive research. Nevertheless, part of the project has been to outline the most important ingredients in an answer to the main problem as it appears in the Danish context, cf. chapter 10.

Competence description

No expectations of "quick fix solutions"

Given the complexity of the problems we are dealing with, one cannot expect to reach quick fix solutions. No matter which measures one chooses to implement, these will all be attempts to influence a process where the nature of the problems, and the notions of what is a step in the right direction, are always changing. We have therefore chosen not to deal with the identified problems one at a time, but have instead tried to reach conclusions along the lines of "if we could only . . . then . . . ".

Competencies as aims for education subject specialisation

The task group has therefore chosen to focus on proposing changes in one of the many areas which influences the way mathematics teaching is carried out, that is to say, the way mathematics teaching is controlled by a determination of its content. In practice we have worked with the utilisation of a competence description of mathematics education subject specialisation as the "aims" for mathematics teaching. For further information about the principles hereof, see chapter 3 and 4.

By choosing problem characteristics and competence descriptions respectively as the two "mainstays" of mathematics subject specialisation, the rest of the clarification work has been determined by two questions: "How would it reasonably be possible to accommodate other aspects of great importance for mathematics teaching in practice given that the 'aims' are described in competence terms?" and "In which areas and to what extent can the potential of the competence approach contribute to a solution of the problems and challenges identified in the project?"

A broad approach

In principle: A broad approach when it comes to educational stage...

Since the project concerns people's mathematical skills over a broad front, the clarification phase started by placing no limitation on the *educational stage* relevant for the project. In principle, all stages of education from the start of primary school, through teacher training and university education have been on the agenda, including e.g. vocational training and adult education. In practice we have naturally had to proceed more modestly and be content with dealing with fewer types of mathematics teaching chosen for their importance and their ability to be exemplary, i.e. representative for a larger class of mathematics teaching.

...as well as when it comes to mathematics and mathematics teaching

Not even when it comes to *content* did we narrow our considerations at the start. This means that the project operates with a broad concept of mathematics and that mathematics is not only regarded as a purely theoretical discipline, but also as one that can be applied in other subject and practice areas and therefore has connections to these as well as being connected to culture and society. Statistics and mathematical history are, for example, therefore included in our concept of mathematics. The same applies to the project's broad view when it comes to teaching in mathematics. We are not only interested in what has traditionally been classified as mathematics teaching, either when it comes to pure or applied mathematics, or mathematics as a support subject, but also in what can be called *mathematics-like* teaching, i.e. teaching which is *de facto* mathematics teaching, but where the word "mathematics" does not, for some reason or the other, appear in the title.

In practice: focus on general teaching

In practice we similarly need to be content with dealing with chosen aspects of mathematics and mathematics teaching in the hope that it will be of inspiration to those contexts we have had to leave untouched. Having a broad view of both "mathematics" and "mathematics teaching" does not mean that we have committed ourselves to covering all manifestations of mathematics teaching or mathematics-like teaching, wherever it appears, merely that we have not limited ourselves to observing the traditionally narrow meaning of the terms mathematics and mathematics teaching. As far as educational stages are concerned, we have in practice primarily focused on teaching concerning society as a whole, including school teaching for children and young people as well as the training of teachers for these schools.

1.3.2 Operationalisation phase

Recommendations

This part of the project has consisted of, on the basis of the clarification work, identifying some concrete input areas and proposing concrete initiatives to be implemented in these areas. The main points of this are comprised in a series of recommendations to different actors and instances which can be found in chapter 11.

The relative concreteness of the proposed recommendations should not be regarded as an expression of any endeavour to produce cut-and-dried solutions just in need of political go ahead before they are implemented. The task group was neither conceived of, nor functioned as, an extension of local government, but rather as a producer of ideas and a creator of inspiration. In the operationalisation phase we have therefore endeavoured to find a suitable balance between, on the one hand, exaggerated generalisations and their concomitant meaningless and harmless recommendations, and, on the other hand, recommendations with an exaggerated degree of detail which would have required a much more delimited focus in the development work and, in the case of certain educational areas, a greater detailed knowledge than the task group either could or wanted to produce.

Two extremes: exaggerated generalisation or exaggerated degree of detail

1.3.3 Implementation phase

Further measures in continuation of the project

If everything goes according to plan, the clarification and operationalisation phase will be followed up by a long series of measures attempting, in different ways, to implement the project's recommendations. As mentioned before, the practical implementation of this third phase of the entire mathematics teaching political "plot" does *not* form part of the actual KOM project. Nevertheless, one can say that the implementation of the project's ideas and recommendations is the most important and the most comprehensive part of the whole process. However, due to a number of different reasons, this will have to be carried out by others than the task group itself.

The fact that the task group will not have any part in the eventual implementation of (parts of) the recommendations proposed, does not mean that the group has remained passive when it comes to this part of the total education and teaching political aspect of the project. On the contrary, one of the driving forces for the task group's engagement in the project has been to contribute, from a certain "aerial perspective", to the education political development process.

Desire to contribute to development is the driving force

We are therefore in a process where we are operating on two fronts to try and influence the implementation phase. Firstly, many resources have, as mentioned previously, been spent on "preparing the ground" of the coming measures by means of contact with and submissions from a large number of partners and interested people in Danish mathematics teaching. The aim of this work is to, among other things, penetrate the "new measures fatigue" syndrome which understandably enough exists in many parts of the education system, and instead, through open and nuanced dialogue, create a "ripple effect in the water" allowing as many as possible to feel themselves part of a common project where the need for change is met with optimism, enthusiasm and commitment.

Counter attempts at "quick fix solutions"

Secondly, there has been a great interest in both the task group and the representatives of the political administrative system who will be responsible for the system and legislation part of the implementation phase, in discussing how this work can and ought to proceed. In the very open and constructive spirit of debate which characterised these discussions, interest from the task group's side was congregated towards the creation of conditions which would avoid the so politically attractive "quick fix solution", where one merely attempts through cheap and easily effected cosmetic changes in e.g. legislation to influence the reality and practice of teaching. Such changes will presumably be meaningless in the long term, while in the short term they will first and foremost result in "change fatigue" among those working with mathematics teaching at the classroom level. They will thereby be directly damaging in relation to more substantial reform measures.

1.4 What the project does *not* aim to achieve

The previous section has aimed to describe what the KOM project aims to achieve, namely, to produce an adequate characterisation of mathematical subject specialisation based on mathematical competences as a means of meeting some of the challenges and dealing with some of the problems.

Experience shows that it can be worthwhile mentioning what the project does *not* aim to achieve, but which could be assumed to be included and which, in addition, could be of importance in itself.

The KOM project is *not a research project* in its actual sense. As such, no research questions have been formulated that need to be answered using existing or newly established research methods. This does not mean that the project foregoes the use of e.g. conceptual or other definitions, nor that it foregoes systematic new thinking. It can best be characterised as *an analytical development project.*

Not a research project

In this way the KOM project does, e.g. *not* purport to be – or to create – a general *mathematics didactics solution* either in a theoretical or a practical sense. This would require infinitely more than the clarifications presented in the project.

Not a total mathematics didactics solution

The project does *not* purport to make a coherent stand for *the justification of mathematics' raison d'être* in the different parts of the education system, even though this issue is touched upon, not least in chapter 10. This too is of primary importance and deserves a project of its own, but due to the scope of our task we have had to refrain from including it in the project. This implies that, in the different relevant stages of the education system dealt with in the project, it has been assumed from the start that mathematics and mathematics teaching does, in principle, have a raison d'etre. What is up for discussion is under what conditions, in what way, and with which organisation mathematics teaching should be situated in the relevant contexts.

Not a project justifying the raison d'être of mathematics

The project does *not* purport to characterise or discuss *general education* or the actual or potential contribution of mathematics as a subject to such education. It is obviously important to have this relationship cleared up, but this cannot transpire in the present context.

Does not focus on general education

Recently the word "competence" has, with its different connotations, become the focus of much attention in educational, political and business circles. There is therefore a great interest in the education system in discussing the diverse forms of *general competencies of an intellectual, personality and social nature.* This refers to competencies like enthusiasm, working capacity, endurance, confidence, the ability to take responsibility - e.g. for your own learning, the ability to make decisions, tolerance, cooperation, empathy, etc. In spite of the importance of these competences, not least for the development of mathematical skills, they are not the focus of this project. The same goes for *labour market and business competencies* of a specific or general nature, like cooperation, adaptability, flexibility,

Not a focus on general competencies or labour market competencies

ICT skills, the use of one or more foreign languages, etc. as they are demanded in business and business organisations. These too are important competencies, but they nevertheless fall outside the scope of the KOM project.

Not an implementation project

Finally, the project does *not*, as emphasised above, purport to understand *the concrete measures needed to implement* its ideas and recommendations, either when it comes to the legal, economic or administrative frames for mathematics teaching in the different stages of the education system, or when it comes to the concrete instructions for teaching, evaluating, producing teaching aids, etc. In this regard “the project only begins in all seriousness when it has finished”.

2 Answers to the terms of reference

2.1 Introduction

The present report is not built up in agreement with the questions in the terms of reference, since the project first involved - in the previous chapter mentioned as the clarification stage - structuring the field, something which the terms of reference could not, by their very nature, be expected to incorporate at the start of the project. Nevertheless most of the questions in the terms of reference have in reality been more or less directly dealt with in the following chapters of the report. This chapter aims at presenting a short summary of what can, on the basis of this background, be said regarding the individual questions in the terms of reference, cf. section 1.2 (page 15).

The report not built up around the questions of the terms of reference

As a matter of course, these questions broadly speaking cover a very large section of the conditions that are essential in relation to mathematics teaching. A satisfactory attempt at answering them all would entail years of study and result in a report in many volumes. It is in recognition of this that the terms of reference do not require answers to the questions, but rather their elucidation, cf. section 1.2.

Questions elucidated, not answered

2.2 The individual points of the terms of reference

2.2.1 a) To what extent is there a need for a renewal of existing forms of mathematics education?

The opening phase of the project, the problem clarification stage, the main results of which are contained in chapter 10 of this report, was first and foremost aimed at uncovering a set of problems for, in and with

Much is going well, but problems and challenges exist mathematics teaching in Denmark. The conclusion of this phase was that in many respects Danish mathematics teaching contains central and worthwhile qualities at all educational stages (albeit the nature of this quality varies from stage to stage), but that there is also a series of problems and challenges which creates a need for renewal of certain aspects of the existing mathematics teaching system.

Justification problems

We have identified a set of problems related to *the justification of mathematics teaching*.

Mathematics courses of education not chosen

One of these is that pupils and students no longer choose courses of education or educational fields that contain mathematics to a sufficient extent, so that a disparity exists between the qualifications children and young people acquire and those that are required in their working, public or private lives. The main problem here is, first and foremost, not failed recruitment to the mathematical sciences in the education system, even though there is a growing lack of competent mathematics teachers in primary and secondary schools, but rather that a large number of young people do not choose other types of courses with a strong mathematical element. Since this is a well-documented international phenomenon and problem, its reasons cannot all be particular to Denmark. This obviously does not prevent these international tendencies either from being added to or reinforced by specific Danish conditions nor does it make it impossible to counter these tendencies and problems via the education system in general or mathematics teaching in particular.

The relevance paradox

Another problem in this set is the so-called “relevance paradox” which covers the disparity between, on the one hand, the objective, though often hidden, relevance of mathematics for society in the broad sense, and, on the other hand, the subjective irrelevance felt by many of the recipients of mathematics teaching regarding their own use of and relations to of mathematics. The relevance paradox is manifested, among other things, as an isolation problem for mathematics teaching, i.e. when it has difficulty being used in the interplay with teaching in other subjects. The motivation problem - which is reinforced by the relevance paradox, but also has a background and life of its own - consists of many pupils finding working with mathematics boring, meaningless, without perspective, or simply too demanding in relation to the expected benefits from the work

Threat to “maths for all”

The final justification problem is the gradually growing threat to “mathematics for all” which is particularly evident in countries like the USA, Japan, Germany and, in a more diffuse form, in Norway and Sweden, but that is also gaining ground in Denmark. In some circles in society, it is simply questioned whether mathematical skills and mathematical competencies are really so important for the whole population to have. Is it not sufficient that such skills and competencies are mastered by a minority while the rest can get by with mathematics propagated and camouflaged by user friendly ICT systems? In other circles, mainly among certain mathematicians and natural scientists, the scepticism is rather that "for all" means "for all *together*" since the fear in these circles is that "the weak students", with regard to acquisition and motivation, will come to lower the standard of mathematics teaching to the extent that it is trivialised and no longer appeals to or is sufficiently rewarding for "the strong student". In this way, none of the parties get anything out of mathematics teaching and society does not get anyone possessing sufficient mathematical competencies. No matter what your opinion on these threats to "mathematics for all" is, they give rise to serious challenges to Danish mathematics teaching and its understanding of itself.

Implementation problems

The second set of problems dealt with in this report, have been gather under the title *implementation problems*.

Maths teacher's qualifications

The first of these is linked to the qualifications of mathematics teachers. Even though the most characteristic feature of the different segments of mathematics teacher is the wide diversity found in each segment, it must be acknowledged that there is, on average, room for improvement of teacher qualifications, including their attitudes either when it comes to the subject mathematics, or when it comes to didactic-pedagogical issues, or both. This is true irrespective of whether we look at primary school teachers, diverse categories of teachers involved in the education of young people, or tertiary mathematics teachers.

Coherence

We have also directed our attention to problems of coherence, transition and progression in mathematics teaching. The coherence problem involves the fact that the subject called mathematics has in reality been planned, interpreted and realised so differently in the different sections of the education system, that it can be hard to point out what is common to the subject. For those pupils who, at different stages in their lives, find themselves in the various educational sections, this leads to confusion and orientation difficulties. These problems are especially heightened when it comes to the transition from one section of the education system to another (e.g. from lower to upper secondary school, or from upper secondary school to tertiary education courses) where both pupils and teachers experience significant uncertainty, a waste of mental and other resources, and a weakening of motivation and interest, etc. The problem with achieving adequate progression in mathematics teaching and acquisition, both between and within educational sections, is part of this context.

Diversity in pupils' benefits

A second implementation problem focuses on the diversity of outcomes pupils achieve from teaching at a given stage. This diversity is very significant in Danish mathematics teaching. Besides making it difficult to maintain a uniform level for the target group in the teaching, this diversity leads to what we can call a declaration problem which arises with "migration" along or across the education system, where those receiving the different groups of school leavers or graduates do not know what to expect in the mathematical baggage of the pupils or students they receive.

Teaching differentiation

The problems to be found with teaching differentiation as practiced in the education system, are also classified as implementation problems. Irrespective of the conceptions or intentions one has towards teaching differentiation (in reality the concept covers many different and sometimes contradictory things), it must be accepted that there are problems in the creation of clear frameworks and in obtaining sufficient resources to realise differentiation; and desired but unsuccessful teaching differentiation is a problem in itself.

Assessment

The last question presented in this set of problems is linked to that of assessment. On the one hand, this points to the problem of disharmony which consists of the fact that many of the traditionally used forms of assessment only to a limited extent allow for assessment of those skills and competencies which one would actually like to promote in mathematics teaching. On the other hand, this underlines the more in-depth problem of interpretation which covers the great difficulties there are in ascertaining that the assessment forms which are actually utilised, both allow for and are used to validly reveal a pupil's mathematical acquisition and mastery.

"The answer"

Renewal that addresses the problems mentioned

The above should not be seen as an exhaustive exploration of the problems for, in and with mathematics teaching in Denmark. However, the problems highlighted here are, according to the task group, among the most important. Here follows a concise answer to the question a): *There is a need to renew the existing mathematics teaching in such a way and to such an extent that the problems mentioned above can be solved or reduced significantly.*

The suggested means to such a renewal are discussed in chapter 11 which presents the KOM project's recommendations.

2.2.2 b) Which mathematical competencies need to be developed in students at the different stages of the education system?

The answer to this question is contained in this report's Part III. Taking into consideration the extent and diversity of the education system, the answer can only be summarised in general terms as follows.

All eight maths competencies on the agenda from the start of teaching

From the beginning of formalised mathematics teaching, all eight mathematical competencies – cf. chapter 4 - are to be on the teaching agenda. It is a crucial point in the KOM project that this happens, because the main idea is to utilise the competencies to create a common frame of reference for all mathematics teaching. On the other hand, in the beginning, only certain of the characteristics of the individual competencies are taken into account. Up through the education system it is assumed that more features of each competence will gradually be added to the competence possessed by the pupil. By the end of secondary school A-levels, students are expected to have acquired each competence at full coverage. The same goes for the courses of further education with a strong mathematical component, including courses for mathematics teachers of children, young people and adults. In this regard, it is to be assumed that the universities' masters and bachelor courses in mathematics, contrary to what used to be the case, aim towards developing the modelling, communication, and aids and tools competencies. With regard to the non-mathematical, yet mathematics-utilising courses which can be found after ordinary school, only certain of the competencies are expected to make up the mathematical baggage the pupil or student is meant to be equipped with. Emphasis here is on the modelling competence and the other aspects needed to handle mathematics in extra-mathematical situations.

Continuous development

In teacher training and from A-level and upwards, each competence is expected to be fully covered

Useful to highlight the maths content in vocational training

There is reason to make special mention of the many vocational courses, as well as short and medium-length tertiary education courses, which do not in themselves aim towards developing mathematics skills in their pupils or students, and which perhaps neither offer any teaching under the title "mathematics", but which, nevertheless, make use of or de facto develop mathematical competencies. In this report the training courses for gastronomists, electricians, and computer programmers are representative of this large class of training programmes. Our conclusion is that it will be of didactic and pedagogic use to elucidate the conditions whereby these courses make use of mathematical competencies, in the same way that it ought to be made clear exactly which competencies are the issue.

Competencies and content material

While the competencies in themselves are to be put on the agenda throughout the whole education system, the way they are manifested in the actual mathematical activities varies a lot from place to place, not least when it comes to the interplay with course content material, as dealt with in the report's part IV. There is also no doubt that the weighting of competencies varies according to educational stage and place.

Three types of overview and judgement

The same is virtually true when it comes to the three types of overview and judgement regarding mathematics as a discipline. Overview and judgement with regard to the actual use of mathematics in other subjects and practice areas, and the nature of mathematics as a subject area respectively, is in principle present from the very start of mathematics teaching, but, as is obvious, in ways are suited to the respective educational stages, as well as the competencies and the content material which is dealt with in the teaching. Overview and judgement regarding the historical development of mathematics is expected to form part of mathematics education at a later stage. Within the previously mentioned vocational courses, it hardly makes sense to try and develop overview and judgement with regard to mathematics as a subject area, besides that which is essential to make it clear to students that the vocation contains and assumes certain mathematical competencies.

2.2.3 c) How does one ensure progression and coherence in mathematics teaching throughout the education system?

Necessity of mental fellowship

If we assume for a moment that we know what we mean by progression and coherence in mathematics teaching, one prerequisite is essential if we are to ensure progression and coherence in mathematics teaching throughout the *entire* education system. This is that the actors in mathematics teaching - i.e. the central and local authorities, teachers, teacher trainers, teaching material producers, etc. - all mean the same subject when they think of mathematics, and not just its outer label, and that they see their task as being, each in his or her own way, to contribute to children and young people, throughout the education system, developing and building on to their mathematical competence. In other words, it is crucial that the actors in mathematics teaching all regard themselves as part of the same overarching teaching project and not as actors in a range of separate projects that do not have any particular link to each other, or that can directly be at cross purposes with each other. Creating such a mental fellowship is a multifaceted problem linked to structural and organisational conditions, to subject and professionally different traditions, to working conditions, and to salary conditions, etc. Many of these relationships do not have anything in particular to do with mathematics teaching, and therefore lie beyond the scope of this project at present.

A feeling of fellowship and cooperation is necessary for coherence

Recent examples of steps to create fellowship

The scope of the project does, however, allow for other contributions to the creation of the above-mentioned fellowship among people professionally engaged in mathematics teaching, regardless of the educational stage. In recent years there have been a number of measures taken which have, on the organisational level, contributed to bringing people involved in mathematics teaching together from most parts of the education system. Of mention in this regard are The Danish National Sub-Commission on Mathematical Instruction, The Forum for the Didactics of Mathematics, work involved in preparing for World Mathematical Year 2000 - in Denmark, the project “Mathematics and natural science in world class” dealing with primary and secondary schools in and around Copenhagen, and work involved in preparing ICME-10 - The Tenth International Congress on Mathematical Education, which is to take place in Copenhagen in 2004 - all recent examples of steps to create fellowship. While the KOM project can assuredly not take the honour for these steps, there is an indication of a fertile breeding ground for fellowship and cooperation. The KOM project itself, with its contacts to all layers in the education system, has been an important accelerator and catalyst in this direction.

Establishment of local contact agencies long overdue

In the recommendations in chapter 11 we have further suggested a series of measures with the aim of strengthening the mental and organisational fellowship in mathematics teaching, not least in connection with the transition between the large sections of the education system, i.e. from lower to upper secondary school, and from secondary school to tertiary education, where we have recommended the establishment of various local agencies to manage and promote contact between the different parts of the education system.

These improved possibilities for the establishment of meeting places for many types of mathematics teachers form a much needed, not to say vital, platform for the creation of awareness of mathematics teaching as a common project. The next question is how this can be translated into a common understanding of what the mutual project actually entails/ought to entail. The KOM project's suggestion is that the project ought to entail imparting to the recipients of mathematics teaching at all stages the mathematical competencies and forms of overview and judgement put forward and discussed in this report.

An understanding of coherence

The way in which progression and coherence are hereafter understood and described, are tied to these competencies, etc. Coherence involves the same competencies being sought after throughout the education system so that the subject mathematics does not just fall into a range of different subjects linked together by the same title. Coherence also involves the same ten mathematical subject content areas from which the curricular topics are to be chosen and made the object of teaching, from primary school through to introductory tertiary education. As emphasised many times in this report, the competencies manifest themselves differently at the different educational stages, just as a further realisation of the mathematical subject matter is obviously defined by this same educational stage.

Progression in the individual's mathematics mastery

Progression in mathematics mastery and teaching

When it comes to progression in the individual's *mastery* of mathematics, this comprises, on the one hand, growth in mathematical competence, overview and judgement, and, on the other hand, gaining ground when it comes to the mathematical subject areas in which the individual is capable of dealing and operating. The competencies are developed and practiced through the use of the content areas. In this project, we recommend that the detection and promotion of an individual's mathematical competence is realised by focusing on its growth in three dimensions, i.e. degree of coverage, radius of action and technical level, cf. afsnit 4.4.4 (page 72). If you view the development of the radius of action and technical level of all the competencies as one, you get a strong connection between the development of competence and the gaining of new ground in relation to content areas. Progression in mathematics *teaching* is thereby synonymous with the creation of progression in the individual pupils' mathematics mastery as described above. Progression in both mathematics mastery and mathematics teaching is part of the same issue within and along the various sections of the education system, and thereby also across the sectional and institutional borders contained in the education system.

Re the question of "how"

If progression is understood in this way, the main pedagogical question is how one can concretely arrange a mathematics course and orchestrate teaching that promotes it. Here it is necessary for the KOM project to be satisfied with a few general considerations, since a more in-depth treatment of this question for all the relevant educational stages, will far exceed the framework of the project.

Firstly, we are of the opinion that the utilisation, in all mathematics teaching, not least the day to day activities, of the competence thinking presented here, will in itself be the first step towards promoting progression in teaching merely by increasing awareness of it.

Secondly, the utilisation of this form of thinking in planning, arranging and implementing teaching will contribute to an orchestration of teaching and learning activities, with the explicit aim of developing the mathematical competencies of the individual. Without going into detail here, it can without further ado be maintained that the orchestration metaphor supposes the need for a rich diversity of such activities that can, each in their own way, in their own place, and at their own pace, contribute to the development or consolidation of a subset of the mathematical competencies.

Thirdly, the arrangement and construction of assessment forms and instruments which aim towards - and are suited to - detecting, characterising and evaluating mathematical competencies, serve to promote competence development in the individual, but also create input to adjust the actual teaching so that it can better promote progression.

2.2.4 d) How does one measure mathematical competencies?

Static and dynamic measurement in three dimensions

The starting point for measuring, i.e. detecting, characterising and evaluating a mathematical competence, is the oft-mentioned three dimensions of a competence: degree of coverage, radius of action and technical level. This means of measurement, further details of which appear in chapter 9, leads to both a static status report and a dynamic development description. In other words, a static or dynamic measurement of a competence involves measuring its degree of coverage, radius of action and technical level.

How?

Even though it has been made clear *what* is being measured, nothing has been said as to *how* this can/should be measured, i.e. under which conditions and with which instruments the measurement is carried out. Large parts of chapter 9 involve going through well-established, new and barely existing assessment forms and instruments respectively, which can either be used for a summative evaluation in the form of examinations at the end of a course, or which teachers can use for formative evaluation during mathematics teaching, in both cases with the mathematical competencies as the assessment object.

A broad spectrum of tools is necessary

One conclusion is that no single assessment tool is sufficient to capture the entire spectrum of competencies (or the three forms of overview and judgement). A broad spectrum of tools is necessary. It has been further concluded that most of the current assessment forms and instruments are actually suitable for the assessment of some of the competencies, in fact, if taken together, for rather a lot of them, if it is supposed that they are "reworked" and oriented towards specifically aiming for the relevant competence. This demands a not negligible, but nevertheless manageable, amount of adjustment and development to bring about. Finally, it has been concluded that even though one can go far towards evaluating the competencies with the adjustment of known assessment tools, there is still a need to develop new tools for this evaluation, not least in relation to the diverse spectra of teaching and activity forms that modern mathematics makes use of today.

2.2.5 e) What should be the content of up-to-date mathematics curricula?

Content as a combination of competencies, content areas and concrete content

One of the starting points of the KOM project was that the content of up-to-date mathematics teaching could not solely be characterised with the aid of the mathematical subject matter dealt with in mathematical teaching. The content also comprises the competencies and the forms of overview and judgement which are on the agenda for teaching and learning, as well as the actual concrete content, including that of an extra-mathematical nature, which is present in the objects, phenomena, situations, problems, questions, etc. which are dealt with in teaching.

This does not in any way imply that mathematical subject matter is of secondary importance in mathematics teaching. Each development and practice of mathematical competencies will, for example, take place while handling mathematical subject areas. Here, the task group has identified ten mathematical subject areas which create the stock of subject matter for most of mathematics teaching from the introductory school years right through to the first year or so of university mathematics courses. This does not prevent one from using or incorporating other subject areas if different contexts call for it. This is obviously the case in tertiary educational stages. We have chosen not to focus on this issue in the project.

Ten subject areas

The ten subject areas are *number areas*, *arithmetic*, *algebra*, *geometry*, *functions*, *calculus and real analysis*, *probability theory*, *statistics*, *discrete mathematics* and *optimisation*. In part IV of the report, a further characterisation, in general terms, of the content of these ten subject areas has been carried out, and similarly the connection between these subject areas, the competencies and the educational stage have been set out in a three dimensional array which, not least, serves to establish that there are, in fact, three entirely different axes at play here.

Classic titles and delimitations

The content areas have deliberately been chosen with rather classic titles and delimitations, one of the reasons being to create conformity with the way mathematics subject content architecture was once developed and constituted. We have decided not to go down the same road as various foreign (especially Anglo-Saxon) educational projects, i.e. by substituting traditional mathematics content areas with infringing thematic or phenomenological categories, e.g. “space”, “form and shape”, “change and growth”, “measurement”, etc. Such categories can certainly be valuable in particular, limited connections, e.g. in relation to a particular form of schooling, but are hardly as useful when one is trying to identify content areas which apply to many, and different, educational stages.

“Application tone” in the choice of content areas

Another guiding factor in the choice of content areas is that these areas reflect the fact that, to some or other extent, the majority of mathematics teaching in Denmark has - or ought to have - an application oriented aim. If mathematics is to be an up-to-date subject not addressed to a narrow circle of theoretical specialists, one needs to take this seriously. This has meant that, besides the basic mathematical content areas, we have emphasised those that, either directly in their theory building, are motivated by the question of application (for example probability theory, statistics and optimisation), or those which are central to the application of mathematics in other subjects or practice areas (e.g. arithmetic, aspects of geometry, aspects of functions, calculus and real analysis, discrete mathematics).

Subject areas as a basis for determining specific content

The fact that the ten content areas were chosen to cover the majority of mathematics teaching, does not mean that all subject areas, or all the mentioned aspects of a given subject area need be scheduled for all the relevant educational stages. We do not, for example, see any point in expecting algebra as a specific subject area to appear before the final years of lower secondary school (7th - 9th grade) or, needless to say, calculus and real analysis to feature in primary school or primary school teacher training.

The point is that the ten content areas are the basis for determining specific content for a given stage. In this regard, the task group had considered it to be crucial for mathematics as an up-to-date subject that there is not - at any educational stage - a too detailed list of the content components forming part of the teaching. The choice of material ought to - so as not to counteract the fundamental considerations and ideas of the current project - be with a great degree of aggregation. Among the forms of counteraction that can be feared with an exaggerated degree of detail in determining content, is an over packed syllabus and the difficulty of assigning responsibility to those involved in mathematics teaching.

Not too much detail in determining content

2.2.6 f) How does one ensure the ongoing development of mathematics as an education subject, and of mathematics teaching and learning?

Promoting the feeling of a common project is part of the answer

Part of the answer to this - very extensive - question has already been given in the answer to question c): One creates conditions, circumstances and organisational frameworks which promote the feeling that all the actors in mathematics teaching see themselves as partners in a common project, regardless of educational stage. This presupposes the creation of platforms and bastions, e.g. as suggested in chapter 11, which can ensure the exchange of experiences and ideas, possibilities and resources for development work, research work, etc.

Better connection between research and practice

Better connections between research and practice

It is particularly important here to create far better connections between the practice of mathematics teaching and research in mathematics education in the country. Such connections should not consist of informing mathematics teachers of research results and expecting them to implement these. Very seldom does research in mathematics education lead to directly transferable "positive" results.

Inspiration from other countries

The task is rather to create connections centred round concrete research and development work between some teachers and some researchers. Successful examples of this form of work community between practitioners and researchers has been seen in, e.g. France (the so-called IREMs), Italy (the so-called NRDMs) and sporadically in the USA. Besides being personally rewarding for those involved, the main benefit of such activities is the creation of "ripple effects" in both the relevant educational institutions and the research environment so that other teachers and researchers receive inspiration for their work from these activities.

Good possibilities for mathematics teachers' in-service training

In-service training and further education

In addition, it is obvious that copious and sufficient in-service training, further education possibilities, conference participation, etc. for teachers at all educational stages is essential both for the continuous development of mathematics as an education subject and for mathematics teaching as a whole as brought to light in this project. Regarding the latter, in chapter 11 we point out that, everything else being equal, it will be more viable as far as the realisation of the ideas in - for example - this project goes, to focus our efforts and resources on better basic education, as well as in-service training and further education for teachers, rather than on increasing the number of mathematics teaching hours in, for example, primary or lower secondary school, even though this would naturally give mathematics teaching more opportunities for development.

Development borne by teachers

An important reason for why one ought to, in many different ways, aim to improve and develop the working conditions of teachers generally, is that it is a precondition for teachers feeling respected and taken seriously by the political and administrative system. If this is not the case, it is impossible to ensure a continuous development of mathematics as an education subject and mathematics teaching since any such development is of necessity borne by its teachers.

Reforms that respect and involve mathematics teachers

Reforms exclusively dictated top down have no chance

The KOM project has invested a lot of effort into being a project in continuous dialogue and development contact with many types of actors in the arena of mathematics teaching throughout the entire education system. Besides the idea that this would contribute towards a better project than would otherwise be the case, this procedure reflects a conscious desire not to be a "top down project". It is, without a doubt, a fact that reforms which are solely implemented via a top down dictate have hardly any chance of working other than in a purely superficial way. If there is not a sufficient number of actors in the field of mathematics teaching who feel joint ownership of a reform, there are countless different ways that this reform can in reality come to nothing without it happening officially.

Ownership and participation of teachers are necessary conditions

The main thing is therefore that any future reform takes place in a manner and at a pace which earns the professional and processional respect, trust, ownership and active participation of teachers, unless it is one's intention to replace entire staffs. This is a necessary, not a sufficient condition for successful development of mathematics as an education subject and for mathematics teaching. This is not a question of placating teachers, but of being realistic, something that we recommend.

2.2.7 g) What does society demand of mathematics teaching?

In general, three demands

The demands of society on mathematics teaching are both explicit, implicit, and often simply unconscious. Briefly, society generally speaking has three different intentions with and, thereby, demands on mathematics teaching (cf. afsnit 10.2).

It has to contribute to the technological and socio-economic development of society as a whole. It has to supply individuals with the tools, qualifications and competencies necessary to help them deal with life's demands and challenges, both as private individuals, in their working lives, and as citizens. It has to contribute to society's political, ideological and cultural maintenance and development in Denmark in a democratic perspective.

Possible mutual conflicts

On an overarching plane, these intentions work perfectly well together and can even be seen as mutually reinforcing. However, when it comes to concretely realising them in mathematics teaching, they can work at cross-purposes, or even directly contradict each other. This is particularly the case if they have to be weighed up or if priorities have to be made between the different intentions and efforts. If one sector of society wants to, for example, train a workforce which can, within a narrow field, rapidly and proficiently carry out certain routine mathematical operations - what has by some been dubbed "the living calculator" - and if mathematics teaching is expected to be organised and carried out so that this task is central, it is highly unlikely that the same teaching will be able to take up the task of contributing to the education of citizens who can, knowledgably, reflectively and analytically, critically relate to the use and misuse of mathematics when it is of significance to decisions made in society.

Danish tradition of prioritising all three demands

However, in Denmark - there has, for a number of years, been a definite wish from society - represented by the political and administrative systems together with the large organisations on the labour market - that all three intentions be emphasised. On the explicit level, as for example in written syllabi, etc. the last two intentions have been particularly articulated recently, although one can just as easily prove that trade and industry and the political system emphasise the first-mentioned intention. This has, not least, been visible in the last couple of decades where the intake into education with a manifest mathematical component has been inadequate to meet the demands of society and, not least, of industry, cf. chapter 10.

Many expectations of maths teaching

This means that society demands that mathematics teaching be well functioning enough to engage the pupils so that they are motivated to take on the task of acquiring mathematical abilities, and choosing courses of education which comprise some or other dealing with mathematics. In this regard, society expects mathematics teaching not to generate too high a drop-out rate, either by too many dropping-out "prematurely" or by too many not being able to pass the tests and examinations. Furthermore, society demands that mathematics teaching is always effective enough to supply pupils and students with the qualifications and mathematical competencies sought by the labour market - without it being specified what these comprise. Recently, there have, e.g. been sections of society that have demanded measurable and documented skills of a much reduced nature as well as short and easily understood declarations of what pupils "know" and "can" do. At other times there has been more of a demand for understanding, insight, creativity and critical sense. Society demands, furthermore, that mathematics teaching supplies children and young people with abilities and skills so that they can hold their own in international comparisons - specifically in relation to the other Nordic countries - in a way that corresponds with national ambitions in combination with national identity. Society also demands that mathematics teaching contributes to the training of citizens who are active, sympathetic, independent, critical, and who can make up their own minds. Finally, society demands that the recipients of mathematics teaching thrive with, and are fond of the subject.

Unclear signals about mutual weighting

These many and different demands can be difficult to meet within one and the same course of mathematics. It is therefore obvious that it is impossible to present a set of simple directions as to how this can be achieved. The fact that no clear signal is given by society as to how these demands can be weighted in relation to each other, is naturally due to Danish society not being a unified organism with one overall head and command centre for the social body. There are different interests, accentuations and priorities which are alternately broken and then coexist in a society such as ours. The mathematics teaching system must therefore size up for itself signals, conditions and circumstances in an effort to make use of them in different types of frameworks and realisations.

Competence thinking a means to articulate intentions

It is the opinion of the task group that competence thinking can contribute, more clearly than before, to an articulation of the intentions with, demands of and priorities within mathematics teaching so that decisions and arrangements can be made and implemented in well thought out and understandable ways. In this connection there is even reason to believe that much of the fruitless and false opposition which can be found with regard to mathematics teaching can be removed or at least reduced with the incorporation of this way of thinking.

2.2.8 h) What would future mathematical teaching materials look like?

Not thoroughly treated in the KOM project

The task group has not been able to deal with this issue particularly thoroughly. It involves so many communicative, technological, media and commercial aspects that it would demand an entirely independent study. We can, however, say something about teaching materials on the basis of the KOM project.

Change is necessary

If the ideas and suggestions incorporated in the KOM project are to be carried out, changes have to be made to much of the material available for teaching. Current textbook are only suitable - used traditionally - for promoting the development of a limited part of the mathematical competencies emphasised in this report, solely because they are not oriented towards working in this way.

Aim to develop the spectrum of competencies

The development of mathematical competencies and the overview and judgement regarding mathematics as a discipline takes place via activities aimed specifically towards this development. For their part, such activities presuppose access to a rich source of very different teaching means which the teacher can use to orchestrate his or her teaching and which the pupils can access on their own while carrying out the activities.

The textual materials

As far as the textual materials go, there is need of textbook elements which systematically build up theoretical constructions; activity descriptions and stimulation material; collections of texts of different orientations and natures; articles on particular topics and issues; collections of texts comprising types of cases, e.g. examples of mathematical models, newspaper, magazine and periodical extracts, or solved mathematical problems; collections of problems ranging from routine exercises to challenging pure or applied mathematical problems; mathematical history books, reference and general works, etc.

ICT side

On the ICT side there is, on the one hand, talk of access to various forms of database, both of data collections, e.g. of a statistical nature, and of libraries (or collections) of mathematical objects like geometric figures and bodies, special functions and their properties, mathematical lexicons and dictionaries, addresses of relevant internet sites and their different contents;

and on the other hand, there is talk of processing software like CAS (Computer Algebra Systems), statistical and differential equation packages, modelling tools, etc. Dynamic visualisation software, interactive media and presentation tools, etc. will also be available to future mathematics teaching. Concrete material like blocks, bricks, rods, games, cards, puzzles, strings, cutting-out paper, programmable robots, etc. will still be included in the repertoire of things mathematics teaching can utilise.

Available to all?

The main question in this regard is to what extent such material will be available to the individual institutions, the individual teacher, the individual class and the individual pupil. At a time where it is difficult to obtain funding to supply teaching with up-to-date "old fashioned" textbooks, one can fear that the new teaching materials will only be available to a minority.

2.2.9 i) How can one, in Denmark, make use of international experiences with mathematics teaching and learning?

If one looks at mathematics teaching in other countries, there are two characteristic features.

Great differences in traditions and conditions

Firstly there are great differences in the traditions, frames, conditions and circumstances for mathematics teaching even in countries with a similar culture to our own. The great differences in teacher training and the organisational frameworks for mathematics teaching alone in, e.g. the Nordic countries, spring to mind.

Similarities in problems and perspectives

Secondly there is, despite the many differences, a great similarity in the problems, perspectives and discussions regarding mathematics teaching in different countries. This is a result, on the one hand, of common characteristics in social development in many countries, and, on the other hand, of there being an extensive international exchange of information, experiences, discussions and ideas about mathematics teaching, something that can easily take place due to the subject's universal character. The long-standing international and bilateral cooperation on the mathematical didactics and mathematics teaching, which is still being developed at many levels and which has, among other things, lead to a wealth of project cooperation and international conferences of many types, has, together with the implementation of large international comparative studies like TIMSS and PISA, contributed to a globalisation - for both good and bad - of the problems, doctrines and procedures of mathematics teaching (e.g. the mathematics part of the OECD's PISA project is strongly marked by competence thinking, partly because the KOM project's chairperson is a member of PISA's mathematics expert group).

Can learn from each other, but concrete "good ideas" can seldom be transferred

All this means that, on the one hand, there are many advantages to being aware of the mathematics teaching conditions in other countries and the problems, ideas and solution measures they have. This is both true of countries with a similar culture to ours and of the USA and, e.g. Asian countries like Japan, Korea and Singapore where there are significant and challenging experiences to understand and relate to. On the other hand, one ought to beware of recklessly transferring promising measures and "good ideas" to the Danish context, exactly because there is first a need to determine whether there is talk of nationally specific conditions and characteristics which are nonexistent in Denmark, or of more general matters which can be transferred with advantage to Danish conditions. Not only can one learn from the good ideas of other countries, but also from their mistakes. In this way, from an international perspective, mathematics teaching is a global laboratory which can be of advantage to Danish mathematics teaching to the extent that the laboratory results are carefully and cautiously examined.

In other words, there is every reason for Danish mathematics teaching to keep itself thoroughly, but critically, abreast with international conditions. It will, for example, be important to get to the bottom of the reasons for the Finnish education system being able to, to a much higher extent than the Danish, reduce the significance of parental education, and social and economic background on pupil performances in PISA2000 (Andersen et al.; 2001; OECD; 2001).

2.2.10 j) How should mathematics teaching be organised in the future?

Central question in continuation of the KOM project

In many respects, this is one of the most important questions with regard to the KOM project. The question deals with both the overarching organisation of the mathematics teaching system as a whole, including teacher training, and the "internal" organisation of mathematics teaching within a given educational sector, and - further - within a given teaching context, e.g. a class or a group.

General discussion of structure not part of the project

As is mentioned in the report, we have mainly desisted in presenting considerations and suggestions regarding the structure of mathematics teaching, because such structural questions are not limited to mathematics teaching alone, but often affect entire sectors in the education system.

We have thus, e.g. not devoted time to the structural or organisational questions in relation to university mathematics teaching. These questions can be answered in many different, though well-justified, ways and it would be a case of futile and damaging harmonisation to suggest these be replaced by a united answer. In the same way, we have not entered a discussion on the opportuneness of the existing senior secondary schools' three (or four) mathematics teaching stages, A (1 and 3), B and C, having the division of labour, division and role that it has. This is a consequence of an overarching way of thinking when it comes to the structuring of subjects in senior secondary school, which in this case needs to be dealt with on a correspondingly overarching level. We have, however, allowed ourselves in one respect to step beyond mathematics teaching's own territory and, finding it crucial, we therefore suggest - also out of consideration to mathematics teaching - that future senior secondary school reform leads to a "set menu secondary school" instead of the existing "buffet style senior secondary school".

Exception: "a set menu secondary school" rather than "a buffet secondary school"

As far as the organisation of mathematics teaching within a given educational section or a given context goes, as a result of the considerations presented in this project, the most important organisational principle is one of *structured diversity.* On the one hand, there is need of mathematics teaching making use of a great number of very different teaching forms and activities to promote pupils' development of mathematical competencies and overview and judgement regarding mathematics as a subject. Hence diversity. On the other hand, there is need of these teaching forms and activities being combined and ordered in a well thought out and planned way, both in relation to the individual "lesson" and in relation to shorter or longer periods of teaching, up to entire blocks, modules, semesters, years, or whatever is relevant, all with the aim of realising a type of teaching which enables the building up and development of mathematical competencies, overview and judgement. Hence the need for structure in the diversity.

Structured diversity as an organisational principle

What types of teaching forms and activities can be used in this regard, is such an extensive question that we have given up any pretence of dealing with it here. As is the case with assessment tools, there is both a need for a reorientation, adaptation and utilisation of the enormous number of known teaching forms and activities which are already in use in mathematics teaching to a greater or lesser extent, and a need for inventing, trying out and implementing new forms and activities.

Need for utilisation of old and development of new forms of activity

This will be one of the most important tasks if the KOM project is to be realised. Solely from this it is clear that the KOM project really first begins when it is finished.

The KOM project seriously begins when it is finished

Part II

Competencies as a means for describing mathematics curricula

3 What is the task?

3.1 Introduction

Inspiration from previous work

The task group's work on the application of mathematical competencies as a means of describing mathematics as an education subject, is extensively based on previous work by the group's chairperson, Mogens Niss (see e.g. Niss; 1999, 2000), who has also, by virtue of his membership of the OECD/PISA project's expert group on mathematics (see e.g. OECD; 1999) exerted influence on this project's use of competencies as a central part of its working foundation. Other members of the task group have, each in their own way, also employed corresponding ways of thinking in their work. Through its work the task group as such has, however, considerably elaborated on and developed the original ideas.

3.2 Tradition

Three components in the curriculum: purpose, syllabus and assessment

Traditionally in Denmark, a mathematics curriculum for any given stage is specified by means of three components:

a) The *purpose* of the teaching.

b) The *syllabus*, i.e. the mathematical content often presented as a list of topics, concepts, theories, methods and results to be covered (possibly supplemented with specific subject related goals).

c) The instruments of *assessment and testing* used to ascertain the extent to which students have mastered the prescribed syllabus (in relation to the subject specific goals mentioned under b).

In certain situations the purposes and goals are determined first, while the syllabus (and possible assessment instrument) is laid down afterwards so that the syllabus is seen to refer to these purposes and goals. Often though, this order is reversed, the syllabus being determined first, after which the purposes and goals are added as a sort of foreword to the curriculum. Depending on the stage and teaching format, internal (teacher controlled) assessment appears to be of secondary importance compared to the syllabus description, while examination assessment instruments and forms normally appear independently of the syllabus itself.

Assessment, tests and examinations play a major role

In practice this means that when it comes to the written syllabus (ministerial notices, study guides, local curricula, etc.) the curriculum plays the predominant role, while purposes, goals and assessment play a secondary role. On the other hand, when it comes to the everyday teaching reality, assessment and final examinations play a much more decisive role in determining the activities and attitudes of both students and teachers.

Problems with this type of curriculum

Serious objections can be raised to this way of specifying a curriculum. The following problems can be identified:

Difficult to clarify what maths ed. is

- On this basis it is *difficult to clarify* in general terms what mathematics education at any given level comprises, without relying on arguments that go round in circles (i.e. "mathematics education at this level consists of studying the following mathematical topics", which basically boils down to saying that mathematics education is about learning mathematics). Reference to the overarching goals and purposes of mathematics education merely helps explain why it should at all take place, and not what it actually comprises in the case that it is at all accepted as relevant.

Trivialisation and reduction of competence

- A syllabus based curriculum description easily leads to mathematical competence being identified with the mastery of the syllabus, i.e. proficiency in the skills and knowledge of the facts related to specific syllabus topics. In view of the fact that everyone professionally involved with the acquisition of mathematical knowledge would agree that a much more profound relationship exists than mere mastery of the syllabus, this identification *trivialises and reduces the notion of mathematical competence* and results in too low a level of ambition for teaching and learning.

Difficult to compare different types of maths ed.

- If we only have syllabus based curriculum specifications at our disposal in mathematics education, *any comparisons between mathematics* at different stages in the education system can only be made via comparisons of the different curricula. One is forced into saying that the difference between mathematics education in X and Y is that in X, curriculum X comprising such and such is used, while in Y, curriculum Y comprising such and such is used instead and the difference is that the following elements appear in curriculum X but not in Y and vice versa. Once again we have a comparison that is superficial and unable to capture the far more essential differences that can be found in the subject's complexity and in the different demands for in-depth study, etc. On the basis of this comparison, two versions of mathematics in upper secondary education would be regarded as equivalent if they covered the same syllabus, while any professional knows that there is a world of difference between the demands made on insight, activities and in-depth study.

Hard to characterise differences in level

In the same way *the difference in level* in mathematics education also depends on the curriculum, such that curriculum X is seen to be at a lower level than curriculum Y if all the elements in curriculum X are comprised in curriculum Y or form a logical notional prerequisite to curriculum Y. Once again anyone involved in this field knows that such a comparison of levels can be utterly misleading. Even though an understanding of natural numbers can be said to be a prerequisite for an understanding of rational numbers, it is easy to identify teaching in these two number domains where that describing natural numbers is of a vastly higher level than that related to rational numbers.

3.3 The task

In search of a common, general means of describing curricula

On the basis of the above, we are left with the following task: we wish to create a general means of specifying mathematics curricula, which, on a common basis pertaining to the majority of the education system, can adequately contribute to

- identifying and characterising, without going round in circles, what it means to *master* (i.e. know, understand, do and use) *mathematics* in itself and in different contexts irrespective of what specific mathematical content or syllabus is involved;
- *describing the development and progression* in mathematics teaching and learning both within and between different curricula;
- characterising different *levels of mastery* so as to describe the development and progression in the individual student's mathematical competence;
- *comparing* different mathematics curricula and different kinds of mathematics education in parallel or different stages of education in a way that goes beyond a mere comparison of curricula.

If this task can be solved, we will undoubtedly be in a better position than at present to discuss with people outside the world of mathematics education the very grounds for its existence, i.e. who and at which stage ought to be able to master mathematics at what level and why.

Competence can be this means

It is our opinion that the notion of mathematical competence can be instrumental in solving the above-mentioned problem.

4 A competence description of mathematical education

> " We should rather enquire who is better wise than who is more wise."[1]

4.1 Introduction

Competence as expertise

A person possessing competence within a field is someone able to master the essential aspects of that field effectively, incisively, and with an overview and certainty of judgement. Among the many various meanings ascribed to the notion of competence, *expertise* rather than the more wide spread *authorisation* has been chosen for this context.

Mathematical competencies in general

Translated into mathematical terms, this means that *mathematical competence* comprises having knowledge of, understanding, doing, using and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role. This obviously implies the presence of a variety of factual and procedural knowledge and concrete skills within the mathematical field, but these prerequisites are not sufficient in themselves to account for mathematical competence.

Definition of *a* mathematical competency

What then is *a* mathematical competence? It is an independent, relatively distinct major constituent in mathematical competence as described above. One could also say that *a mathematical competency is a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge.* The fact that such competencies are independent and relatively distinct does not imply that the different competencies are unrelated to each other or that they are so sharply defined that there is no overlap. Let us instead think of a competency as a "centre of gravity" in a "cluster" of things that are dense near the middle and sparser towards the edges, and which is partly interwoven with other clusters.

[1] Montaigne: *Of pedantisme*, Essays, 1st book, chapter 24.

This also means that any one competency cannot normally be acquired or mastered in isolation from the other competencies.

4.1.1 About the characterisation

Eight mathematical competencies make up mathematical competence

We have reached the stage where we can gainfully identify eight central mathematical competencies. They will be dealt with in detail in the following section. The competencies are, as stated above, mutually connected, but they nevertheless each have their own identity. None of the competencies can be reduced to the remaining ones. Keeping in mind all the above-mentioned exceptions caveats, it can be useful to think of the eight competencies as making up a set of well-defined dimensions, which together encompass mathematical competence. Quite obviously it is impossible to produce scientific documentation that this is theoretically and empirically the case. Rather, there is a pragmatic assertion that these competencies as a whole encompass and encapsulate the essence of mathematical competence. Whether or not this claim can be upheld in practice is first and foremost dependent on its ability to withstand clarifying considerations and concrete use.

In the characterisations of the individual competencies below, the word "ability" is sometimes used. It must be pointed out that this is merely a linguistic substantivation of "being able to", and by no means a psychological term aimed at referring to a person's mental personality traits general mental faculties.

Three types of overview and judgement

Besides the competencies themselves, three types of overviews and judgements pertaining to mathematics as a subject are also dealt with. As will become apparent, they are important for building up insight into the character of mathematics and its role in the world, and such insight does not automatically follow from mastery of the eight competencies.

4.1.2 Two groups of competencies

As intimated above, each of the competencies enables one, based on factual knowledge and concrete skills (which are not generally described in the actual characteristics of the competency) to carry out certain types of mathematical activities. The eight competencies have been divided into two groups, the first referring to *the ability to ask and answer questions in and with mathematics* and covering the first four competencies. The next refers to *the ability to deal with mathematical language and tools* and covers

Ability to ask and answer

Ability to deal with language and tools

the remaining four competencies.

If it is not over interpreted, a visual representation, as in figure 4.1, can be used to support an understanding of the competencies as well as giving one the possibility of remembering them.

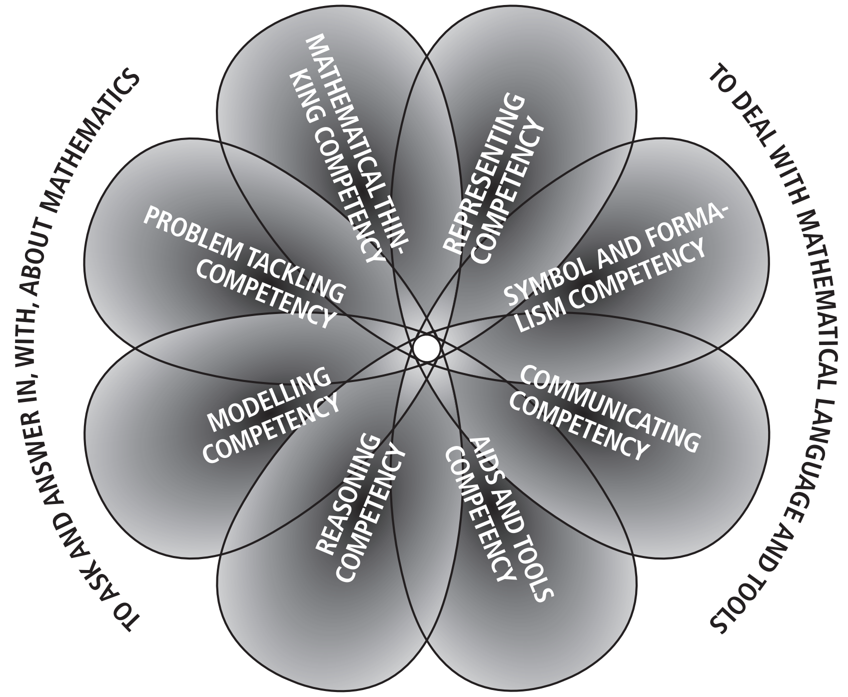

Figure 4.1 A visual representation of the eight mathematical competencies.

It is primarily on account of presentation considerations that we will operate with two groups of competencies here. From an overall point of view, the ability to cope with and in mathematics can be said to consist of exactly these two capacities or "super competencies", each of which, on closer examination, contains a set of specific competencies.

More precisely, the ability to ask and answer questions about and by means of mathematics is, to put it simply, the ability to (a) pose such questions and be aware of the kinds of answers available (mathematical thinking competency), (b) answer such questions in and by means of mathematics (problem tackling competency and modelling competency

The ingredients of being able to ask and answer respectively), as well as (c) the ability to understand, assess and produce arguments to solve mathematical questions (reasoning competency).

The ingredients of being able to handle language and tools

Similarly, being able to cope with mathematical language and tools implies (a) being able to deal with different representations of mathematical entities, phenomena and situations (representing competency), (b) being able to deal with the special symbolic and formulaic representations in mathematics (symbol and formalism competency), (c) being able to communicate in, with and about mathematics (communicating competency), as well as (d) being able to make use of and relate to the diverse technical aids for mathematical activity (aids and tools competency). These eight competencies will be characterised in more detail below.

The division should not be over-interpreted

Dividing the competencies into two groups should not be seen as an indicator that competencies from each of the two groups are less connected to each other than to those from the same group. Other possible layouts than that chosen could show just as close a relationship between two competencies from different groups. For example, having the competency to deal with mathematical symbols and formalisms is often a critical prerequisite for being able to answer questions, i.e. have the competency to pose and solve mathematical problems.

Aspects of competencies are not sub-competencies

In describing the individual competencies below, a range of aspects and components has been used. It is not intended that these aspects and components be seen as independent sub-competencies. They merely serve to describe what the competency is all about. To an even greater extent, the same applies to the examples which are merely present to illustrate the point. In this connection, more examples are used with those competencies which are not entirely self-explanatory than with the remaining ones.

4.2 Being able to ask and answer questions in and with mathematics

4.2.1 Mathematical thinking competency – mastering mathematical modes of thought

Characteristics

The nature of questions and answers

This competency comprises, first of all, *an awareness of* the types of questions that characterise mathematics, an ability to *pose such questions*, and *an insight into the types of answers* that can be expected. Of particular importance here is the mathematical endeavour to find necessary and sufficient conditions for the specific properties of an object.

Scope of the concepts

Furthermore, this competency comprises being able to *recognise*, *understand* and *deal with* the *scope of given mathematical concepts* (as well as their limitations) and their roots in different domains; extend the scope of a concept by *abstracting* some of its properties; *understand* the implications of *generalising* results; and be able to generalise such results to larger classes of objects.

Different types of mathematical statements

This competency also includes being able to *distinguish*, both passively and actively, between *different types of mathematical statements* and assertions including "conditional statements", "definitions", "theorems", "phenomenological statements" about single cases, and "conjectures" based on intuition or experience with special cases. Of particular importance is an understanding of the role played by explicit or implicit "quantifiers" in mathematical statements, not least when these are combined.

Comments

Not the content of questions and answers, but their nature which is central

The core of this competency is the actual nature of mathematical questions and answers. The issue here is not the factual content of the questions or answers themselves, nor the way this activity is carried out, or even whether the answers are correct or not. The procedure of attaining an answer is a core element in the mathematical problem tackling competency (dealt with below), while the correctness of the answer is the core of the mathematical reasoning competency (also dealt with below).

It is perhaps necessary to emphasise that in this context there is first and foremost talk of questions and matters of a mathematical nature, even if these may originate in conditions outside mathematics as a subject, i.e. from the natural world or other subject areas. The ability to translate these non-mathematical conditions into mathematical concepts is an independent competency which will be dealt with below under the mathematical modelling competency.

Exemplification

Typical questions

Typical questions in mathematics often follow a prototypical form along the lines of "Is there. . . ?", "How many. . . ?", "Is it possible that. . . ?", "Is the statement necessary or sufficient or both?", "Can the assumptions be weakened without affecting the conclusion?"

Typical answers

The answers typically take the form of "Yes, because. . . ", "No, because. . . ", "The statement is necessary but not sufficient as the following example shows. . . ”, “It depends on the situation since. . . ”, “It is an open question. . . ”, “If. . . then. . . ”, “We have that . . . if and only if . . . ”.

Concrete illustrations of *typical questions and answers* from different educational levels could, *for example*, be:

A: “How many different ways can the number 3 be expressed as the difference between two natural numbers?”

B: “Infinitely many.”

A: “If you played chess on a board with $11 \cdot 11$ squares, would there also be an equal number of black and white squares just like there is on a normal chessboard?”

B: “No, because the total number of squares is odd.”

A: “Is it true that amongst rectangles with a certain area you can obtain arbitrarily large circumferences?”

B: “Yes.”

A: “Is it also true that amongst rectangles with a certain circumference you can obtain arbitrarily large areas?”

B: “No. The greatest area can be obtained with a square with the given circumference.”

A: “How many different rows can you actually fill in on a soccer lottery coupon of thirteen matches?”

B: “3^{13}.”

A: “Is the range of a polynomial of degree 3 always the set of real numbers?”

B: “Yes.”

A: “Does the same apply for all polynomials?”

B: “No, not for those of even degree.”

A: “Are there any polynomials which have asymptotes?”

B: “Yes, but only polynomials of degree 1 (whose graphs are asymptotes themselves); no other polynomials have asymptotes.”

A: “Isn’t 0.99999 . . . the last number before 1?”

B: “No, 0.99999 . . . is equal to 1”.

A: “Which quadrangles have circumscribed circles?”

B: "That's not so easy to answer out of hand. A definite answer would require a slightly longer explanation."

A: "Can the trigonometric equation $\sin x = a$ be solved?"

B: "It partly depends on what a is, and partly on what is meant by 'solve'. If a lies in the closed interval from -1 to 1, there are approximate solutions with arbitrary accuracy, but for most values of a you cannot write an exact solution involving expressions with only fractions and roots."

4.2.2 Problem tackling competency – formulating and solving mathematical problems

Characteristics

This competency partly involves being able to *put forward*, i.e. detect, formulate, delimitate and specify different kinds of mathematical problems, "pure" as well as "applied", "open" as well as "closed", and partly being able to *solve* such mathematical problems in their already formulated form, whether posed by oneself or by others, and, if necessary or desirable, in different ways.

Put forward and solve problems

Comments

A (formulated) mathematical problem is a particular type of mathematical question, namely one where mathematical investigation is necessary to solve it. In a way, questions that can be answered by means of a (few) specific routine operations also fall under this definition of "problem". The types of questions that can be answered by activating routine skills are not included in the definition of mathematical problems in this context. The notion of a "mathematical problem" is therefore not absolute, but relative to the person faced with the problem. What may be a routine task for one person may be a problem for someone else and vice versa.

Notion of a "problem" is relative

Not every mathematical question poses a mathematical problem. For example, the question "What does it mean when there is a 0 in 406?" is not one that demands mathematical investigation, but is instead a question of understanding mathematical concepts and language. However, since many questions do actually pose a problem, the ability to formulate mathematical problems is intimately tied to being able to pose mathematical questions and being aware of the types of answers possible. In this regard, see the competency regarding mathematical thinking above. However, these two

Not every question poses a problem

Delimitation with regard to other competencies

competencies are by no means identical. Being able to solve a mathematical problem is not included in the competency of mathematical thinking. On the other hand, being able to distinguish between definitions and theorems in the mathematical thinking competency is not part of mathematical problem tackling competency in itself, even though this distinction can, in practice, be an important prerequisite for the competency.

The boundary between dealing with applied mathematical problems and active mathematical model building is fluid. The more important it is to take into consideration specific features of the elements comprised in the problem, the more there is talk of model building.

Being able to detect and formulate mathematical problems and being able to solve already formulated mathematical problems is not the same. It is quite possible to formulate mathematical problems without being able to solve them. One can even put forward a problem with an elementary concept apparatus without it being possible to reach the solution using this concept apparatus. In the same way, it is possible to be good at problem solving without being good at finding or formulating them.

Exemplification

Considering how central posing, formulating and solving problems are in mathematical activities at any stage, an endless number of examples of problems and their solutions can be given. Since solving problems is often a complicated and lengthy affair, there is a limit to how detailed the examples here will be. A few examples will have to suffice.

Only "small" examples

A: "Can you make a triangle out of three sides of arbitrary length?"

B: "No. If we have e.g. side lengths 3, 5, and 10, and start by placing the two short sides each at an endpoint of the long side, the two short sides will not reach each other. Therefore no triangle will be formed."

A: "Is there an equal number of black and white squares on a normal chessboard?"

B: "Yes, because in each row there are four black and four white squares."

A: "If you only had coins with a value of 3 and 5, which amounts would you be able to pay?"

B: "Clearly we can only talk of integer amounts where we obviously cannot pay the amounts 1, 2, 4 and 7. However, all other integer amounts are possible. Let us first ascertain that 6 can be achieved with two coins to the value of 3, 8 can be achieved with one of each of the coins, 9 with three coins to the value of 3, and 10 with two coins to the value of 5. Once we ascertain that every amount between 10 and 14 can be achieved, we have finished because all greater amounts can be achieved in the following way:

Every natural number, n, has a unique remainder among the numbers 0, 1, 2, 3, and 4 when divided by 5. This means that if n is at least 15, there will be exactly one integer $p > 2$ and a remainder r among the numbers 0, 1, 2, 3, 4, so that $n = 5p + r$. If we rewrite this, setting $n = 5(p - 2) + 10 + r$, then $p - 2$ will be a positive integer while $10 + r$ will be an integer from 10 to 14 inclusively. Since the amount $5(p - 2)$ can be paid with a coin to the value of 5 ($p - 2$ pieces) and the amount $10 + r$ lies between 10 and 14 and thus, by our assumption, can also be paid with the 3 piece and 5 piece coins, all amounts from and including 15 can be paid with these coins.

The fact that the amounts 11, 12, 13 and 14 can be paid can be seen by simple inspection ($11 = 2 \cdot 3 + 5$, $12 = 4 \cdot 3$, $13 = 3 + 2 \cdot 5$, $14 = 3 \cdot 3 + 5$). The problem is hereby solved."

A: "If you cut out and rolled a piece of cardboard so that it looked like an obliquely cut off circular cylinder (like the Planetarium in Copenhagen, or the Museum of Modern Art in San Francisco), what boundary curve should the one end of the cardboard have?"

B: "Let us suppose that the finished cylinder has a radius r and that the lowest point on the oblique plane "roof" is the distance m from the bottom plane, while the highest point is the distance M. Let us then place a three-dimensional coordinate system in the cylinder such that both the roof's low point and high point lie on the xz-plan and have the coordinates $(r, 0, m)$ and $(-r, 0, M)$ respectively, and such that the cylinder's axis is the z-axis.

The intersection of the axis with the roof will therefore have the coordinates $(0, 0, (m + M)/2)$, while $(M - m, 0, 2r)$ will be a normal vector to the roof plane. If we represent the typical point on the intersection curve between the cylinder and the roof by the coordinates $(r \cos t, r \sin t, h(t))\,, t \in [0, 2\pi[$, it is essentially the height function h that should be determined. This is done by requiring that the vector from the cylinder axis' intersection with the roof to the point on the boundary curve is perpendicular on the chosen normal vector to the roof plane, i.e.

$$\left(r\cos t, r\sin t, h(t) - \frac{m+M}{2}\right) \cdot (M-m, 0, 2r) = 0,$$

that is

$$(M-m)r\cos t + 2rh(t) - r(m+M) = 0.$$

From this we can determine h:

$$h(t) = \frac{m+M}{2} - \frac{M-m}{2}\cos t\ , \quad t \in [0, 2\pi[\,.$$

If we prefer to parameterise the height as a function of the arc length s (corresponding to the lower side of the piece of cardboard) instead of as a function of the angle of rotation t, we finally get ($s = r\,t$)

$$H(s) = h(s/r) = \frac{m+M}{2} - \frac{M-m}{2}\cos(s/r)\ , \quad s \in [0, 2\pi r[\,.$$

This solves the problem."

4.2.3 Modelling competency – being able to analyse and build mathematical models concerning other areas

Characteristics

Model analysis

This competency involves, on the one hand, being able to *analyse* the foundations and properties of existing models and being able to assess their range and validity. Belonging to this is the ability to "*de-mathematise*" (traits of) existing mathematical models, i.e. being able to decode and interpret model elements and results in terms of the real area or situation which they are supposed to model. On the other hand, the competency involves being able to *perform active modelling* in given contexts, i.e. mathematising and applying it to situations beyond mathematics itself.

Modelling

Elements in modelling

Active modelling contains a range of different elements. Firstly, there is the ability to *structure* the real area or situation that is to be modelled. Then comes being able to implement a *mathematisation* of this situation, i.e. translating the objects, relations, problem formulation, etc. into mathematical terms resulting in a mathematical model. Then one has to be able to *work with* the resulting model, including solving the mathematical problems that may arise as well as *validating* the completed model by assessing it both internally (in relation to the model's mathematical properties) and externally (in relation to the area or situation being modelled). Furthermore, there is the ability to *analyse the model critically* both in relation to its own usability and relevance, and in relation to possible alternative models, as well as to *communicate* with others about the model and its results. Finally, also included in active modelling is being able to *monitor* and *control* the entire modelling process.

Comments

Incorporation of the modelled situation

Even though in principle we are concerned with mathematical modelling each time mathematics is applied outside it's own domain, here we use the terms model and modelling in those situations where there is a non-evident cutting out of the modelled situation that implies decisions, assumptions, and the collection of information and data, etc.

Delimitation to problem tackling competency

Dealing with mathematics-laden problems which do not seriously require working with elements from reality, belong to the above-mentioned problem tackling competency. Those aspects of the modelling process that concentrate on working within the models are closely linked to the above-mentioned problem tackling competency. However, the modelling competency also consists of other elements which are not primarily of a mathematical nature, e.g. knowledge of non-mathematical facts and considerations as well as decisions regarding the model's purpose, suitability, relevance to the questions, etc.

Exemplification

Model analysis

With regard to *the analysis of existing (or proposed) models*, one can, e.g.

- consider a model that operates with exponential growth of the world's population in the period $1900 - 2000$ and compare it to the available population data.
- study the prescriptive body-mass-index model for under weight, normal weight, over weight and obesity in people (BMI = weight [kg]/(height)^2 [m^2]).

Modelling

When it comes to *active modelling*, one can e.g. set up models to work out the below-mentioned challenges. In all cases it is necessary to carry out delimitations, make assumptions, and collect data before completing the modelling.

- A study of what the floor plan of a house can be if its area is 120 m^2.
- A study of how expensive it is to use a cell phone.

- "By how much is one euro, earned by a wage earner, effectively taxed if VAT and duties, etc. are also taken into account?"
- A determination of the optimal shape of a tin can.
- An evaluation of how much of the Danish energy consumption can be covered by windmills, and how many windmills this will require.
- "How are the number of AIDS cases developing in Denmark?"
- "Is it possible for 35 to be the average age of a population at the same time as at least 40% of the population is 60 or above?"

4.2.4 Reasoning competency – being able to reason mathematically

Characteristics

Follow and assess reasoning

This competency consists of, on the one hand, the ability to *follow* and *assess mathematic reasoning*, i.e. a chain of argument put forward by others, in writing or orally, in support of a claim. It is especially about knowing and *understanding* what a mathematical *proof* is and how this differs from other forms of mathematical reasoning, e.g. heuristics based on intuition or on special cases, and it is also about understanding how and when mathematical reasoning actually constitutes a proof, and when it does not. This includes an understanding of the logic behind a *counter example*. Furthermore, the competency comprises being able to *uncover the basic ideas in a mathematical proof*, including distinguishing between main lines of argument and details, between ideas and technicalities.

Understanding what a proof is

Devise and carry out

On the other hand, it consists of the ability to *devise and carry out* informal and *formal arguments* (on the basis of intuition) and hereby transform heuristic reasoning to actual (valid) proofs.

Comments

Not just justification of theorems, also of answers to questions and problems

Many regard mathematical argumentation, even mathematical reasoning in general, as first and foremost a justification of mathematical theorems, more often than not as a sheer reproduction of finished proofs. The reasoning competency includes this aspect, but goes further in that it is constantly relevant in assessing the validity of mathematical claims, including convincing yourself and others of the possible validity of such. It can refer to both the correctness of rules and the theorems, but also to establishing whether a given answer to the question, assignment or problem is correct and adequate. By this inclusion of the justification of answers and solutions, the reasoning competency is closely linked to both the problem tackling and modelling competencies. It comprises the "legal" side of these competencies.

Delimitation with regard to other competencies

In principle, the ability to carry out pure routine operations, e.g. calculations, may be said to fall within the reasoning competency since it involves the justification of a calculation's result. However, what one person may regard as a routine operation, another may regard as an insurmountable problem. The actual carrying out of these operations is therefore included under the below-mentioned competency dealing with mathematical symbols and formalisms, while being able to activate the operation belongs under the reasoning competency if this activation demands creativity, analysis or overview.

Exemplification

Follow and assess arguments

Some examples of the ability *to follow and assess a mathematical argument* can be mentioned:

A: "When you square a number the result is always greater than the original number. This applies to all the infinitely many integers, so it must also apply to all other numbers."

B: "No, first of all this assertion is wrong because e.g. $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2}$. Secondly, it is not possible to transfer all the properties of the set of integers to properties of a greater set of number, e.g. the rational numbers."

A: "Each odd number is composite. For if n is odd, then $n = ((n+1)/2)^2 - ((n-1)/2)^2$ where both $(n+1)/2 = k$ and $(n-1)/2 = m$ are integers (since n is odd). However, since $k^2 - m^2 = (k-m)(k+m)$, n is composite."

B: "The argument is incorrect because $k - m = 1$, so the claim is merely that $n = 1 \cdot n$ which does not make n a composite number."

- The proof of the irrationality of $\sqrt{2}$.

Know and understand what a proof is (not)

To illustrate what it means *to know and understand what a proof is (not)*, the following suggested proof is proposed:

A: "If f has the limit b for x tending to a, and g has the limit c for y tending to b, then the composite function $g \circ f$ has the limit c when x tends to a. Because when x is tending to a, per assumption $f(x)$ is tending to b which, by further assumption on g, leads to $g(f(x))$ tending to c, which is exactly the assertion."

B: “This is not a valid proof because the treatment of the concept of limit is too loose and unfocussed. The claim that the assertion is ‘proved’ is in fact incorrect unless g satisfies further assumptions. The problem is that the range of f can be contained in a part of the definition set for g in such a way that the composite function cannot approach c. An example would be with f and g defined by $f(x) = 0$ for all x, and $g(0) = 1$, but otherwise $g(y) = 0$. With $a = 0$, $f(x)$ will then tend to $b = 0$ for x tending to a. Furthermore $g(y)$ will tend to $c = 0$ for y tending to $b(= 0)$. However, $g(f(x)) = 1$ for all x. It is therefore not true that $g \circ f$ has the limit $c(= 0)$ for x tending to a.”

Reveal the basic ideas in a proof

Revealing *the basic ideas in a (correct) proof* can be illustrated as follows:

- “Gauss’ proof that $1 + 2 + \ldots + n = n(n + 1)/2$ is based on the idea that you can determine the sum by means of an equation. By adding the number $n + \ldots + 2 + 1$ to the left hand side, you get, on the one hand, the sum in question twice, and on the other hand, n parentheses each consisting of two numbers, the sum of which is $n+1$. Using this to express the sum is subsequently a standard technique (multiplication of n by $n + 1$ followed by solving a simple equation.)

  This proof has an advantage over the usual proof by induction, which has the weakness that the sum has to be ’guessed’, which is not, however, a prerequisite in Gauss’ proof where the sum is actually determined.”

Devising independent proofs

Finally, *devising independent proofs*, from heuristic to formal proofs, can be illustrated by the following example:

- “7 must be the most frequently occurring sum of the pips in a two-dice throw, because 7 is the number amongst the possible sums which can be attained in the largest number of ways. To formulate this more precisely: If the outcomes of two dice are assumed to be mutually independent, then the combined set of possible outcomes will be 36 combinations of pips since each dice can give 6 different outcomes. This can be illustrated in a square table. Of these 36 combinations, the sum 7 can be achieved in 6 ways, i.e. $1 + 6$, $2 + 5$, $3 + 4$,$4 + 3$, $5+2$, $6+1$ (the numbers in each sum representing the number of pips on the first and second dice respectively). This corresponds to the number of ways 7 can be divided as the sum of two natural numbers. None of the other possible sums can even be achieved in 6 ways. For sums less than 7, the number of partitions will obviously be less than the number of partitions for 7. For sums from 8 to 12, only some of

> the possible partitions correspond to those possible with the throw of the dice, i.e. $2+6$, $3+5$, $4+4$, $5+3$, $6+2$ in the case of 8, and so on until 12 which can only be obtained by $6+6$."

One could also have mentioned a "repair" of the assumptions and arguments in the above "proof" regarding composite functions. If, for example, g is assumed to be continuous at b, the assertion is correct and the sketch of the proof can be extended and formulated more explicitly into a correct proof.

4.3 Being able to handle mathematical language and tools

4.3.1 Representing competency – being able to handle different representations of mathematical entities

Characteristics

Understand and utilise different representations

This competency comprises being able to, on the one hand, *understand* (i.e. decode, interpret, distinguish between) and *utilise* different kinds of representations of mathematical objects, phenomena, problems or situations (including symbolic, especially algebraic, visual, geometric, graphic, diagrammatic, tabular or verbal representations, but also concrete representations by means of material objects) and, on the other hand, being able to understand the reciprocal *relations* between different representational forms of the same entity, as well as knowing about their strengths and weaknesses including the loss or increase of information. It also comprises being able to *choose* and *switch* between different representational forms for any given entity or phenomenon, depending on the situation and purpose.

Choose and switch between representations

Comments

Delimitation with regard to other competencies

Symbolic representations are of special importance in mathematics. There is therefore a close connection between the present competency and the following symbol and formalism competency, which focuses on, among others, the "rules" for using mathematical symbols, but also deals with parts of mathematic formalism which are not connected to symbolic representation. Since representing mathematical entities and phenomena is closely related to communicating in, with and about mathematics, there is a clear connection to the communicating competency which will be dealt with later.

Representation with the aid of material objects creates the link to the last of the eight competencies, i.e. the aids and tools competency.

Exemplification

The examples below have been taken from different educational levels.

Elementary representations are also representations

An elementary example of this competency would be the ability to represent a natural number with dots or bricks of the same shape or size, or rewrite a number in the position system with the help of Cuisenaire rods, centicubes, abacuses or similar, or with the help of symbols in Hindu-Arabic notation, roman numerals, cuneiform writing, etc. as well as with verbal representations (e.g. five million, one hundred and twenty-six thousand, nine hundred and thirty-seven).

Another elementary example would be time expressions where analogue and digital watches are equivalent, but with completely different representations of the time. A further example would be understanding and utilising different representations of the object π and the relations between them. The representations could, for example, be

- the symbol π.
- an infinite decimal expansion $3,14159265\ldots$.
- a rational approximation (with its concomitant inexactness) by e.g. the fractions 22/7 or 223/71.
- geometrically as the circumference of a circle with a diameter of 1.
- the sum of the infinite series $4 - 4/3 + 4/5 - 4/7 + 4/9 - 4/11 + \ldots$.

Another example is the concept of linear function (in the school mathematics sense) which can be represented

- as an explicit algebraic expression, e.g. $f(x) = 3x - 7$.
- algebraically as the set of solutions to an equation, e.g. $2y-6x+14 = 0$.
- as a parameterised point set in a coordinate system, e.g. $\{(x, y)|x = t,\ y = 3t - 7, t \in \mathbb{R}\}$.
- by drawing a graph in a coordinate system.
- as a geometric object, e.g. the straight line in the plane that passes through the points $(2, -1)$ and $(0, -7)$.
- by a table of corresponding values of x and y (with its concomitant loss of information if it is not known that the table represents a linear function, or with its concomitant excess of information if it is known

that the function is linear and the table contains more than two different corresponding pairs).

A further example would be an ellipse, which can be represented

- geometrically as a cross-section of a cone or a cylinder.
- as the shadow of a sphere.
- as the locus OF all the points whose distance to two given points has a constant sum.
- as the set of point pairs in a coordinate system which satisfies the equation $(x/a)^2 + (y/b)^2 = 1$ $(a, b \neq 0)$.

For all these examples, the representing competency is about understanding the representations, being clear about the connections between them, including information loss and increase during the transfer from one to another, and of the strengths and weaknesses of the individual representations, and being in a position to choose (between) one or more of them.

4.3.2 Symbol and formalism competency – being able to handle symbol and formal mathematical language

Characteristics

This competency comprises, on the one hand, being able to *decode* symbol and formal language; being able to *translate* back and forth between the mathematic symbol language and natural language; and being able to *treat and utilise* symbolic statements and expressions, including formulas. It also, on the other hand, comprises *having an insight into* the nature of the "rules" of formal mathematical systems (typically axiomatic theories).

Decode, translate and handle symbolic statements

Formal mathematical systems

Comments

This competency differs from the above-mentioned representing competency with which it is closely related, in that it focuses on the character, status and meaning of the symbols as well as the way these are used, including the rules for such usage. Added to this is the fact that it is also about the handling of formal mathematical systems whether or not these have a symbolic form.

Delimitation with regard to the representing competency

Mathematical symbols are not just the special symbols of advanced mathematics, but also number symbols and the basic signs used in arithmetic. Similarly, this handling of symbols is not just about "algebraic manipulations", "calculus" and the like, but also the formal side of elementary arithmetic.

Exemplification

Also elementary symbols and forms

On the elementary plane, this competency can be illustrated by e.g. numbers and the handling of numbers. For example

- an understanding that 406 stands for four hundreds, no tens and 6 ones.
- that it is not permissible to write $6 + \cdot 5$ or $6 - -3$ (while $6 + +3$ is not meaningless, but bad syntax).
- that $5 \cdot (3 + 4)$ is not the same as $5 \cdot 3 + 4$.
- that $4 < 7$ is a statement which should be read as "4 is less than 7".

At a higher level, it could be an understanding of

- $\{(x, y)|x = t, y = 3t - 7, t \in \mathbb{R}\}$ denotes the set of all real pairs of numbers where the first coordinate assumes an arbitrary real value, while the second coordinate is bound to be exactly three times this value minus 7.
- the content of what has been called "the world's most beautiful formula": $e^{i\pi} + 1 = 0$.

Handling symbol and formal language

Being able to decode symbol and formal language can be exemplified by an ability to say that the above-mentioned set describes the straight line in a right-angled coordinate system which intersects the y-axis at -7 and has a slope of 3.

On the other hand, the ability to, e.g. write the collection of all natural numbers that, divided by 5, give a remainder of 4, in symbol language as $\{p \in \mathbb{N}|\exists k \in \mathbb{N} : p = 5k + 4\}$ is an example of a translation from natural language to symbol language. The same applies to $(a + b)(a - b) = a^2 - b^2$ as a translation into symbols of the previously "popular" rule that "the sum of two numbers times their difference, equals the difference between their squares".

When it comes to *the handling of symbol language and formalism*, there are countless examples, e.g.

- being able to handle manipulations like $3x^3 - 2x^2 - x = x(3x^2 - 2x - 1) = x(x - 1)(3x + 1)$ where the last step also involves solving the quadratic equation $3x^2 - 2x - 1 = 0$.
- immediately being able to see that $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{1}{5}$.
- being able to rearrange $P(A|B) = P(A \cap B)/P(B) = P(B|A)P(A)/P(B)$ for events A and B, where $P(A), P(B) \neq 0$.

- being able to conclude that the equation $x\,(y+z) = x\,y+z$ is satisfied for all tuples of the form $(x, y, 0)$ for arbitrary x and y or the form $(1, y, z)$ for arbitrary y and z, but not by others.

Finally, *the ability to handle formal mathematical systems* can be illustrated by an insight into what it means to make a geometric construction based on Euclid's axioms, including understanding in which sense it is impossible to trisect an angle using a compass and a ruler.

Axiomatic Euclidean geometry can also serve as an example of mathematical formalism which does not have to be conveyed in symbolic form.

4.3.3 Communicating competency – being able to communicate in, with, and about mathematics

Characteristics

This competency consists of, on the one hand, being able to *study and interpret* others' written, oral or visual mathematical expressions or "texts", and, on the other hand, being able to *express oneself* in different ways and with different levels of theoretical or technical precision about mathematical matters, either written, oral or visual, to different types of audiences.

Understand and interpret expressions and texts

Express oneself about mathematics

Comments

Since written, oral or visual communication *in* and *with* mathematics makes use of diverse forms (and media) of representation, this competency is closely linked to the above-mentioned representing competency. Such communication often makes use of mathematic symbols and terms, which serves to highlight the link to the symbol and formalism competency.

delimitations with regard to other competencies

The communicating competency, however, goes further than the others since the communication happens between the sender and receiver, and their situations, backgrounds and prerequisites need to be taken into account for communication in the same way that purpose, message and media are.

There is also reason to note that communication *about* mathematics does not necessarily need to include specific mathematical forms of representation.

Exemplification

Any written or oral presentation of a mathematical activity can serve to exemplify the *expression side* of the communicating competency. For example, being able to state a mathematical consideration, e.g. solving an

Many forms of expression

Decoding and interpreting

exercise or a problem, falls under this. Similarly, decoding and interpreting mathematical presentations, e.g. in a textbook or in a lecture, serve as examples of what it is to be at the *receiving* end of the communicating competency.

The ability to enter into discussions with others about mathematical topics also demands the communicating competency. One could, for example imagine the following dialogue between two students in the final grade of lower secondary or in upper secondary school:

E_1: “We’re always told that we may not divide by 0. Why not; is it just a rule or what?”

E_2: “Yes, I suppose it is.”

E_1: “But where does it come from? There must be a reason.”

E_2: “Well, let’s try and see what division is all about. If we divided a by 0, we should get the number that, multiplied by 0 gives us a. But a number multiplied by 0 gives us 0 and not a. So the division just doesn’t work. Maybe that’s why it is forbidden.”

E_1: “Hey, if a is 0, then it does work. So you could multiply 0 by, for example 1 and get the right answer, namely 0.”

E_2: “Well yes, but we could also have multiplied by 10^{10} and still got 0. So then $0/0$ would be 10^{10}.”

E_1: “Yes, we could multiply by anything and still get the right answer.”

E_2: “But then you may as well say that division doesn’t give any definite result, if we can get anything we like out of it. And doesn’t that make it impossible?”

E_1: “OK, so it’s forbidden to divide by 0 because we’d never get any definite result. In most cases we would get absolutely nothing out of it, and if $a = 0$, we can get anything.”

4.3.4 Aids and tools competency – being able to make use of and relate to the aids and tools of mathematics (incl. ICT)

Characteristics

Knowing possibilities and limitations of, and being able to use aids and tools

This competency consists of, on the one hand, *having knowledge of* the existence and properties of the diverse forms of relevant tools used in mathematics and having an insight into their *possibilities and limitations* in different sorts of contexts, and, on the other hand, being able to reflectively *use* such aids.

Comments

Mathematics has always made use of diverse technical aids, both to represent and maintain mathematical entities and phenomena, and to deal with them, e.g. in relation to measurements and calculations. This is not just a reference to ICT, i.e. calculators and computers (including arithmetic programmes, graphic programmes, computer algebra and spreadsheets), but also to tables, slide rules, abacuses, rulers, compasses, protractors, logarithmic and normal distribution paper, etc. The competency is about being able to deal with and relate to such aids.

Not just ICT

Since each of these aids involves one or more types of mathematical representation, the aids and tools competency is closely linked to the representing competency. Furthermore, since using certain aids often involves submitting to rather definite "rules" and rests on particular mathematical assumptions, the aids and tools competency is also linked to the symbol and formalism competency.

Delimitations with regard to other competencies

Exemplification

There is no limit to the number of examples that can be given to show the use of aids in mathematics. In the lower grades, it would involve the ability to use concrete materials to support conceptualisation, the study of connections and patterns, the verification of hypotheses, the teaching of basic skills, etc. Geoboards, centicubes or other block, brick or rod systems, abacuses, geometric templates, Spiro graphs, rulers, compasses, protractors, dice, specially lined paper, cardboard for folding or cutting out all belong to this category.

Infinite number of examples

Mention can also be made of the reflective use of calculators and computers, together with ICT software like LOGO, Cabri-Géomètre, spreadsheets, MathCad, Maple, etc. used for calculations as well as graphic representations, empirical studies and visualisations, etc.

4.4 Five comments

4.4.1 About the relationship between competencies

As has already been made apparent, many of the competencies are closely related to each other. This is true of e.g. the representing competency,

Links and differences: Representing, symbol and formalism, and communicating competencies the symbol and formalism competency as well as the communicating competency which are, furthermore, also in the same group together. Nevertheless, they emphasise different aspects. In the representing competency, the emphasis is on the *actual representation* of a mathematical entity or phenomenon, and the different possibilities there are when choosing a representation. One could say that representation was a semantic activity. Some of these representations can be symbolic, but they do not have to be. The symbol and formalism competency, on the other hand, accentuates the *"rules"* when dealing with symbolic language and formal systems (axiomatic theories), and can be regarded as a mainly syntactic activity. Finally, the focus in the communicating competency is on how one generally communicates in, with and about mathematics. Representations, together with symbols and formalisms, each have their place here, but there is much more at stake, not least the incorporation of the sender and the receiver of the communication.

Thinking, reasoning and problem tackling competencies

Similarly, the thinking, reasoning and problem tackling competencies are closely related, but the emphases are once again different. In the thinking competency, the weight is on the *questions* mathematics deals with, while the problem tackling competency focuses on the *strategies* one can use to answer the questions, and the reasoning competency is about the *justification* of assertions, including whether a given approach or procedure can actually produce a correct solution to a problem arising from a mathematical question. Naturally, the representing competency and the symbol and formalism competency are also involved here, but rather as tools than as the heart of the matter.

Modelling, problem tackling, and representation competencies

Finally, among the many relationships between the competencies, the connection between the modelling, problem tackling, and representation competencies can be highlighted. For instance, both the representation and problem tackling competencies are vital for carrying out the modelling competency. However, the competencies have different focuses. We have already mentioned the different emphases in the representation and problem tackling competencies. In the modelling competency, it is the use of mathematics to understand and treat matters outside mathematics itself that is the focus.

4.4.2 About the dual nature of competency characteristics

An "investigative" and a "productive" side

As is apparent in the characterisations, all the competencies have both an "investigative" and a "productive" side. The "productive" side of a competency consists of being able to, by oneself, *carry out* the processes covered by the competency. The "investigative" side comprises an *understanding, analysis and critical assessment* of the processes already carried out and the products thereof.

Competencies are *behavioural*, but not behaviouristic

It ought to be emphasised that the investigative element (reflection, analysis and assessment) is active in nature, even though this action takes place on the mental plane. It is merely a different type of activity than that involved in carrying out the current processes which lead to products which are, in some or other form, "visible". Both the investigative and the productive side of the competencies concern mental or physical activities which are *behavioural* in nature. The focus is on the person possessing the competency being able to *carry out* the relevant activities.

The fact that the competencies are behavioural in nature, definitely does not mean that they should be understood behaviouristically, i.e. that they can necessarily be read from the outside as clearly limited and well defined activities which can be understood as an individual's response to given stimuli.

4.4.3 About intuition and creativity as common traits in the competencies

Some may bemoan the lack of certain mathematical competencies in the list. These may be the use of mathematical intuition or mathematical creativity. In the ideas which form the basis for the current presentation, neither intuition nor creativity is regarded as an independent competency, but rather as a combinations of traits in the competencies mentioned. Thus "intuition" can be found in the thinking, reasoning, problem tackling and representation competencies.

Intuition

Creativity

"Creativity" can virtually be regarded as the essence of all the productive sides of the competencies. In other words, creativity is being able to pose good external or internal mathematical questions and formulate the problems arising therefrom; thereafter, with the aid of intuition, abstraction, generalisation, choice of suitable representations, symbol and formalism handling, as well as the possible use of aids, to solve these problems; thereafter to present correct and complete arguments (proofs) for the suggested solutions to actually work; and finally to be able to communicate both the process and the product in a clear and convincing manner to a target audience.

4.4.4 The three dimensions with which a competency is mastered

It appears meaningful to work with the idea that a person's mastery of a competency has three dimensions which we, lacking any better expressions, will call 'degree of coverage', 'radius of action' and 'technical level'.

Degree of coverage

Degree of coverage concerns the aspects of a competency

The *degree of coverage* a person has of a competency is used to indicate the extent to which the person masters those *aspects* which characterise the competency, i.e. how many of these aspects the person can activate in the different situations available, and to what extent independent activation takes place.

For example, a person who is often able to understand the proofs of others, but seldom able to think out or carry out satisfactory proofs him- or herself, has less of a degree of coverage of the reasoning competency than the person who can more often do both. Similarly, a person who is able to, clearly and in ordinary language, state the thought processes behind the solution to a mathematical problem and who is also able to state the solution in technical terms, has a greater degree of coverage of the communicating competency than someone who is only able to do the latter.

Radius of action

Radius of action concerns contexts and situations

A person's *radius of action* of a competency is the spectrum of *contexts and situations* in which the person can activate the competency. This is first and foremost in relation to mathematical contexts and situations (both internal mathematical ones and applied topics), but also in relation to contexts and situations determined by problem formulations and challenges.

If a person's problem tackling competency, for example, could be activated successfully both within arithmetic, algebra, geometry and probability theory, he or she has a greater radius of action than a person who can only successfully activate it in arithmetic and algebra. Similarly, a person who can apply mathematics to his or her daily economy, cooking or D.I.Y constructions, has a greater radius of action to his or her modelling competency than the person who can only apply it while shopping in a supermarket.

Technical level

Technical level concerns the substance in the situation

The *technical level* of a person's competency is determined by how *conceptually and technically advanced* the entities and tools are that can be activated in the relevant competency.

A person who is only able to calculate correctly in situations involving two- or three-digit numbers, has a lower technical level of his/her symbol and formalism competency than that person who can also cope with multi-digit numbers or decimals. The person who can sketch graphs for real functions of one variable, but not for real functions of two variables, has a representing competency at a lower technical level than the person who can attain both.

The dimensions as partial non-quantitative ordering principles

Neither a total nor quantifiable order

It is important to emphasise that even though we have chosen terms which imply the possibility of simple quantitative measurement, there is no such assumption in the following considerations. The only thing we are implying in this regard is that each of the dimensions allows us some kind of ordering, i.e. that one version of a competency can, in relation to a specific dimension, be more or less comprehensive than another version of the same competency. Since this is only a partial ordering, it may well happen that two arbitrary versions of the same competency cannot be compared in this way.

It is, therefore, meaningless to allege that the level of coverage of the problem tackling competency in a person who can only solve problems within algebra, geometry and probability theory, is less than in a person who can solve problems within probability theory, functions, calculus of infinitesimals and optimisation. Similarly, it is also meaningless to compare the technical level of the symbol and formalism competency in a person who is a master at handling expressions within trigonometry, with the technical level of a person who is a master at calculations in probability distributions.

4.4.5 The competencies as subject specific, but content general

General and comprehensive, but mathematics specific

The fifth and final comment is perhaps the most important: Each of the eight competencies is of a general and comprehensive nature. They make sense for (and are therefore independent of) each *concrete* piece of mathematical subject matter, just as they make sense for each stage of education. However, they are also specific to mathematics. It may be possible to formulate similar competencies for other subject areas, perhaps even using many of the same words, but the elements in the mathematics competencies all refer to matters of a specific mathematical nature.

4.5 Overview and judgement regarding mathematics as a subject

The above-mentioned competencies are all characterised by being action orientated in that they are directed towards handling different types of challenging mathematical situations. Besides the mathematical mastery we have tried to capture with these competencies, we have also found it desirable to operate with types of "active insights" into the nature and role of mathematics in the world, and which are not directly behavioural in nature. While these insights enable the person mastering them to have a set of views allowing him or her *overview and judgement of the relations between mathematics and in conditions and chances in nature, society and culture*, and thereby can also be said to have a type of competency character, though directed towards mathematics as a subject area rather than towards mathematical situations, we decline to call them competencies so as to avoid any confusing mix up with the competencies already dealt with.

"Active insights" regarding the nature and role of mathematics in the world

Object is mathematics as a whole

Having an overview and being able to exercise judgement are of significant importance for the creation of a balanced picture of mathematics, even though this is not behavioural in any simplistic way. The point is that the object of this judgement is mathematics as a whole and not specific mathematical situations or problems.

Three types of overview and judgement

It is all about having, based on knowledge and ability, an overview and sense of judgement when it comes to a) the actual application of mathematics in other subject and practice areas, b) the historical development of mathematics, both internally and from a social point of view, and c) the nature of mathematics as a subject area.

Points at infinity

You may possibly think that choosing terms such as "overview" and "judgement" is rather pretentious, not least since we insist that they are relevant on all levels of education and teaching. However, since one would never speak of "complete overview" or "total judgement" which are virtually "points at infinity", the concepts need to be understood relatively. In this way they are not different from the eight competencies, which cannot be mastered completely either. The crucial thing is that the mathematical perspectives we are dealing with are made the object of explicit treatment, reflection and articulation.

4.5.1 The actual application of mathematics in other subject and practice areas

Characteristics

The object of this form of overview and judgement is the actual application of mathematics to extra-mathematical purposes within areas of everyday, social or scientific importance. This application is brought about and expressed through the creation and utilisation of mathematical models.

Who uses mathematics for what?

Comments

While the previously mentioned modelling competency is concerned with the ability to deal with concrete extra-mathematical situations and problems where mathematics is brought into play, here there is rather talk of a broad and generalised form of overview and judgement of an almost sociological or science philosophical nature. Obviously a well-developed modelling competency will contribute to a concrete entrenchment and consolidation of overview and judgement, but such overview and judgement is not automatically a result of having a well-developed modelling competency.

Delimitations with regard to modelling competency

Exemplification

The case can be exemplified by questions such as:

- "Who, outside mathematics itself, actually uses it for anything?"
- "What for?"
- "Why?"
- "How?"
- "By what means?"
- "On what conditions?"
- "With what consequences?"
- "What is required to be able to use it?" Etc.

4.5.2 The historical development of mathematics, both internally and from a social point of view

Characteristics

The object of this form of overview and judgement is the fact that mathe-

Mathematics' development in time and space matics has developed in time and space, in culture and society.

Comments

Not to be confused with the history of mathematics

This form of overview and judgement should not be confused with a knowledge of "the history of mathematics" viewed as an independent topic. The focus is on the actual fact that mathematics has developed in culturally and socially determined environments, and subject to the motivations and mechanisms which are responsible for this development. On the other hand it is obvious that if overview and judgement regarding this development is to have any weight, it must rest on concrete examples from the history of mathematics.

Where the former type of overview and judgement can be said to have a corresponding competency, the same is not true in this case. We have not dealt with a "mathematical historical competency" as an ingredient in general mathematical competency. It would actually be possible to identify and characterise a mathematical historical competency, but this will be regarded as too peculiar to fit into the general context.

Exemplification

Of interest are questions like:

- "How has mathematics developed through the ages?"
- "What were the internal and external forces and motives for development?"
- "What types of actors were involved in the development?"
- "In which social situations did it take place?"
- "What has the interplay with other fields been like?" Etc.

4.5.3 The nature of mathematics as a subject

Characteristics

Characteristics of mathematics in relation to other subject areas

As a subject area, mathematics has its own characteristics. These characteristics are the subject of the present type of overview and judgement. Mathematics has some of these characteristics in common with other subject areas, but some of them are unique.

Comments

All eight competencies contribute to the establishment of this form of overview and judgement and to giving it flesh and blood. It is hereby probably the one out of the three forms of overview and judgement which lies most in continuation of the competencies. The point, however, is that only if the particular nature of mathematics as a subject area is itself made the object of elucidation and consideration, is a conscious and articulated form of overview and judgement created.

Demands articulation and reflection

If you had to point out any of the competencies as contributing particularly to the creation of a basis for overview and judgement when it comes to the *particular* traits of mathematics, then it must be the mathematical thinking, reasoning, and symbol and formalism competencies.

Exemplification

The following questions are relevant:

- "What is characteristic of mathematical problem formulation, thought and methods?"
- "What types of results are produced and what are they used for?"
- "What science philosophical status does its concepts and results have?"
- "How is mathematics constructed?"
- "What is its connection to other disciplines?"
- "In what ways does it distinguish itself scientifically from other disciplines?" Etc.

4.6 Further comments

The types of overview and judgement which have been considered cannot, as has been made apparent, automatically derive from the eight competencies dealt with above, but on the other hand, to be properly entrenched, they need to rest on a foundation of these competencies. In other words, to have overview and judgement regarding mathematics, it is insufficient merely to have heard (stories) of mathematical application, historical development and its particular nature.

Not automatically a consequence of the competencies

It can possibly seem to be slightly difficult to distinguish between, on the one hand, the "investigative" side of the eight competencies and, on the other hand, the stated types of overview and judgement, especially perhaps the first and last mentioned ones. The main difference, as intimated, is that the competencies must always be considered as utilised on concrete mathematical objects, problems or situations, while overview and judgement are concerned with mathematics as a *whole* subject area having a particular nature, history, and social placement, and as something which can be applied outside its own terrain for purposes which are not of a mathematical nature in themselves.

General and comprehensive, but mathematics specific

As in the case of the eight competencies, the types of overview and judgement mentioned are also of a general and comprehensive nature. They too make sense for (and are therefore independent of) all *concrete* mathematical content, and likewise make sense for every educational level while at the same time being specific to mathematics.

4.7 The application of the competency description of mathematics

An infinite spectrum of levels of mastery

The individual competency can be viewed as an infinite, three-dimensional, continuous spectrum of levels of mastery. The same is true when it comes to overview and judgement in mathematics. Mastery of a competency, overview or judgement is not an either/or question. Just taking the degree of coverage, one can master a competency on a very elementary plane that only comprises its most basic aspects. The more aspects of a competency one can activate and combine, the more contexts and situations one can apply it to, i.e. the greater one's range in a competency, and the more conceptually and technically advanced entities one can deal with, the greater the level with which one masters this competency.

On the other hand, one cannot master a competency, overview or judgement completely, since they are boundless, no matter to what depth, level of summary or complication one is concerned with an entity. All this does not, however, necessarily mean that it is impossible in practice to pragmatically divide up a competency into a smaller number of mastery levels, if one should choose to do so.

4.7.1 The concept apparatus applied normatively and descriptively

The competency description of mathematics can now be used to describe the subject in two different ways.

Normative use, e.g. in determining curricula

It can be applied *normatively*, i.e. for decisions about the weight and level of mastery the individual competency should have on the agenda in any curriculum context at a given educational level. In this way the competencies become the main instruments in determining curricula (though not the only instruments). This use is evidently closely related to notions of aims and purposes in education.

The normative use of competencies can in principle lead to a decision that one or more of the competencies either should not even be developed at a particular educational stage, or that only certain aspects of them need be on the agenda. It could, for example, be that some of the competencies only feature in their investigative forms, while their productive sides are toned down. In this connection it is not impossible that, in the actual *wording* of the curriculum, it would be appropriate to amalgamate some of the competencies or aspects of them, so as not to have to operate with a greater degree of detail in the description than is actually desirable in the situation.

The competencies can also be applied *descriptively*, i.e. to describe and analyse what is actually on the go in any given mathematics education, both on the curriculum level and when it comes to everyday teaching. It can, furthermore, be used as an aid to detecting and characterising mathematical acquisition in the individual pupil.

Descriptive use, e.g. to reveal the reality of mathematics teaching

4.7.2 The competency description as metacognitive support

Besides being used for a *subject description*, the competency description can also be used as a *metacognitive support*, i.e. as an aid to everyday teaching, both descriptively and normatively. The teacher can, on the one hand, use it when planning and implementing his or her teaching, and use it as the subject for dialogues and discussions between teacher and pupil and between pupils reciprocally about what the teaching is about, about what is actually taking place, and about what ought to be taking place, both on the teaching and acquisition planes.

Support tool in teaching for teachers and pupils

Finally, the competencies can be used in the teachers' subject specific, didactic and pedagogical discussions with colleagues.

4.7.3 The competency description as focal point, not standing alone

It is not the intention that the competency descriptions are the only things to be said when it comes to a concrete description of the subject in a given context. A given set of competencies can be promoted and activated by employing a lot of different mathematical material, like there has been intimated in the examples used to illustrate the individual competencies. What exactly this material should be, cannot be determined merely by taking the competency alone into consideration.

Choice of material rests on more than the competencies

If we add the fact that they are the same competencies – though with different weightings and priorities – which are involved at every level of education, and that teaching cannot incessantly make use of the same material, it is clear that the *choice of the subject matter by which the competencies have to manifest themselves* must be done with the incorporation of other points of view than that of the competencies alone. On the other hand, it is crucial that the considerations given to the suitability of the teaching material to promote the competencies that have been included in the programme play an important role in the choice of material.

Similarly, the choice of the *assessment instruments*, including examination forms, is also relevant. One cannot derive these instruments from the competencies, but many of the assessment instruments prevalent only allow a very limited section of the given competency to be evaluated. The main task is, therefore, to construct and implement assessment instruments and frameworks which are suitable for evaluating the competencies.

Teaching activities to promote competence

Finally, nothing has been said so far about that which is often the most important: *The activities which are brought into play in any given teaching situation.* There are activities which only to a limited extent serve to promote the development of the entire spectrum of competencies, while others have a greater range as far as the competencies are concerned. The way these relevant activities can be thought up, combined and implemented is the main task in everyday teaching, no matter what the level. It is an issue we will touch on again at the end of this report.

Part III

The education of mathematics teachers

5 Introduction to part III

5.1 Need for interplay between the different types of competencies

A discussion about the training of mathematics teachers is normally – and in all countries – a minefield when it comes to weighing the baggage of mathematical mastery and that of didactics and pedagogics, and, not least, when it comes to attitudes to the interplay between these.

A minefield

Firstly, all research and other experience points to the fact that, generally speaking, preparation for a mathematics teaching profession is completely insufficient if it is just about acquiring mathematical mastery, no matter at what level this occurs. It is true that there are many examples of excellent mathematics teachers whose only basis for their teaching is their mathematical mastery, but since this is the exception rather than the rule, it cannot form the basis of a generally viable strategy. Experience shows even more convincingly that preparation for the profession is, similarly, entirely insufficient if it exclusively rests on an acquisition of general didactic and pedagogical baggage without the inclusion of mathematical competencies.

Neither mathematical nor general didactic competence is sufficient

Secondly, much experience shows that it is also – generally speaking – completely insufficient to have attained mathematical mastery and general didactic/pedagogical baggage if these two components are possessed in an isolated form without being brought into interplay with each other. In other words, to be a good teacher it is necessary to possess both mathematical didactics and pedagogical competence, i.e. a competence that brings the mathematics competence into play with the issues regarding the teaching and learning of mathematics.

5.2 The structure of the following section

In the same way that we based the description of mastery in mathematics on the competencies, we would therefore like to carry out a corresponding description of mathematics teachers' competence. This description consists of two components.

"What does it mean to be a good mathematics teacher?"

The first component comprises the competencies required to exert the actual profession. One could say that it seeks to answer the question "What does it mean to be a good mathematics teacher?", which is a mathematical didactic/pedagogical question. It is entirely different to the question "What does it mean to be a good teacher who can also do maths?" We do not regard the posing of the latter question as relevant in this context.

"What mathematical competencies does a good mathematics teacher ought to have?"

The second component comprises a description of the mathematical mastery which a mathematics teacher ought to have, i.e. those (aspects of) mathematical competencies, as well as the forms of overview and judgement regarding mathematics as a subject, he or she ought to have in his or her baggage, not because of being a citizen or a teacher in general, but because he or she is a mathematics teacher. Here we are asking "What mathematical competencies does a good mathematics teacher ought to have?"

Despite the fact that in the following chapters we describe these two components individually, it cannot be stressed enough that, in a good teacher, they are integrated in the sense that he or she can both apply competent mathematical points of view to every didactic or pedagogical problem, and relate to the didactic/pedagogical potential in the mathematical abilities and insights he or she possesses, as well as being able to bring these two components together in an integrated manner in teaching.

"The how" is not dealt with here

In continuation of the considerations of the competencies a good mathematics teacher ought to have, another question becomes important: How should these competencies be acquired and developed during teacher training on the one hand, and while practising one's profession on the other? This question is both very complex and very important, but on the whole, we cannot deal with it in the report. It can, however, be pointed out that in consideration of this part of the report, we have not related to *how* mathematics teacher training can be arranged and organised. This can take place in many different and fruitful ways, and, among the many ways under discussion, we have not chosen one above the other in this project.

The characterisation of the didactic and pedagogical components linked to exert the profession is sufficiently common to different teacher training programmes, irrespective of the teaching level being addressed, that the characterisation can be carried out under one umbrella. On the other hand, it is out of consideration for mathematical mastery that we have divided up the teacher training programmes along the lines of those used in Part VII for other sections of the education system.

6 A competency based description of the profession of mathematics teachers: Didactic and pedagogical competencies

6.1 Introduction

A good teacher possesses a variety of general teaching competencies. A good *mathematics* teacher furthermore possesses, regardless of educational stage, a range of specific mathematical didactic and pedagogical competencies. They will be detailed below where we, moreover, use the word "student" as a common indicator for a person in the process of learning (mathematics) regardless of whether we mean a school student or a university student.

Not a characterisation of general teaching competencies

On the other hand, we do not see it as our task in this project to go into details with the characterisation of general teaching competencies. So even though the characterisation below of the competencies concerned may often make use of terminology which does not specifically refer to mathematics, it is nevertheless this subject that is the issue. These are not general didactic and pedagogical competencies. The extent to which it may be possible to describe subject mastery competencies in other subjects using similar terminology is something which is up to the other subjects themselves.

The task of mathematics teacher training is to instil in teachers the following didactic and pedagogical competencies which are characterised in the following sections:

- Curriculum competency.
- Teaching competency.
- Competency of revealing learning.

- Assessment competency.
- Cooperation competency.
- Professional development competency.

6.2 Curriculum competency – being able to evaluate and draw up curricula

Study and analyse

This competency comprises, on the one hand, being able to *study, analyse and relate to* every current or possible future framework curriculum for mathematics teaching at the relevant educational stage, and being able to *evaluate* these plans and their significance for one's actual teaching.

Draw up and implement

On the other hand, it comprises being able to *draw up and implement* different types of curricula and course plans with different purposes and aims at different levels taking into consideration the overarching frameworks and terms which may exist, both under current conditions and those in the expected future.

6.3 Teaching competency – being able to think out, plan and carry out teaching

Think out, plan and carry out concrete courses of teaching

This competency comprises being able to, with overview and together with the students, think out, plan and carry out concrete *teaching sequences* with different purposes and aims.

This involves the creation of an abundant spectrum of *teaching and learning situations*, including the planning and organisation of activities for students and student groups with consideration being given to their *characteristics and needs.* It also covers the selection and presentation of tasks as well as the other *assignments* and challenges of the students' activities. In addition it comprises being able to find, judge, select and produce different types of *teaching means and material.* Furthermore, the competency involves being able to *justify* and *discuss* with the students the content, form and perspectives of the teaching, and being able to *motivate and inspire* students to become engaged in mathematical activities, as well as being able to create room for students' own initiatives in mathematics teaching.

6.4 Competency of revealing learning – being able to reveal and interpret students' learning

This competency comprises being able to *reveal and interpret* students' actual mathematical *learning* and mastery of mathematical competencies as well as their *conceptions, beliefs* of and *attitudes* to mathematics, and it includes being able to identify the *development* of these over time.

Reveal and interpret learning, conceptions and attitudes

Part of the competency is being able to *get behind the facade* of the ways in which the individual's mathematics learning, understanding and mastery is expressed in concrete situations, with the intention of understanding and interpreting the cognitive and affective background for these.

6.5 Assessment competency – being able to reveal, evaluate and characterise the students' mathematical yield and competencies

This competency comprises being able to select or construct, as well as utilise, a broad spectrum of forms and *instruments to reveal and evaluate* a student's or student group's mathematical yield and competencies, both *during* the course of teaching and at the *end* of it, and both in *absolute* and *relative* terms.

Select or construct, as well as utilise assessment instruments

Included in this is the ability to *critically relate to* the validity and extent of the conclusions reached via the use of the individual assessment instruments. This competency is a precondition for continuous assessment, i.e. assessment carried out during the course of teaching, and includes the ability to *characterise* the individual student's yield and competencies and the ability to be able to *communicate* with the student about the observations and interpretations made, and then *help* him or her to correct, improve or further develop his or her mathematical competencies. The same is true for final assessments, including examinations, even though the guidance in this situation is often of a different nature.

6.6 Cooperation competency – being able to cooperate with colleagues and others regarding teaching and its boundary conditions

Colleagues

This competency comprises, first of all, being able to cooperate with colleagues, both *subject colleagues* and *colleagues in other subjects*, about matters of significance to mathematics teaching. Included in this is the ability to bring the above-mentioned four competencies into play in mathematical, pedagogical and didactic cooperative projects and in discussions with different types of colleagues.

Parents, administration, authorities, etc.

Secondly, the competency includes the ability to cooperate with *people beyond the staff room*, e.g. the parents of students, administrative agencies, the authorities, etc. about the boundary conditions of teaching.

6.7 Professional development competency – being able to develop one's competency as a mathematics teacher

This competency comprises being able to develop ones competency as a mathematics teacher. In other words, it is a kind of *meta-competency.*

Enter into and relate to developmental activities

More concretely it involves being able to *enter into and relate to activities* which can serve the development of one's mathematical, didactic and pedagogical competency, taking into consideration changing conditions, circumstances and possibilities. This is about being able to *reflect on one's teaching* and discuss it with mathematics colleagues, being able to *identify a developmental need*, and being able to select or arrange as well as evaluate activities which can promote the desired development whether or not there is talk of external *in-service training and further education courses*, *conferences* or *projects with colleagues* and activities like, e.g. participation in study groups and research projects. It is also about *keeping oneself up-to-date* with the latest trends, new material and new literature in one's field, about benefiting from relevant research and development contributions, and maybe even about *writing* articles or books of a mathematical, didactic or pedagogical nature.

7 The mathematical competencies of mathematics teachers

7.1 General remarks

7.1.1 Training of teachers for primary and junior secondary school

In Denmark, teacher training for primary and junior secondary school teachers is a professional education. That is, teacher training is a specific preparatory course (moreover at a special type of educational institution) for a particular profession, not a general course of study that can, among other things, lead to a position at a primary and junior secondary school as just one of a range of occupational possibilities.

A professional course of education

As is known, this is not the only possible way of organising teacher training. One can, as is the case in some countries and in Denmark when it comes to senior secondary school teachers, have a more or less general subject specific training which, along with different didactic, pedagogical or practice teaching components as additions, can qualify one to become a teacher. There are many large issues of a cultural, political, economic, organisational, didactic and pedagogical nature when it comes to discussing the planning and placement of teacher training. It will be useful for Danish society if such a basic discussion was on the agenda, with due respect to the time, efforts and resources needed if this discussion is to be qualified.

Useful to discuss the structural boundary conditions

We have not seen it as our task in the context of this project to take up this discussion. The broad organisational lines when it comes to the arrangement and institutional affiliation in primary and junior secondary school teacher training have therefore been taken for granted. However, we are of the opinion that the considerations in this report will stand up to even the most comprehensive organisational changes in the teacher training

. . . but this will not be done here

system in Denmark.

Competencies have full degree of coverage

As is generally the case with mathematics competencies at the higher stages of the education system, we hold the view that primary and junior secondary school teachers in mathematics ought also to master the eight mathematics competencies and the three forms of overview and judgement regarding mathematics as a subject, and with a total degree of coverage. Nothing has hereby been said of the competencies' radius of action or conceptual or technical level. Neither has anything been said about the subject matter within which the competencies will be practiced. This is a case requiring independent deliberation as is addressed in Chapter 8.

7.1.2 Teacher training for senior secondary and tertiary education

A general university degree

The training of mathematics teachers for senior secondary and tertiary education in Denmark takes place to all intents and purposes at university. In the previous four decades, approximately, future teachers for these educational stages have typically followed a general university course of education with mathematics as a subject, and teaching as a profession has just been one of the many possibilities of a future job. Previously the courses of education have only occasionally (e.g. at Roskilde University and Aalborg University) contained features that could specifically train students for the teaching profession, but such features have increasingly gained a footing in the universities as a whole. At the same time there has been a development on the pedagogical front towards upgrading not just practical teacher training, but also subject specific didactics, just like there has, from the Ministry of Education, been a formulation of certain content demands for university education as a precondition for conferring formal competence (in the authorisation sense of the word) as a senior secondary and tertiary education teacher. At business and technical colleges, the employment criteria are broader and it is, to a large extent, up to the college principal as to what is necessary to carry out teaching in these institutions.

Didactic features gain footing

University teaching has traditionally not imposed any pedagogical or didactic demands on its teachers. However, in recent years, junior lecturers have had to complete a particular course in university pedagogics in order to apply for senior lectureship positions.

The structural parameters are not discussed here

There are in this regard also many possibilities, each equally sensible, for organising the training of future mathematics teachers. One can both arrange excellent training courses where the didactic-pedagogical components are part of the actual university education, and training programmes where such components are only dealt with in add-on courses. We will desist from putting forward an opinion about the structural parameters allow for the best linking of didactic and pedagogical competencies to mathematical competence. The most important issue here is that, as emphasised in the previous chapter, this is something which is actually taken seriously.

It goes without saying that senior secondary and tertiary education teachers ought also to master the eight competencies and three forms of overview and judgement regarding mathematics with a total degree of coverage.

Competencies have full degree of coverage

7.1.3 Back to the more general

In relation to the profession of mathematics teacher, regardless of educational stage, the competencies have to, first and foremost, be expressed in contexts and situations which actually have, or potentially could have, relevance in relation to mathematics teaching. We have therefore chosen, when going through the teachers' mathematical competencies below, to combine the complete competency characteristics with comments and examples, partly of teachers' own competencies, and partly of their exercise of the individual competency in relation to their profession as a mathematics teacher. This is on the assumption that it is self-evident that the mathematical competencies of a mathematics teacher in all dimensions have to include the competencies of the students he or she will be teaching, and that he or she, furthermore, has to have such an excess of competence ability so as to be able to exercise the mathematical teaching competencies as described in the relevant chapters. The examples elucidated have deliberately been chosen to cover teachers at different stages of education.

Competencies manifested in teaching

What has often been discussed is the extent to which mathematics teachers ought to have an excess of subject mastery to enable them to exert their profession satisfactorily. This is, not least, a political and economic question regarding teacher training and conditions of employment. From the point of view of the task group it is, nevertheless, essential that the subject ballast of mathematics teachers extends significantly beyond the ability to teach the material currently on the agenda for the individual educational stages, as is, for example, expressed in current (and occasionally criticised) textbooks. It is only with a considerable excess of subject mastery, formulated here with the help of the mathematical competencies which the

An excess of subject mastery is essential teacher ought to have, that he or she is able to master the tasks covered in the mathematics teaching profession as described above.

The objective of mathematics teacher education is, irrespective of how this takes place, to equip teachers with the below-mentioned mathematics competencies. These will, as usual, be exemplified individually. Besides the given examples, there are also examples given in Part VII in the chapters on primary and lower secondary schools, senior secondary schools, and university courses in and with mathematics, which will be relevant in this context.

7.2 Mathematical competencies of mathematics teachers

7.2.1 Mathematical thinking competency

Characterisation

The nature of questions and answers

This competency comprises, first of all, *being aware of* the types of questions which are characteristic of mathematics, being able to *pose such questions* oneself, and *having an eye for the types of answers* which can be expected. Of special significance here is the mathematical endeavour to find necessary and sufficient conditions for specific properties of an object.

The scope of concepts

Furthermore, it comprises being able to *recognise*, *understand* and *deal with* the *scope of given mathematical concepts* (as well as their limitations) and their roots in different domains; extend the scope of a concept by *abstracting* some of its properties; *understand* the implications of *generalising* results; and being able to generalise such results to larger classes of objects.

Different types of mathematical statements

This competency also includes being able to *distinguish*, both passively and actively, between *different types of mathematical statements* and assertions including “conditional statements”, “definitions”, “theorems”, “phenomenological statements” about single cases, and “conjectures” based on intuition or experience with special cases. Of particular importance is an understanding of the role played by explicit or implicit “quantifiers” in mathematical statements, not least when these are combined.

Didactic and pedagogic comments

Basic overview of the subject

To be able to operate as a mathematics teacher at any educational stage, which always leads to the adoption of a range of different views on the subject, it is important to have a basic insight into the types of questions and answers that specifically belong to mathematics as a subject for the relevant stage. Furthermore, in connection with, e.g. implementing mathematically relevant tasks for the students, it is important not only to be able to formulate such questions oneself and work out possible answers, but to have a feeling for the types of answers that can be expected of the students at the given stage.

Abstractions and generalisations

Students' own observations and results are often linked to concrete situations and individual cases. It is therefore important that the teacher, taking as a starting point such situations and individual cases, is able to help students to progress in their work by being able to make conceptual abstractions as well as deduce and emphasise general properties and connections. With the aim of creating clarity in teaching and learning, the teacher must be able to determine when existing conditions are necessary and/or sufficient for an object to have a given property. Furthermore, it is important that the teacher is able to determine whether a student is, e.g. in the process of naming a mathematical object or talking about the properties of such an object.

Distinguish between types of student statements

Exemplification

Characteristic questions relevant to teaching

Examples of characteristic questions relevant to teaching which are important for the teacher to be able to relate to – and express, though not necessarily be able to answer – could be:

A: "Are there fractions which cannot be rewritten as decimals?"

B: "No."

A: "Are there decimals which cannot be rewritten as fractions?"

B: "Yes, e.g. π."

A: "In an ordinary box, the sum of the three angles of a corner is 270°. Is this always the case with a body 'made up of' six parallelograms?"

B: "No. The sum of the angles in such a '3-corner' can be as small as you want as long as the sum is positive, and as close to 360° as you want as long as the sum is below – and not equal to – 360°."

A: "How many rational numbers are there compared to the natural numbers?"

B: "Just as many (once we have reached a specific decision as to what 'just as many' means)."

A: "Is there, like for equations of the 1st and 2nd degree, a corresponding formula for the solution of a general equation of the n'th degree?"

B: "No, there is not."

A: "Why are hyperbole functions actually called hyperbolic, and why are cos, sin, etc. included in the notation used?"

B: "This is, firstly, because of the 'hyperbolic idiot equation' $\cosh^2 x - \sinh^2 x = 1$ which, on the one hand shows that the point $(\cosh x, \sinh x)$ always lies on the hyperbola, and, on the other hand, reminds one of the usual idiot equation for cos and sin (which, by the way, is the reason why these are, in certain connections, called circle functions). Secondly, $\cosh' = \sinh$, $\sinh' = \cosh$, $\tanh' x = \frac{1}{\cosh^2 x}$, etc. These geometric and trigonometric relationships give the background for the name."

It is equally important that the teacher is able to make it clear that, e.g. any realistically expected answer to a question about the population's average income for a particular year must be produced by calculation, that is by a mathematical procedure and not by measurement. In addition to this, a question like "When was World War I?" or "How many species of birds are there in the world?" supposes an answer involving numbers, but neither the questions nor the answers are characteristic of mathematics.

Surplus of mathematical insight

An example of a surplus of mathematical insight in this connection, is the need for the teacher to be able to ask him or herself questions like "Why can one be allowed to take 30 10-samples out of a set of 10 000 without replacement, and then do calculations from a binomial distribution which presupposes that the samples have been taken with replacement? Is this because 30 10-samples represent such a small part of 10 000 that it does not play any role whether they are replaced again or not?" Another example could be "Why is it that for probabilities on infinite probability fields one demands σ-additivity and not just additivity?".

Recognise, understand, and deal with the scope and limitations of given mathematical concepts

The importance of the teacher being able to *recognise, understand, and deal with the scope and limitations of given mathematical concepts* as well as their abstraction, can be illustrated by this imaginary dialogue between a teacher and a student:

S: "We know that a square has four equal sides. If all four sides of a rectangle are equal in length, can we be certain that it is a square?"

T: "Let's take a look at this. What does it take before we call a figure a square?"

S: "It has to be a rectangle where all the sides are equal?"

T: "Yes, and you touched on something besides the sides being equal. What was that?"

S: "The figure has to be a rectangle."

T: "And what does that mean?"

S: "All four angles are right angles."

T: "So, what do we need altogether for it to be a square?"

S: "The figure must have four equal sides and four right angles."

T: "Yes. Can you imagine a figure with four equal sides but without four right angles?"

S: "Oh yes, if you squash a square together in the opposite corners, without changing the sides – do you understand what I mean? – then the figure is no longer a square, but it still has four equal sides."

T: "Exactly. So we can have rectangles with four equal sides without them having to be squares. We call them rhombuses."

S: "So a square is a rhombus, but a rhombus isn't necessarily a square?"

T: "Exactly!"

Here is another example:

S: "I've heard that you can talk about the exponential function of a complex number. Is this really true?"

T: "Yes, it is. You decide that if z is the complex number $z = x + iy$, then $\exp(z) = \exp(x)[\cos x + i \sin y]$ where all the functions on the right are the known real functions, and where i is the imaginary unity. However, you can't go around calling a newly defined function as something known unless there is a link to the known. And there is one actually. Firstly, if z is real, i.e. if $y = 0$, then the new function equals the old one. Secondly, one can prove that the new function has a number of properties that are characteristic of the real exponential function, the most important being that $\exp(z+w) = \exp(z)\cdot\exp(w)$."

An example at upper secondary school level of the expansion of the scope of a mathematical concept, is when we go from defining sine, cosine and tangent for angles in right-angled triangles, to angles situated in a unit circle, and then to arguments in $\mathbb{R}$.

Promoting understanding of the implications of generalising

When it comes to *promoting understanding of the implications of generalising*, the teacher could

- start with simple examples like $1+3=4$, $3+6=9$, and $6+10=16$, and encourage the students to generalise so as to reach the supposition that "the sum of two consecutive triangle numbers is always a square number".
- work with generalising (not necessarily proving) the assertion $0.99999\ldots = 1$ to $T.99999\ldots = T+1$ for every natural number T.
- get his or her students to find out how many of the properties of the real exponential function are also valid for the complex one.

Distinguish between different types of mathematical statements

It is important that the teacher can help the students to *distinguish between different types of mathematical statements*, e.g. in relation to being able to

- distinguish between the definition of the heights of a triangle and the theorem (correct assertion) that the heights of a triangle always intersect each other at the same point; and then between this assertion and the assertion (also correct) that the medians, too, intersect each other at the same point.
- understand that assertions like "If $-1 \leq x \leq 1$, then $1 - x^2 \geq 0$" and like "A function that is differentiable at a point is also continuous at the point" are assumptions until proven.
- determine which of the assertions are necessary and sufficient in statements like "As rectangle $ABCD$ is a parallelogram, the diagonals of the rectangle bisect each other" (to know/ realise that the assertions are equivalent, demands specific geometric skills which are not solely covered by the mathematical thinking competency) and "Since the angle v lies in the interval $]\pi/2,\ \pi[$, we get $\cos v = -\sqrt{1-\sin^2 v}$."

7.2.2 Problem tackling competency

Characterisation

Put forward and solve problems

This competency both involves being able to *pose*, i.e. detect, formulate, delimitate and specify different kinds of mathematical problems, "pure" as well as "applied", "open" as well as "closed", and being able to *solve* such mathematical problems in their already formulated form, whether posed by oneself or by others, and, if necessary or desirable, in different ways.

Didactic and pedagogical comments

Own confidence

To be able to implement and guide learning processes of an investigative, experimental or problem solving nature among the students, it is first of all important that the teacher is confident with such processes his or herself. In the beginning, many teachers and university students find it difficult to work in a problem solving way, not to mention posing problems. On the basis of this, the formulation and solution of mathematical problems ought to be a general trait in preparing for the mathematics teaching profession.

Help students

As the organiser of teaching and learning activities, the teacher has to be able to pose and formulate problems and questions which can lead to problem solving activities among the students. Involved in this is being able to point out, select, formulate and define a variety of mathematical problems which can, in relation to different groups of students, give rise to such activity. It is obvious that the teacher has to be confident in solving such problems. Considering that the students often have different intellectual, socio-economic and cultural backgrounds, it is also particularly important that the teacher is able to set up different strategies for dealing with the problem concerned and for helping the students to approach them from a range of different angles, depending on their backgrounds.

Exemplification

Solve already formulated, closed problems

When it comes to the ability to *solve already formulated, closed problems* for potential use in mathematics teaching, it is important that the teacher has a great fund of experience, also over and above what is/can be expected at a given stage. Examples of this are:

- "Express as a number the ratio between the pieces of a line AC divided by point B so that the ratio between the whole line and the largest of the pieces is the same as the ratio between the largest and the smallest of the pieces" (the golden mean).
- "Which is more advantageous when buying discount goods: to get the discount before or after paying VAT?"
- "Find the sum of all the four digit numbers that can be made of different digits choosing from among $1, 2, 3, 4, 5, 6, 7, 8, 9$."
- "When will the big hand and the little hand first lie in extension of each other after 12 o'clock?"
- "Find the dimensions of a piece of paper in A-format when these are defined by $A0$ being 1 m^2, and $A(N + 1)$ is obtained from AN by perpendicular bisection of the longest side."
- "Let the point P be an inner point in a half disc. Construct P's projection on the diameter using only a ruler."

- "Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$ which satisfy $f(x) - f(y) \leq (x - y)^2$ for all $x, y \in \mathbb{R}$."
- "A textbook contains the following task: 'We have a right-angled triangle with hypotenuse 8 and height 5. What is the area?' The task contains an error. Find it."

Formulation and specification of a problem

An example of a teacher's *formulation and specification of a problem* which can create the starting point for students' investigative work with tile patterns could be: "What needs to apply for a regular polygon to make up the only base element in a covering tile pattern?", and further "What can be said, e.g. about the number of sides in the polygon?".

A more *open problem* as the starting point for student activity, could be any of the questions:

- "How many different rectangles with an area of 2 can be constructed on a nail board?"
- "What would be a suitable position for a cultural house in the country if the distance between the inhabitants on farms and in houses with a known position has to be as small as possible?" (an example of a problem that demands specification from the students' side).
- "What can be said about the relationship between two consecutive numbers far out in a so-called Fibonacci sequence $a, b, a+b, a+2b, 2a+3b, \ldots$?"

Deal with in many different ways

The following could serve as an example of a pure and relatively closed mathematical problem which the teacher ought to have the competency *to deal with in many different ways*:

- "How big an error is one making by approximating the circumference of a circle with the circumference of a regular polygon with n sides"?

which can, e.g. be specified

- "What is the difference between the circumference of a circle with radius r and the circumference of an inscribed regular polygon with n sides by means of r and n, and when is the error of substituting the circumference of the circle with the circumference of the n-sided figure less than, e.g. 1%?"

Another example of a range of different treatment possibilities being mentionable for setting up a problem, is to ask the students to, starting with $1, 2, 3 \ldots n$ points in a plane of which no set of three points lie on the same line, are collinear, answer the question "How many different triangles with corners in the n points can be drawn?". Here it is important that the teacher can use or relate to many different solution strategies (simple counting, systematic counting, actual combinatoric counting methods, deduction of "patterns" in the number of triangles for concrete increasing values of n).

Another example:

- "Anne is 8 years old, and her mother is 38. Their birthday is on the same day. How old is the mother when she is three times as old as the daughter?"

The problem can either be solved with the equation $38 + x = 3(x + 8)$, where x is the age of the daughter; or the equations $y = 3x$ and $y - x = 30$, where y is the age of the mother, x that of the daughter. One could also choose to start with the constant age difference of 30 years and determine that when the mother is 3 times as old as the daughter, the difference must be double that of the daughter's age, and she must therefore be 15. The teacher's problem tackling competency ought also to cover being able to assist the student to deal with expansions of the original problem, e.g. "When will the mother be five times as old as the daughter?"

7.2.3 Modelling competency

Characterisation

Model analysis

This competency involves, on the one hand, being able to *analyse* the foundations and properties of existing models and being able to assess their range and validity. Belonging to this is the ability to "*de-mathematise*" (traits of) existing mathematical models, i.e. being able to decode and interpret model elements and results in terms of the real area or situation which they are supposed to model. On the other hand, the competency involves being able to *perform active modelling* in given contexts, i.e. mathematising and applying it to situations beyond mathematics itself.

Modelling

Elements in modelling

Active modelling contains a range of different elements. Firstly, there is the ability to *structure* the real area or situation that is to be modelled. Then comes being able to implement a *mathematisation* of this situation, i.e. translating the objects, relations, problem formulation, etc. into mathematical terms resulting in a mathematical model. Then one has to be able to *work with* the resulting model, including solving the mathematical problems that may arise as well as *validating* the completed model by assessing it both internally (in relation to the model's mathematical properties) and externally (in relation to the area or situation being modelled). Furthermore, there is the ability to *analyse the model critically* both in relation to its own usability and relevance, and in relation to possible alternative models, as well as to *communicate* with others about the model and its results. Finally, also included in active modelling is being able to *monitor* and *control* the entire modelling process.

Didactic and pedagogical comments

Analyse and evaluate others' use of mathematics

It is crucial that a mathematics teacher can initiate and guide students' work both with established models (e.g. from textbooks or other publications, e.g. articles) as well as with active modelling of simple situations. The teacher must therefore be able to analyse and evaluate other people's use of mathematics for the purpose of application, and be able to apply mathematics him or herself in relation to the problems and situations in the surrounding world.

Decode and interpret models and model elements

An important, though traditionally also very difficult aspect of mathematical modelling in a teaching context is the students' establishment and recognition of the relationships between, on the one hand, existing models or elements of such, and, on the other hand, the situations and circumstances which are being modelled. It is therefore important that the teacher is confident in being able to decode and interpret models and model elements.

Master the modelling process

When it comes to the students' own work with mathematical modelling, it will in many situations often be necessary or appropriate to select certain parts of the modelling process (e.g. decoding elements in an existing model in relation to a given situation) as the object of teaching. The teacher must therefore master the part processes (structuring of the situation, mathematising, interpreting, validating, etc.) that make up a modelling process, and he/she must, with a view to selecting and evaluating the degree of difficulty of the individual processes, be able to have a general overview of the total process in concrete situations.

Exemplification

Analyse the basis for and properties of models

When it comes to the teacher's ability to *analyse the basis for and properties of established (or suggested) models* of potential relevance for the school, and his or her ability to judge their scope and validity, one can, e.g. look at whether it is reasonable to use a "heads-and-tails"-based experiment to simulate distributions of boys and girls in families with children, or the binomial distribution $b_{13,\frac{1}{3}}$ as a basis for evaluating the chances of different types of prizes in the Danish soccer lottery.

Other examples could be to understand and judge the basis for using the prescriptive body-mass-index model (BMI = $\text{weight}[kg]/(\text{height})^2[m^2]$)

for underweight, normal weight, overweight and obesity in people. Or the prescriptive models for taxi fare systems, postal rates for letters, the calculation of income tax, the determining of the true rate of interest when buying bonds, etc.

An example of *de-mathematising a model* could be to interpret the relationship between precipitation and wind conditions from a correlation coefficient calculated from observations of wind conditions and precipitation over a particular period. Or being able to explain the relationship between, on the one hand, the elements of the formula

De-mathematising a model

$$f(n) = y \cdot \frac{(1+r)^n - 1}{r}$$

and, on the other hand, an annuity policy.

The assessment of the properties of the existing model is especially important for a well thought through evaluation of their validity, e.g. sensitivity to changes in the parameters contained in Danish Statistics' model for a long term population forecast for Denmark.

As far as *active modelling* is concerned, one can, e.g. imagine project style activities starting from the following example:

Active modelling

A's car has an actual value of 20 000 DKK. To get it into good working condition, it will have to be repaired for 10 000 DKK. after which it will probably last for another two years. Is it a good solution to have it repaired, or should A rather buy a new car right away? The model building will include being able to

- structure the situation, i.e. identify the crucial components of significance for the model building like the prices of new cars, financing conditions, depreciation on new and old cars respectively, etc.
- carry out a mathematisation of the situation and its treatment which could comprise calculating expenses for the next two years for a new and an old car respectively, and calculating the depreciation in value of both cars after two years, and then comparing these calculations.
- validate the model by, e.g. evaluating if any inexpedient simplifications have occurred along the way.
- analyse the model critically, e.g. by evaluating the possibilities of further repair expenses on the old car, including the questions of petrol consumption, safety, comfort, etc.
- communicate with others about the model and its results, e.g. by being able to justify the problem formulation, the chosen form of

investigation, and by being able to give reasons for the conclusion by incorporating relevant reservations for its scope.

Other examples could be setting up a model to answer questions like the following:

- "How far is it to the horizon when one is standing looking out over the ocean?"
- "How tall is 'this' chimney?"
- "What could the floor plan of a house be like if its area had to be 120 m^2?"
- "How expensive is it to use a cell phone?"
- "Is it possible that the average age of a population can be 35 at the same time as at least 40% of the population is 60 or above?"
- "Bakers' boxes are folded from or cut out of a rectangular piece of cardboard. What dimensions should it have to have the greatest volume?"

7.2.4 Reasoning competency

Characterisation

Follow and assess reasoning

This competency consists of, on the one hand, the ability to *follow* and *assess mathematic reasoning*, i.e. a chain of argument put forward by others, in writing or orally, in support of a claim. It is especially about knowing and *understanding* what a mathematical *proof* is and how this differs from other forms of mathematical reasoning, e.g. heuristics based on intuition or on special cases, and it is also about understanding how and when mathematical reasoning actually constitutes a proof, and when it does not. This includes an understanding of the logic behind a *counter example*. Furthermore, the competency comprises being able to *uncover the basic ideas in a mathematical proof*, including distinguishing between the main lines of an argument and the details, and between ideas and technicalities.

Understand what a proof is

Devise and carry out

On the other hand, it consists of the ability to *devise and carry out* informal and *formal arguments* (on the basis of intuition) and hereby transform heuristic reasoning to actual (valid) proofs.

Didactic and pedagogical comments

Generally, a central part of what it means to be a (mathematics) teacher is being able to familiarise oneself with the students' ways of thinking and, not least, reasoning. When students have to learn to carry out independent mathematical reasoning of the kind encompassed by the reasoning competency, it is important that the teacher is able to transcend his or her own arguments, both with the intention of following, characterising, commenting and evaluating the students' reasoning, and with the intention of helping them to develop their own mathematical reasoning ability.

Transcending own arguments

In everyday teaching, the students' reasoning and conclusions will often be fragmentary. To allow for the constructive inclusion in the wider teaching of such student contributions, it is important that the teacher, having a general overview and flexibility, is able to, if necessary, rephrase and tighten reasoning and, where appropriate, incorporate elements of real proof into the picture.

Rephrase and tighten reasoning

In primary and junior secondary school it is not always possible or topical to prove mathematical assertions in a strict sense. Instead, many assertions are explained, illustrated and made plausible. In this connection it is essential, both that the teacher is able to distinguish between when an assertion has the nature of a proof, and when it is "merely" a good explanation or illustration, and that he/she is able to help the students to reach this distinction.

Proofs

At primary and junior secondary school there is a limit to how in-depth and detailed one can work with mathematical proving. It is therefore important that the teacher in his/her lessons can focus on and explain the basic ideas of proofs without necessarily taking all the details into account. This demands that he/she in these situations is able to reveal the basic ideas and control the degree to which the more technical side of the proof should be included in a concrete teaching context.

Up through secondary school and tertiary education, dealing with proofs and actively proving occurs with more intensity and emphasis. Here it is, among other things, an important task for the teacher to help the students understand and take a stance about when a proof suggestion is correct and complete according to the given criteria.

Exemplification

A teacher's ability to *follow and judge students' mathematical reasoning* could be, e.g. illustrated with the following imaginary dialogue with a student:

Follow and judge mathematical reasoning

S: "I've discovered that a number is divisible by 3, 7 and 9 if these divide the sum of the digits of that number."

T: "Oh, yes? And how can you convince me?"

S: "Well, if you take for example 642 and 231, the sum of the digits, 12 and 6, are divisible by 3 and so are the numbers themselves. And 133, 455, and 511 are divisible by 7, as is the sum of their digits. And if you take the numbers 297 and 2376, then they are divisible by 9, as is the sum of their digits."

T: "Your examples are fine, but is this always true?"

S: "Yes, isn't it? I have more examples, but I just haven't shown them to you."

T: "Let's look at the number 16. Is your assertion still true?"

S: "No, the sum of the digits is divisible by 7, but 16 is not."

T: "Can you say that you've convinced me?"

S: "No, I haven't, but perhaps that's because 16 is a 2-digit number. All my examples have more than two digits. Maybe it works for these types of numbers."

T: "What about 1616?"

S: "Well, no. The sum of the digits is divisible by 7, but the number is not. So it only sometimes works with 7, not all the time. What about 3 and 9 though? Can one prove that?"

T: "Yes, actually one can, but it's a bit complicated to explain all the details in general."

The teacher's own competence should make it possible for him/her to continue the dialogue as follows, even though this is highly unlikely in many junior secondary schools:

T: "Let's restrict ourselves to 3-digit numbers. One can approach it in the same way with other numbers of digits. What is a 3-digit number $a_2a_1a_0$ actually? Well it's a number which in reality has the following form: $a_2 10^2 + a_1 10 + a_0$. What does it mean when we say a number is divisible by, e.g. 9?"

S: "The number can be written as 9 times another number."

T: "Exactly. For the sum of the digits to be divisible by 9, it means that $T = a_2 + a_1 + a_0 = 9d$ where d is some or other natural number; the same with 3. Let's look at our 3-digit number. We can write it in a sneaky way, like this:

$$\begin{aligned}a_2 10^2 + a_1 10 + a_0 &= a_2(99+1) + a_1(9+1) + a_0\\ &= 99a_2 + a_2 + 9a_1 + a_1 + a_o\\ &= 9\cdot(11a_2 + a_1) + T\end{aligned}$$

If T is divisible by 9, as we assume, then we have

$$\begin{aligned}a_2a_1a_0 &= a_2 10^2 + a_1 10 + a_0\\ &= 9\cdot(11a_2 + a_1) + T\\ &= 9\cdot(11a_2 + a_1) + 9d\\ &= 9\cdot(11a_2 + a_1 + d).\end{aligned}$$

But this shows that the number can be written as 9 times a natural number. In other words, the number is divisible by 9. If we'd taken 3 instead, we could have used the same argument, because $9\cdot(11a_2+a_1)$ is divisible by 3 (it is also divisible by 9), and if T is divisible by 3, then the sum of the digits is as well."

S: "I'd never have been able to work that out myself."

T: "No, and I definitely don't expect you to do so; that would be expecting too much. But perhaps you could now try with a 4-digit number?"

The teacher's ability to activate a counter-example "Socratically" is central in the first part of this dialogue. Something similar could occur with a student's suggestion that if two triangles have an angle, an adjacent side, and an opposite side in common, then they must be congruent.

Understand what a proof is and when something is not a proof

Enabling the students to *understand what a proof is and to determine when mathematical reasoning does (or does not) constitute a proof* can be exemplified by asking the students to point out – and possibly correct – mistakes in reasoning.

- "The fact is that almost 70% of the population do not visit libraries, namely 39% of the men and 30% of the women."
- "Studies show that 60% of high school students are girls. In other words 60% of the girls of the relevant age group are in high school."
- "It is true that $1 = 0$. We have the formula $(a+b)(a-b) = a^2 - b^2$, and if we here divide both sides by $a-b$, we get $a+b = {(a^2-b^2)}/{(a-b)}$. If we now take $a = b = 1/2$, then the left hand side is obviously 1 (since $\frac{1}{2}+\frac{1}{2}=1$), while the right hand side is obviously 0, because

the numerator is 0, since it is true that $a^2 - b^2 = 0$, when $a = b$. Ergo $1 = 0$." (The reasoning is incorrect because $a - b = 0$ when $a = b$, and it is prohibited to divide by 0. The suppression of this problem is the point of the "trick").

Another example could be to show where and how a classical geometric proof for the three altitudes or the three medians in a triangle intersect each other at the same point, differs from an illustration produced by a dynamic geometry programme on a computer.

Uncovering the basic ideas in a (correct) mathematical proof

The teacher's ability to assist the students with *uncovering the basic ideas in a (correct) mathematical proof* can be illustrated as follows:

- "Gauss's proof of $1 + 2 + \ldots + n = \frac{1}{2}n(n + 1)$ rests on the idea that one can determine the sum by means of an equation. By adding the number $n + \ldots + 2 + 1$ to the left hand side, you get on the one hand the sum in question, and on the other hand n parentheses each comprising two numbers whose sum is $n + 1$. Utilising this to express the sum is thereafter a routine technique (multiplication of n by $n+1$ followed by solving a simple equation)."
- "There are many different proofs of the fundamental theorem of algebra (which states that every complex polynomial has a (complex) root). One of the most prevalent ones rests on the idea that a polynomial $P(z)$ of degree n, regarded as a (continuous) complex function, must satisfy $|P(z)|$ tends to ∞ for $|z|$ tending to ∞. This results in there being a closed disc $S = \{z \,|\, |z| \leq r\}$, such that P outside this only assumes values which are numerically greater than a suitably chosen positive constant. Hence it cannot have any roots outside the disc. On S, which is compact, $|P|$, which is continuous, assumes a minimum value. If this minimum value was positive, one can obtain a contradiction with the facts at hand, since one can at the same time realise – though it demands a bit of technical expertise – that $\inf \{|P(z)| \,|\, z \in S\} = 0$. Ergo, the assumed minimum value has to be 0, which shows that P has a root in S and therefore in $\mathbb{C}$."

Independent proofs

Finally, a teacher's role with carrying out *independent proofs* from heuristic to formal proofs, is illustrated in the efforts to prove the following assertions:

- "The sum of two consecutive triangular numbers is always a square."
- "7 must be the most frequently occurring sum of the pips in a two-dice throw" and to specify the assumptions on which the statement and the proof rest.

- "The number of permutations of a set of objects must be greater than the number of combinations of the same set" (an example of a correct proof not needing symbolic manipulation).
- "There are infinitely many rectangles where the circumference and the area have the same value. Any such rectangle must have side lengths all of which are greater than 2."
- "We have a rectangular piece of paper $ABCD$ which is folded over a normal through the midpoint M of the side BC. Thereafter the paper is folded along an as yet undetermined diagonal line such that the corner A ends in M. The fold line goes through a point F on side AB. Hereby a right-angled triangle $\triangle FBM$ is created. This triangle is always a 3-4-5-triangle."
- "The functions $f : \mathbb{R} \mapsto \mathbb{R}$ that satisfy the inequality

  $$f(x) - f(y) \leq (x - y)^2$$

  for all $x, y \in \mathbb{R}$ are precisely the constant functions."
- "The three medians in an arbitrary triangle have a common intersection" built up as a geometric proof, e.g. starting with observations of individual cases.

7.2.5 Representing competency

Characterisation

This competency comprises being able to, on the one hand, *understand* (i.e. decode, interpret, distinguish between) and *utilise* different kinds of representations of mathematical objects, phenomena, problems or situations (including symbolic, especially algebraic, visual, geometric, graphic, diagrammatic, tabular or verbal representations, but also concrete representations by means of material objects) and, on the other hand, being able to understand the reciprocal *relations* between different representational forms of the same entity, as well as knowing about their strengths and weaknesses including the loss or increase of information. It also comprises being able to *choose* and *switch* between different representational forms for any given entity or phenomenon, depending on the situation and purpose.

Understand and utilise different representations

Choose and switch between representations

Didactic and pedagogical comments

If a teacher is to carry out qualified mathematics teaching, it is important

Diversity of the students' backgrounds

that he/she, with the intention of taking into consideration a student group composed of very different backgrounds and premises, can shed light on and handle mathematical concepts, topics and problem formulations in many different ways and initiate student work on this basis. An important precondition for this is that the teacher him or herself knows about and can make use of a broad spectrum of mathematical representations, and that he/she can relate and contribute to the development of the students' use of such representations.

Judge the strengths and weaknesses of the forms of representation

The teacher's repertoire of representation forms is, however, not only important in relation to the diversity in the students' backgrounds. It is also important to bring into play a range of representations with the aim of shedding light on a given case and, in this relation, be able to judge the strengths and weaknesses of the representation, e.g. with the aim of being able to prioritise among them, also in a teaching context.

"A leitmotiv" in the teaching

Furthermore, it is important that the teacher, in relation to the often motley crowd of students, and in consideration of the all-round elucidation of concepts and topics, is able to retain "a leitmotiv" in his/her teaching by creating links between the different forms of representation.

Exemplification

Primary school

An example of this competency of relevance to the primary school could be the teacher's ability to, for or together with the students, find representations of natural numbers in the form of drawn lines, dots, etc. or bricks of the same shape and size, or writing up numbers in the position system with the help of cuisenaire rods, centicubes, abacuses or similar, or with the help of symbols in usual Hindu-Arabic notation, roman numerals, cuneiform writing, etc. as well as with verbal representations (e.g. five million, one hundred and twenty-six thousand, nine hundred and thirty-seven).

Another example from the world of small children would be time expressions where analogue and digital watches provide equivalent, but completely different representations of the time.

Further along in the education system

Further along in the education system an example could deal with understanding and handling different representations of the object π and the connections between them. The object can both be represented by this symbol itself, e.g. on a calculator, and by an infinite decimal expansion $3.14159265\ldots$, or a rational approximation (with its concomitant inexactness) by e.g. the fractions $^{22}/_{7}$ or $^{223}/_{71}$. However, π can also be represented geometrically as the circumference of a circle with a diameter of 1. At an advanced level π is, e.g. represented as the sum of various infinite series, or as a value obtained from various inverse trigonometric functions, e.g. $4 \arctan 1$. In such cases it is important that the teacher can clarify to the students as to which representation is being used when they, in different connections, are presented with other references to the object π.

Another example could be to distinguish between and compare the representations of a parabola as the graph of a polynomial of degree two, a plane geometric locus given by directrix and focus, and a certain plane section of a mathematical cone and a light cone respectively, e.g. brought forth by a torch held in a suitable position in relation to a wall.

Understand and handle reciprocal connections

As a general example of the importance of being able to understand and handle reciprocal connections between different forms of representations for the same case, and being able to point out their respective strengths and weaknesses, can be mentioned the representation of functions by explicit algebraic expressions, graphs, tables, spreadsheets, etc.

An illustration can be a function which, represented in different ways, describes the development of a bank deposit of 100 DKK. with interest to be paid at the rate of 5% p.a.:

- As a table:

x	0	1	2	...	...
y	100	105	110.25	...	...

- As an algebraic expression: $f(x) = 100 \cdot 1.05^x$.
- As an exponentially increasing graph.
- In a spreadsheet based on the generating recursive relation $f(x+1) = f(x) \cdot 1.05$.

A related illustration can be that the teacher, in a given context, can judge the usefulness of, and if necessary choose between, the direct formula $y = G \cdot {}^{r}/_{[1-(1+r)^{-n}]}$ and the recursive formula $G_{NEW} = G_{OLD} + G_{OLD} \cdot r - y$ to represent the important connections between the central quantities in a debt annuity.

Consolidate understanding of mathematical results

An example of a teacher's switching between representations which can be used to consolidate understanding of mathematical results, is the connection between transcriptions like $(a+b)^2 = a^2 + b^2 + 2ab$ and considerations of areas based on the division of a square with the side $a+b$ in sub squares and rectangles.

In a similar way, the inequality between the geometric and the algebraic mean of two non-negative numbers x, y can be illustrated geometrically in the following way: If x and y are drawn as line segments in extension of each other, a half circle with the combined segments as its diameter is formed, and if the diameter normal is erected from the dividing point between the line segments to the intersection with the circle at a point P, a right-angled triangle is formed. The altitude h from P is then the mean proportional between x and y, i.e. h is the geometric mean of x and y. The altitude is obviously at most the radius of the circle, i.e. $\frac{x+y}{2}$. Hence $\sqrt{xy} \leq \frac{x+y}{2}$. This is exactly the content in the inequality for two variables. This example is further suited to a discussion of the pros and cons of such representations in relation to each other.

7.2.6 Symbol and formalism competency

Characterisation

Decode, translate and handle symbolic statements

This competency comprises, on the one hand, being able to *decode* symbol and formal language; being able to *translate* back and forth between mathematic symbol language and natural language; and being able to *treat and utilise* symbolic statements and expressions, including formulas. It also, on the other hand, comprises *having an insight into* the nature of the "rules of the game of formal mathematical systems" (typically axiomatic theories).

Formal mathematical systems

Didactic and pedagogical comments

Decode symbol and formal language

At all levels, the teacher needs to be able to, through his or her own descriptions, explanations, illustrations and concrete examples, as well as by creating situations for students to work on, support the students' work with more or less abstract symbolic statements. As the basis for this, the teacher needs to be competent at decoding symbol and formal language him or herself.

"Translate"

To a great extent, this needs to be built on the teachers' and students' use of natural language. It is therefore important that he/she is able to translate back and forth between symbolic mathematical language and natural language, and is able to stimulate corresponding translation among the students.

Set up, treat and utilise symbolic statements

Being able to set up, treat and utilise (including translate) symbolic statements is notoriously difficult for many students at nearly all levels. To be able to understand the students and their often very different difficulties, and to be able to help them to overcome these, the teacher must be able to deal with such statements with considerable assurance and excessive knowledge, and with a general overview that enables him/her to act flexibly and effectively in teaching situations. This is not least necessary when the teacher has to be able to understand, assist and stimulate students with a precarious or unorthodox, ingenious or downright creative relationship to symbolic manipulations.

The rules of formal mathematical systems

The importance of the teacher having insight into the nature of the rules of formal mathematical systems, lies, at primary and junior secondary school level, first and foremost with the teacher being able to draw on his or her knowledge of what can, "at the end of the day" create the basis for acquiring the mathematical concepts in the curriculum. On further educational levels, such systems are the actual objects of teaching, so the teacher obviously has to master those traits of the competency concerning the formal mathematical systems.

Exemplification

The elementary level

On the elementary level it is important that the teacher helps the students reach clarity about the conditions for handling numbers, including arithmetic calculations. This could be

- to get them to understand the position system in writing up and reading a number like 406.
- the conventions that prevent one from writing $6 + \cdot 5$ or $6 - -3$ (while $6 + +3$ is not meaningless, just bad syntax).
- that $5 \cdot (3 + 4)$ is not the same as $5 \cdot 3 + 4$.
- that $(2^3)^4$ is not the same as 2 raised to the power of 3^4, but the same as 2^{12} and, generally, that x^{y^z} by convention means $x^{(y^z)}$.
- that $4 < 7$ is a statement and not an expression.
- what is meant by $n!$, with $7/9$ or with $7,423423423\ldots$.
- why one cannot (in school!) find the square root of a negative number, while this is possible by using complex numbers.

Later stage

At a later stage this could involve the teacher getting the students to understand the rules for handling coordinate systems, which, e.g. includes

- interpreting $\{(x, y)|x = t, y = 3t - 7, t \in \mathbb{R}\}$ as the set of all real pairs of numbers where the first coordinate assumes an arbitrary real value while the second coordinate has to be (exactly) three times this value, minus 7.

- analysing the function $f(x) = ax^2 + bx + c$ algebraically for all sets of coefficients a, b and c in order to formulate the role the coefficients play for the location of the graph of the function in a coordinate system.

Of special importance here, is the teacher's ability to get the students to understand that, except for a few fixed symbols, the symbolic naming of mathematical entities is completely open just as long as different entities are not called the same thing, the same entity is (rather) not called by different names within the same symbol context. At the same time there are nevertheless a few, in principle completely arbitrary conventions which result in particular entities often being called something particular, as when coefficients, constants and parameters are often named with letters from the beginning of the alphabet, while variables are called x, y, z, etc. It is, however, important for teachers to also be able to deal with situations where these conventions are completely broken.

"Translation"

Translation between symbolic mathematical language and natural language can be illustrated by the teacher's explanation

- that the identity $(a + b)(a - b) = a^2 - b^2$ states, on one hand, that the sum of two arbitrary numbers times their difference equals the difference between their squares and – on the other hand – that the difference between the squares of two numbers equals the product of their sum and their difference.
- that $P(n, r) = K(n, r) \cdot r!$ states that the number of ways in which r elements can be chosen amongst n, taking into account the order of the elements, is the same as the number of ways, in which one can choose r elements amongst the n, not taking into account the order of the elements, multiplied by the number of ways in which r elements can be rearranged in different orders.
- that the formula $K(n, r) = K(n, n - r)$ simply states that the number of ways, in which one can choose r elements amongst n is the same as the number of ways, in which one can choose $n - r$ elements amongst the n.
- in natural language, what formulas like

$$K_n = K(1 + r)^n \quad \text{and} \quad \frac{d \ln(\sin^2 x)}{dx} = 2 \cot x$$

express, and what the difference is between

$$\int_0^1 \sin t \exp t \, dt \quad \text{and} \quad \int \sin t \exp t \, dt \, .$$

A widespread problem at high school and further educational levels is the students' tendency to cheerfully and regularly use incorrect logical symbols to create links between mathematical statements in symbolic or natural language forms. Something similar is true for symbols from set theory. The teacher ought to be able to help students to achieve a correct and appropriate use of such symbols.

The treatment and utilisation of symbolic statements and expressions can, e.g. comprise the teacher, with the help of suitable questions, assisting the student towards

- from the formula for the volume of a circular cylinder – $V = \pi \cdot r^2 \cdot h$ – being able to decide, whether a doubling of the radius or of the height leads to the biggest change of the volume.
- being able to rearrange the quadratic equation $ax^2 + bx + c = 0$ such that possible solutions

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  appear.
- generally recognising $\int f(g(t))g'(t) \; dt$ as the function $t \mapsto F(g(t))$, where F is an indefinite integral of f, and in particular, that $\int \frac{f'(t)}{f(t)} \, dt = \ln |f(t)|$ (assuming that the integrands which appear are integrable).

Finally, the teacher's ability to help students to deal with formal mathematical systems can be illustrated by the conditions and rules for dealing with real numbers, including the solution of equations and inequalities, and with an insight into what is involved in carrying out geometric constructions on the basis of Euclid's axioms, including an understanding of in which sense (not why!) it is impossible to trisect an angle using a compass and a ruler. Furthermore, axiomatic Euclidian geometry can serve as an example of a mathematical formalism which does not have to be represented symbolically.

Deal with formal mathematical systems

7.2.7 Communicating competency

Characterisation

This competency consists of, on the one hand, being able to *comprehend*

Understand and interpret expressions and texts *and interpret* written, oral or visual mathematical expressions or “texts” of others, and, on the other hand, being able to *express oneself* in different ways and with different levels of theoretical or technical precision about mathematical matters, either written, oral or visual, to different types of audiences.

Express oneself about mathematics

Didactic and pedagogical comments

By its very nature, an absolutely central qualification in a teacher is being able to enter into a communicative dialogue on his or her subject with the students.

Comprehend and interpret

First of all this means that the teacher must be able to comprehend and interpret both written and oral expressions from the students; he/she must also be able to relate to the students’ different ways of expressing themselves, something which can vary significantly in relation to, e.g. language use, theoretic level and precision. Secondly, this means that the teacher must be able to express him or herself about mathematical matters in a variety of ways, either written, orally or visually since he/she needs to suit his/her expression to the students and other circumstances in the teaching situation. The variation may, amongst other things, include subject specific terminology and its level of precision, combinations or shifts between text, speech, visual illustrations and other forms of expression.

Express oneself

Furthermore, it is important that the teacher can help the students to develop and expand their communicating competency by making different communication means and forms objects of teaching.

Evaluate teaching materials

The teacher must also be able to evaluate the presentation and communicative quality of mathematical topics in a variety of teaching materials, not least in textbooks.

Communicate with colleagues and others

Finally it is important that the teacher can communicate in a flexible way with colleagues both within and beyond the subject and, e.g. with parents, about the perspectives, problem formulations, contents and methods of mathematics teaching.

Exemplification

Communicate with students

To illustrate the teacher’s ability to *communicate with students about basic mathematical matters* one could, e.g. imagine the following dialogue taking place between a teacher and a student in the junior secondary school or the start of senior secondary school.

S: "You re always telling us that we may not divide by 0. But I don't think you have told us why not; is it just a rule or what?"

T: "Yes, it is actually."

S: "But where does it come from? There must be a reason."

T: "Well, let's try and see what division is all about. If we divided e.g. 1 by 0, we should get the number that, multiplied by 0 gives us 1. But a number, no matter which, multiplied by 0 gives us 0 and not 1. So the division just doesn't work. Not even if I had selected all other possible numbers besides 1."

S: "Oh, yes. But what if you divide 0 by 0, then it does work. You could then multiply 0 by, for example 10 and get the right answer, namely 0."

T: "You re right, I forgot about that one. We could also have multiplied by 10^{10} and still got 0. So then $0/0$ would be 10^{10}. And just previously it would have been 10."

S: "That must be rubbish!"

T: "Exactly. The division of 0 by 0 doesn't give any definite result, so we regard it as being impossible. Can you now formulate the reason why we are not allowed to divide by 0?"

S: "OK, so it's forbidden to divide by 0 because we'd never get any definite result. In most cases we would get absolutely nothing out of it, and if we divide 0 by 0, we can get anything."

One could imagine analogous dialogues about why the square root of a positive number a is always positive, despite the equation $x^2 = a$ having both a positive and a negative solution; about why it is forbidden – in school – to operate with the square root of a negative number; about why general power functions are only defined for positive values, since there is no problem in taking the cubic root of any real number.

Comprehend the oral statements of others

An example of the need for teachers to *be able to comprehend the oral statements of others* could be: "1 dm^3 is the same as 1 litre. Since there are 1000 cm^3 in 1 dm^3, why are there not 1000 cL in 1 litre, but only 100 cL?", where the teacher needs to be able to enter into a dialogue with the students about the meaning of "deci", "centi", "milli", etc. in relation to the different unit systems.

Express him or herself in different ways

An illustration of the teacher being able to *express him or herself in different ways*, could comprise the teacher, at a later educational stage, referring to the commutativity of multiplication, while he or she would, to a 3rd grade class, refer to the same phenomenon using an everyday statement like "it doesn't matter which one we take first when we multiply two numbers" and illustrate this with explanatory examples, i.e. supported by drawings. At a further educational stage, the teacher can refer to the definite integral as a linear functional, while at high school level, this would be limited to stating that $\int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx$.

7.2.8 Aids and tools competency

Characterisation

Knowing possibilities and limitations of, and being able to use aids and tools

This competency consists of, on the one hand, *having a knowledge of* the existence and properties of the diverse forms of relevant tools used in mathematics and having an insight into their *possibilities and limitations* in different sorts of contexts, and, on the other hand, being able to reflectively *use* such aids.

Didactic and pedagogical comments

Since one of the aims of mathematics teaching is to foster the students' competency in dealing with current and relevant mathematical aids and tools, the teacher also has to be in possession of this competency to a reasonable extent.

Students' different backgrounds

Furthermore, a mathematics teacher will typically have a heterogeneous group of students when it comes to maturity, background, interests, "learner styles", etc. It is therefore necessary that the teacher knows and can use a broad repertoire of tools and aids which are typical and accessible for the mathematical activities at the level he or she teaches.

Insight into didactic potentials

The teacher can make use of different aids and tools, e.g. computers, as a means of initiating or fostering the students' learning processes. This means that the teacher's own knowledge and mastery of such tools and aids must be combined with an insight into what the individual tool can convey to mathematics teaching with regard to content and working methods at different levels and in different contexts.

Exemplification

Lowest classroom levels

At the lowest classroom levels, mention can be made of the teacher's ability to reflect on and make use of concrete materials like counting blocks, centicubes or other block, brick or rod systems, abacuses, etc. in support of the development of concepts among individual students, the study of connections and patterns, the examination of hypotheses, the laying down of basic abilities, etc.

Higher levels

At the higher levels, there is talk of the incorporation of geometric templates, spirographs, rulers, compasses, protractors, dice, specially ruled paper, cardboard for folding or cutting, etc.

Mention could also be made of the teacher's reflective initiation of student activities that imply the utilisation of calculators. It is vital that the teacher knows how such use of calculators can influence the students' conception and understanding of numbers as well as their ability to calculate.

In the same way, the teacher needs to decide whether to use specific computer programmes, e.g. to create new ways of working with subjects and concepts. This could refer to

- promoting work with probability with the help of simulations or statistical investigations.
- increasing the possibility of graphical support for problem solving in relation to working with functions.
- judging whether a dynamic geometry programme is suited to explorative investigations, e.g. of what happens to the area of a triangle if one retains the base and "pulls" the opposite angle, e.g. parallel to the base or perpendicular to it, or what happens if one makes different plane sections in a spatial figure.
- judging the usefulness of spreadsheets to calculate many different values of formula expressions and for the production of diverse types of diagrams, or for model building and treatment (e.g. of the type $balance_{n+1} = balance_n + balance_n \cdot r + y$ for the modelling of an annuity investment).
- relating to the possibilities of being able to visualise mathematical objects, phenomena and situations both statically and dynamically with the help of computer systems.
- judging the usefulness of different computer-based mathematical packages to solve equations, including differential equations, symbolic algebra, numerical analysis, graphic representation, as well as packages which include special modelling tools, etc.

7.3 Overview and judgement regarding mathematics as a discipline among mathematics teachers

Relate to the subject as a whole

It is important that mathematics teachers possess overview and judgement regarding mathematics as a discipline in such a way that they cannot only relate to specific teaching situations, but also to the subject as a whole.

The teacher has to, him or herself, possess and be able to communicate to the students an adequate picture of mathematics as it is manifested in relation to the students' current and future worlds, and it is here that overview and the exercise of judgement regarding the subject plays a key role.

7.3.1 The actual use of mathematics in other subjects and practice areas

Characterisation

Who uses mathematics for what?

The object of this type of overview and judgement is the actual use of mathematics for non-mathematical purposes within areas of everyday, social or scientific importance. This usage is brought about and expressed via the building and utilisation of mathematical models.

Didactic and pedagogical comments

Justify own teaching

It is important for a teacher to have an overview of when, for what purpose, and by whom mathematics is actually and can be used, and also why it is significant to learn mathematics in a modern society. This is one of the preconditions for a teacher to be able to justify his or her teaching both to him or herself, to the students and parents, and to other colleagues.

Interdisciplinary teaching

Interdisciplinary teaching of subjects and themes, often in the form of project work, is becoming more widespread at more and more stages of education. So as to be able to take mathematics learning into consideration in this regard, it is important that the teacher has a large store of knowledge about when mathematics can be used and to what purpose, and about the (possible) connections between mathematics and other subject areas. Related to this is the ability to judge when an activity will be unable to result in a satisfactory mathematical outcome.

In a broader perspective, it is important that the teacher is aware of the fact that a significant part of mathematical application is nowadays hidden in complicated models which are often further disguised by a strong ICT component. This utilisation has, not infrequently, significant social consequences both of a positive and negative nature, and based on a foundation that is as often as not unjustified, even though the mathematics is often assigned an authoritative role in this connection if, that is, it is at all visible. Having an insight into this sort of relationship is a precondition for the teacher being able to be both critical of mathematical application and being able to see under which conditions the application is justifiable, both of these so as to contribute towards the students reaching an adequate and nuanced view of the conditions.

Be critical of mathematical application and the circumstances justifying this application

7.3.2 The historical development of mathematics, both internally and from a social point of view

Characterisation

The subject of this form of overview and judgement is the fact that mathematics has developed in time and space, as well as in culture and society.

Development of mathematics in time and space

Didactic and pedagogical comments

It is important that students achieve a knowledge of the fact that mathematics is not immutable, not something that has always been as it is, but something that has developed through time by virtue of human activity and in step with different social developments, and that it will continue to do so. This can contribute towards students reaching a nuanced perspective of mathematics and mathematical activities. Such knowledge is achieved by, among other things, incorporating relevant historical aspects of mathematics in the teaching. This presupposes that the teacher him or herself has an overview of the main trends and points in historical mathematical development.

Mathematics is not immutable

Besides this, the teacher's use of well-chosen historical points and illustrations can serve the didactic and pedagogical purpose of showing that some of the students' difficulties with acquiring mathematical concepts have also been the difficulties that mathematics has needed to conquer throughout its history.

Historical parallels to students' difficulties

7.3.3 The nature of mathematics as a discipline

Characterisation

Nature of mathematics in relation to other subject areas

Mathematics has its own characteristics as a discipline. It is these characteristics which form the basis of the above-mentioned types of overview and judgement. Mathematics has some of these characteristics in common with other subject areas, and others not.

Didactic and pedagogical comments

When and why specific areas can be incorporated

Part of the mathematics teacher's job is to develop students' understanding of the characteristic features of mathematical thought processes and activities. Included here are the traits by which mathematics differs from other subject areas. This obviously presupposes that the teacher has an insight into specific mathematical traits. To be able to judge the purpose to which, when and in what way specific areas of mathematics can be incorporated into teaching, the teacher has to have an overview of the problem formulations, thought processes and methods which characterise the subject.

This is, for example, important when it comes to organising teaching situations where the students are inspired to find patterns and structures, set up and test hypotheses, discover examples, solve problems, attempt to generalise and set out justifications for the results found; processes which often call for qualification via the teacher's considered use of questions and suggestions.

Part IV

Competencies and subject matter

8 The interplay between subject matter and the competencies at different levels

8.1 Introduction

Basically there are two types of links between the competencies and subject material:

A competency is practised and developed in relation to the subject material

- A competency can *be practised* in relation to the given subject material, i.e. come into play and be expressed in relation to this subject material.
- A competency can *be developed*, i.e. created or consolidated in relation to the given subject material.

As far as the latter is concerned, it is fundamental to the way of thinking of the KOM project that the competencies can only be developed through close contact with and in interaction with concrete mathematical material, and not merely by hearing or reading about them, or experiencing them through some or other form of general context independent drill.

There is no doubt that some types of material are more suitable than others when it comes to promoting the development of a certain competency. For example, prolonged use of arithmetic is undoubtedly necessary for the development of the modelling competency in relation to a variety of everyday problem situations, but it hardly makes for the best foundation for the development of the full reasoning competency, including proof and proving. Similarly, an absorption in abstract algebra is a significant aid to the development of the symbol and formalism competency, but is an insufficient means by which to develop the modelling competency.

With this said, there is much reason to believe that each of the competencies in all essentials can be developed by the use of a broad spectrum of quite different subject materials. This is because, in general, the same mathematical points of view and methods are repeated and are the mainstay in all work with mathematical material.

Competencies alone cannot decide the choice of material

These considerations imply that it is only to a lesser extent that decisions regarding the choice of subject material to be on the agenda of the various educational and teaching stages is based on consideration of the competencies. This choice must therefore be based on other considerations. These could be the significance and placement of the individual subject areas in the creation of the understanding of mathematical concepts, or in the building up of mathematics as a discipline. It could also refer to the relevance of the subject area in relation it makes to the use of mathematics for extra-mathematical purposes for the target groups at which it is aimed.

Focus on exercising a competency in a chosen subject area

In the light of this, our task must be seen as, on the one hand, carrying out an identification of a few large and general mathematical subject areas to be included in mathematical teaching at the different levels, and, on the other hand, putting forward some considerations as to how the individual competencies can be *exercised* in relation to these subject areas. The desire to keep the number of subject areas relatively small is due to a need to avoid an inexpedient over-particularisation which can so easily lead to attention being directed to the absence or presence of individual points in the curriculum.

Our focus here on the way in which the competencies are exercised in relation to the subject areas does not mean that the development of the competencies via association with subject material is relegated to secondary importance. On the contrary, this matter is of the greatest importance for the organisation and implementation of concrete teaching. However, this is a theme which will not be dealt with in the present context.

8.2 A matrix structure

Subject $\times$ competency creates a matrix

It would be obvious for each teaching level to think by means of a matrix structure where the mathematical subject areas comprise, for example, the rows, and the eight competencies the columns. The matrix could then be regarded as a statement of how the individual competencies are practised in relation to the individual subject areas. This means that the unit of consideration is the individual subject area, corresponding to the rows, while the focus is on the manifestation of the competencies herein, corresponding to the columns.

Competency/ Subject area	Math. thinking comp.	Problem tackling comp.	Model-ling comp.	...	Aids and tools comp.
subject area 1					
subject area 2					
...					
subject area n					

One can imagine different models for completing and utilising such a matrix structure. We have chosen a model where, for each individual cell, "the specific correlation" between the occurring subject area and the occurring competency is concretely determined. The nature of this correlation can consequently vary from cell to cell. For some cells there is perhaps the possibility that the relevant competency (nearly) plays no role in the use of the subject area concerned. For others it could be that the correlation between the competency and the subject area is of a different nature to that in neighbouring cells.

Each cell: Interplay between subject area and competency

However, with this model the content of each cell in the matrix has to be decided on individually for each teaching level. In practice, it will be necessary to make do with illustrating the procedure by describing a representative selection of cells for the different teaching levels. To illustrate this, a sketch of a possible description of the way in which the competencies can be expressed in chosen subject areas at specified levels appears later in this text.

Illustration of a representative selection of cells

8.3 Choice of subject areas

The first task is to decide on the degree of detail in the subject areas we select. The task group has been unanimous in deciding on a number of main subject areas which, to avoid syllabus entrenchment, will *not* be subdivided but instead described in a short text, the sole idea of which is to make it possible to understand what is meant by the main headings. These texts consequently do not serve to prescribe a syllabus.

Avoid exaggerated detailing of subject

We have concentrated on ten subject areas which are on the agenda in the school system or in introductory further education. Subject areas with a more specialised position, e.g. at university level, have not been taken

Choice of ten mathematical subject areas into consideration here. It needs to be emphasised that we have chosen only to select *proper mathematical subject areas.* This means that specific utilisation areas like coinage, weights and measures, standard models and the like are not mentioned specifically, though this is not to disparage them. They are supposed to be activated in relation to the relevant mathematical subject areas and competencies. Finally, it must be noted that not all subject areas necessarily appear at all teaching levels and that, even though the description of the individual subject area is the same for all the levels concerned, this does not mean that all the characteristics which appear in the description necessarily ought to be included on the agenda of each level. Which subject areas and characteristics are dealt with at each level is a question requiring a specific stand for the level concerned.

8.3.1 A combination of "classic cornerstones" and newer subject areas

The choice of subject areas depends partly on the view of the classic cornerstones of mathematics as a discipline and a teaching subject, and partly on a selection of those modern subject areas which are significant for the use of mathematics within a broad spectrum of other subjects or practice areas.

The classic cornerstones

The classic cornerstones comprise numbers, algebra, geometry and functions, the first three of which have constituted the core of mathematics for a couple of millennia. These four are all included in the following list of subject areas (and in lists of subject areas all over the world) with the single modification that the "numbers" area has, due to its importance, been divided into two: one that focuses on the actual concept of numbers and number areas, and one that focuses on the use of numbers for applicational goals, e.g. arithmetic. A distinction has furthermore been made between the concept of functions as such and particular special functions on the one hand, and the analytical study of functions (calculus of infinitesimals) on the other hand.

Newer subject areas of significance for, e.g. applications

As regards *the newer subject areas*, the main aim, as mentioned, has been to choose aspects with a broad significance, both in relation to their application and to the teaching level under consideration. In this case the choice has fallen on probability theory, statistics, discrete mathematics and optimisation, all of which play significant roles within a variety of scientific, professional and practical domains.

The number domains: By this is meant the concept of numbers and the classical main number domains: the natural numbers, the integers, the rationals, the reals, and the complex numbers. There is, in addition, the notation of numbers, including the position system, fractions, decimal numbers, etc.

Arithmetic: By this is meant the four species addition, subtraction, multiplication and division as applied to concrete numbers, as well as diverse algorithms to carry out the calculations. Included in this subject area are also the calculation of percentage as well as estimation and approximation.

Algebra: By this is meant the formal characteristics of compositions applied to various sets of objects such as compositions and their interplay, including general operation rules, equations and the solving of equations, algebraic structures (groups, rings, fields, vector spaces, etc.), algebraic investigations of geometrical objects.

Geometry By this is meant the whole spectrum of geometrical problems, points of view and disciplines such as descriptive geometry concerning planar and spatial objects, geometrical measurement, coordinate systems and analytic geometry, deductive geometry (on a global or local axiomatic foundation), curves and surfaces, differential geometry, geometrical investigations of algebraic objects.

Functions: By this is meant both the concept of functions itself, including the notion of variables and graphs of functions, as well as the basic special real functions: linear and other polynomial functions, rational functions, trigonometric functions, power functions, exponential and logarithmic functions.

Calculus: By this is meant classical real analysis concerning topics such as continuity and limits of functions, differentiability and differentiation, extrema, integrability and integration, differential equations, and convergence and divergence of sequences of numbers and series as well as numerical analysis.

Probability: By this is meant the actual concept of randomness and probability, combinatorial probabilities and finite probability spaces, stochastic variables and distributions, including the usual standard distributions and axiomatic probability theory.

Statistics: By this is meant the organising, interpretation and drawing of conclusions concerning quantitative data, such as uncertainty, descriptive statistics, empirical distributions, estimation of parameters, test of hypotheses, planning of experiments and inference.

Discrete mathematics: By this is meant the investigation of finite collections of objects (or infinite collections which do not constitute a continuum): counting methods and combinatorics, classical (elementary) number theory, graphs and networks, codes and algorithms.

Optimisation: By this is meant determining local or global extrema for real functions with or without calculus, such as maxima and minima for real functions of one or more variables, optimisation under constraints, including linear programming.

8.4 Subject areas and educational and teaching levels

Subject areas and teaching levels: What and when?

In the following diagram we have suggested at which educational and teaching level the individual subject areas ought, *at the latest*, to be dealt with explicitly and in relation to systematic steps in some or other way.

Ed. level/ Subject	Grade 1–3	Grade 4-6	Grade 7-9	Grade C (10)	Grade B (11)	Grade A (12)	Junior sec. teacher	Univ. ed.
Number domains	•	•	•	•	•	•	•	•
Arithmetic	•	•	•	•	•	•	•	•
Algebra			•	•	•	•	•	•
Geometry	•	•	•	•	•	•	•	•
Functions		•	•	•	•	•	•	•
Calculus					•	•		•
Probability		•	•	•	•	•	•	•
Statistics		•	•	•	•	•	•	•
Discrete math		•	•	•	•	•	•	•
Optimisation					•	•		•

It must be emphasised that this is not a suggestion that the subject area *first* ought to be incorporated at this level. Good teaching over time implicitly builds up to an explicit coverage of the subject area at a later stage.

For example, it would be obvious in grades 1 – 3 to work with intuitive considerations of chance and risk in connection with games, without the concept of probability as such being on the agenda, or to work with the question of what happens to a particular magnitude if another amount is changed without the concept of functions being dealt with expressly. In the same way, in grades 7 – 9(10) and at the lover stream in high school/senior secondary school it makes sense to deal with certain *problems* regarding optimisation without optimisation as a *subject area* being on the agenda. In other words, the lack of any marking in a cell in the matrix does not necessarily mean that a subject area cannot by rights be taught at the level in question.

Finally, it must again be emphasised that the presence of a subject area at a particular level says nothing about the extent or way in which it must be represented.

8.5 Examples of the interplay between competencies and subject areas

Two examples for illustration

When it comes to the actual description and formulation of the interplay between competencies and subject areas, one could imagine something along the following lines:

8.5.1 Example: Geometry for grades 1-3

Expectations when handling the material

Geometry for grades 1-3

At the end of grade 3, the students are expected to *know the names of* elementary geometric objects and basic shapes (such as e.g. point, line (segment), circle, ring, triangle, quadrangle, sphere, cube, box etc.) and to be able to use these for *idealised descriptions* of physical objects (e.g. drawing of a car or a bus as circles with quadrangles of varying sizes on top, a house as a quadrangle with a triangle on top, etc.). They are furthermore expected to have an idea of the (approximate) *measurement* of lengths, areas and volumes by means of given units. They are assumed to have gained experience with the *representation* of elementary geometric basic shapes, situations and phenomena by means of *concrete material* such as e.g. drawings, puzzles, rods, hinges, strings, geoboards, screen images, icons etc. They can, when using concrete geometric objects, relate to and *ask questions* of the type “What is a thing like that called…?”, “How many corners does a triangle, a square, a cube have?” or “Can a 1-sided (or 2-sided) object exist, and, if so, how many corners does it have?” On the basis of inspection of individual cases, they can also *answer such questions*. In this connection, they can support their answers with simple *reasoning* resting upon concrete experiences. They are furthermore expected to be able to *communicate* with their friends and close adults about their experiences and encounters with geometric objects and phenomena using everyday spoken language and drawn pictures.

How are the competencies practiced?

Which competency is activated?

From this it is evident that the basic features of the *mathematical thinking competency* are expressed in connection with the formulation of questions regarding the names and properties of geometric objects. The *modelling competency* is expressed by seeing geometric objects as idealisations of physical objects, while the *problem tackling* and *reasoning competencies* appear in connection with the study of basic geometric phenomena and the justification of the answers achieved. The *representing* and *aids and tools competencies* come directly into play in relation to the actual way in which the students meet geometric objects, while the *communicating competency* is directly practiced in the students' work together with each other and in conversation with adults. On the other hand, the *symbol and formalism competency* does not seem to play an independent role in geometric activities at this level.

8.5.2 Example: Functions for grades 7-9

Expectations when handling the material

Functions for grades 7-9

At the end of grade 9, the students are expected to be able to understand, activate and *utilise the concept of functions* in their normal form in internal mathematical contexts and in connection with various mathematical applications including modelling. In this connection the students are able to *choose or identify the dependent and the independent variables* in the given situations. They are furthermore expected to be able to understand, interpret, utilise and construct different *representations* of functions, including rules of calculation, tables, spreadsheets, etc., and, not least, *graphs of functions*. They are especially expected to be able to deal with *linear functions*, both algebraically and graphically, and to be aware of the properties of these functions and the connection between the properties and the characteristic parameters of linear functions. In this regard, they are able to raise, and often answer, questions about the presence or absence of *proportionality* and *linearity* as well as relate to the use made of these in mathematical models. This comprises, amongst other things, that the students, in theory as well as practice, can differentiate between linear and other functions.

How are the competencies practiced?

Which competency is activated?

To be able to deal with the concept of functions in general and linear functions in particular, the students need to be able to activate the *mathematical thinking competency*. To be able to formulate and solve "pure" or "applied" problems involving the concept of functions, they have to master the *problem tackling* and *modelling* competencies. In order to be able to justify interpretations concerning statements about functions and to be able to verify the correctness of solutions to problems about functions, the students must have a certain amount of *reasoning competency*. It is obvious that the *representing competency* is central for dealing with nearly all the aspects of the concept of functions, also at this level. *The symbol and formalism competency* appears in relation to the manipulation of algebraic (mainly linear) functions, used to solve equations or to draw conclusions of an analytic-geometric type. Every time the students, written, spoken or with figures have to show, describe, explain or discuss the use of, presence of, or characteristics of one or more functions, the *communicating competency* is clearly activated. The *aids and tools competency* plays a role especially when the students make use of calculators or computers for visual or tabular representation of graphs of functions, variations of parameters, reading of function values, intersections etc.

8.6 Overview and judgement in relation to subject areas

As to the connection between the competencies and the subject areas to apply, the competencies, as mentioned, need to both be developed and practised in relation to the work on the subject areas. The same applies to the connection between overview and judgement on the one hand, and the subject areas on the other hand.

Development and practice of overview and judgement in relation to math. subject areas

It is certainly true that overview and judgement are *developed*, among other things, by dealing with the subject areas, something which is a significant question for concrete teaching. Conversely, the *practice* of overview and judgement regarding the actual application of mathematics, its historical development and special nature as a subject respectively, contributes to the motivation and colouration of the activities within subject areas, and to the extended understanding of the organisation and nature of such areas.

The means by which this takes place, depends on both the level and the subject area. Overview and judgement already developed in relation to the *application of mathematics* will play a role in the use of subject areas like arithmetic, functions, probability, statistics, discrete mathematics and optimisation. Overview and judgement regarding *the historical development of mathematics* will enrich and support the dealing with all the subject areas, not least the number domains, arithmetic, algebra, geometry, probability and calculus. The same is also true of overview and judgement when it comes to *the special nature of mathematics* which adds to insight and understanding of the central subject areas like algebra, (deductive) geometry, calculus and probability.

8.7 Directions for a natural continuation of the work

Invitation to further development work

The structure of the interplay between competencies and the subject material, which we have presented in this chapter, may be seen as an invitation to engage in a larger analytical development work. In the task group, we have along the way had ideas for a range of elements which ought to be included in any continuation of the present analysis which, considering the framework of this project, we have only been able to sketch briefly.

"Completion" of cells in the matrix

The most suitable form of continued analysis would be, for each teaching level, to *"complete" all the relevant cells in the competency/subject material matrix*. For each cell, this means trying to answer questions like:

- Which role does this combination of competency and subject material ought to play on each specific teaching level?
- If the answer is "none ", why is this? (In other words, it makes sense to comment on the cells which, on a first "completion" of the matrix, remain blank.)
- If the combination does have a role to play, what is fruitful about exactly this combination?

The two examples of a completion of a cell given above are meant to be an inspiration for answering these questions for the other cells in the matrix.

Weighing up of the content elements

A natural continuation of this comprises *weighing up* the different content elements in relation to each other.

- Which elements ought to be heavily weighted and which ought to play a more humble role given that there is a limited amount of time and mental energy available? (One can imagine "large" and "small" dots in the matrix – "either or" is soon seen to be a too simplified approach.)
- Are there specific competencies which, across the subject areas, ought to have a particularly prominent position on a specific teaching level?
- Is there anything similar which is true for specific subject areas?

Analysis of the matrix by column and row

The last two questions suggest to depart from the "cell level" and analyse the individual *columns and rows* of the matrix:

- What connection can one expect between the way a given competency can/ought to be developed or practiced in relation to the different subject areas?
- What connection can one expect between the way a given subject area contributes to the development of the different competencies?

Teaching level as the third dimension

In relation to the last question, one can start by looking at the treatment of a single concept: How can work with, e.g. the concept "circles" be organised/-approached/ challenged if one wishes to contribute to the development of the students' reasoning competency/modelling competency/aids and tools competency, etc? All the considerations mentioned above have only been in relation to one given matrix and, thereby, one given teaching level. The structure we have sketched here is, however, three-dimensional. With the matrix as the starting point, the teaching level forms the third "depth dimension" which is easily overlooked. Analysis within this dimension deals with *longitudinal subject areas and their connections to competencies*:

- What should the *longitudinal* connection and development be like in the education system with regard to working with competency X in relation to subject area Y?

In relation to the implementation of the KOM project, these are perhaps the most important questions of all.

Part V

Progression in and assessment of the development of mathematical competence

9 Assessment of competencies

9.1 Competencies are manifested in activities

A competency is an insight-based readiness to act and is manifested in mathematical activities

Mastering one of the eight mathematical competencies (to a greater or lesser extent) comprises, as has been emphasised many times in this report, being prepared and able to carry out certain mathematical actions on the basis of insight. The core of a competency, in other words, is an insight-based readiness to act, where "the action" can be both physical and behavioural – including oral – as well as mental. A valid and comprehensive assessment of a person's mathematical competencies must therefore, as a starting point, be based on identifying the presence and extent of these features in relation to the mathematical activities in which the respective person has been/is being involved.

About mathematical activities

A mathematical activity consists in a set of conscious and goal oriented mathematical actions in a situation. The fact that the action is goal oriented does not mean that it is a foregone conclusion. A mathematical activity can, for example, be to solve a pure or applied mathematical problem, to understand or construct a concrete mathematical model, to read a mathematical text with the view of understanding or acting on it, to prove a mathematical theorem, to study the interrelations of a theory, to write a mathematical text for others to read, or to give a presentation, etc.

Carrying out any mathematical activity demands the exercise of one or more mathematical competency. Let us for a moment presume that for a specific activity it is, on the one hand, possible to identify (respectively necessary and sufficient) competencies in advance and, on the other hand, to observe in which sense and to what extent a person carrying out the activity brings the different competencies into play. It is hereby possible to detect and judge the competencies of the person concerned in relation to the specific activity undertaken.

Study of the competency content of activities

An preliminary study of the competencies in a given activity is, first and foremost, a theoretical and analytical undertaking, although its empirical moments do exist. Of crucial importance here, is being able to define and characterise the activity and its component parts and demands in a relatively well-demarcated and clear way. A study of which competencies a person actually brings into play in a given activity is above all an empirical enterprise. It can only be realised if the content of the competency in a person's actions while carrying out the activity and the results of such actions are detectable in a valid, reliable and clear way.

Different activities involve different competencies

It is expected that a given activity only can provide opportunity for the use of a subset of competencies. Different activities will hereby also provide opportunity for the involvement of different sets of competencies. It is therefore reasonable to suppose that a spectrum of different natured mathematical activities is necessary to master a comprehensive and rich representation of the full set of mathematical competencies. Similarly, it is to be expected that for one to achieve a comprehensive and rich picture of a person's mathematical competencies, one must study that person's actions within a broad range of mathematical activities.

9.2 The task

Until now we have had as our starting point mathematical activities and looked at their competency content both theoretically and empirically. Actually, however, the task is the opposite: partly to find a way of evaluating the individual person's mastery of a given mathematical competency, and partly to get an overall picture of the respective person's mathematical competency profile. Since the competencies are expressed via mathematical activities, the task can be defined as follows:

- To find – or construct – *types* of mathematical activities suitable for a valid, reliable and clear way of demonstrating the presence of a given mathematical competency in a person involved in the activity. This presupposes that instruments exist, or can be created, that make it possible to detect, characterise and judge the extent and depth of competency mastery in the way it is expressed in the individual activity.

To produce activities which demonstrate a competency or competency profile

- To find – or construct – *a set* of mathematical activities which together are suitable for a valid, reliable and clear way of illustrating a person's total mathematical competency profile, e.g. a person's mastery of the complete competency spectrum.

- On the basis of this, to find a way of identifying, characterising and judging progression in a person's mastery of one or more of the mathematical competencies.

We have, in section 4.4.4 (page 72), introduced three dimensions in a person's mastery of a competency: *degree of coverage*, *radius of action* and *technical level*. With their help, the task can be further defined to cover detection, characterisation and judgement of the degree of coverage, radius of action and technical level respectively whereby a person can activate a given mathematical competency in a variety of mathematical activities. While the first two parts of this task involve finding out the state of a person's competency mastery, in other words a static picture, the third part involves describing the development over time of this competency mastery, in other words a dynamic picture. The three pictures hereby become the key in the description of the progression of a person's competency mastery.

Static and dynamic description of the dimensions in competency mastery

It is vital to remember that in this task we are not only thinking of the final – summative – assessment in the form of different tests, examinations and the like, but just as importantly of the continuous assessment during teaching with the aim of supplying information about and for the individual student – formative assessment – or for the teacher about the status and development of the teaching.[1]

Focus on both formative and summative assessment

9.3 Progression

One of the most important tasks of the KOM project has been to investigate the possibilities of identifying, characterising and judging progression in a student's development of mathematical competencies as he or she progresses through the education system. By incorporating the *degree of coverage*, *radius of action* and *technical level* dimensions to characterise a student's actual mastery of a specific competency, we have obtained a tool for dynamically describing how the respective student develops the respective competency over time. Since it is the same competency in question the whole way through the education system, the progression description is not limited by what happens at an individual teaching level. A student's mathematical competency is developed by being expanded with the reclamation of "new land", i.e. by its degree of coverage, radius of action or technical level being expanded over time. We assume that as long as one is doing mathematics in the education system, there can normally only be stagnation or growth

A competency's degree of coverage, radius of action and technical level is expanded over time

[1] In Niss (1993) you can read more about the aims of and components in assessment.

in a dimension and not reduction. The fact that this assumption does not hold true in the case of any lengthy break in using mathematics must be viewed as less important in this context.

Figure 9.1 is an attempt to illustrate the growth of the competencies. The figure is, further more, suitable for illustrating that a competency can be improved in different ways depending on which dimensions are touched on during development. For example, one can imagine that degree of coverage and radius of action are expanded, while the technical level remains unchanged. Or that degree of coverage and radius of action are unchanged while the technical level improves, etc. In principle one can imagine the development of a certain mathematical competency in a student being followed and registered throughout his or her course of education.

Characterisation of progression promotes development of competencies

If and when we are able to characterise progression in the development of a competency, we also have a tool to actually promote this development, as it is possible for the teacher to give attention to his or her students on points where there is room for further reclamation of new land. It is hereby possible to achieve conditions to programme arrangements and activities whereby such expansions can take place.

If it is possible to describe a student's progression in the mastery of each of the mathematical competencies as indicated here, we have automatically achieved a description of progression in the whole set of competencies, in other words in the mathematical competency profile of the student concerned. We hereby also obtain a tool to promote the actual development of the competency profile as a whole.

9.4 Assessment forms and instruments

Need for research and development work

It must be stressed that the solution of this task exacts considerable further research and development work, not least when it comes to the second and third part of the task. A range of the existent, more or less current, assessment forms and instruments can be suitable to detect and judge certain of the competencies.

Use of well-known assessment forms

Solving *written exercises or problems* can particularly be used to evaluate parts of the problem tackling, reasoning, representing, symbol and formalism, communicating, and aids and tools competencies. The same is true, depending on the specific framework, of *oral exercises, quizzes and interviews* which can also be used to evaluate the mathematical thinking competency and the three types of overview and judgement. *Essays* can

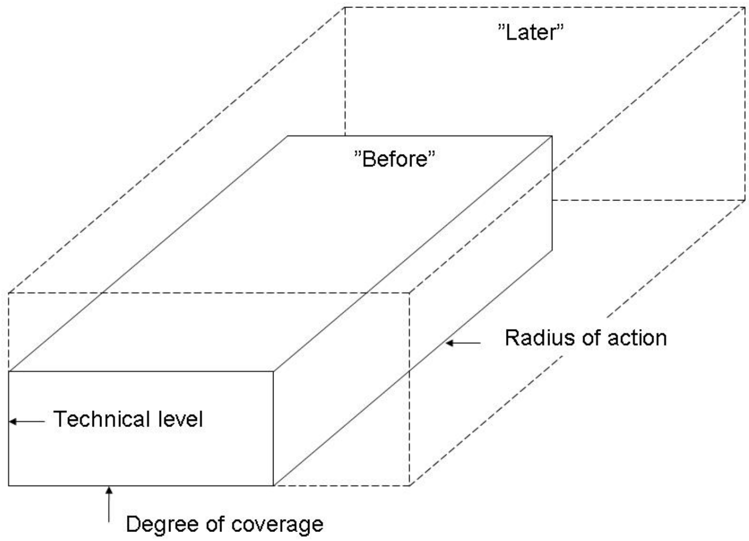

Figure 9.1 Visual presentation of progression in a person's mastery of a mathematical competency.

be a suitable means of evaluating the mathematical thinking and communicating competencies as well as overview and judgement. *Projects* can – depending on their type and form – serve to evaluate the whole spectrum of mathematical competencies, including overview and judgement. Projects are, not least, especially well suited to evaluating the modelling competency. The same is true of thorough *observations of students at work*, and of their *logbooks* (i.e. a type of notebook-*cum*-diary used by students to note down their activities and their considerations regarding what they have done), and *portfolios* (i.e. files containing the student's written work). *Actively telling others* in the form of articles, lectures, posters and media products can be suitable for the evaluation of the mathematical thinking, representing, communicating, and aids and tools competencies.

Some of the assignments described are first and foremost designed to be evaluation instruments (e.g. oral exams and portfolios), while others serve a mixed purpose being, at one and the same time, both an evaluation instrument and a learning tool (e.g. written assignments and essays), and still others are first and foremost tools used for teaching and learning which can also be used for evaluation purposes (e.g. projects and actively telling others). Finally some serve as both an evaluation instrument and as a support for students' reflection on their own learning, otherwise known as metacognition; this is the case for logbooks.

This should not be regarded as an exhaustive overview of competency relevant evaluation forms, let alone of evaluation forms in general. The aim is merely to point out that well-known evaluation forms can be suitable as instruments of evaluation of one or more of the competencies. Neither is there, on the other hand, talk of each of these evaluation forms and instruments being in any case equally suitable for evaluating the given competencies either individually or together. Considerable development and design work needs to be done in order to make them suitable for this purpose. This work has to comprise incorporating different types of relevant mathematical activities within different fields into the relevant evaluation forms. Nevertheless, as a start, there is good reason to use the existing forms with the intention of seeing how far one can go with them when evaluating the competencies so as to better utilise the investment of effort in thinking out new evaluation forms for this purpose.

Known evaluation forms to be further developed

9.4.1 Tests and examinations – old and new forms

Classical forms and their variations

The evaluation forms and instruments which are most often in use in Danish mathematics teaching constitute a rather small spread. When it comes to tests and examinations, individual oral and written tests dominate the picture. The written tests normally comprise pre-formulated pure or postulated/stylised applied mathematical tasks which, under invigilation, have to be solved within a set time limit from a few minutes to 4-5 hours typically sat at the school or institution. The oral tests which are mainly used at the end of the year or at exams, normally take place with the student drawing lots for one or more question which then has to be presented and treated during an oral séance, after which it is possible for the teacher and possibly the external examiner to pose more in-depth questions. It is very common that the student at such an exam is given a certain amount of preparation time to eliminate pure memory problems in the presentation.

Tests and exams dominated by individual written and oral test forms

In recent years these "pure" forms have in many places been relaxed in different ways. For example, a written examination can now take the form of a "take home exam", where the student has a couple of days at his

Relaxation of the classic forms

or her disposition to work out the given assignments, and where a solemn declaration is regarded as ensuring that the student did so without outside assistance. The intention with this modification of written examinations is mainly to reduce the distorting effect of a restrictive time factor on the quality of the answers. Sometimes an oral examination can include a greater or lesser element of presentation of material prepared at home, e.g. of larger assignments or projects which the student has worked on during the course of teaching, or that the examination is no longer individual but covers a group of students.

In their classic pure forms there are narrow limits for which (aspects of) the mathematical competencies the written or oral examinations can evaluate. It is especially difficult to incorporate the creative, in-depth and time-consuming sides of the competencies in these types of exams. The relaxations mentioned in written and oral examinations open up for the evaluation of more facets of the competencies than possible with the pure forms, but there are still limits to what one can achieve in this regard. For example, the evaluation of a student's ability to carry out whole complex modelling tasks, to find out and implement non-routine problem solving or mathematical proofs, or to produce larger coherent pieces of mathematical text demanding a different framework than that available in the modified test and examination formats.

Opening up for evaluation of more competency facets

Arrival of new forms

Gradually, as mathematics teaching over the course of the last two to three decades has changed so that a broader spectrum of teaching and working forms have acquired a certain foothold in the practice of mathematics teaching, there has also been a certain development in the test and examination forms employed round about in the education system. Not least the increasing incorporation of project work, first at some universities, later at other further educational institutions and in primary and junior secondary schools and high schools, has lead to these changes.

New test and examination forms

Project work

In many places it has also become more normal to arrange for group examinations of project work carried out by smaller groups of students either over a long period of time, or on the spot in the exam situation, as it is used in the final examinations in primary or junior secondary school where the students in small groups work with a two hour assignment under observation and questioning of the teacher and an external examiner. In view of the possibilities for complexity in mathematical related project work, it is obvious that evaluation of project work opens up for many possibilities to incorporate an evaluation of a whole lot of mathematical competencies.

High school third year assignment

In general high school students' individual work with their so-called third year assignment, if carried out in mathematics, has also opened up for new assessment forms at examination level. The fact that students hand in a whole, well-balanced mathematical text for evaluation, presents direct challenges to the evaluation of a whole lot of the competencies according to the nature and theme of the third year assignment. In any case the mathematical thinking, representing, symbol and formalism, plus the communicating competencies are central in answering the third year assignment.

The classic forms dominate

In the meantime, it should not be forgotten that these newly developed test and examination forms still only occupy a modest place in overall evaluations connected to mathematics teaching in Denmark. Added to this is the fact that neither the classic nor the more "modern" assessment forms have yet served to clarify or articulate mathematical competencies. However, together they offer a potential for moving significantly closer to this goal.

Continued need for new test and examination forms

Continued need for new developments

However, there is still a great need for continuously devising, testing and evaluating new test and examination forms. For the sake of illustration and inspiration, let us take an, in principle, arbitrarily chosen example of such a test which was introduced in a mathematics course at Roskilde University around 1997. Let us set the record straight by emphasising that actual competency terminology has only been used to a limited extent in connection with this examination form, and that only in very recent years.

An example: written exam with oral "defence"

The examination of the two particular subject fields (linear algebra with supplements, and mathematical analysis respectively) take place as follows: the students first fetch an assignment set at a particular time which has to be answered individually in written form within three working days. The assignments are typically very complex, often with open elements. Besides common questions like "Prove that … ", "Determine … ", "Find … ", there can also be questions like

The written part

- "Investigate whether… "
- "Formulate some hypotheses about the connection between objects of category A and objects of category B, and try to confirm or disprove

the hypotheses through proven claims, example-based conjectures, counter examples or the like."

- "Do objects exist, which satisfy property P, but not property Q? Do objects exist, which satisfy Q, but not P? Substantiate the results you reach by proven claims or examples."
- "Think of and comment on an example which illustrates the point in ..."

Questions like these are included to explicitly enable evaluation (and thereby also development) of the mathematical thinking, problem tackling, reasoning, and communicating competencies; the latter though – in this case – only in written form. Often some questions will, in addition, provide opportunity to evaluate the representing and aids and tools competencies. There are also usually some questions which demand a degree of creativity and inventiveness, and some which require a high level of technicality and trenchancy. The latter makes possible a more rigorous evaluation of the student's symbol and formalism competency. On the other hand, these examination assignments are unlikely to contain modelling problems, as this competency is evaluated in a different connection in the mathematics programme at Roskilde University.

After the students have handed in their written answers (accompanied by a solemn declaration that he/she did not receive outside assistance), these are sent to the examiner and the external examiner who then judge the answers.

The oral part

After about 14 days, an oral defence is held. Here the student explains and defends his or her answer orally to the examiner and external examiner, who, on the basis of the written material ask questions about the more obscure or weak points in the presentation. They can also challenge the student with the following types of questions:

- "On page ... you answered question x under these conditions. What would happen if ...? Would this result in a different answer? If not, why not? If so, how and why?"
- "You have not answered question y. Why not? Can you explain what the hurdle was? Can you say more about the question today, or possibly give an answer to it?"
- "Your use of concept z is a bit woolly. Can you tell us exactly what it is you understand by z and how you bring it into play in the present case?"

Besides the fact that an oral examination serves the purpose of ensuring that the student truly owns and can vouch for his or her own answers, the examination is also suited to evaluating, partly new, characteristics of the mathematical thinking, problem tackling, and reasoning competencies (often the representing, and symbol and formalism competencies too), but most obviously the oral part of the communicating competency.

One combined grade

When the oral examination is over, the student is given one combined grade, comprising an integrated weighing up of the written and oral examination. The grade and the performance on which it is based, is presented to and discussed with the student by the examiner and external examiner in a short follow-up session.

Demanding, but relevant examination form

It is possibly worth noting that the examination form described here is very often experienced by the respective student as intense and demanding, but at the same time very relevant, and something which covers the actual intentions and purpose of the course being examined.

Similar examples from technical and teacher training colleges

Similar types of examination forms had, by the end of the 90s, secured a footing in the teacher training mathematics exam (where an individual six hour test takes place after students have had a chance to work on preparatory material handed out 48 hours in advance), and the technical training college's mathematics B-level examination (where as an experiment, the examination comprised project work covering a three week period and resulting in a report, which is then defended in a ten minute oral test).

9.4.2 Continuous assessment

Classical forms

Greater freedom and flexibility

If we take a look at the continuous assessment of students undertaken by the teacher within the framework of everyday teaching, there is in general more freedom in the assessment forms and instruments available than in the final examination. To the extent that the teacher utilises tests, the situation has been dealt with in the foregoing discussion.

Written homework assignments

In Denmark, there is a solid tradition of seeing and judging a student's knowledge, proficiency and abilities by getting him or her to complete written homework assignments which typically involve working with exercises/problems in a spectrum ranging from the routine to (more rarely) complex problems which require an non routine-like overview, combinatory abilities and inventiveness. Answering such questions does not only serve an assessment purpose, but is also part of the learning and training process. The assignments are marked and commented on by the teacher which is one of the most important ways in which various interested parties can be informed about the teacher's view of the students' proficiency and development. At school, the teacher's judgement of the quality of an answer is often a part of the basis for awarding proficiency marks, while in further education systems, the quality of the answer is either a part of the decision basis for whether a course has been passed or not, or it has no consequences at all as far as the system is concerned. Except for the fact that the assignments are not completed under the same time restraints as for tests and examinations, and that the student can have had access to different forms of help in the process of answering, the assignments are similar in nature to those appearing in tests and examinations. Therefore it is mainly the same competencies that are expressed and revealed in these two cases.

Presentations at the board

Besides answering written homework assignments, it is normal practice in Danish mathematics teaching for the students to go up to the board for the opportunity to orally, with written support, present material and answers to tasks. These activities also serve more than one purpose, assessment being just one of them. The assessment purpose is served by the teacher (or, in rare cases, also classmates) commenting on the quality of a student's presentation and – if this is part of the game – allowing this to serve as a basis for grading the student. It is obvious that working at the board allows for an assessment of parts of the competencies which cannot so easily be evaluated from written work. This mainly refers to the communicating competency, but also to the thinking, reasoning, representing and symbol and formalism competencies.

Incorporation of new forms

Along with the developments which have taken place in the forms of mathematics teaching over the past three decades, there is also a need and possibility for continuous assessment. This has first and foremost happened in conjunction with students working in groups where the teacher's role is a combination of supervisor and observer, which allows for the inclusion of new possibilities for assessment of the mathematical competencies which are otherwise difficult to see in traditional written or oral work. On the other hand, the use of essays, logbooks and portfolios, as mentioned above, are not, as far as is known, widely used in Danish mathematics teaching.

New forms

Even in continuous assessment, there is a need for a development of new forms and instruments that go together well with claiming new territory within teaching and learning.

Student construction of tasks

An example could be to ask students to construct tasks that meet certain pre-set specifications. This could be that the task has to illustrate a specific theoretical point, for example:

- "Construct a mathematics exercise related to the real world, which on the one hand shows the difference between growth in percentages and growth in percentage points, and on the other hand demonstrates that one cannot always simply add up percentages."
- "Construct an exercise which shows that proportionality is not always a characteristic feature of problems from the real world."
- "Construct a mathematics exercise which illustrates why one, in the mean value theorem of calculus, cannot weaken the assumption about the continuity of the function on the closed interval."

The students exchange tasks according to one or other procedure, solve their classmates tasks and hereafter discuss the differences, similarities, points, good and bad ideas either in small groups or in plenum with the whole class. The evaluation of the results therefore takes place together with the students and the teacher. Once again, a large number of the competencies are highlighted in this type of evaluation, but the activity is especially suited to focussing on the thinking, reasoning, and communicating competencies.

Students correcting assignments

Another possibility is to ask pairs of students to produce written comments to and corrections of each other's answers, after which the commented answers are handed in to the teacher for further comments and evaluation.

Another example is especially aimed at evaluating (and developing) the mathematical thinking competency together with the formalism part of the symbol and formalism competency, in connection with the reasoning and representing competencies. This form has over the years been used extensively with mathematics course teaching at Roskilde University, where the course has been centred on working through a textbook. At intervals after the completion of larger segments of the book (chapters, sections, or whatever is relevant) e.g. around 50 pages or so, the students, individually or in groups depending on the arrangement, are asked to condense the leading or key concepts, constructions and results from the relevant segment in an organised and consistent form covering no more than say five pages. Being able to decide on what is essential and what is less important, and differentiate between general concepts and examples, etc. has proved to be a demanding, yet valuable, task for the students. The students' often different and, not least, competing contributions are all presented to the group after which a teacher-lead discussion takes place to ensure a common summary of the segment. This form especially gives the teacher good opportunities

Clarification of the principal constructions in mathematical subject matter to gain insight into not only the individual student's development of the competencies under discussion, but also of the success of his or her own teaching in promoting the conscious acquisition of the vital characteristics of mathematical theory by the students.

9.4.3 Adapting known forms of assessment to competency purposes

As appears, there is, in reality, a large repertoire of traditional and innovative assessment forms and instruments available to assess the mathematical competencies. The main task, however, is to adapt and focus the relevant forms and instruments so that they specifically aim at assessing these competencies. This demands that it is made clear for each assessment form and instrument exactly which competency it is going to evaluate, as well as specifying how this is to be done. This implies that a considerable, but not insurmountable developmental work ought to be done.

Adequate assessment forms often time and resource consuming

Besides this, it is necessary to be aware of the fact that some of the adequate evaluation forms described are very time- and resource-consuming. It is nevertheless a cost that needs to be met if one is serious about implementing a suitable, valid and reliable assessment of the students' mathematical competencies.

9.5 Assessing the individual competency

Research contributions

Research in mathematics education has, for quite a while now, been concerned with ways of assessing some of the competencies, though without necessarily employing the specific term. In this regard, there is talk of the problem tackling, reasoning, representing and symbol and formalism competencies, as well as some parts of the thinking competency. More recently the modelling and tools and aids competencies, as well as the communicating competency to a certain extent, have been the focus of research attention. On the other hand, however, very little has been done to describe and assess combined competency profiles. The closest attempt made in this direction is probably the current PISA investigations[2], which entails an international comparison of what one could refer to as the mathematical competency profiles of 15 year olds in a broad range of countries.

OECD – PISA

[2] See e.g. Andersen et al. (2001) and OECD (1999, 2001).

Characterising a person's mastery of the three dimensions of a competency

So far, we have looked at evaluation forms and instruments in the light of whether they were more or less suited to assessing parts of the mathematical competency spectrum. However, we still need to go into detail on how one can detect, characterise and assess a person's mastery of the three dimensions of a given competency. Let it be made clear now that there is no final answer to this task. To achieve such an end is no negligible research and development task, and it lies beyond the scope of this project.

We do not, however, have to start completely from scratch. Using specific examples, we are able to illustrate how to approach this task. What the examples have in common is that they concern the mastery of a specific mathematical competency by a specific student at a specific stage of education. A complete survey of the field would involve not only considering all the competencies combined with all the stages of education, but also all the different categories of students in relation to the combination at issue; all in all a rather extensive set of examples.

9.5.1 Example 1: Mathematical thinking competency in an 8th grade student

A specific 8th grader

Let us imagine a specific primary school student, a boy (let us call him B) in an 8th grade class whose mathematical thinking competency we would like to describe. Let us furthermore imagine that the following description is based on many different observations of his work in group work situations, in doing written assignments, and in his answers to questions and his contributions to discussions in the class.

Degree of coverage

Focus on quantitative questions

In connection to counting and calculating in pure or applied mathematical context, B is able to, by himself, pose mathematical questions of a quantitative type, e.g.

- "How many… are there?"
- "What do we get when…?"
- "How much does that make?"
- "What percentage is…?"
- "What situation gives the most…?"

In this regard he is aware that one often with mathematical methods – not least those involving a calculator – can expect to reach an answer in the form of an unambiguous number. On the other hand, he is not aware that there may be situations where quantitative questions do not necessarily have an unambiguous answer, if one at all.

B's mathematical concepts are primarily bound to natural and positive rational numbers, except for the fact that he knows the names of common geometrical figures. The range of his concepts is bound to the actual number domain regulated by the rules of arithmetics, and to situations from everyday life where numbers appear along with units. He has no understanding of the implication of a definition. For example, he can only say what a fraction is by naming some concrete examples. To all appearances, he has no idea of what abstraction of mathematical concepts or generalisation of mathematical results mean and comprise. In this regard he has difficulty distinguishing between assertions about a few individual cases and assertions about a whole class of situations, i.e. mathematical propositions. He also has trouble distinguishing between definitions and theses as he seems to understand the defining properties of, e.g. geometrical figures, as being propositions about them. The fact that most mathematical statements are conditional, i.e. rest on expressed or unexpressed assumptions, does not seem to have influenced his understanding. Neither is he aware of the fact that a given assertion can therefore be true in some cases and not in others.

Focus on concrete examples

To sum up, the degree of coverage of B's mathematical thinking competency, is rather modest since it primarily covers understanding of the fact that certain types of quantitative questions and answers are characteristic of mathematics. Furthermore, his competency has a very limited conception of the range of mathematical concepts by their being based on concrete situations, typical everyday ones, when they came into play.

Modest degree of coverage

Radius of action

It is easy for B to relate his otherwise limited conception of mathematical thinking as consisting only of posing, and expecting answers to, quantitative questions – to many different types of situations, particularly, though not entirely, outside mathematics. He can, for instance, see quantitative questions in all sorts of everyday situations where it could be interesting to ask about the first, biggest, fastest, strongest, oldest, youngest, richest, smallest, longest object, etc. He is especially likely to formulate quantitative questions when it comes to games and betting. In addition, he realises that many everyday situations require a mathematical calculation to reach an answer, not least when monetary transactions are involved. He can also pose quantitative questions in everyday situations where measuring

Insight into quantitative conditions

and weighing, etc. are involved, and similarly questions of a descriptive nature to do with social statistics are also part of his repertoire. In a mathematical context he can easily pose questions to do with numerical calculations where he would expect to get an unequivocal answer. These can almost always be produced with a calculator, he believes. He can, however, also pose questions about how many numbers there are in the world, and which is the smallest and largest of these respectively. However, he does not have any idea that the answers to these questions are not of a simple quantitative nature, but rather take forms such as

- "Infinitely many."
- "That depends on which number domain one considers. If you take, e.g. all rational numbers (or all positive rational numbers), there is neither a smallest nor a largest. If you take, e.g. only the non-negative rational numbers, then 0 is the smallest, while there is no largest number."

Very broad radius of action

To sum up, B's mathematical thinking competency, given its degree of coverage, has a very broad radius of action when it comes to contexts and situations from everyday life and reality, while his internal mathematical radius of action is limited to questions and answers concerning numbers and number manipulation.

Technical level

As is evident, B's mathematical thinking competency is first and foremost related to quantitative matters. When it comes to the technical level at which he can activate this competency, there are contexts covering natural numbers (and 0) which B can identify with their representation in the decimal position system, decimal fractions with only a few decimal points, and standard fractions. He thus takes it for granted that a natural number is simply identical to its representation in the decimal position system. On the other hand, he can clearly and correctly explain what the digits in a multi-digit number stand for, as he can with the first 3-4 decimal points in a decimal fraction. However, he is not aware of the relationship between fractions and decimal fractions. Further more, he can only seldom pose questions or imagine answers relating to negative numbers, infinite decimal fractions, periodicity of decimal fractions, just like he seldom incorporates powers of numbers in his use of mathematical thinking. Irrational numbers are not taken into consideration, and neither are algebraic features, nor representations of numbers expressed by means

Focus on concrete numbers, arithmetic and use of a calculator of symbols naturally incorporated in his field of vision. The same goes for the concept of functions. The forms of questions and answers with which he operates are almost entirely related to classical rational arithmetic, i.e. the four species, as well as percentages carried out on concrete rational numbers, not least those that appear in his surroundings and in everyday life. In this regard, he can formulate questions which incorporate simple statistical descriptions like averages and "the top 10%". The answers to these types of questions are, according to B, to be found best and rapidly by using a calculator. He is aware of the fat that in principle he can attain these answers himself, but that this takes too long and he can too easily make mistakes. Within the above-mentioned limited area, he shows a considerable agility in his activation of the mathematical thinking competency.

Geometric names

While B knows the concepts and names of current geometric figures like triangles, rectangles, squares, circles, etc., he does not pose geometric questions himself. Similarly, he has difficulty seeing a geometric statement as an answer to a – possibly implicit – question. He apparently views geometric issues as ones of naming, not as questions about properties about which one can ask questions and expect answers. His understanding of the conceptual logical hierarchies in this area, e.g. square-rectangle-quadrilateral, is limited.

Modest technical level

To conclude, B can activate his mathematical thinking competency at a modest technical level, one that is dominated by rational arithmetic supported by the use of a calculator. Within this area, his mathematical thinking competency is fairly well developed. Beyond this area, he can only activate his competency sporadically and unsystematically, even when these are areas in which he has actually received teaching, e.g. functions, geometry and equations.

9.5.2 Example 2: Modelling competency of a 2nd year high school student

An imagined 2nd year high school student

Let us imagine a girl, G, who is a second year high school student taking mathematics at B-level, the second highest high school level. We would like to describe G's modelling competency on the basis of observations of her activities in everyday classroom situations, including her answers to written work and tests, her contributions to group work and smaller projects, and her oral explanations while up at the board and in the class as a whole.

Degree of coverage

G is able to thoroughly analyse, with precision and clarity, the basis, range and validity of the concrete mathematical models she has had the opportunity to work with during high school. In this regard, she is good at seeing the assumptions and premises on which a model is based, and being able to relate them to what the model can express as well as what has not been taken into consideration. Furthermore, she is usually able to decode and interpret model elements and results in relation to the situations being modelled. On the other hand, she finds it hard to uncover and judge a model's more principle mathematical properties besides those that immediately appear in the results brought about by concretely using the model.

Focus on the model-reality relations

When it comes to active model building, G is able to structure the situation to be modelled, including being able to choose the elements and links between them that need to be taken into consideration. On the other hand, she often has great difficulty carrying out a profitable mathematisation that could lead to the setting up of a mathematical model. In this regard she has problems noticing and choosing the idealisations and limitations needed, as well as dealing with the loss of information which is unavoidable with any mathematisation. Once, however, a model is available, often with the help of others, she can deal with it mathematically as long as the treatment can be achieved using familiar methods. She can hereby achieve mathematical results and conclusions which she is more than able to interpret and judge in relation to the situation the model tries to describe. In this regard, she is able to apply *common sense* considerations to validate the model, while she does not master the more in-depth or sophisticated means, e.g. of a theoretical or statistical nature, to carry out a more thorough validation of the model. She has difficulty modifying or improving an inadequate model herself, as well as imagining or suggesting alternatives. She has a well-developed overview of the total modelling process, even though, as mentioned, she has difficulty controlling the entire process because she has problems with some of its important points – something she is quite aware of herself. G is good at communicating with others about a model she has been part of setting up, including those points she has difficulty carrying out herself.

Difficulties with mathematisation

Focus on the familiar

In conclusion, G's modelling competency has a rather high degree of coverage as far as the basic features in analysing and building mathematical models are concerned, but important points regarding mathematisation and

Rather high degree of coverage in the basics the handling of more sophisticated, non-routine mathematical treatment of models are only poorly covered by her competency.

Radius of action

G's modelling competency is first and foremost concerned with problems regarding growth and decline. She has experience with active modelling of, firstly, the growth of human and animal populations (among other things the spread of epidemics), secondly, changes in economic and financial magnitudes (including loans and savings), and finally radioactive decay. She can deal with such growth and decline problems with an overview certainty when it comes to active modelling tasks, though especially if they are rather standard in nature. On the other hand, she is not good at handling other types of modelling situations, e.g. regarding geometric forms, technical, physical and chemical relations, random phenomena, etc. The situations she can handle are preferably such that are partly structured and prepared in advance so that it is unnecessary to start right from scratch, e.g. by collecting basic information. Faced with completely new growth or decline situations which do not contain many familiar traits, she has great difficulty getting started with building a model.

Focus on growth and decline problems

In summary then, G's modelling competency has a modest radius of action, which is largely dependent on her previous experience with models and model building. To all appearance, her radius of action can only be expanded gradually via thorough working with new types of modelling situations.

Modest radius of action

Technical level

G can first and foremost activate her modelling competency in contexts where linear or other polynomial functions, power functions, or exponential and logarithmic functions, preferably in their pure form, are central to the situation. She has difficulties incorporating trigometrical functions in modelling situations because she doesn't find it easy to relate their periodic oscillations to the growth questions of which she has modelling experience. With models characterised by the fore mentioned class of functions, she can confidently and with a fair degree of certainty utilise her thorough phenomenological knowledge of the relevant functions in modelling. This is also true when questions of the speed or rate of growth demand concepts and results from differential calculus. However, she has no technical basis for understanding or dealing with models based on differential equations. Neither is she good at constructing other equation formulated models –

Focus on "growth"-functions

Difficulties with equations

setting up equations as a means to model a situation generally gives her trouble. On the other hand, if a model is given in terms of explicit equations, she can usually solve the equations involved if they are similar to those she has met in earlier mathematics classes.

She is virtually incapable of utilising statistical models for parameter estimation, regression analysis and tests in the validation of model results compared to existing empirical data. This is, among other things, due to the fact that these questions have only been briefly touched on in class. On the other hand, she can easily use graphic methods based on *common sense* considerations to carry out a comparison of model results and data.

Relatively high technical level with a limited repertoire

In summary then, G's technical level of her modelling competency is relatively high, given the material her teaching has covered. She can, however, only activate her technical abilities regarding geometry, trigonometric functions, equations and statistics to a limited extent when it comes to modelling purposes. In these respects there is room for improvement of G's technical level.

9.5.3 Example 3: Symbol and formalism competency of a student

Imagine a tertiary science student at the introductory stage

Finally, let us imagine a female student, F, doing her first or second year of a science degree which uses mathematics, but does not aim at a degree in mathematics. She could, e.g., be a student within the fields of biology or chemistry, or a basic science degree, or engineering, and is now doing an introductory mathematics course. The idea is to characterise F's symbol and formalism competency on the basis of observations of her individual answers to written exercises and informal tests, of her activity in class and teacher supervised group work, as well as of reports from short group assignments to which she has contributed.

Degree of coverage

F shows a considerable flexibility and trenchancy in her oral use of symbols and formalisms, both when it comes to following others' use of symbols and formalisms, and when it comes to having to carry out a goal directed, correct and effective manipulation of symbol rich statements and formula. In this regard, she is very oriented towards and has a good grasp of the rules for writing up and manipulating symbol rich expressions and formula, and, due to a well-developed technical routine, she seldom makes mistakes in this regard. However, her understanding and interpretation of what

Good at symbol and formalism rules, but not at interpretation symbols and formalisms cover is not so well developed. In other words, she is very good at playing symbol and formalism games, also result oriented ones, but not so good at seeing the point of the games, something she does not seem to feel the need to do. When it comes to formal mathematical systems, typically built up as a whole theory, F has difficulty understanding the point of it all, except when theory leads to calculatory results and methods which she then concentrates on mastering, often without a clear view of their role in the total system, something she tends to ignore or sees as frightening, irritating or as a pedantic aside.

In summary then, F is a very competent symbol and formalism manipulator, but not that good at understanding and interpreting the meaning behind the use of symbols and formalisms. Her eye for the character and meaning of formal mathematical systems is almost nonexistent. Her symbol and formalism competency therefore has a clearly limited degree of coverage which would entail a solid effort to extend.

Limited degree of coverage

Radius of action

F's well-developed ability to use symbols and formalisms contributes to her being able to cope both in routine-like situations and in those unfamiliar to her, but which can be dealt with with the apparatus she has available. This is true both of internal mathematical situations and of applied situations. As far as the latter is concerned, a precondition is that the mathematics part is either set out or self-evident. Her trenchancy is greatest in situations where the treatment of symbols and formalisms is aimed at giving a well-defined – though not necessarily unambiguous – result, preferably of a calculatory nature. She does not suffer from the otherwise prevalent misconception of identifying a mathematical symbol with the entity it symbolises. Therefore she is also able to handle situations where common symbols are replaced with other, less common ones, but if the tradition is more broken than this, e.g. when variables are named a, b or c, while constants are called t, x, y and functions called m, n and p – she becomes less certain. This uncertainty can usually be overcome if she first takes the time to inculcate the roles these symbols now represent.

Can handle new situations where symbols are introduced

F does not find it easy to introduce symbolic appellations in situations where they do not appear already, unless the situation is familiar, like when naming coordinates, functions, matrices, eigenvalues, etc. following tradition. The fact that an important decision is made in a mathematical situation when deciding which objects play a role big enough to be ascribed a symbol, is not encompassed by F's symbol and formalism competency.

Difficulty introducing symbols herself

Relatively large radius of action

In summary then, F's symbol and formalism competency has a rather Large radius of action relative to its degree of coverage, i.e. in relation to contexts and situations which do not involve an interpretation of symbolic activity nor require handling of formal mathematical systems.

Technical level

Focus on real functions and linear algebra in standard situations

The technical level of F's symbol and formalism competency is linked not only to her high school proficiency, but also to real functions of one or two variables, as well as linear algebra in number spaces. She can cope with both symbol and formula rich problems relating to concrete exemplars of functions, vectors, matrices, eigenvalues, etc. and to general examples, though the latter only if the framework for symbol and formalism treatment is clear and does not involve the incorporation of more intricate theoretical problems and considerations. She has a solid grasp of symbol and formula based handling of differentiation, integration, series expansion, series convergence, determination of eigenvalues, diagonalisation of matrices, etc. in theoretically unproblematic standard situations, where conditions for utilising well-known procedures are met. However, if originality and inventiveness are the order of the day because the situation is not completely covered by familiar conditions and procedures, she can normally not work out what to do.

Difficulty with formal mathematical systems

F can only to a limited extent cope in formal mathematical systems. This means that she has difficulty deciding what are, respectively are not, legal consequences of what in such a system, unless the considerations are attached to symbol and formulae treatment. This is true both when following other people's presentations and, especially, when producing one herself. She has a tendency to learn the conditions for application of rules and procedures off by heart, something she is good at.

High technical level

In summary then, the technical level of F's symbol and formalism competency is high, seen in relation to the competency's limited degree of coverage. If she was better able to cope in situations demanding theoretical overview or originality and inventiveness, it would be very high indeed.

9.5.4 The "volume" of a competency

If one, for whatever reason, wanted to carry out a comprehensive assessment of a given student's mastery of a particular mathematical competency, e.g. for use in a summative assessment of the student's standing, maybe in the form of a grade, the dimensional description introduced here makes it possible to weigh up the relative strengths of all three dimensions in a competency description. Thus a marked fulfilment in one of the dimensions could counterbalance weaker fulfilment in the other two.

To use a metaphor: One can picture the mastery of a competency "measured" as the "product" of the fulfilments of the three dimensions, a kind of "volume". Implicit in this metaphor is that mastery of a competency as a whole is ascribed the measure zero if one of the dimensions is not present with the student in question, which is a natural result of the competency concept itself. If either degree of coverage, radius of action or technical level are non-existent, there is absolutely no competency. Furthermore, the metaphor implies that different students can have the same volume of a competency, even though the dimensions included have different formats.

"Volume" = "product" of dimensions

Part VI

Further ahead: Challenges and recommendations

10 A characterisation of selected central problems regarding Danish mathematics teaching

10.1 Introduction

Mathematical knowledge and skills (in a broad sense) of the population, and mathematics teaching which should be the cornerstone of the development of both, are traditionally given great importance in all types of societies, but not least in the technologically and economically advanced ones. In the 20th century, these aspects received, from time to time, considerable societal attention, in discussions, through developmental work and in relation to reforms and their implementations. The main questions have been the following: Who in society should acquire what mathematical knowledge and skills, and why? To what extent does the education system in general and mathematics teaching in particular supply the target groups with the desired knowledge and skills? To the extent this does not happen in a satisfactory way, what could be done to improve the situation?

The mathematical knowledge and skills of the population are considered important

10.1.1 A lot goes well, but here the focus is on problems and challenges

In Denmark, these questions – maybe in a new form – have once again become pressing. "Something" in relation to the connection between the actual or desired mathematical knowledge and skills of the population and the underlying mathematics teaching does not seem to be the way it should.

"Something" ought to be improved

At the same time, it is important to bear in mind that many aspects of mathematics teaching in Denmark work very well; e.g. the students in primary and lower secondary school are generally content with the

A lot goes very well

mathematics teaching, a large proportion of the students in upper secondary school chooses mathematics at the highest level available, and Denmark obtains acceptable scores in international comparative mathematics surveys such as TIMSS and PISA (see e.g. Allerup et al. (1998) and Andersen et al. (2001)), on the highest educational level even good scores.

Focus on challenges

To keep such positive elements in mind, and in practice in future reform initiatives is a significant part of the challenge. This also becomes important in this project, where we deliberately avoid a complete charting of the situation, but focus instead on certain aspects that we, or others, find give rise to particular challenges which presumably can be successfully dealt with.

10.1.2 Our perspectives on the problem area

A fully objective analysis is not possible

With our point of departure in broadly formulated and normatively toned questions, as is the case here, it is naive – and basically an indication of disrespect for the complexity of the problems – to believe that it is possible to carry out an objective, complete analysis of all significant problems. A partly subjective and more or less deliberate demarcation of the problem area will always be present. This is said in order to stress that the following is *our* perspectives on the matter at hand, which will be expressed though the analytical categorisations according to which we have chosen to structure our work.

Three kinds of problems: “why?”, “what?” and “how?”

The exposition is thus founded on the belief that it is beneficial to view as being the *problématique* connected to (mathematics) teaching composed of three main problem types: the “why”, “what” and “how” questions. These problems we identify in this respect are categorised as *problems of justification*, *problems of content*, and *problems of implementation*. In the analysis in this chapter, we have chosen to focus on the “why” question and the “how” question, while problems related to content have been dealt with earlier in this report.

Within the issues “why” and “how”, we try to identify and characterise a set of more specific problems and challenges, as they are experienced by different groups in and around mathematics teaching and learning. To provide an overview, we in this connection introduce a coarse-grained distinction, as we operate with three types of educational programmes involving mathematics:

General education involving mathematics: This includes education programmes containing mathematical elements which aim, as a constitutive feature, at contributing to students' general educational development ("bildung") as an individual, knowledge and skills with "the many" (as opposed to "the few") as the target group, see Niss (2000). As general education is a constitutive aim, this type of educational programmes mostly, but not exclusively, comprise the mathematical activities in primary, lower secondary, upper secondary, and general adult education.

Educational programmes in which mathematics is a key service subject: Thereby we mean educational programmes, that authorise and aim to qualify the students to professions, in which the application of mathematics (to a varying degree) is significant, but where the profession cannot be described as fundamentally mathematical. Architects, chief clerks in a bank, biologists, electricians, pharmacists, industrial operators, crane operators, politicians, nurses, carpenters, and economists are examples of professions and occupations, which, as we see it, fall under this category.

Mathematical professions programmes: By this we mean those educational programmes, which authorise and aim to qualify the students to professions, whose professionalism can be characterised as being fundamentally mathematical. As the most obvious cases, this applies to professions as research mathematician and mathematics teacher at all levels, but also statisticians, physicists, chemists, astronomers, computer scientists, actuaries, land surveyors, and many kinds of engineers, as well as university instructors of these subject areas, fall under this category.

Who experience problems and challenges?

Furthermore, we find it appropriate to add yet another dimension by taking into account *who* experience problems and challenges (both of a justificational and an implementational nature) in each of the three kinds of educational programmes: Is it the *recipients*[1] of the graduates from those programmes? Is it the *organisers*[2] of the mathematical parts of the educational programmes? Is it the *teachers/instructors*? Is it the *students*?

[1] Ranging from educational institutions admitting graduates from "feeding line" programmes over employers to "society" seen as the ill-defined entity that general politicians are elected to represent.

[2] Ranging from education politicians responsible for the overall boundary conditions to educational planners working on curriculum development, to teacher trainers working on more specific implementation related issues associated with organising.

Or is it the future *users*[3] of the expected outcomes of the mathematical training?

If the two analytical dimensions "type of mathematical education" and "interest group" are crossed, the following 3×5 matrix is obtained:

Interest group/ Education	Recipients	Organisers	Teachers	Students	Users
General ed. involving math					
Ed. programmes in which math is a key service subject					
Mathematical professions educations					

Problem space of mathematics education programmes

If – without offering yet another illustration – we introduce a third analytical dimension focusing on the problem (as either related to justification, content, or implementation), we have in total 45 ($3 \times 5 \times 3$) "cells" in the resulting three-dimensional[4] *problem space of mathematics educational programmes*.

Establishing such a problem space and using it as a tool for analysis does not imply that we believe it is possible to view each cell in isolation. This is neither analytically nor practically possible, the situations and circumstances that exist in the system are far to complex, howenver, the structure can help to cover the problem area and thus to *discover* problems experienced locally and then subsequently, to analyse the range of these problems. Furthermore, the structuring may help to counteract the temptation to remain at such a general level of description that nobody really experiences that the description is relevant to him/her. Both here, in relation to the clarification of the problem area, and in the analyses offered in the remainder of this report, we have obviously tried to keep these two aspects in mind.

[3] Ranging from professional mathematicians over the "mathematical consuming" professionals to the users of general educations involving mathematics, mentioned above, sharing the distinctive feature that they have left (the mathematical parts of) the education system and now want to profit from their participation.

[4] The three dimensions "education type", "interest group" and "problem type" focus on "where", "for whom", and "what" in relation to didactical analyses, not to be confused with the why-what-how-distinction concerning mathematics education per se.

10.2 Justification problems

Reasons and justifications

By a *reason* for offering mathematics teaching to students within a segment of the educational system, we understand a driving force, which *in reality* has motivated and contributed to causing the existence of mathematics teaching within that segment. By a *justification* for offering mathematics educational programmes, we understand the activity of proposing arguments for supporting the existences of such programmes. In practice, such justification attempts will often reflect one or several reasons why mathematics teaching actually exists, but this does not need to be the case.[5] Below, when analysing problems of justification, we examine whether – and in the affirmative case, under what forms – a particular mathematics education programme should exist, and which issues and arguments can be identified in that connection.

Three types of reasons

As part of the development of a useful framework for our characterisation of actual problems related to such reasons, Niss (1996, p. 13) concludes that, fundamentally speaking, there are only three types of *reasons* for mathematics education covering the entire international scene:

- The economico-technological reason: To contribute to the technological and socio-economical development of society as a whole (hereafter referred to as the economico-technological reason).
- The individual-oriented reason: To equip individuals with tools, competencies and qualifications so as to help them deal with the challenges of life at large (hereafter referred to as the individual-oriented reason).
- The politico-cultural reason: To contribute to the political, ideological, and cultural maintenance and development of society (hereafter referred to as the politico-cultural reason).[6]

[5] See Niss (1996, p. 12f).

[6] This categorisation can be seen as an elaboration of a classical and more general approach, according to which, the role of the education system in society, on the one hand, is to introduce students to the many different facets and ways of thinking in society, that is to initiate them to the particular culture(s) of society, and, on the other hand, to equip them with techniques and methods needed for coping with and participing in the many functions of society, e.g. to be able to read, write and do arithmetic. These two roles are sometimes called *the socialising* and *the qualifying* roles, respectively. From this perspective, the politico-cultural reasons represent a wish for socialisation, while the economico-technological and individual-oriented reasons represent a wish for qualification. For analyses with a Danish perspective: see e.g. Christiansen (1989) concerning mathematics teaching at the secondary educational level, Jensen & Kyndlev (1994) containing many references on this topic and Undervisningsministeriet (1978)

These categories of reasons become relevant because many problems of justification originally arise when focusing on one of the three categories. This also applies to the three problem areas we have chosen to elaborate on below.

10.2.1 Distortion of the qualifications of the workforce

A need for a mathematically well-educated population

During the last fifty years, a main *justification* for the existence of mathematics teaching from the point of view of the recipients has been that a mathematically (and technically scientifically) well-educated population has always been a prerequisite for first – the establishment and – then – the maintenance of the welfare societies in most Western European countries.[7] This justification contains two aspects; one refering to the economico-technological reason, and one refering to the individual-oriented reason. The latter reason, which primarily concerns life as a citizen in a democratic society, is treated in section 10.2.3. The rationale behind the economico-technological justification can briefly be stated:

The driveng force behind the establishment of the Danish welfare state[8] in the 1950's and 1960's was the increasing economic growth. As is clear from today's political debate, such growth is also a prerequisite for the maintenance of the welfare state, at least in the leading layers of society. A large and growing gross domestic product is the best way to ensure the maintenance of a high level of activity. In that context, there has, in recent times, been a broad political agreement that a large *amount* of manpower does not in itself ensure this growth, but that the *knowledge level* of the workforce is equally important. Thus, a crucial qualification demand is the ability to develop and utilise production conditions that allows for increasing productivity. This resembles an optimistic perception of technology that places technology as "something man puts between himself and nature" in order to enhance his power over it and make use of

[Ministry of Education] and Undervisnings- og forskningsministeriet (1990) [Ministry of Education and Ministry of Research] considering multiple educational levels.

[7] See Gregersen & Jensen (1998) for a fuller historical account of these arguments.

[8] The notion *welfare state* is used about societies characterised by a high level of social security, that is societies where a social stratification does exist, but is *relatively* low. Some use the term *social state* about this type of society. In a welfare state the welfare of the population is an explicitly formulated goal in public policy, and the state actively takes initiatives to achieve these goals. A welfare state is thus an *active* state, that *intervenes* in the free market forces with the intent to redistribute resources.

it.[9]

In this connection, mathematical knowledge and skills – and hence teaching designed to create them – appear as a central player for two reasons. Firstly, because such knowledge and skills often are necessary to be able to utilise technology in the traditional sense, and secondly, because mathematics by means of mathematical models, in itself constitutes such technology. This completes the reasoning which from a recipient's point of view justifies the importance of mathematics teaching: An educational policy placing mathematics teaching at the front is, amplified by the labour market policy, essential for the maintenance and development of the welfare state. This can so to speak be seen as a politico-cultural reason carrying with it an economico-technological reason.

Math. knowledge and skills and technology

Against this background, it is a problem when insufficient numbers of students choose those kinds of education that society would prefer them to choose. There is a general tendency, that the influx of applicants to different educational programmes correlates negatively with the emphasis put on mathematical insight and skills. Several studies (e.g. Simonsen & Ulriksen; 1998) indicate that the crucial criterion for the choice of a field of study is to many studetns, that the studies can contribute to their continuing self-realisation project, that they see as necessary in an ever more complex world. This may well bring mathematics educational programmes in a difficult position.

Insufficient influx of applicants to mathematics educational programmes

This problem might originate from different sources, for instance from within mathematics as a discipline, or from the fact that mathematics educational programmes traditionally have high demands in terms of workload and psychological strength in a broad sense. In this report, we will not go deeper into this question.

The influx of applicants to some engineering programmes have decreased severely in the last few years, while several of the other mathematical profession programmes have not been directly weakened – but on the other hand a growth resembling the demand of society has not happened either.

At the general educations involving mathematics, education planners find themselves in an uncomfortable dilemma: As a consequence of the afore-mentioned negative correlation, they can either announce what level of mathematics is actually needed and plan the syllabus accordingly, which may easily entail that the programme ends up being placed in the same category as the mathematical professions, or they can tone down the

Uncomfortable dilemma

[9] See Jensen & Skovsmose (1986).

mathematical level in order to increase the influx of applicants, and then try *ad hoc* to correct the ensuing imbalance in applicants' competency profiles.

Both choices lead to imbalances in the qualifications of the work force as indicated in the title of this section. From the recipients' points of view too few students have a positive attitude towards embarking on mathematics education at a more advanced level.

10.2.2 The relevance paradox and the problem of motivation

The objective relevance of math. knowledge and skills

On the one hand, all societies like ours assign key importance to mathematical knowledge and skills (and to the mathematics teaching aiming at producing them). For societal reasons, it is thus *objectively relevant* that "some" members of society possess such knowledge and skills. During the last century the tendency has been that "some" is to be understood as "still more", and in the last quarter of the 20th century simply as "everybody".[10]

Subjective relevance

On the other hand, evidence from all countries and all educational levels show that large groups of students in the education system find it difficult to see any *subjective relevance* of the mathematics teaching they receive, and overall to be engaged in mathematics. Several explanations for this are possible. One could be that the students do not have access to ways to become familiar with the objective relevance of mathematical knowledge and skills, or because they, in spite of such access do not feel convinced about the strength of the relevance. Another possibility is that the students are in fact convinced about this relevance on a societal level, but nevertheless do not feel any personally usefulness or relevance of mathematical knowledge and skills in relation to their beliefs about future careers and life prospects. On the individual level, some might express this belief as follows: "I know that I am probably *useless* without mathematics, but mathematics is useless for me."

Relevance paradox

In both cases, the contradiction between the objective relevance and the subjectively experienced irrelevance creates a paradox, the so-called *relevance paradox*. If the relevance paradox concerns sufficiently large groups

[10] By saying that practically everybody in society should possess some kind and degree of mathematical knowledge and skills, it is not said what *sorts* of mathematical knowledge and skills different categories of members of society should be equipped with, and how they should come to acquire them. This is precisely the question, which has been examined earlier in this report.

of students it becomes a societal problem and thus a challenge, which the education system in general and mathematics teaching in particular have to deal with. Many things suggest that such a societal problem exists both internationally and in Denmark these years.

The relevance paradox does not only manifest itself at a societal and an individual level, but is also seen at an institutional level. At primary and secondary levels, as well as at tertiary institutions, problems concerning bringing mathematics into play within other subject areas are being encountered. Often teachers of other subject areas (but sometimes mathematics teachers as well) find it difficult to see what mathematics is good for, either in the institution as a whole or in relations to their subject areas. Inspite of this, more and more subject areas contain mathematical elements to a still increasing extent, even though the mathematical nature of these elements not always are recognised, e.g. because of the terminology or the conceptual framework. This manifestation of the relevance paradox results in an *isolation problem*, which is damaging both to mathematics teaching and to those subject areas, which could benefit from a deliberate inclusion of mathematical components in their activities.

The isolation problem

It might, however, also be that the students are convinced about the relevance of acquiring mathematical knowledge and skills, e.g. in relation to their education and career plans, and for that matter also find activities which aim at making them acquire such knowledge and skills subjectively relevant, but that they, nevertheless, find it boring, meaningless, irrelevant or lacking in perspective to study mathematics, or simply too demanding compared with the expected or achieved outcome. In this situation, no relevance paradox exists, but there exists a significant *motivation problem*, which – if it is of a substantial magnitude (and also this seems to be the case these years, at least at some educational levels) – can be as fatal as the relevance paradox to the endeavour to equip the population with functional mathematical knowledge and skills. To bring matters to a head, if the teaching of mathematics is not capable of producing a minimum amount of enthusiasm for the subject amongst the recipients, even the best founded, designed and implemented educational plans will fall short.

The motivation problem

It is the belief of the task group that both the relevance paradox (including the manifestation as an isolation problem) and the motivation problem are present to an extent which creates substantial challenges for the organisation and implementation of successful and fruitful mathematics teaching.

The relevance paradox and the motivation problem create challenges

10.2.3 A possible threat against "mathematics for all"?

As mentioned earlier, politicians and other decision-makers, together with the general public and the mathematics teaching environments in all countries, have become accustomed during the 20th century to taking it for granted that mathematical knowledge and skills and mathematics teaching are given substantial and growing importance at all educational levels. Many countries have invested great efforts and resources in establishing mathematics teaching in places where it did not exist earlier, and in solidifying and expanding it in places where it already did exist. In this way, new categories of recipients of mathematics teaching who previously did not have access to such teaching now got an opportunity to receive it. It is not an exaggeration, internationally speaking, to characterise the main evolution of mathematics teaching in the second half of the 20th century as a development towards "mathematics for all".

Previously: Focus on "mathematics for all"

As was suggested in section 10.2.1, all the countries – both industrialised countries in the East and the West and in the Third World – have been convinced about the crucial importance of their citizens' mathematical knowledge and skills for the perposeful functioning and development of society in relation to technological, socio-economic and cultural welfare. The more mathematically competent the general population in a country is, the better this country is positioned with respect to material and immaterial wealth and growth, the doctrine says.

In other words, while mathematics teaching (beyond elementary calculus) before around 1960 was reserved for a rather limited part of the population, at the end of the 20th century mathematics teaching was expanded to also address many groups of students, who would not have chosen mathematical studies if personal inclination and interest at the time of the choice were the only significant factors.

Signs of a change of direction

From a global point of view, it can no longer be viewed as a matter of course that society will continue to view mathematics teaching for all as something highly important that has to be improved and expanded continuously.

In Japan, it was rather recently decided to reduce the amount of general mathematics teaching in primary and lower school considerably. Some years ago – under the headline "Sieben Jahre sind genug" – a debate took place in Germany, about how much mathematics the individual in reality needs to acquire.[11] In Norway and Sweden voices from general pedagogues, which

[11] The debate, which took place in both newspapers and journals, was initiated by a

advocate for a significant reduction or outright removal of mathematics from general elementary schools have been raised.

In Denmark, a proposal put forward by the secondary schools teachers' union suggests that mathematics should not be a common subject at the socalled HF (general higher preparatory exam). From time to time newspaper articles are published which promote a drastic down playing of the mathematics teaching for ordinary students. One example is an article in the influential Danish newspaper *Politiken* on October 24 2001, by technical manager Rasmus Wiuff (member of The Board of Technical Education), who among other things writes that "mathematics for far too long has been allowed to stand untouched as an indisputable necessity at both tertiary education and in secondary and primary schools." These are just singular examples, but fundamental features of the organisation and mode of operation of modern society suggest that they might not be incidental but are part of deeply lying processes. We shall try to point out some of the features of these processes.

The concrete utility of mathematics is difficult to verify

Firstly, in the complex society of our time, it is not so straight-forward to give a direct and concrete demonstration of exactly which mathematical knowedge and skills are necessary for an individual to cope with life and sections of society. This makes it difficult, cf. the relevance paradox, to relate the important issues in the mathematics education with matters of societal relevance, at least for the majority of the population who are not going to work in strongly mathematically oriented professions. If it continually appears to be difficult to demonstrate the immediate relevance of primary, secondary and tertiary mathematical knowledge and skills to the world outside the mathematics classrooms, some infers that then mathematics *cannot* be that important for the majority of people.[12]

Too many get too little of their maths teaching

Secondly, we must admit that despite considerable efforts in teaching development and research in mathematics education, there are limits to how well most of the countries have succeeded in providing the majority of the students in the education system with as solid, convincing and useful mathematical knowledge and skills as intended. Too many recipients of mathematics teaching get too little out of it. Would it, then, not be more rational and humane to focus the efforts on teaching only those who are able to benefit from the teaching? Reserving substantial mathematics

dissertation by Hans Werner Heymann (Heymann; 1996).

[12] See e.g. The National Commission on Mathematics and Science Teaching for the 21st Century (2000), an American commission report.

teaching for those students might create a higher return on the investments in education than by maintaining "mathematics for all" as the primary task, some might argue. Frustrated about the lack of possibilities to do something effective for the last-mentioned group, many mathematics teachers and professional mathematicians around the world have began to promote such points of view.

ICT as a substitute

Thirdly, it may appear as if large parts of the knowledge and skills which traditionally have been the cornerstones in mathematics teaching at different levels, can now be handled faster and more securely using ICT. Furthermore, ICT can handle tasks which were completely inaccessible in those days where mathematics teaching focused a lot on educating "the human calculator". Continuing the above line of thought, would it not be more effective to focus the educational energy of society on making the general population capable of handling ICT in a competent and flexible manner, also as regards mathematics, instead of spending considerable resources on carrying through mathematics teaching for large groups of students who experience severe difficulties at learning it? Of course, society still has a need for a large group of people who really know and master a significant amount of mathematics indepth. But this group will, at any rate, be of a considerably smaller size and presumably easier to teach than the majority.

General reduction of research and teaching

Many countries have initiated a slowdown of the funds allocated to teaching (and research) that is lacking a clear and immediate outcome, which emphasises the tendency described above. Reductions in a system easily lead to failing enthusiasm and demoralisation amongst those working within the system, and hence to the emergence of mal functions and inefficiency.

Focus on "the easy" problems

This, in turn, leads to further demands for rationalisation (trimming and reorganisation) of the system, which, in spirit of broad indifference, ends up focusing its efforts on the least difficult problems. As a consequence in many places in the education system the efforts are invested in the more clever students, who are doing well and make progress without too much assistance, while the students who need a lot of help are neglected and hence left behind.

These tendencies have reached Denmark

There are reason to suspect that the tendencies described will also take a stronger hold in Denmark than the case is today. We do not claim that this will happen fast, or that these tendencies will become the dominating ones in the coming years. Nevertheless, we find it necessary to make an effort to consider the issues very carefully. Should these considerations result in the conclusion that these tendencies ought to be favoured, this will lead to a significant change of the frames and the perspectives for mathematics teaching at all levels as well as of the specific design and implementation of it. If on the other hand, the conclusion is that the tendencies should be counteracted, it is necessary to devise and launch more effective strategies than the ones available at the moment.

A closer analysis is needed to uncover whether one or the other conclusions prevail. In any case, the latent threat against "mathematics for all" represents a significant challenge to the present project and to mathematics teaching in Denmark from the highest to the lowest level. We do want to stress that we – if only for the sake of a wish to enhance the democratic competence of the population – consider it vital to maintain "mathematics for all" as a key task for mathematics teaching in Denmark. How this task is to be interpreted and solved is another matter.

10.3 Problems of implementation

Distance between intended and realised outcomes

There is a distance between the intended and the realised outcome of different kinds of mathematics education. Outside a utopian world of ideas, this will of course always be the case, but for some students the distance is larger than is necessary and desirable, and too few students at any given educational stage reach the highest level possible.

This characterisation constitutes a common feature of what we call problems of implementation. Such problems are inevitably connected to ideas about the content intended in a particular education programme and the reasons for the way in which this content is being brought into play, but what is in focus here is the widespread experience that "it does not work (well enough)". All the aspects mentioned below can be seen as part of an attempt to propose mutually supplementary explanations as to why this experience is prevalent in different segments of the education system.

10.3.1 Problems with teachers' qualifications

As is clear from Chapter 6 dealing with mathematics teachers' competences, to be a good mathematics teacher is a complex challenge, to which many teachers at all stages rise splendidly and experience as exciting and personal fulfilling.

Great variability in teacher qualifications

As in Denmark alone there are a five-digit number of people who in their daily occupation are faced with this challenge, it goes without saying that there will be some who do not completely meet the challenge. The individual and the environment he or she is a part of, may experience this as problematic and try to do something about it, but the presence of less good teachers is something that the education system at all times has to accept. If the number is limited these teachers will not constitute a serious societal problem.

Some do not meet the challenge

What is problematic – as we see it – is that a non-negligible part of the mathematics teachers at different levels unfortunately do not meet the challenge to a sufficient extent, and that this is likely to be caused by inadequacies in the spectrum of competencies which, according to our analysis, are needed for carrying out quality mathematics teaching. The fact that it is possible to detect a certain regularity – see below – in the types of competencies mathematics teachers at the different educational stages seem to be lacking, is disturbing, but it also provides substantiation of the belief that the problem can be remedied by political and organisational means.

In a later section we put forward some concrete suggestions as to initiatives it would be prudent to launch. Here we summarise some of the considerations about what constitutes "the good teacher", that were stated in Chapter 7 about mathematics teachers, in order to point out the regularity in the nature of the problem.

The good mathematics teacher

The good mathematics teacher is didactico-pedagogically reflected

On the theoretical front, the good mathematics teacher has to be *didactico-pedagogical reflected*, which we use as a unifying term for the ability to consider and reflect – and see this as a natural part of teaching – upon the three categories of fundamental didactico-pedagogical questions and their mutual connections, see section 10.1: *Why* is mathematics teaching being offered to this group of students – what is the objective societal justification, and to what extent can I identify myself with it? *What* are the students supposed to learn by attending the teaching, and what competencies are they to develop through their attendance, ranging from completely general educational and meta-disciplinary aspects to specific concrete skills? *How* can I, as a teacher, contribute to students' developing of the competencies desired, and to their learning of what they are supposed to learn. What are the optimal "conditions for growth" in the situation at hand, and what are difficulties and obstacles I have to be aware of? To stress a preveous point once again, it is neither sufficient solely to be able to reflect on the subject area itself, nor to be able to carry out general didactical reflections only, in order to possess the ability to reflect upon the didactico-pedagogical questions in a qualified manner.

The good teacher can "reach out" to the students

As regards the practice part of the profession, the good teacher has to be able to "reach out over the desk" to the students. The ability to do this is above all a question of personal traits, style of communication, and life experience of the individual, and only in the second instance a question of schooling in the sense of "systematically acquired competencies and pedagogical techniques". Schooling has a potentially central role when the individual teacher is to become capable of making his or her practice-oriented capacity and didactico-pedagogical insights work together in a constructive manner so as to form an integrated whole, which we see as fundamental.[13]

In summary, we characterise the ideal *good mathematics teacher* as a didactically reflected person, who on, this ground, is capable of practising his or her profession, and who recognises and is in constantly dialogue with him- or herself (and others) about, the complementary relation between the didactically reflected and the practice sides of good teaching.

With this point of departure we are now able to identify the following general pattern:

Primary and lower secondary school

In primary and lower secondary school – or more generally at those educational programmes, where teaching authorisation presupposes a teacher training diploma from a teacher training institution – it cannot be taken for granted that any mathematics teacher is well familiar with the essence of mathematics, since fairly many practising mathematics teachers do not teach on a basis of a subject specific training in mathematics. On the other hand, most are good practitioners, because they are in general, among other things, fond of teaching and educating children and youngsters. As a main rule, the practice takes place in a qualified interplay with reflections about general how-oriented problems, which, as its most characteristic feature, entails a focus on the planning of activities that establish a basis for learning in a diverse student population.

Focus on practice and methodology: "how?"

It is, however, a problem that the great preoccupation with and knowledge about discussions and problems concerning methods prevails at the expense of *subject-didiactic* reflections of justification, content, and learning-

Less emphasis on subject-didactic reflections

[13] The importance of this interplay is well-described in Ramsden (1999, p. 139ff.), who uses it to choose among three theories of teaching. The perception among university

related types, as can be seen in the discussions carried out. When reflections related to justification, content and learning are occasionally put forward, general subject-didactic perspectives are practically speaking missing: Discussions related to justification are carried out on a general pedagogical basis with reference to general disciplinary aspects of teaching. Reflections about the content of the teaching are often fragmented and closely related to practice-oriented considerations about what is possible in relation to single notions and skills, while discussions related to learning are carried out at a general level, in which all members of the teaching staff group can participate on an equal footing regardless of their individual scientific backgrounds.

Upper secondary education and tertiary institutions

In upper secondary education and at tertiary institutions – or more generally within such educational programmes in which teacher authorisation presupposes a master's degree in mathematics – the mathematics teachers' affection for mathematics teaching cannot be taken for granted. This does not mean that there are not lots of good practitioners among them, only that many teachers on these levels are fascinated by *the subject matter* of mathematics rather than by didactical issues and problems. For others, mathematics comes as a secondary subject compared to their major, which might not even have any relation to mathematics or science. Some teachers have chosen to study mathematics as a career furthering means rather than because they find mathematics as a subject area attractive. In either case, teachers' perspectives on the profession of mathematics teacher tend to become dominated by content-related reflections ("what-problems"), primarily with a focus on internal mathematical problems which most teachers at these educational levels feel capable of handling in a proper way. The teacher's teaching and scientific overview become the main focal point.

Focus on the subject matter, especially on the issues of "what?"

In relation to the characteristics of "the good mathematics teacher", the problem at the upper secondary level is that internal content-related problems, which are eagerly debated (the debate in the magazine published by the maths and science teacher association is a good indicator), overshadow discussions with more nuances. Often the interest in reflections of a justificational and a broader implementational nature – whether it concerns problems departuring from this or concerns adopting a more broadly meta-disciplinary perspective on content related issues – is pretty

Less interests in broader perspectives on the subject area

teachers is his point of departure, but this is not pivotal for the analysis.

limited. This might be related to the fact that the mathematics teachers we are talking about here, often feel less competent in these areas compared to their, in many cases, quite solid internal scientific knowledge and skills.

Summary

In an undoubtelly very simplified summary, there seems to be two different – and partly opposite – problems in relation to the qualification patterns among mathematics teachers teaching in different types of educational programmes: A *method fetishism problem*, which – bringing the matters to the head – consists of making the attention to students' possibilities to learn through activities the issue that all other problems revolve around. A *subject fetishism problem* consisting of assigning this pivotal status to teachers' attention to mathematics as a discipline. These two very different approaches give rise to different cultures among mathematics teachers, resulting from the fact that both problems exist in various places in the Danish education system at the same time, probably provide a fair part of the explanation for many of the other implementational problems to be mentioned below.

"Method fetishism" vs. "subject fetishism"

10.3.2 Problems related to coherence, transition and progression

Among those problems and challenges that led to the establishment of the KOM project and the task group, several are mutually related to an extent which almost makes them form a complex. For the sake of convenience, we refer to them as "problems" even though they often take the shape of challenges. One type of problems concern *coherence* in the mathematics teaching and learning taking place at different stages in the education system. The second type of problems concern the *transition* between different education types and levels, e.g. from lower secondary to upper secondary school or from upper secondary school to tertiary educational programmes. The third and last type of problems concern *progression* in the learning of mathematics – including growth of knowledge, skills and basic abilities – throughout the education system, both in longitudinal and in transverse terms (within a given level).

Problems related to coherence, transition and progression form a complex

Problems related to coherence

A frequently made observation within different educational programmes and types of institutions concerns the very different perceptions of what

Different perceptions of the subject area at different levels the subject of mathematics is all about, what mathematical knowledge and skills consist of and how they are acquired and developed, and of how mathematics teaching should be carried out. Roughly speaking, we see the following pattern: In primary school and in teacher training institutions great emphasis is often placed on developing conceptual structures and understanding by means of investigative and exploratory student activities within informal disciplinary boundaries. Less emphasis is put on, say, drilling of skills – in the case of primary and lower secondary school – on formal symbolism. In generally oriented upper secondary schools emphasis is often placed on developing conceptual structures and understanding through task solving in which the handling of formal symbolism is a pivotal point. In vocation upper secondary schools the applicability of the concepts and methods to other subject areas and areas of practice is accentuated. In mathematics teaching (including teaching of mathematics as a service subject) at university the focus is often on the rigorous development of theory and on mathematical proof.

The problem is not so much the diversity in accentuation within the different educational programmes, which might very well be well-founded (taking into consideration that these programmes do not have the same goals or conditions), but rather that this diversity can be perceived by teachers and students as almost pointing opposite directions. Many of those involved experience that the educational levels do not work together to the common task of teaching students mathematics, but enforce so different perceptions and traditions that instead of contributing to solving a common task using different approaches and perspectives they end up obstructing each others' work. When entering new educational levels, many students experience that the agenda of mathematics has changed and that they now have to work in a completely different manner than before. What before was seen as unimportant now becomes important and vice versa. It is important to realise that the problem of coherence is not a transitional problem (see below) in itself, even though it tends to become particularly visible at the transitions between two forms or stages of education. First and foremost, the question concerns more fundamental *differences* between the forms and stages of education.

Absence of a common task

Lack of coherence

Sometimes the differences of opinion, traditions and culture between the educational levels lead to some degree of mutual disrespect for others' work and institutions. Problems at one level are often blamed on the previous (or subsequent) levels, because emphasis and energy are wasted on the "wrong content", others have insufficient prerequisites etc., such that they do not "deliver" or, in contrast, have unrealistic expectations of students. When such a "them-and-us" syndrome is being created and developed, it is often accompanied by a lacking urge and will to acquaint oneself properly with the conditions and reality of the others' world. Instead of images and descriptions based on knowledge of this world, caricatures and distorted images are created so as to increase the distance between the educational environments.

A danger of "them-and-us" syndromes

Due to the nature of things, it is difficult to get an overview of the degree to which the situation actually is as sketched above. Undoubtedly, there is some thruth in it, and to the extent there is, one likely result is confusion and lack of coherence and consistence in the individual students' perceptions of mathematics, and probably also in their mathematical knowledge and skills. From the perspective of the KOM project, using a common set of mathematical competencies, as the ones suggested here, as a main tool in the description and the planning of the teaching across programmes and stages may be a first – and big – step towards doing something effective about the problems related to coherence.

Competencies can bridge the gap

Problems related to transition

This type of problems relates to difficulties concerning subject specific discontinuity, adaptation and planning problems, and hence to student and teacher insecurity that arise in connection with students' transition from one educational level to a subsequent one. Furthermore, such problems lead to a waste of mental and economic resources, and to weakening of student motivation and interest in becoming engaged in mathematical activities, as well as to reduced pace and progression in the learning of mathematics.

The sources of the transitional problems lie partly in the fact that the change from one form of institution to another with different tasks, perspectives and conditions always will cause friction and not the least so as students' personal maturation comes into play over a longer period of time. But they are also to be found in the above mentioned differences between attitudes, cultures and traditions among the educational levels, both with respect to these levels in general and to those of mathematics in particular.

The sources of the problems

Problems of progression

A long line of participants, observers and recipients related to mathematics teaching and learning in Denmark feel that too little progression takes place in the growth, development and consolidation of the individual student's

Unsatisfactory progression

mathematical insight, knowledge and skills along his or her pathway through the education system. Both within and across individual programme or levels of education (e.g. the primary school or Higher Preparatory Examination Programme) there is some talk of unsatisfactory academic progression, that is, not much new land is really being reclaimed along the road.

Part of the problem stems from the afore-mentioned difference *between* the different types of education, but part of it also concerns the conditions *within* a particular type of education. To the extent insufficient progression takes place within one particular education programme, the problem could be caused by insufficient attention being paid to the need for progression, or by the conditions and tools available for advancing it.

Competency considerations can be helpful

Progression problems have been addressed previously in this report. Here, too, we find that a focus on the development of students' mathematical competencies across the boundaries of the different educational levels could contribute to remedy or counteract those progression problems that are not, first and foremost, due to general education-sociological circumstances.

10.3.3 Problems of variation at the same level

Variation in the students' mathematical "luggage"

A field of problems connected to, but not coinciding with, the one treated above is often brought to the light both by central politico-administrative quaters and by those who receive students or graduates from a particular level of the education system. This is *the variation* in the mathematical luggage each student carries with him or her from (previous) mathematics education. Most probably, the group of students leaving a certain section of the education system, – e.g. the ninth grade with a final exam in mathematics, students with an A-level diploma from upper secondary school, newly educated primary and secondary school teachers or university graduates majoring in mathematics – have been exposed to a great diversity of mathematics teaching within the educational level at issue. This results in marked variations in the mathematical experiences, competencies and abilities at school leavers or graduates carry with them from a particular level, despite the fact that the "academic level" in principle and allegedly is meant to be the same for that level. This is especially the case at the various 10-12 grades programmes, where the alleged academic level is even given a certain classification (A-, B-, C-level) across the different educational programmes, even though it is generally known that, say, a B-level covers a very wide range of different outcomes depending on its origination in a Higher Preparatory Examination or in the mathematics stream in the general upper secondary school or in Higher Commercial Examination Programme or in Higher Technical Examination.

There are two main reasons why many stakeholders find these variations within the same level problematic:

The "declaration of contents" problem

First, there is *the declaration of contents problem* primarily experienced by the recipients of the different categories of students and graduates. At a first glance, the problem might look like a transition problem, but it really is not. It is not the actual transition that is the problem (even though, as was the case with the problem of coherence, it particularly manifests itself at transitions between the educational levels), but exactly the variation that exists within a given declaration of contents.

The grades 10-12 programmes do not exactly know what to expect from the lower secondary school students concerning knowledge, insight and abilities. Likewise, tertiary educational programmes do not exactly know what to expect from the students who have a completed upper secondary education, and the same goes for those employers who receive the graduates from these programmes.

The local authorities employing primary and lower secondary school teachers often experience a similar problem, which does not so much concern the details of the candidates' mathematical insights and abilities as the very existence of them. The institutions employing upper secondary school teachers usually notice the presence or absence of larger mathematical components at a topic or discipline level, where the focus first and foremost is on whether or not the applicant has been exposed to what it takes to acquire a formal teaching authorisation in mathematics. Other categories of employers appear to have lesser problems with the labelling as long as "relevant goods are available". To the extent such problems do exist, they typically arise from an uncertainty about what sorts of job people can manage.

Problem concerning the teaching planning

The declaration of contents problem primarily appears to be a problem concerning the planning of teaching taking previous educational levels into consideration. If the variation of the mathematical prerequisites is large among those people who are to be subjected to mathematics oriented teaching, problems can occur in relation to the identification of these prerequisites. Problems can also occur in relation to making adjustments or more fundamental changes of teaching plans. All of this can be time-consuming and troublesome for those involved who either have to invest work to chart student prerequisites and revise plans, or run a risk of missing out in their communication with students.

Variation cannot be avoided

The declaration of content problem is difficult to overcome, because within any population, there will be variation which cannot be captured by a brief description. Also, for instance, within the former classical elitist diploma of upper secondary education restricted to only a few percent of the population, or within the classical elitist university programmes in mathematics for only a fraction of one percent of the population fairly large variations existed. But naturally, the declaration of contents problem increases with the size of the population and the variation within it.

The problem of target group levels

Content related consequences of the variation problem for the target group outcome

While the declaration of contents problem focuses on the time consumption and difficulties caused by a large variation within the spectrum of knowledge, insight and abilities of students and graduates, the *target group level problem* focuses on content related consequences of this variation. This problem, which mainly but not exclusively, is experienced in the educational institutions, consists in the fact that if a given teaching programme is to be received by a very heterogeneous group of students, it is difficult to plan the teaching in such a way that everybody gets a fair outcome of it (unless additional resources are allocated in order to treat different subgroups differently – see the section below about differentiated teaching – which give rise to problems of its own) let alone reaches the same "level" (if, for a moment, we assume that this concept has been defined). As is mostly the case, if the teaching is designed to address "the average attendee", a lowering of the originally intended level can happen. Conversely, if the teaching is aiming to reach the students having the most abundant knowledge, insight and abilities, the risk is that the teaching will pass over the majority, with the formal or actual consequences this may entail; consequences that we and many others consider to be ethically, politically, and economically unacceptable.

The background to the problem

The target group level problem stems from different sources, where the main one is of a general education-political nature. This source is to do with the fact that the various sections of the education system are frequented by a considerable fraction of the age cohorts they are addressing. Another source is related to the fact that we, in Denmark, wish to avoid a very fine-grained division and selection of students in primary and lower secondary school and at the educational programmes for 16-19 year olds in a host of streams, branches, specialisations and levels, at least when the subject matter is mathematics. The fact that the sources of the target group problem are of a general nature does not necessarily mean that reasonable initiatives are out of reach.

10.3.4 Problems with differentiated teaching

In recent years, Danish mathematics teaching has, especially at primary and lower secondary school, focused on differentiated teaching within the individual classes. This is partly meant to answer the problems related to heterogeneity, caused by a large variation in the background, prerequisites and interests of the students in the same class, which in turn follows from the aforementioned resistance to an extensive division of the students in the Danish school system. Differentiated teaching aims to design and adapt the teaching in any given class so that special attention is paid to the individual student's background, prerequisites, interests and presumed needs.

Differentiated teaching as a means to handle heterogeneity

There are different problematic aspects attached to the question about differentiation. The first of these concerns what the notion of differentiated teaching actually means. The second aspect concerns the relationship between ideals and reality, that is between different perspectives on and approaches to differentiation, on the one hand, and the reality in the institutions and in the classrooms on the other hand.

What does differentiated teaching mean?

Clarification of concepts

Looking at usual, non-differentiated teaching for a moment, all students in a class receive the same teacher attention, both quantitatively and qualitatively. This means that the teacher divides his or her time equally among the students and expose them to practically the same teaching and learning activities, whether the teacher is addressing the whole class or is using other sorts of activities. No special attention is paid to the situations of the individual student. A certain limited individualisation of the relationship with a single student might occur, e.g. by commenting and grading written assignments and group work individually, or by giving individual feedback on students's achievement, development and so on. What is on the agenda in the teaching addresses all the students to the same extent, and in the same manner.

Characteristics of non-differentiated teaching

Multiple evidence suggests that such a non-individualised teaching leads to variation in students' learning outcomes, even though the teacher seeks to treat them as equal. Some students in a class are able to profit considerably

Uniform treatment leads to differentiated results from the teaching, while the outcome for others might be more modest. In other words, it is a well-known fact that equal treatment almost always leads to varying results, that is to outcome differentiation.

Quantitative and qualitative teaching differentiation

An alternative to the traditional approach is differentiated teaching, in which the students in a class receive some degree of different teaching. The difference can be of a quantitative nature, where the amount of teacher attention made available to students can vary considerably, or of a qualitative nature, where the activities offered to the individual students vary in kind and content. Naturally, a combination of both quantitative and qualitative variation may also occur. As defined here, differentiated teaching is intentional and is implemented based on the teacher's judgement of the situation, the capacities and needs of the individual students, with the aim to support the students' learning of mathematics. Hence, we are not in this definition concerned with unintended differences.

Intentions of differentiated teaching

Different intentions with differentiated teaching

There might be completely different *intentions* of differentiated teaching. Thus the intention might be that all students in a class are to achieve practically the same goals and results, while it is realised that for this to happen they need very different support. Some students are able to achieve the goals intended with only limited support from the teacher, whereas others might be able to reach the goals only if they receive a more extensive support specially designed for them. We might call this type of differentiated teaching *differentiation with the intent to impart the same outcome*. Even though it is the intention to make the students reach the same level, this is not so easy to accomplish in practice. If this intention fails, the students will end up profiting rather differently from the teaching.

Goal: The same outcome

Goal: To meet the students' different needs and potentials

The intention to differentiate teaching may also originate from the point of view that each student has a certain rather stable core of needs and possibilities that demand different kinds and amounts of teacher efforts to be supported and brought to maturitation. For instance, students experiencing difficulties at learning mathematics as well as students with special interests in and capacity for mathematics would need focused teacher attention. This line of thinking presupposes that some students are equipped with a particular urge, need and capacity to reach a high mathematical level, whereas others do not have the potential to go this far. On the basis of this intention it is a natural consequence that the students will profit very differently from the mathematics teaching they receive, even though the intention is not in itself to create such variation. We might call this sort of teaching differentiation *differentiation with the intent to realise the individual student's potential.*

Difficulties related to differentiated teaching

What difficulties arise with differentiated teaching? As a starting point, let us emphasise that the task group behind this report supports differentiated teaching of mathematics. The students' situations are so different that it becomes nearly impossible to conduct non-differentiated teaching without achieving large unwanted and undesirable differences in student outcome. Thus the problem is of a concrete not of a principal nature.

The task group supports differentiated teaching

Firstly, it is a problem if there is a wish or a decision to carry through differentiated teaching even though there is a lack of the teacher or teaching resources needed to allow for differentiation. This appears to be the case many places at primary and lower secondary school. This might easily lead to a situation where differentiated teaching either fails to be implemented, or, if attempted to be carried through anyway, leaving some students alone with very limited teacher attention only. Both situations lead to outcome differentiation whether it is intended or not. This problem becomes particularly visible in classes with a large variation amongst students.

Insufficient resources for differentiation

A second and more basic problem arises if differentiated teaching aiming at realising the individual student's potentials occurs on the ground of a misjudgement of the students' potentials, opportunities, and needs. It makes considerable demands on the teacher to clarify a student's real potentials, not at least so as these are not static but undergo continual development. By containing some students in certain roles and treating them accordingly, mathematics teaching might end up giving them stones for bread. It is reasonable to ask if mathematics teaching in Denmark too large an extent accepts unnecessarily large differences in student outcomes while interpreting as differences in student capacities. Differentiated teaching might thus enhance previously existing differences in students' social, economical and educational environments instead of contributing to even them out. On the other hand, it is of course also a problem if differentiated teaching implemented with the intention to make the student outcomes more or less equal, falls short because the variation is too big or the conditions to difficult.

The danger of misjudging the individual student's needs and possibilities

Stances towards the impact of these problems are closely connected to general views of human nature as well as general political and social attitudes. An equity-oriented view-point will find outcome differentiation problematic whether it is intended or not. A non-equity-oriented view

Views on differentiated teaching reflect general attitudes point will not find outcome differentiation problematic in itself if only differentiated teaching reflects what is considered to be real differences in the students' situations and capacities.

To further uncover this problem area through studying the existence and extent of all the problems mentioned would demand independent research projects, which fall outside the frames of this project.

10.3.5 Problems of assessment

In all mathematics teaching the issue of assessment occupies a key position whether it concerns different forms of final assessment, including tests and exams, or continuous assessment attached to the teaching. An overwhelming body of scientific evidence underpins that no matter what forms of assessment are used, the assessment has a significant retroactive ("backwash") impact on teaching and learning processes.[14] In some places this insight is formulated briefly and in slogan form as "what you assess is what you get" (and also what you discover).

Assessment affects teaching and learning

The existence of effects of assessment on teaching and learning processes is not a problem in itself. On the contrary, this connection can be viewed as a potentially useful tool for teaching planning and implementation. The possession of such a powerful tool demands an attention to its application: In a system using assessment, competencies, knowledge and skills which are not assessed become invisible if they not simply wither away altogether. In other words, the competencies desired not only have to be put on the teaching agenda, they also have to be put on the assessment agenda. The forms of assessment deliver a much more effective tool for pointing out what is considered to be important and unimportant at a certain educational level than all the world's formulation of goals and aims, teaching guides and presentations to teachers etc.

What is not assessed is missed

On this basis it is possible to point to two variants of the assessment problem especially connected to mathematics teaching in Denmark.

The mismatch problem

Mismatch between the forms of assessment and what is desired to advance and measure

The first variant, which could be called *the mismatch problem* arises, because many of the forms of assessment traditionally used in Danish mathematics

[14] See e.g. Niss (1993a,b) and Clarke (1996), both containing analyses and examples with an international perspective, and Barnes et al. (2000), taking as their starting point Australian experiences, argue for the necessity to include new forms of assessment in relation to curriculum reforms.

teaching only to a limited degree allow for assessment of the mathematical competencies, knowledge and skills that the mathematics teaching is aimed to foster and advance. This is not the least the case with the prevailing forms of final examination, which for instance – because of limited time frames – do not allocate space for serious work on mathematical modelling and more profound problem solving.

To the extent every day teaching seeks to develop experience, insight, knowledge, skills, and basic abilities, which cannot be properly taken into consideration by the forms of assessment employed, the mismatch not only exists between these forms and the knowledge and skills which basically are demanded at the end of the day, but also between the forms of assessment and the teaching actually carried out. Since the assessment, as mentioned above, usually exerts a greater influence on the teaching than vice versa, the mismatch problem gives rise to a distortion of the mathematics teaching and the learning in relation to what was intended.

Mismatch between the forms of assessment and the teaching

Even though there has been a lot of developmental work in Denmark concerning assessment concurrently with a blurring of the traditional frameworks for assessment, the problem of mismatch is still rather manifest, in particular as regards high stakes testing and exams. There is a continuing need for new initiatives that may contribute to reducing the mismatch problem.

The problem of interpretation

The second variant is *the problem of interpretation*. Whether the forms of assessment employed are suitable for assessing what is important or not, difficulties at securing that they assess, in a reliable and adequately way, what is actually intended often occur. It is difficult to secure that the interpretations of and conclusions about students' mathematics learning and mastery, obtained through the use of a given assessment tool are actually robost towards a closer and more thorough examination. Many research results[15] show the occurrence of misleading assessment conclusions in cases when control questions and follow-up questions on the answers provided are not possible, as for instance is characteristic of many forms

Problem to interpret assessment result validly and adequately

Danger of misleading conclusions

[15] See e.g. Bodin (1993) who with reference to a French study discusses this issue in relation to different forms of assessment, or Jakobsen et al. (1999) reporting from a developmental study at the Danish Technical Univerity, where 10 teams of students having received the same teaching were exposed to two different forms of assessment which gave completely different results.

of written assignments, in particular those based on the type “solving of routine problems”.

Clarifying the real extent of this problem in the practice of mathematics teaching would demand research rather than developmental work. In the mean time, see Chapter 9, it would be appropriate to stress that assessment tools allowing for a fair degree of validity as well as of reliability of the interpretations, do in fact exist but these are usually resource- and time-consuming both for teachers and students.

Explanation for the assessment problem

Causes: Resource limitations, inertia and ignorance

There are at least three reasons why the two variants of the assessment problem are difficult to get rid of. Firstly, resource and time limitations, including difficulties in balancing assessment activities and teaching. Secondly, the general inertia at all levels of the education system and scepticism towards new initiatives, not the least in a traditionally sensitive high stakes area such as assessment, which among other things plays a decisive role in determining students’ future lives and careers. And last, but not least, ignorance of alternative forms of assessment, of their potentials, ranges, and limitations.

11 Recommendations

11.1 Introduction

It was always the purpose and the nature of the KOM project to be exploratory, development oriented, creative and exemplifying. The project was neither meant to be a research project in the traditional sense nor a decision or implementation project.

Not a decision or implementation project

It was not part of the terms of reference for the work, nor of the background characteristics of the KOM task group for that matter, that proposals should be put forward concerning executive orders or changes thereof, concrete curricula, or the like, for different kinds of established mathematics teaching in Denmark. This is in line with the fact that the task group has not been provided with a mandate involving transition of the power of formal authority to the task group. Thus, it is up to the relevant formal authorities to decide whether, and if so in what way, the thoughts and recommendations put forward in this report should be made specific and carried out in real life. Presumely, the realisation of the recommendations would require a non-negligible extent of work.

Realisation of the recommendations is demanding and labour intensive

Listed below are the overarching recommendations the task group wants to put forward in this connection. First, an overview of the recommendations in bullet form is given. The overview is structured according to the recipient body of the recommendations, that is, the decision makers requested to take responsibility for the recommendation at issue. Secondly, the task group's comments to and justifications of the recommendations are presented.

It is necessary to underline that the recommendations are not of an either-or nature. That is, it is not the case that only a complete realisation of the recommendations makes sense. Also a partial realisation can contribute to promoting thoughts and intentions from the KOM project. However, the range of such promotions is, of course, closely related to the extent to which the recommendations are implemented.

Not an either-or question

11.2 Overview of the recommendations

*-marked recommendations

Some of the recommendations below have been marked with a star, *. These recommendations are the ones to which the task group ascribes special importance, recommendations which should be implemented immediately. This does not mean, however, that the rest of the recommendations lack importance. Rather, they are partly derived recommendations of a more long-ranging and multifaceted nature.

11.2.1 The Ministry of Education is recommended to

Curriculum and course plans

1.1. * devise consistent and coherent *curriculum and course plans* in mathematics based on the competencies identified and dealt with in this report, for primary and lower secondary school, the grades 10-12 programmes, the vocational programmes, and for non-tertiary adult education. This could happen, for instance, to establishing committees responsible for each of the educational programmes concerned, as well as a coordination committee responsible for overseeing coherence along and across the educational levels considered.

Programmes involving math. competencies

1.2. ensure that the relevant *mathematical competencies are included in those educational programmes* which are not specifically mathematical but do involve some degree of mathematical competencies.

Experiments for innovation

1.3. take initiative to launch *systematical, and centrally monitored experiments*, as well as ensure *legally-administrative latitude* for locally initiated experiments, for the advancement of competency-based mathematics teaching. Selected experiments should be monitored, surveyed and evaluated scientifically.

Forms of tests and exams

1.4. * launch a study to *revise and supplement existing forms and instruments of tests and exams*, with the intent to assess the whole range of mathematical competencies in an adequately and reliable way.

   – This can both be accomplished by obtaining information about and adobting forms and instruments already tested at other institutions at home and abroad, and by creating and developing new ones.

Teacher training

1.5. * ensure that *teacher training* at all educational levels under the jurisdiction of the Ministry be designed and structured so that future teachers are equipped with the mathematical, didactic and pedagogical competencies presented in this report.

1.6. * devise guidelines and measures so as to make sure that mathematics teaching in primary school and lower secondary school *is only managed by teachers with subject-pedagogical education* in mathematics. Subject-pedagogical education is necessary

1.7. * develop, implement and (co-)finance a broad spectrum of *in-service training and further education courses in mathematics and its teaching and learning* for teachers at all relevant teaching and educational levels. In-service training and further education courses

1.8. ensure an *active involvement of the entrants of mathematics teaching*, first of all the teachers, in all ministerial initiatives aiming to implement this report and its recommendations. Involve the entrants

1.9. take initiative to design and carry through a *high school reform* operating with carefully thought out and balanced "subject packages" so that extensive cooperation between the different subject areas becomes possible. High school reform

11.2.2 Universities and institutions of higher educations are recommended to

2.1. consider *revision of curricula and course plans* for the mathematical subjects with the intent to base these on the complete set of mathematical competencies as presented in this report. This should happen with the considerations in mind concerning the interplay between competencies and subject matter as well as the possibilities of assessing competencies presented in this report. (By the term "mathematical subjects" we mean mathematics as a scientific discipline (the study of mathematics), as an applied discipline (as support and tool for mathematically based subject areas), and as an educational subject (in teacher training programmes)). Curricula and course plans

- This might take place through the establishment of task forces for the educational programmes concerned.

2.2. take initiative to *didactic and pedagogical developmental work* concerning the teaching offered in the mathematical subjects provided by each institution. Developmental work

2.3. * contribute to ensure that *mathematics teacher training programmes* for upper secondary and tertiary education are designed and planned such that future teachers are prepared to carry out teaching aiming Teacher training at equipping its recipients with the proposed mathematical competencies.

– To realise this, future teachers should be equipped with those mathematical, didactic and pedagogical competencies that are presented in this report.

Subject didactic and pedagogical upgrading

2.4. make sure that the teachers of mathematical subjects are ensured continued *reinforcement of their subject didactic and pedagogical competencies.*

– This can take place though in-service training and further education courses for teachers, and as special courses for newly appointed assistant professors, among others.

Assessment

2.5. carry out a *systematic inspection of the forms and instruments of assessment* used for continuous assessment with the intent to uncover these tools' capacity to assess the different mathematical competencies in an adequate and well-founded way, and to adjust and focus them so that this capacity can manifest itself and be unfolded.

Forms of tests and exams

2.6. launch a work *to revise and supplement the forms of exams and exam instruments employed* such that the whole range of competencies can be assessed in a valid and reliable way.

– This can happen by gathering information about and adobting forms and instruments already tested elsewhere at home and abroad and by inventing and developing completely new ones in case this proves desireble.

Contact and cooperative bodies

2.7. take initiative, at a regional level, to establish *contact and cooperative bodies* with upper secondary schools in the region with the intent to launch and maintain discussions and projects about transition problems in mathematics from the grades 10-12 programmes to tertiary level education.

11.2.3 Teacher training institutions/Centres for further education are recommended to

Revision of the teacher training programmes

3.1. * *revise the design of the mathematics teacher training programmes for primary and lower secondary school* in cooperation with the Ministry of Education with the intent to equip a satisfactory number of teachers with the mathematical, didactic and pedagogical competencies presented in this report. This should take place by including the considerations put forward in this report concerning the interplay between the competencies and subject matter, and about the possibilities of evaluating the competencies. The revision should include the requirement that teacher students should have at least a B-level certificate from upper secondary school.

3.2. ensure that *future mathematics teachers* for primary and lower secondary school are prepared to *carry out teaching* aiming at equipping the recipients with the mathematical competencies identified in this report.

Comp. oriented teaching

3.3. ensure that their own teacher educators in mathematics are provided with opportunities to *strengthen their mathematical, didactic and pedagogical competencies.*

Upgrading of teacher educators

- This can take place through in-service training and further education courses for tenured teacher educators and through specially designed courses for newcomers in the profession.

3.4. launch a *systematic inspection of the forms and instruments of assessment* employed in the institution to conduct continuous assessment with the intent to establish their capacity of assessing the different mathematical competencies in an adequate and sound manner, and to adjust and focus them such that this capacity can be expressed and unfolded.

Assessment

3.5. begin to *revise and supplement existing test and exam forms and instruments* such that the complete set of competencies can be evaluated in a valid and reliable way.

Test and exam forms

- This can take place both by gathering information about and adopt forms and instruments which have been tested elsewhere, both at home and abroad, and by inventing and developing new ones if desireble.

11.2.4 The vocational programmes are recommended to

Uncover math. comp.

4.1. *uncover and articulate those mathematical competencies which they attempt to develop* as parts of the individual vocational programmes, and to consider the optimal educational conditions for such development. Here we are specifically thinking about those programmes that actually entail fostering certain mathematical competencies with students, but where mathematics as a subject area is not in itself on the agenda in any explicit way, but also of course, programmes where mathematics teaching is explicitly on the agenda.

11.2.5 Municipalities and other local school authorities are recommended to

Curriculum and course plans

5.1. ensure and advance the preparation of supplementary local *curriculum and course plans* in mathematics, based on the competencies treated in this report, for the schools under their jurisdiction.

   – This can, e.g. happen through the establishment of committees for the programmes concerned.

Time for cooperation

5.2. make sure that the teachers at the schools with their jurisdictions have *time for cooperation* about the development of the mathematics teaching in accordance with the ideas expressed in this report.

Experimental teaching

5.3. ensure financial and administrative *latitude for experimental teaching* initiated locally with the aim to (further) develop ways in which to place mathematical competencies on the teaching agenda.

Mathematico-pedagogically competent teachers

5.4. make sure that mathematics teaching in primary and lower secondary school *is only attented to by teachers with mathematico-pedagogical education.*

In-service training and further education courses

5.5. * ensure the conditions and the funding for a broad spectrum of mathematical and pedagogico-didactical *in-service training and further education courses* – of a proper quality – in mathematics and its teaching for all mathematics teachers within their jurisdiction.

Students' and parents' acceptance

5.6. strive to achieve *student and parent acceptance* of the demands put on the students to develop the mathematical competencies treated in this report.

11.2.6 Mathematics teachers and their associations are recommended to

Experimental teaching

6.1. launch, at all educational levels, both large and small scale *teaching experiments* with the intent to (further) develop ways to put the mathematical competencies on the teaching agenda.

Assessment

6.2. inspect and develop those *assessment and test forms and instruments* that teachers use to conduct continuous assessment with the intent to uncover the capacity of these forms and instruments to assess the different mathematical competencies in an adequate and sound way, and to adjust and focus them such that this capacity can be expressed and unfolded.

6.3. take part in a broad variety of *in-service training and further education courses* with the intent to strengthen teachers' capacity to plan and carry through competency oriented mathematics teaching.

In-service training and further education courses

6.4. cooperate with researchers in mathematics education about the initiation of mathematics education *research projects* to describe and analyse initiatives aiming at developing competency oriented mathematics teaching.

Research projects

6.5. establish permanent *contact and cooperation bodies* for and across mathematics teachers in primary and lower secondary schools and in other programmes, and for the mathematics teachers at upper secondary and tertiary education, respectively, with the aim to initiate and maintain discussions and projects about transition problems in mathematics from one educational level to the next.

Contact and cooperation bodies

11.2.7 Textbook authors and publishing companies are recommended to

7.1. *develop and produce materials for learning* which can provide a platform for teaching seeking to impart students with the whole range of mathematical competencies as well as the forms of overview and judgement concerning mathematics as a discipline that have been put forward in this report.

Materials for learning with a focus on competencies

11.2.8 Researchers in mathematics education are recommended to

8.1. launch, in cooperation with teachers, institutions and authorities, proper mathematics education *research projects* to describe and analyse initiatives aiming at developing competency oriented mathematics teaching.

Research projects about competencies

11.3 Comments on and reasons for the recommendations

11.3.1 Recommendations about curriculum and course plan revisions

It is the conviction of the task group that initiatives leading to the realisation of considerations made in the KOM project may, in many places, contribute significantly to raising the level of ambitions in the mathematics teaching.

Re-orientation and re-formulation

In connection with re-orientating and re-formulating the aims and goals of mathematics teaching, as well as the design hereof, there will, in most places, probably be a need for further clarification concerning students' possession of the different competencies, as well as the interplay between the competencies and mathematical subject matter, probably to a much larger extent than we have been able to present in this report. If such a clarification process is initiated it is important to try to avoid two possible pitfalls. First, an understandable wish to make such a clarification as concrete and specific as possible may result in too comprehensive and detailed a specification and "singularisation" of the individual features and elements of the mathematical competencies in question. This will tend to jeopardise the main underlying idea, which is to identify a smaller number of major components in the mastery of mathematics. Secondly, it might, for some, be tempting to convert the competencies, each of which represents an infinite spectrum of mastery which can never be acquired completely, to a set of clearly delineated, recognisable, and behaviouristically described skills to handle a set of well-defined situations and tasks.

Avoid over-detailing

Avoid simplification

But this would lead to a dramatic reduction of the level of ambition attached to the suggested characterisation of mathematical mastery. Hence, it is crucial that "clear goals" are not to be identified with "simplistic" or "simplified goals".

These considerations lie behind the recommendations 1.1, 1.2, 1.5, 2.1, 3.1, 4.1 and 5.1. First of all, it is recommended here that the relevant authorities, bodies, associations and institutions consider – e.g. through the establishment of working committees – in what respects and in what ways, curricula and course plans in mathematics as part of the respective educational programmes can be based on a mathematical competency description as offered in this report. This should happen by involving the considerations presented in this report about the interplay between competencies and mathematical subject matter and the possibilities of assessing these competencies. Supposing that these considerations at a given educational level lead to the conclusion that such a description is possible and desirable, the working committees should, as their task, have to devise drafts of executive order(s) or analogues codes of practice on the basis mentioned. Secondly, it is recommended to ensure exchange of information, consistency, and, to the extent needed, co-ordination along and across mathematics teaching at the above mentioned educational levels. This follows from one of the main goals of the KOM project which is to contribute to creating coherence and progression in mathematics teaching throughout the educational system by way of the development of a common conceptual and descriptive framework.

Integrated approach

Regarding vocational programmes and tertiary educational programmes without a distinctly mathematical nature, but with mathematics as an important key service subject, we want to stress the traditional significance of the modelling competency, the representing competency, the symbol and formalism competency, but also the aids and tools competency, where weight should be given to, e.g., computer algebra systems, spreadsheets and statistical software. However, it becomes ever more important that both the problem tackling competency and the communicating competency are developed to the benefit of the educational programmes in which mathematics is a key service subject.

Educational programmes in which mathematics is a key service subject

Mathematics teachers and users of mathematics often discuss how much weight should be assigned to the thinking and reasoning competencies. It is the viewpoint of the task group that the more powerful the ICT-tools available are for the technical treatment of mathematical issues, the more important it is to be able to relate to the questions in focus, and to the answers given, which is exactly the main focus of the mathematical thinking and reasoning competencies.

ICT

Regarding tertiary mathematics programmes it is the position of the task group that the entire spectrum of competencies should be assigned weight, also those which – in some places – have not traditionally been on the agenda, such as the modelling, communicating, aids and tools competencies. In circumstances where this is possible, the modelling competency should to the extent possible be developed in close cooperation with neighbouring subjects like physics, economics, biology, computer science etc.

All comp. should be assigned (varying) weight

11.3.2 Recommendations regarding contact and cooperation between different educational levels and programmes

Gaps and transition problems

One of the overarching problems in Danish mathematics education mentioned in Chapter 10 concerns the wide gaps in culture, perspectives, content, demands and teaching practice occurring at the transition from one part of the education system to the next – from primary school to the grades 10-12 programmes, from the grades 10-12 programmes to tertiary education – gabs which, among other things, create transition problems for the students leaving one segment of the system and entering another.

A softening is necessary

It is one of the main points of this project that some of these problems can be remedied and reduced through acceptance and application of a common, overarching understanding of the aims and goals of mathematics teaching as offered by a competency based approach. Since the lines of fracture between the different parts of the education system are so well-established it is not realistic to await significant progress in the softening of them by means of only one initiative. A variety of initiatives at all levels are needed to achieve this.

Contact and cooperative bodies

Some of these initiatives are included in the recommendations 1.2, 2.7 and 6.5. There it is recommended to establish permanent contact and cooperative bodies given the task to initiate and maintain discussions and projects concerning transition problems in mathematics. The task group believes that the competency approach can provide an important common basis for such discussions and projects. Due to the contentual and the geographical complexity of parts of the education system, we cannot suggest a common, simple model for such bodies. In some cases it would be natural for the individual educational institution to take initiatives to establish such a body, while in other cases it should be a task for the schools and the mathematics teachers within one geografic region to devise useful models for contact and cooperation.

11.3.3 Recommendations concerning experimental teaching

The KOM project has been focused on the overall characterisation of mathematics mastery in relation to different parts of the education system, and, consequently, on the design of the boundary conditions for mathematics teaching, the main problems that mathematics teaching is faced with, as well as on the mathematics teachers' professional competence. As recommended above, this can lead to new ways to describe curricula and course plans. It also creates a number of instruments applicable for teachers to develop and re-orientate their teaching so as to really put focus on the development of the students' mathematical competencies. It is not obvious that all sorts of form and content of the mathematics teaching currently practiced in Denmark are equally well-suited for this purpose.

It falls outside the scope of the KOM project to specify how teaching can concretely be designed and carried through in order to contribute to the solution of this task, what student activities are appropriate, and how textbooks and other types of teaching materials can be designed to advance the project, and so on and so forth.

It goes without saying that these issues are, in many ways, the most important ones when it comes to the realisation of the thoughts behind the KOM project, and simple and quick answers do not exist. Answers must be gained through experimental teaching both within and across the official framework of mathematics teaching.

No simple, quick answers – experiments are necessary

This constitutes some of the background to the recommendations 1.3, 2.2, 5.2, 5.3, 6.1, 7.1 and partly 8.1. On the one hand, it is recommended there that the Ministry of Education, local authorities, as well as educational institutions provide room for manoeuvring with respect to administrative rules, time and resources for a variety of experimental teaching aiming at (further) developing ways in which the mathematical competencies can be on the teaching agenda. On the other hand, it is recommended, that mathematics teachers and other relevant parties actively becomes involved in such experimental teaching. It would, furthermore, be desireble if particularly engaged and interested mathematics teachers at all levels cooperated with researchers in mathematics education in research projects proper concerning competency based mathematics teaching. This is exactly the content of the recommendation 6.4.

The road for experiments must be cleared

11.3.4 Recommendation concerning textbooks and other teaching materials

Competency based mathematics teaching must obviously employ textbooks and other teaching materials that are in concord with the overall intention. It is clear that only a smaller part of the competency spectrum is represented, not to speak of accentuated, in most existing textbooks and materials. This is the background for recommendation 7.1.

The need for new teaching materials

11.3.5 Recommendation concerning in-service training and further education courses for teachers

If the KOM project should stand a chance, mathematics teachers at all levels must be equipped with the competence to design, organise, and carry through teaching aiming at developing students' mathematical competencies. With the intent to accommodate this, undoubtedly profound, need for professional competence development with current teachers, it is necessary to initiate a range of in-service training and further education activities in all educational sectors and on all teaching levels. This is the content of the recommendations 1.7, 2.4, 3.3, 5.5 and 6.3.

Professional comp. development

In-service training and further education activities can take place at different levels of ambition ranging from collegial study groups at a local level, meetings and conferences, through to courses and extensive local, regional or national developmental projects. Here, for the allocation of time, room and resources to ensure the activities a satisfactory extent and impact becomes important.

Obviously, the more comprehensive upgrading of the mathematics teacher corps is demanded, the larger the scale of the in-service training and further education activities. *In this context, it is the view of the task group that far more positive effects can be gained through a substantial, general upgrading of mathematics teachers in accordance with the thoughts included in the KOM project than by allocating more hours to teaching*, although an increased number of teaching periods will, of course, also provide mathematics teaching with enhanced opportunities.

Comp. development of teachers is most important

11.3.6 Recommendations concerning the pre-service training of mathematics teachers

Regarding prospective mathematics teachers, the realisation of the thoughts and ideas in the KOM project is a far more extensive and complex task with significant educational, political, structural, and economic implications of a general nature.

Initiatives leading to changes in the mathematics teacher training programmes cannot, however, be viewed as isolated from general issues concerning the education of teachers in general. As was discussed in some details in Chapter 10, at each level there are different problems and challenges related to the work and working conditions of Danish mathematics teachers, and to the backgrounds against which they carry out their profession.

Re: teacher training programmes in general

The most striking and pre-dominant feature of the mathematics teachers working at a given educational level is the immense plurality and diversity characteristic of them. This means that any (other) generalisation will be misleading with regard to large sub-groups of teachers. Nevertheless, we are able to identify some general problematic features that should not be subject to taboo and suppression, and which call for changes.

Difference and diversity

Let it be clearly stated that these problematic features are not the result of flaws and deficiencies residing with individuals, single groups or single institutions. Rather, they are a result of profound systemic problems related to the entire Danish education system, which is why they should be dealt with as such and not just by initiatives concerning single elements in the system.

Systemic problems

The recommendations 1.5, 2.3, 2.4, 3.1, 3.2, and partly 2.1 and 5.4 suggest that the training of mathematics teachers for all levels, from primary school to university, should be designed or orchestrated such that future teachers are prepared to conduct teaching aiming at equipping its recipients with the mathematical competencies suggested. As is evident from part III, this clearly entails that teachers must be trained so as to possess these competencies themselves.

Focus on comp.

While the possible changes of the different teacher educational programmes may come in many different forms and shapes, the design of the programmes should primarily be entrusted to the teacher training education institutions themselves, eventhough the call for change may come from the politico-administrative quarters.

Mathematics teachers in primary and lower secondary school

Especially regarding the education of mathematics teachers for primary and lower secondary school the current situation is such that many students only have or obtain very modest mathematical and didactical prerequisites for performing the tasks the profession involves, see part III. In this connection the task group found it necessary to recommend – in items 1.6 and 5.4 – that only teachers with a mathematico-pedagogical education should teach mathematics. It is still unclear to what extent the recent and ongoing structural modifications of the teacher training programmes for this level will remedy this problem, but one might fear that the modifications will prove insufficient compared to the need.

Only mathematically and pedagogically trained mathematics teachers

Whether the modifications will entail more fundamental changes of the structure and the organisational position of the teacher training programmes must be further examined and discussed very carefully in relevant fora which naturally include the teacher training institutions.

No equalisation

It should be stressed that there is no recommendation to equalise the primary and lower secondary school teacher training programmes across the institutions which offer them. It is possible to have several, and each one valuable, ways to design teacher education. We hope such discussions will not be avoided, in spite of all the difficulties, problems, controversies and disagreeableness they will certainly lead to.

Mathematics teachers in upper secondary schools

Regarding mathematics teachers at the grades 10-12 programmes the variation is so large that we restrain ourselves to looking at the upper secondary schools only. Here only minor parts of the problem concern the mathematics teachers' mathematical subject knowledge in a traditional sense (although they may, in some cases and in some respect, lack parts of mathematical competencies as defined in this report), and usually this is also true for their teaching competence in a traditional sense, since it is developed through experience and practice. The main problem is that quite a few of the mathematics teachers teaching in upper secondary schools lack sufficient subject-didactical preparation to handle all the challenges produced by continuing alterations of the frameworks, terms and circumstances for teaching.

Reinforcement of the subject-didactical preparation

As recommended, there is a need here for visible reinforcement of the subject-didactical preparation in the education of teachers for upper secondary school. Reinforcement can take place in many different ways within or after university studies, as long as it is kept in mind that the reinforcement required will be unsatisfactory if it is only sought through teaching practice or activities related to didactic and pedagogical questions of a general, non-subject-specific nature.

No unification

Nor here do we recommend unification of the ways in which to achieve the desired subject-didactical reinforcement. Many different roads lead to this goal.

Mathematics instructors in universities and other tertiary institutions

Finally, regarding instructors in mathematics in the universities and other tertiary institutions the main problem is usually not the instructors' mathematical subject knowledge (although some teachers may have problems with some of the mathematical competencies). On the other hand, the problems can be of both a subject-pedagogical and a subject-didactical nature.

Part of tertiary mathematics teaching often takes place in forms, e.g. lectures for large groups of students, in which many instructors have limited contact with the students, and hence limited possibilities for obtaining continuous feedback from the students on teaching and learning. But also at the tertiary educational programmes changes of framework, terms and circumstances do entail a need to considerably reinforce many instructors' subject-didactical and pedagogical competence. Again, initiatives to this end should not be constrained to include only general didactical and pedagogical issues or to take place soely through practice.

Subject-didactical and pedagogical upgrading

As the case was for the pre-service teacher training programmes, we do, of course, not want to recommend any form of uniformisation of initiatives across departments or institutions.

No uniformisation

11.3.7 Recommendations concerning assessment and exams

One of the most important factors in successful implementation of a (new) initiative in mathematics teaching is assessment, including continuous assessment as well as test and exams. In an education system which makes use of organised assessment as a main means for monitoring, regulating and controling teaching and its outcomes, no new initiative will take a hold among teachers and students if the components involved are not subjected to assessment. In this regard, it is crucial that the tools of assessment chosen depict the aims and goals pursued in the teaching, and that they are in a reasonable degree of accordance with the activities and working forms used.

Accordance between goals, work forms and assessment

In the mathematics teaching in today's Denmark, it is the case that any form of assessment employed focuses on a limited part only of the mathematical competencies identified in this report, and this takes place somewhat indirect without explicit uncovering and articulation of the competencies in play.

However, if used in combination the forms and tools traditionally available to the teacher for conducting continuous assessment are considered to have a potential for allowing assessment of the competencies to a rather large extent. But to realise this potential the assessment forms and tools must be checked, adjusted and focused with the intent to aim them explicitly towards to identification and evaluation of students' acquisition of the different mathematical competencies.

Potential of prevalent assessment forms and tools

In contrast, when it comes to the prevalent written and oral test and exam forms and related instruments employed in assessment in mathematics, these allow for assessment of a limited range of competencies only. Here, too, there is a need to investigate, adjust and focus to unfold the potentials such that the forms and tools can be used to assess the competencies at large. This is precisely the content of the recommendations 1.3, 2.5, 3.4 and 6.2.

Investigate and adjust

As to the most frequently used test and exam forms and instruments, the task group finds that important parts of the competencies *cannot* be captured in an adequate and well-founded way by these forms and instruments. With a view to changing this situation we have put forward the recommendations 1.4, 2.6, 3.5, and partly 1.3 and 6.2, to the Ministry of Education and to all the educational institutions, from primary school to universities, to initiate revisions of and amendments to the test and exam forms and instruments employed.

Need for innovation

In this context it is crucial that the resulting assessment tools are in the abovementioned accordance with the aims and goals of the teaching as well as with the activities and working methods used.

Focus on larger assignment tasks

Future assessment methods at the tertiary educations may benefit from being dominated by different larger assignment tasks with subsequent interviews, and with a larger element of oral exams than is typically the case today.

11.3.8 Recommendations concerning the stakeholders in the implementation of the KOM-ideas

It has been pointed out several times in this report that no mathematics teaching reform, not even the one suggested here, has a chance to succeed – or for that matter even to be carried out – if the main stakeholders do not have a fair amount of conviction of the idea and the realism of the project and feel a certain amount of joint ownership and enthusiasm. In relation to the present project, the main stakeholders are the mathematics teachers at all levels, as well as the curriculum planners, policy makers and administrators, and the controlling authorities responsible for setting the boundary conditions for mathematics teaching and learning and the controlling the outcomes.

Joint ownership and enthusiasm

Especially the teachers

To facilitate this, the chairman and secretary of the project as well as other members of the task group have, throughout the duration of the project, participated in a significant number of meetings, education days, conferences, etc. all over the country to present and discuss the project with

The KOM project and the outside world

practicing (and prospective) mathematics teachers, and representatives of the educational authorities at all levels of the education system, as well as many representatives from other subject areas who wanted to become aquainted with the project. Furthermore, the task group has held regular meetings with the sparring group mentioned in the introduction which has followed the project closely and contributed with substantial comments and suggestions. Finally, the chairman of the project has presented and discussed the project at meetings and conferences in Japan, Sweden, Germany and the United States with the intent to test its solidity vis-á-vis international experts.

Although the experiences gained from all these contacts with the world outside can only be described as highly encouraging and promising for the continuation and implementation of the project, we find it of pivotal importance, also in future implementation phases, that extensive and close contact and dialogue with the stakeholders in mathematics education are maintained, e.g. through involving a large number of them in all the phases of the aforementioned work, in committees and project activities etc.. This is the content of recommendation 1.8.

Continuing contact, dialogue and participation

Among the most important stakeholders in the primary and lower secondary school are, obviously, the students, but also their parents. Considering that the development of mathematical competencies is notoriously a very effort and time consuming process for practically every student, it is crucial that the school on every level – from the individual teacher over the school administration and school board to the school authorities – are actively trying to develop the students' and their parents' understanding of this fact. This is the content of recommendation 5.6.

Parents as stakeholders

It is, of course, also important that the students perceive the teaching they receive to be competent, relevant and engaging, but even the most stimulating and convincing mathematics teaching cannot obviate the students' own serious work on the subject matter in a broad sense.

11.3.9 Considerations concerning the general frameworks and structures in the education system

It is clear from the above, that we have, by and large, refrained from stating recommendations concerning the general frameworks and structures of the education system, although these obviously exert a significant influence on the possibilities of realising the thoughts and ideas of the KOM project. There are several reasons for this choice.

(Almost) no recommendations concerning the structure of the education system

First, it is hardly realistic to imagine that a limited project such as this one can have a significant impact on vital educational, political, structural and economic concerns more far reaching than mathematics education, despite its extent and distribution, and therefore affects a large and complex group of interested parties and stakeholders. Thus it might appear futile if we took the opportunity to make comments and recommendations of a very general nature.

Improvements can be realised within the boundaries

Second, it is important to make it clear that significant parts of the perspectives and recommendations of the project can be realised within the existing frameworks and structures. Thus, the project does not depend upon the allocation of more teaching periods to the mathematics teaching in primary and lower secondary school. Nor does it depend upon all the students in tertiary education in need of mathematical competencies, to be exposed to an increased amount of formalised mathematics teaching, or that future teacher training takes place in universities etc [as is generally not the case in Denmark]. As is clear from this report and the recommendations above, there is a need for change, also of a more extensive nature. But this does hardly call for revolutionary transformations of the entire education system.

Exception: Upper secondary school

On one issue we shall, however, move beyond the horizon of mathematics education and touch upon a general structural problem: The design and structure of the general upper secondary school. The development of the mathematical competencies at this level is far too important to be entrusted to mathematics teaching alone. This is particularly significant when it comes to the application of mathematics within other subject areas – where the modelling competency is central. There is a need to ensure that mathematics teaching can work on topics and problems from other subject areas and that other subject areas can treat mathematical problems that occur in their contexts. This demands some degree of cooperation between the different subject areas. However, cooperation between different subject areas has very bad conditions in the existing "options based high school", which in practice hinders cooperation.

This problem is not only internal to high schools. It also has severe consequences for the students' future lives in further educational programmes, in which mathematics is a key service subject, if their mathematical competencies and the relations of these with other subject areas are as diverse and sporadic as the case is.

"A set menu high school" as opposed to a "buffet high school"

Here it would be recommendable to instigate a high school reform, that would replace the options-based high school – the "buffet high school", as it is sometimes called – with a high school operating with well thought out and carefully balanced subject area packages making a cooperation between the subject areas possible – a “set menu high school”. This is the content of recommendation 1.9.

As is well known, a reform in this direction can be more or less far-reaching, ranging from one common subject area package for all, with no or only a few choices, to a streamed high school with a number of streams each with its own subject area package as was the case before 1988. This is not the place to suggest which of these or alternative designs a new high school structure should have.

Bibliography

Allerup, P., Bredo, O. & Weng, P. (1998). *Matematik og naturvidenskab i ungdomsuddannelserne – en international undersøgelse*, DPI (TIMMS), København.

Andersen, A. M., Egelund, N., Jensen, T. P., Krone, M., Lindenskov, L. & Mejding, J. (2001). *Forventninger og færdigheder – danske unge i en international sammenligning*, AKF, DPU og SFI (OECD – PISA), København.

Barnes, M., Clarke, D. & Stephens, M. (2000). Assessment: the engine of systemic curricular reform?, *Journal of Curriculum Studies* **32**: 623–650.

Bishop, A. J., Clements, K., Keitel, C., Kilpatrick, J. & Laborde, C. (eds) (1996). *International Handbook of Mathematics Education*, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Bodin, A. (1993). What Does to Assess Mean? The Case of Assessing Mathematical Knowledge, *in* Niss (1993b), pp. 113–141.

Christiansen, B. (1989). Konferencens tema i fagdidaktiske perspektiver, *in* G. Nissen & J. Bjørneboe (eds), *Gymnasiets matematikundervisning mellem studie- og erhvervskrav og demokratikrav*, IMFUFA, Roskilde Universitetscenter, Roskilde, pp. 33–92.

Clarke, D. (1996). Assessment, *in* Bishop et al. (1996), pp. 327–370.

Gregersen, P. & Jensen, T. H. (1998). *Problemløsning og modellering i en almendannende matematikundervisning*, number 353 in *IMFUFAtekst*, IMFUFA, Roskilde University, Roskilde, Denmark. To be ordered from imfufa@ruc.dk.

Heymann, H. W. (1996). *Allgemeinbildung und Mathematik – Studien zur Schulpädagogik und Didaktik*, Beltz, Weiheim und Basel, Tyskland.

Jakobsen, A. et al. (1999). Kvalitetsudviklingsprojektet "Faglig Sammenhæng" – Hovedrapport, *Technical Report 1*, Center for Didaktik og Metodeudvikling, Danmarks Tekniske Universitet.

Jensen, H. S. & Skovsmose, O. (1986). *Teknologikritik – et teknologifilosofisk essay*, Systime.

Jensen, P. H. & Kyndlev, L. (1994). *Det er ikke til at se det, hvis man ikke lige ve' det! Gymnasiematematikkens begrundelsesproblem*, number 284 in *IMFUFAtekst*, IMFUFA, Roskilde Universitetscenter, Roskilde.

Niss, M. (1993). Assessment in Mathematics Education and its Effects: An Introduction, *in* Niss (1993b), pp. 1–30.

Niss, M. (1996). Goals of Mathematics Teaching, *in* Bishop et al. (1996), pp. 11–47.

Niss, M. (1999). Kompetencer og uddannelsesbeskrivelse, *Uddannelse* **9**: 21–29.

Niss, M. (2000). Gymnasiets opgave, almen dannelse og kompetencer, *Uddannelse* **2**: 23–33.

Niss, M. & Jensen, T. H. (eds) (2002). *Kompetencer og matematiklæring – Ideer og inspiration til udvikling af matematikundervisning i Danmark*, number 18 in *Uddannelsesstyrelsens temahæfteserie*, The Ministry of Education, Copenhagen, Denmark. Cf. http://nyfaglighed.emu.dk/kom.

Niss, M. (ed.) (1993a). *Cases of Assessment in Mathematics Education*, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Niss, M. (ed.) (1993b). *Investigations into Assessment in Mathematics Education*, Kluwer Academic Publishers, Dordrecht, The Netherlands.

OECD (1999). *Measuring Student Knowledge and Skills – A new Framework for Assessment*, OECD, Programme for International Student Assessment (PISA), Paris, France.

OECD (2001). *Knowledge and Skills for Life – First Results from PISA 2000*, OECD, Programme for International Student Assessment (PISA), Paris, France.

Ramsden, P. (1999). *Strategier for bedre undervisning*, Gyldendal, København. Oversat efter "Learning to Teach in Higher Education", Routledge, UK, 1992.

Simonsen, B. & Ulriksen, L. (eds) (1998). *Universitetsstudier i krise: Fag, projekter og moderne studenter*, Roskilde Universitetsforlag, Roskilde.

The National Commission on Mathematics and Science Teaching for the 21st Century (2000). *Before It's Too Late ("The Glenn-report")*, U.S. Department of Education, Washington DC, USA. Se http://www.ed.gov/americacounts/glenn/.

Undervisnings- og forskningsministeriet (1990). *Matematik. Undervisningen i matematik i det danske uddannelsessystem. En beskrivelse og vurdering af mål og indhold*, Undervisnings- og forskningsministeriet, København.

Undervisningsministeriet (1978). *U90 – Samlet uddannelsesplanlægning frem til 90'erne, bind 1 og 2*, Undervisningsministeriet, København.