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Chapter 1
Hesitating progress – the slow development toward algebraic symbolization in abbacus-and related manuscripts, c. 1300 to c. 1550

Jens Høyrup

Ian Mueller in memoriam

Abstract From the early fourteenth century onward, some Italian Abbacus manuscripts begin to use particular abbreviations for algebraic operations and objects and, to be distinguished from that, examples of symbolic operation. The algebraic abbreviations and symbolic operations we find in German *Rechenmeister* writings can further be seen to have antecedents in Italian manuscripts. This might suggest a continuous trend or perhaps even an inherent logic in the process. Without negating the possibility of such a trend or logic, the paper will show that it becomes invisible in a close-up picture, and that it was thus not understood – nor intended – by the participants in the process.

Key words: Abbacus school, Algebra, Symbolism

1.1 Before Italy

Ultimately, Italian abbacus algebra[1] descended from Arabic algebra – this is obvious from its terminology and techniques. I shall return very briefly to some of the details of this genealogy – not so much in order to tell what

[1] The "abbacus school" was a school training merchant youth and a number of other boys, 11-12 years of age, in practical mathematics. It flourished in Italy, between Genoa-Milan-Venice to the north and Umbria to the south, from c. 1260 to c. 1550. It taught calculation with Hindu numerals, the rule of three, partnership, barter, alligation, simple and composite interest, and simple false position. Outside this curriculum, many of the abbacus books (teachers' handbooks and notes, etc.) deal with the double false position, and from the fourteenth century onward also with algebra.

happened as to point out how things *did not* happen; this is indeed the best we can do for the moment.

First, however, let us have a look at Arabic algebra itself under the perspective of "symbolism".[2]

The earliest surviving Arabic treatise on the topic was written by al-Khwārizmī somewhere around the year 820.[3] It is clear from the introduction that al-Khwārizmī did not invent the technique: the caliph al-Maʾūn, so he tells, had asked him to write a compendious introduction to it, so it must have existed and been so conspicuous that the caliph knew about it; but it may have existed as a technique, not in treatise form. If we are to believe al-Khwārizmī's claim that he choose to write about what was subtle and what was noble in the art (and why not believe him?), al-Khwārizmī's treatise is likely not to contain everything belonging to it but to leave out elementary matters.

It is not certain that al-Khwārizmī's treatise was the first of its kind, but of the rival to this title (written by the otherwise little known ibn Turk) only a fragment survives (ed. Sayılı, 1962). In any case it is clear that one of the two roughly contemporary treatises has influenced the other, and for our purpose we may take al-Khwārizmī's work to represent the beginning of written Arabic algebra well.

Al-Khwārizmī's algebra (proper) is basically a rhetorical algebra. As al-Khwārizmī starts by saying (ed. Hughes, 1986, p. 233), the numbers that are necessary in *al-jabr wa'l-muqābalah* are *roots*, *census* and simple numbers. *Census* (eventually *censo* in Italian) translates Arabic *māl*, a "possession" or "amount of money", the *root* (*radix/jidhr*, eventually *radice*) is its square root. As al-Khwārizmī explains, the root is something which is to be multiplied by itself, and the *census* that which results when the root is multiplied by itself; while the fundamental second-degree problems (on which presently) are likely to have *originated* as riddles concerned with a real amount of money and its square root (similar to what one finds, for instance, in Indian problem collections),[4] we see that the root is on its way to take over the role as basic unknown quantity (but only on its way), whereas "dirham" serves in

[2] I shall leave open the question of what constitutes an algebraic "symbolism", and adopt a fairly tolerant stance. Instead of delimiting by definition I shall describe the actual character and use of notations.

[3] The treatise is known from several Arabic manuscripts, which have now appeared in a critical edition (Rashed, 2007), and from several Latin translations, of which the one due to Gherardo of Cremona (ed. Hughes, 1986) is not only superior to the other translations as a witness of the original but also a better witness of the original Arabic text than the extant Arabic manuscripts as far as it goes (it omits the geometry and the chapter on legacies, as well as the introduction) – both regarding the grammatical format (Høyrup, 1998) and as far as the contents is concerned (Rashed, 2007, p. 89).

[4] Correspondingly, the "number term" is originally an amount of *dirham* (in Latin *dragmata*), no pure number.

al-Khwārizmī's exposition simply as the denomination for the number term, similarly to Diophantos's *monás*. In the first steps of a problem solution, the basic unknown may be posited as a *res* or *šayʾ*, "a thing" (*cosa* in Italian); but in second-degree problems it eventually becomes a *root*, as we shall see.

As an example of this we may look at the following problem (ed. Hughes, 1986, p. 250):[5]

> I have divided ten into two parts. Next I multiplied one of them by the other, and twenty-one resulted. Then you now know that one of the two sections of ten is a thing.[6] Therefore multiply that with ten with a thing removed, and you say: Ten with a thing removed times a thing are ten things, with a *census* removed, which are made equal to twenty-one. Therefore restore ten things by a census, and add a *census* to twenty-one; and say: ten things are made equal to twenty-one and a *census*. Therefore halve the roots, and they will be five, which you multiply with itself, and twenty-five results. From this you then take away twenty-one, and four remains. Whose root you take, which is two, and you subtract it from the half of the things. There thus remains three, which is one of the parts.

This falls into two sections. The first is a rhetorical-algebraic reduction which more or less explains itself.[7] There is not a single symbol here, not even a Hindu-Arabic numeral. The second section, marked in sanserif, is an unexplained algorithm, and indeed a reference to one of six such algorithms for the solution of reduced and normalized first- and second-degree equations which have been presented earlier on.

Al-Khwārizmī is perfectly able to multiply two binomials just in the way he multiplies a monomial and a binomial here; slightly later (ed. Hughes, 1986, p. 249) he states that "ten with a thing removed" multiplied by itself yields "hundred and a *census* with twenty things removed". He would thus have no difficulty in finding that a "root diminished by five" multiplied by itself gives a "*census* and twenty-five, diminished by ten roots". But he cannot go the other way, the rhetorical style and the way the powers of the unknown are labeled makes the dissolution of a trinomial into a product of two binomials too opaque either for al-Khwārizmī himself or for his "model reader". In consequence, when after presenting the algorithms al-Khwārizmī wants to give proofs for these, his proofs are geometric, not algebraic – geometric proofs not of his own making (as are his geometric illustrations of how to deal with binomials), but that is of no importance here.

It is not uncommon that rhetorical algebra like that of al-Khwārizmī is translated into letter symbols, the *thing* becoming x and the *census* becoming

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[5] My translation, as everywhere in the following when no translator into English is identified.

[6] This position was already made in the previous problem about a "divided ten".

[7] However, those who are already somewhat familiar with the technique may take note of a detail: we are to restore ten things with a *census*, *and then* add a *census* to 21. "Restoring" (*al-jabr*) is thus not the addition to both sides of the equation (as normally assumed, in agreement with later usage) but a reparation of the deficiency on that side where something

x^2. The above problem and its solution thereby becomes

$$\begin{cases} 10 = x + (10 - x) \\ x(10 - x) = 21 \end{cases}$$
$$10x - x^2 = 21$$
$$10x = 21 + x^2$$
$$x = \frac{10}{2} - \sqrt{\left(\frac{10}{2}\right)^2 - 21}$$

To the extent that this allows us to follow the steps in a medium to which we are as accustomed as the medieval algebraic calculators were to the use of words, it may be regarded as adequate. But only to this extent: the letter symbolism makes it so much easier to understand the dissolution of trinomials into products that the need for geometric proofs becomes incomprehensible – which has to do with the theme of our meeting.

Geometric proofs recur in many later Arabic expositions of algebra – not only in Abū Kāmil but also in al-Karajī's *Fakhrī* (Woepcke, 1853, pp. 65–71), even though al-Karajī's insight in the arithmetic of polynomials[8] would certainly have allowed him to offer purely algebraic proofs (his *Al-Badīʿ* explicitly shows how to find the square root of a polynomial (ed. Hebeisen, 2008, p. 117–137)). What is more: he brings not only the type of proof that goes back to al-Khwārizmī but also the type based directly on *Elements* II (as introduced by Ṯẖābit ibn Qurrah, ed. (Luckey, 1941)).

Some Arabic writers on algebra give no geometric proofs – for instance, ibn Badr and ibn al-Bannāʾ. That, however, is because they give no proofs at all; algebraic proofs for the solution of the basic equations are absent from the entire Arabic tradition.[9]

This complete absence is interesting by showing that we should expect no direct connection between the existence of an algebraic symbolism and the creation of the kind of reasoning it seems with hindsight to make possible. It has indeed been known to historians of mathematics since Franz Woepcke's work

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is lacking; this is followed by a corresponding addition to the other side.

[8] Carried by a purely rhetorical exposition, only supplemented by use of the particle *illā* ("less") – still a word, but used contrary to the rules of grammar in the phrase *wa illā*, "and less" – to mark a subtractive contribution. As pointed out by Abdeljaouad (2002, p. 38), this implies that *illā* has become an attribute (namely subtractivity) of the number.

[9] An interesting variant is found in ibn al-Hāʾim's *šarḥ al-Urjūzah al-Yasmīnya*, "Commentary to al-Yāsamin's *Urjuza*" (ed., trans. Abdeljaouad, 2004, pp. 18f). Ibn al-Hāʾim explains that the specialists have a tradition for giving geometric proofs, by lines (viz, as Ṯẖābit) or by areas (viz, as al-Khwārizmī), which however presuppose familiarity with Euclid. He therefore gives an arithmetical argument, fashioned after *Elements* II.4. For use of this theorem he is likely to have had precursors, since Fibonacci also seems to model his first *geometric* proof after this proposition (ed. Boncompagni, 1857, p. 408) (his second proof is "by lines").

فإذا قيل لك اضرب ثمانية أشياء إلّا أربعة من العدد في ستة أموال إلّا ثلاثة أشياء فأنزل ذلك هكذا 8ش الا 4 في 6م الا 3ش .

ثم اضرب الثمانية في الستة يخرج لك ثمانية وأربعون (171) كعبا ، لأن أس المضروبين 6م و8ش ثلاثة ، احفظها أولا ، ثم اضرب الثمانية في الثلاثة يخرج لك أربعة وعشرون مالا ، وهو ناقص ، لأنه من ضرب زائد في ناقص ، احفظه بحرف الاستثناء ، ثم اضرب الأربعة في الستة يخرج لك اربعة وعشرون مالا ناقصا أيضا ، ضعه مع نظيره ، ثم اضرب أيضا الاربعة في الثلاثة يخرج لك اثنا عشر شيئا زائدا لأنه من ضرب [/33و] ناقص في مثله ، اجعله مع المحفوظ الأول ، فيكون الخارج اثني عشر شيئا وثمانية واربعين كعبا إلّا ثمانية واربعين مالا هكذا : 12ش 48ك الا 48م

Fig. 1.1: Al-Qalaṣādi's explanation of how to multiply "8 things less 4" by "6 census less 3 things" in Souissi's edition (1988, p. Ar. 96) – symbolic notations in frames (added here).

in (1854) that elements of algebraic symbolism were present in the Maghreb, at least in the mid-fifteenth century (they are found in al-Qalaṣādi's *Kašf*,[10] but also referred to by ibn Khaldūn). Woepcke points to symbols for powers of the unknown and to signs for subtraction, square root and equality; symbols for the powers[11] are written above their coefficient, and the root

[10] The use of the symbols can thus be seen in Mohamed Souissi's edition of he Arabic text (1988). His translation renders the same expressions in post-Cartesian symbols; edition as well as translation change the format of the text (unless this change of format has already taken place in the manuscript he uses, which is not to be excluded). Woepcke's translation (1859) renders the formulae more faithfully (using **K** for the cube, **Q** for the square and **C** for the unknown itself), and also renders the original format better (putting the symbolic notations outside the text). Figures 1.1 and 1.2 confront Woepcke's translation with Souissi's Arabic text.

[11] There are individual signs for the *thing*, the *census* and the *cube*. Higher powers are represented by products of these (the fifth power thus with the signs for *census* and *cube*, one written above or in continuation of the other, corresponding to the verbal name *māl kaʿb*. However, the arithmetization of the sequence of "powers" (i.e., exponents) was present. Ibn al-Bannāʾ must have known it, since he says (he was a purist) that it is not "allowed" to speak of the power of the *māl* (as 2), viz because it is an entity of its own; ibn Qunfudh (1339–1407), in the commentary from which we know this prohibition, states that other writers on algebra did not agree, and speaks himself of the power of the *number* as "nothing", that is, 0 (Djebbar, 2005, pp. 95*f*). The individual names for the powers should thus

Donc si l'on vous dit : multipliez huit choses moins quatre en nombre par six carrés moins trois choses, posez cela ainsi :

		C
4	moins	8
C		Q
3	moins	6

Ensuite multipliez le huit par le six. Vous aurez pour résultat quarante huit cubes , parce que le fond des deux facteurs est trois. Réservez cela. Après cela multipliez de nouveau le huit par les trois choses. Vous aurez pour résultat vingt quatre carrés, ce qui est négatif, parce que cela (provient) de la multiplication du positif par le négatif. Réservez cela (en le plaçant) après la particule de l'exception. Puis multipliez le quatre par le six. Vous aurez pour résultat vingt quatre carrés. Mais cela est de nouveau négatif. Placez-le avec son analogue (***). Ensuite multipliez encore le quatre par le trois. Vous aurez pour résultat douze choses positives , parce que cela (provient) de la multiplication du négatif par le négatif. Réservez cela avec le premier (produit) réservé. Le résultat sera douze choses et quarante huit cubes moins quarante huit carrés, ainsi :

Q		K	C
48	moins	48	12

Fig. 1.2: The same in Woepckes translation (1859, p. 427)

sign above the radicand. He shows that these symbols (derived from the initial letters of the corresponding words, prolonged so as to be able to cover composite expressions, that is, to delimit algebraic parentheses[12]) are used

not have been a serious impediment to the development of algebraic proofs, had the intention been there to develop them.

[12] Three points should perhaps be made here. One concerns terminology. "Parenthesis" does not designate the bracket but the expression that is marked off, *for example* by a pair of brackets; but pauses may also mark off a parenthesis in the flow of spoken words, and a couple of dashes may do so in written prose. What characterizes an *algebraic parenthesis* is that it marks off a single entity which can be submitted to operations as a whole, and therefore has to be calculated first in the case of calculations. When division is indicated by a fraction line, this line delimits the numerator as well as the denominator as parentheses if they happen to be composite expressions (for instance, polynomials). Similarly, the modern root sign marks off the radicand as a parenthesis.

The remaining points are substantial, one of them general. The possibility of "embedding" parentheses is fundamental for the unrestricted development of mathematical thought, as I discuss in (Høyrup, 2000). An algebraic language without full ability to form parentheses and manipulate them is bound to remain "close to earth".

The last point, also substantial, is specific and concerns the Maghreb notation. It did not use the parenthesis function to the full. The fraction line and the root sign might mark off polynomials as parentheses; the signs for powers of the unknown, on the other hand, to write polynomials and equations, and even to operate on the equations. Making the observation (p. 355) that

> la condition indispensable pour donner à des signes conventionnels quelconques le caractère d'une notation, c'est qu'ils soient toujours employeés quand il y a lieu, et toujours de la même manière

he shows that one manuscript at his disposal fulfils this condition (another one not, probably because of "la negligence d'un copiste ou d'une succession de copistes").

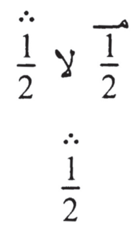

Fig. 1.3: Ibn al-Yāsamin's scheme for multiplying $\frac{1}{2}$ māl less $\frac{1}{2}$ šaiʾ by $\frac{1}{2}$ šaiʾ

Ibn Khaldūn's description made Woepcke suspect that the notation goes back to the twelfth century, as has now been confirmed by two isolated passages in ibn al-Yāsamin's *Talqīḥ al-afkār* reproduced by Mahdi Abdeljaouad (2002, p. 11) after Touhami Zemmouli's master thesis and corresponding exactly to what al-Qalaṣādi was going to do – one of them is shown in Figure 1.3.

Though manuscripts differ in this respect (as observed by Woepcke), the symbolic calculations appears to have been often made separate from the running text (as shown in Woepcke's translation of al-Qalaṣādi), usually preceded by the expression "its image is". They illustrate and duplicate the expressions used by words. They may also stand as marginal commentaries, as in the "Jerba manuscript" (written in Istanbul in 1747) of ibn al-Hāʾim's *šarḥ al-Urjūzah al-Yasmīnya*, "Commentary to al-Yāsamin's *Urjuza*" (originally written in 1387 – manuscripts preceding the one from Jerba are without these marginalia) (ed. Abdeljaouad, 2004), of which Figure 1.4 shows a page. According to ibn Munʿim (†1228) and al-Qalaṣādī, these marginal calculations may correspond to what was to be written in a *takht* (a dustboard, in particular used for calculation with Hindu numerals) or a *lawha* (a clayboard used for

might at most mark off a composite numerical expression – see (Abdeljaouad, 2002, pp. 25–34) for a much more detailed exposition. This should not surprise us: even Descartes

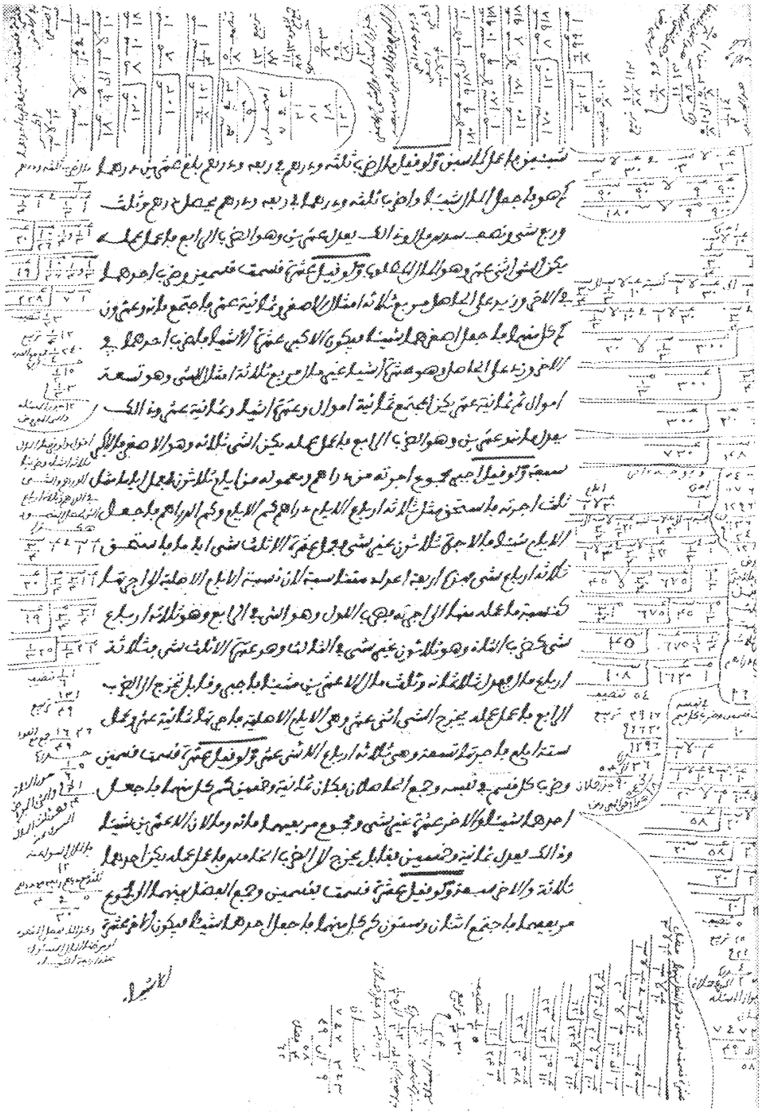

Fig. 1.4: A page from the "Jerba manuscript" of ibn al-Hā᾿im's *Šarḥ al-Urjūzah al-Yasminīya* (ed. Abdeljaouad, 2004, p. Ar. 45)

eschewed general use of the parenthesis – for instance, expressions like $(y-3)^2$, as pointed out by Michel Serfati (1998, p. 259).

temporary writing) – see (Lamrabet, 1994, p. 203) and (Abdeljaouad, 2002, pp. 14, 19*f*). The use of such a device would explain that the examples of symbolic notation we find in manuscripts normally do not contain intermediate calculations, nor erasures (Abdeljaouad, 2002, p. 20).

We are accustomed to consider the notation for fractions as something quite separate from algebraic symbolism. In twelfth-century Maghreb, the two probably belonged together,[13] and from al-Ḥaṣṣār's *Kitāb al-bayān wa'l-ta<u>dh</u>kār* onward Maghreb mathematicians used the various fraction notations with which we are familiar from Fibonacci's *Liber abbaci* (and other works of his): simple fractions written with the fraction line, ascending continued fractions ($\frac{e\ c\ a}{f\ d\ b}$ meaning $\frac{a}{b}+\frac{c}{bd}+\frac{e}{bdf}$), and additively and multiplicatively compounded fractions – see (Lamrabet, 1994, pp. 180*f*) and (Djebbar, 1992, pp. 231–234).

1.2 Latin algebra: *Liber mahamaleth*, *Liber abbaci*, translations of al-Khwārizmī – and Jordanus

The earliest documents in our possession from "Christian Europe" which speak of algebra are the *Liber mahamaleth* and, with a proviso, Robert of Chester's translations of al-Khwārizmī's *Algebra* (c. 1145); slightly later is Gherardo da Cremona's translation of al-Khwārizmī's treatise. All of these are from the twelfth century. From 1228 we have the algebra chapter in Fibonacci's *Liber abbaci* (the first edition from 1202 was probably rather similar, but we do not know *how* similar). In his *De numeris datis*, Jordanus de Nemore presented an *alternative to algebra*, showing how its familiar results could be based in (rather) strictly deductive manner on his *Elements of Arithmetic*, but he avoided to speak about algebra (hinting only for connoisseurs at the algebraic sub-text by using many of the familiar numerical examples) – see the analysis in (Høyrup, 1988, pp. 332–336). Finally, around 1300 a revised version of al-Khwārizmī's *Algebra* of interest for our topic was produced (ed. (Kaunzner, 1986), cf. (Kaunzner, 1985)).

The *Liber mahamaleth* and the *Liber abbaci* share certain characteristics, and may therefore be dealt with first.

All extant manuscripts of the *Liber mahamaleth*[14] have lost an introductory systematic presentation of algebra, which however is regularly referred to.[15]

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[13] Cf. the hypothesis of Mahdi Abdeljaouad (2002, pp. 16–18), that "l'algèbre symbolique est un chapitre de l'arithmétique indienne maghrébine".

[14] I have consulted (Sesiano, 1988) and a photocopy of the manuscript Paris, Bibliothèque Nazionale, ms. latin 7377A.

[15] Thus fol. 154^v, "sicut docuimus in algebra"; fol. 161^r, "sicut ostensum est in algebra".

There are also references to Abū Kāmil,[16] and a number of problem solutions make use of algebra. Fractions are written in the Maghreb way, with Hindu numerals and fraction line;[17] there are also copious marginal calculations in rectangular frames probably rendering computation on a *lawha*. However, one finds no more traces of algebraic symbolism than in al-Khwārizmī's and Abū Kāmil's algebraic writings.

Fibonacci uses Maghreb fraction notations to the full in the *Liber abbaci* (ed. Boncompagni, 1857), writing composite fractions from right to left and mixed numbers with the fraction to the left – all in agreement with Arabic custom. Further, he often illustrates non-algebraic calculations in rectangular marginal frames suggesting a *lawha*. That systematic presentation of the algebraic technique which has been lost from the *Liber mahamaleth* is present in the *Liber abbaci*; there is no explicit reference to Abū Kāmil, but there are unmistakeable borrowings (which could of course be indirect, mediated by one or more of the many lost treatises). When the "thing" technique is used in the solution of commercial or recreational first-degree problems,[18] it is referred to as *regula recta*, not as algebra. But in one respect their algebras are similar: they are totally devoid of any hint of algebraic symbolism.[19] Inasfar as the *Liber mahamaleth* is concerned, this could hardly be otherwise – it antedates the probable creation of the Maghreb algebraic notation.

Equally devoid of any trace of symbolism is Gherardo's translation of al-Khwārizmī, which is indeed very faithful to the original – to the extent indeed that no Hindu numerals nor fraction lines occur, everything is completely verbal.

Robert does use Hindu numerals heavily in his translation (as we know it), but apart from that his translation is also fully verbal. It has often been believed, on the faith of Karpinski's edition (1915, p. 126) that his translation describes an algebraic formalism. It is true that the manuscripts contain a final list of *Regule 6 capitulis algabre correspondentes* making use of symbols for *census*, *thing* and *dragma* (the "unit" for the number term, we remember);

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[16] Thus fol. 203^r, "modum agendi secundum algebra, non tamen secundum Auoqamel"; cf. (Sesiano, 1988, pp. 73f, 95f). We may observe that the spelling "Auoqamel" reflects an Iberian pronunciation.

[17] However, ascending continued fractions are written in a mixed system and not in Maghreb notation – e.g., "$\frac{4}{5}$ et $\frac{2}{5}$ unius sue $\frac{e}{5}$" (fol. 167^{rl}. – 9) for $\frac{4}{5}+\frac{2}{5}\cdot\frac{1}{5}$ ($\frac{e}{5}$ means "quinte").

[18] The *Liber mahamaleth* contains several pseudo-commercial problems involving the square root of an amount of money, leading to second-degree problems – see (Sesiano, 1988, pp. 80, 83). The *Liber abbaci* contains nothing of the kind, and no second-degree problems outside the final chapter 15.

[19] Florian Cajori (1928, I, p. 90) has observed a single appearance of ℞ in the *Pratica geometrie* (ed. Boncompagni, 1862, p. 209). Given how systematically Fibonacci uses his notations for composite fractions we may be sure that this isolated abbreviation is a copyist's slip of the pen (the manuscript is from the fourteenth century, where this abbreviation began to spread). Marginal reader's notes in a manuscript of the *Flos* are no better evidence of

they are classified as an appendix by Barnabas Hughes (1989, p. 67), but even he appears (p. 26) to accept them as genuine. However, the symbols are those known from the southern Germanic area of the later fifteenth century,[20] and all three manuscripts were indeed written in this area during that very period (Hughes, 1989, p. 11–13). The appendix has clearly crept in some three centuries after Robert made his translation.

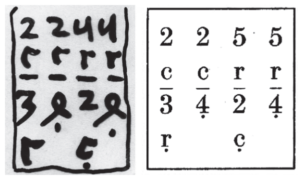

Fig. 1.5: From Oxford, Bodleian Library, Lyell 52, fol. 45^r (Kaunzner, 1986, pp. 64f)

Far more interesting from the point of view of symbolism is the anonymous ak-Khwārizmī redaction from around 1300. It contains a short section *Qualiter figurentur census, radices et dragma*, "How *census*, roots and *dragmas* are represented" (ed. Kaunzner, 1986, pp. 63*f*).[21] Here, *census* is written as *c*, roots as *r*, and *dragmata* (the unit for number) as *d* or not written at all. If a term is subtractive, a dot is put under it. These symbols are written below the coefficient, not above, as in the Maghreb notation. In Figure 1.5 we see (redrawn from photo and following Wolfgang Kaunzner's transcription) "2 *census* less 3 *roots*", "2 *census* less 4 *dragmata*", "5 *roots* less 2 *census*, and "5 *roots* less 4 *dragmata*". Outside this section, the notation is not used, which speaks against its being an invention of the author of the redaction; it rather looks as if he reports something he knows from elsewhere, and which, as he says, facilitates the teaching of algebraic computation. He refers not only

what Fibonacci did himself.

[20] One of them is an abbreviation of the spelling *zenso/zensus*, the spelling of many manuscripts from northern Italy (below, note 86). The spelling *zensus* as well as the abbreviation were taken over in Germany (as the north-Italian spelling *cossa* was taken over as *coss*); the spelling was unknown in twelfth-century Spain, and the corresponding abbreviation could therefore never have been invented in Spain in 1145.

[21] This redaction is often supposed to be identical with a translation made by Guglielmo de Lunis. However, all references to this translation (except a false ascription of a manuscript of the Gherardo translation) borrow from it a list of Arabic terms with vernacular explanation which is absent from the present Latin treatise. It is a safe conclusion that Guglielmo translated *into Italian*; that his translation is lost ; and that the present redaction is to be considered anonymous.

to additive-subtractive operations but also to multiplication, stating however only the product of *thing* by *thing* and of *thing* by *number*. He can indeed do nothing more, he has not yet explained the multiplication of binomials. The notation is certainly not identical with what we find in the Maghreb texts; the similarity to what we find in ibn al-Yāsamin and al-Qalaṣādī is sufficiently great, however, to suggest some kind of inspiration – very possibly indirect. However that may be: apart from an Italian translation from c. 1400 (Vatican, Urb. lat. 291), where *c* is replaced by *s* (for *senso*) and *r* by *c* (for *cose*), no influence in later writings can be traced. A brief description of a notation which is not used for anything was obviously not understood to be of great importance (whether the redactor believed it to be can also be doubted, given that he does not insist by using it in the rest of the treatise).

Jordanus de Nemore's *De numeris datis* precedes this redaction of al-Khwārizmī by a small century or so.[22] It is commonly cited as an early instance of symbolic algebra, and as a matter of fact it employs letters as general representatives of numbers. At the same time it is claimed to be very clumsy – which might suggest that the interpretation as symbolic algebra could be mistaken. We may look at an example:[23]

> If a given number is divided into two and if the product of one with the other is given, each of them will also be given by necessity.
>
> Let the given number *abc* be divided into *ab* and *c*, and let the product of *ab* with *c* be given as *d*, and let similarly the product of *abc* with itself be *e*. Then the quadruple of *d* is taken, which is *f*. When this is withdrawn from *e*, *g* remains, and this will be the square on the difference between *ab* and *c*. Therefore the root of *g* is extracted, and it will be *b*, the difference between *ab* and *c*. And since *b* will be given, *c* and *ab* will also be given.

As we see, Jordanus does not operate on his symbols, every calculation leads to the introduction of a *new* letter. What Jordanus has invented here is a symbolic representation of an *algorithm*, not clumsy symbolic algebra.

The same letter symbolism is used in Jordanus's *De elementis arithmetice artis*, which is presupposed by the *De numeris datis* and hence earlier. In the

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[22] As well known, the only certain date *ante quem* for Jordanus is that all his known works appear in Richard de Fournival's *Biblionomina* (ed. de Vleeschauwer, 1965), which was certainly written some time before Richard's death in 1260 (Rouse, 1973, p. 257). However, one manuscript of Jordanus's *Demonstratio de algorismo* (Oxford, Bodleian Library, Savile 21) seems to be written by Robert Grosseteste in 1215–16, and in any case at that moment (Hunt, 1955, p. 134). This is the revised version of Jordanus's treatise on algorism. In consequence, Jordanus must have been beyond his first juvenile period by then. It seems likely (but of course is not certain) that the arithmetical works (the *Elements* and the *Data* of arithmetic) are closer in time to the beginning of his career that works on statics and on the geometry of the astrolabe, and that they should therefore antedate 1230.

[23] Translated from (Hughes, 1981, p. 58) (Hughes' own English translation is free and therefore unfit for the present purpose). Juxtaposition of letters is meant as aggregation, that is, addition (in agreement with the Euclidean understanding of number and addition).

algorithm treatises, letters are used to represent unspecified digits (Eneström, 1907, p. 146); in the two demonstrations that are quoted by Eneström (pp. 140*f*), the revised version can be seen also to use the mature notation, while it is absent from the early version. The assumption is close at hand that Jordanus developed the notation from the representation of digits by letters in his earliest work; it is hard to imagine that it can have been inspired in any way by the Maghreb notations. This representation of digits *might* have given rise to an algebraic symbolism – but as we see, that was not what Jordanus aimed at. Actually – as mentioned above – he did not characterize his *De numeris datis* as algebra even though he shows that he knows it to be at least a (theoretically better founded) *alternative to algebra.*

There are few echoes of this alternative in the following centuries. When taking up algebra in the mid-fourteenth century in his *Quadripartitum numerorum* ((ed. l'Huillier, 1990), cf. (l'Huillier, 1980)), Jean de Murs borrows from the *Liber abbaci*, not from Jordanus. Somewhere around 1450, Peurbach refers in a poem to "what algebra calculates, what Jordanus demonstrates" (ed. Größing, 1983, p. 210), and in his Padua lecture from 1464 (ed. Schmeidler, 1972, p. 46) Regiomontanus refers in parallel to Jordanus's "three most beautiful books about given numbers" and to "the thirteen most subtle books of Diophantos, in which the flower of the whole of arithmetic is hidden, namely the art of the thing and the *census*, which today is called algebra by an Arabic name". Regiomontanus thus seems to have been aware of the connection to algebra, and he also planned to print Jordanus's work (but suddenly died before any of his printing plans were realized).[24]

Two German algebraists from the sixteenth century knew, and used, Jordanus's quasi-algebra: Adam Ries and Johann Scheubel. The codex known as Adam Ries' *Coß* (ed. Kaunzner and Wußing, 1992) includes a fragment of an originally complete redaction of the *De numeris datis*, containing the statements of the propositions in Latin and in German translation, and for each statement an alternative solution of a numerical example by cossic technique; Jordanus's general proofs as well as his letter symbols have disappeared (Kaunzner and Wußing, 1992, II, pp. 92–100). From Scheubel's hand, a complete manuscript has survived. It has the same character – as Barnabas Hughes says in his description (1972, pp. 222*f*), "Scheubel's revision and elucidation [...] has all the characteristics of an original work save one: he used the statements of the propositions enunciated by Jordanus". Both thus did to Jordanus exactly what Jordanus had done to Arabic algebra: they took over his problems and showed how their own technique (basically that of Arabic algebra) allowed them to deal with them in what *they* saw as a more satisfactory man-

[24] As we shall see, these prestigious representatives of Ancient and university culture had no impact on Regiomontanus's own algebraic practice.

ner. Jordanus's treatise must thus have had a certain prestige, even though his technique appealed to nobody.[25]

I only know of two works where Jordanus's letter formalism turns up after his own times, both from France. One is Lefèvre d'étaples' edition of Jordanus's *De elementis arithmetice artis* (Lefèvre d'étaples, 1514) (first edition 1494). The other is Claude Gaspar Bachet's *Problemes plaisans et delectables, que se font par les nombres* (1624) (first edition 1612), where (for the first and only time?) Jordanus's technique is used actively and creatively by a later mathematician.[26]

1.3 Abbacus writings before algebra

The earliest extant abbacus treatises are roughly contemporary with the al-Khwārizmī-redaction (at least the originals – what we have are later copies). They contain no algebra, but their use of the notations for fractions is of some interest.

Traditionally, a *Livero dell'abbecho* (ed. Arrighi, 1989) conserved in the codex Florence, Ricc. 2404, has been supposed to be the earliest extant abbacus book, "internal evidence" suggesting a date in the years 1288–90. Since closer analysis reveals this internal evidence to be copied from elsewhere, all we can say on this foundation is that the treatise postdates 1290 (Høyrup, 2005, p. 47 n. 57) – but not by many decades, see imminently.

The treatise claims in its incipit to be "according to the opinion" of Fibonacci. Actually, it consists of two strata – see the analysis in (Høyrup, 2005). One corresponds to the basic abbacus school curriculum, and has nothing to do with Fibonacci; the other contains advanced matters, translated from the *Liber abbaci* but demonstrably often with scarce understanding.

The Fibonacci-stratum copies his numbers, not only his mixed numbers with the fraction written to the left ($\frac{2}{7}10$ where we would write $10\frac{2}{7}$) but also his ascending continued fractions (written, we remember, in Maghreb notation, and indeed from right to left, as done by al-Ḥaṣṣār, cf. above). However, the compiler does not understand the notation, at one place (ed.

[25] Vague evidence for prestige can also be read from the catalogue the books belonging to a third Vienna astronomer (Andreas Stiborius, c. 1500). Three neighbouring items in the list are *dedomenorum euclidis. Iordanus de datis. Demonstrationes cosse* (Clagett, 1978, p. 347). Whether it was Stiborius (in the ordering of his books) or Georg Tannstetter (who made the list) who understood *De numeris datis* as belonging midway between Euclid's Data and algebra remains a guess.

[26] In order to discover that one has to go to the seventeenth-century editions. Labosne's "edition" (1959) is a paraphrase in modern algebraic symbolism. Ries and Stifel were not the last of their kind.

Arrighi, 1989, p. 112), for instance, he changes

$$\frac{33 \quad 6\ 42\ 46}{53\ 53\ 53\ 53}$$

standing in the *Liber abbaci* (ed. Boncompagni, 1857, p. 273) for

$$\frac{46+\dfrac{42+\dfrac{6+\dfrac{33}{53}}{53}}{53}}{53}$$

into $\frac{3364246}{53535353}$. It is obvious, moreover, that he has not got the faintest idea about algebra: he mostly omits Fibonacci's alternative solutions by means of *regula recta*; on one occasion where he does not (fol. 83^r, ed. Arrighi 1989: 89) he skips the initial position and afterwards translates *res* as an ordinary, not an algebraic *cosa*.[27]

The basic stratum contains ordinary fractions written with a fraction line but none of the composite fractions. Very strange is its way to speak of concrete mixed numbers. On the first few pages they look quite regular – e.g. "d. $6\frac{27}{28}$ de denaio", meaning "*denari* 6, $\frac{27}{28}$ of a *denaro*". Then, suddenly (with some slips that show the compiler to copy from material written in the normal way) the system changes, and we find expressions like "d. $\frac{2}{7}4$ de denaio", "*denari* $\frac{2}{7}4$ of a *denaro*" – obviously a misshaped compromise between Fibonacci's way to write mixed numbers with the way of the source material, which hence can *not* have been produced by Fibonacci (all his extant works write simple and composite fractions as well as mixed numbers in the same way as the *Liber abbaci*). All in all, the *Livero dell'abbecho* is thus evidence, firstly, that the Maghreb notations adopted by Fibonacci had not gained foothold in the early Italian abbacus environment (which it would by necessity have, had Fibonacci's works been the inspiration) ; secondly, that the aspiration of the compiler to dress himself in the robes of the famous culture hero was not accompanied by understanding of these notations (nor of other advanced matters presented by Fibonacci).

The other early abbacus book is the *Columbia Algorism* (New York, Columbia University, MS X511 AL3, ed. (Vogel, 1977)). The manuscript was written in the fourteenth century, but a new reading of a coin list which it contains dates this list to the years 1278–1284 (Travaini, 2003, pp. 88–92). Since the shapes of numerals are mostly those of the thirteenth century (with occasional slips, where the scribe uses those of his own epoch) (Vogel, 1977, p. 12), a dating close to the coin list seems plausible – for which reason we

[27] This total ignorance of everything algebraic allows us to conclude that the treatise cannot be written many decades after 1290.

must suppose the Columbia Algorism to be (a fairly scrupulous copy of) the oldest extant abbacus book.

There is no trace of familiarity with algebra, neither a systematic exposition nor an occasional algebraic *cosa*. *A fortiori*, there is no algebraic symbolism whatsoever, not even rudiments. Another one of the Maghreb innovations is present, however (Vogel, 1977, p. 13). Ascending continued fractions turn up several times, sometimes in Maghreb notation, but once reversed and thus to be read from left to right ($\frac{1}{4}\!\!-\!\!\frac{1}{2}$ standing for $\frac{3}{8}$). Nothing else suggests any link to Fibonacci. Moreover, the notation is used in a way never found in the *Liber abbaci*, the first "denominator" being sometimes the metrological denomination – thus $\frac{1}{gran}\!\!-\!\!\frac{1}{2}$ being used for $1\frac{1}{2}$ *gran* (or rather, as it would be written elsewhere in the manuscript, for 1 *gran* $\frac{1}{2}$). Next, the *Columbia Algorism* differs from all other Italian treatises (including those written in Provence by Italians) in its formulation of the rule of three – but in a way which approaches it to Ibero-Provençal writings of abbacus type – see (Høyrup, 2008, pp. 5*f*). Finally, at least one problem in the *Columbia Algorism* is strikingly similar to a problem found in a Castilian manuscript written in 1393 (copied from an earlier original) while not appearing elsewhere in sources I have inspected – see (Høyrup, 2005, p. 42 n. 32). In conclusion it seems reasonable to assume that the *Columbia Algorism* has learned the Maghreb notation for ascending continued fractions *not* from Fibonacci but from the Iberian area.

1.4 The beginning of abbacus algebra

The earliest abbacus *algebra* we know of was written in Montpellier in 1307 by one Jacopo da Firenze (or Jacobus de Florentia; otherwise unknown as a person). It is contained in one of three manuscripts claiming to represent his *Tractatus algorismi* (Vatican, Vat. lat. 4826; the others are Florence, Riccardiana 2236, and Milan, Trivulziana 90).[28] As it follows from in-depth analysis of the texts (Høyrup, 2007a, pp. 5–25 and *passim*), the Florence and Milan manuscripts represent a revised and abridged version of the original, while the Vatican manuscript is a meticulous copy of a meticulous copy of the shared archetype for all three manuscripts (extra intermediate steps not being excluded, but they must have been equally meticulous if they exist);

[28] The Vatican manuscript can be dated by watermarks to c. 1450, the Milan manuscript in the same way to c. 1410. The Florence manuscript is undated but slightly more removed from the precursor it shares with the Milan manuscript (which of course does not automatically make it younger but disqualifies it as a better source for the original).

this shared archetype could be Jacopo's original, but also a copy written well before 1328.[29]

Jacopo may have been aware of presenting something new. Whereas the rest of the treatise (and the rest of the vocabulary in the algebra chapter) employs the standard abbreviations of the epoch and genre, the algebraic technical vocabulary is never abbreviated.[30] Even *meno*, abbreviated m in the coin list, is written in full in the algebra section. Everything here is rhetorical, there is not the slightest hint of any symbolism. We may probably take this as evidence that Jacopo was aware of writing about a topic the reader would not know about in advance (the book is stated also to be intended for independent study), and thus perhaps that his algebra is not only the earliest extant Italian algebra but also the first that was written. As we shall see, however, several manuscripts certainly written later also avoid the abbreviation of algebraic core terms – even around 1400, authors of general abbacus treatises may have suspected their readers to possess no preliminary knowledge of algebra.

Not only symbolism but also the Maghreb notations for composite fractions are absent from the treatise, even though they turned up in the Columbia Algorism. None the less, Jacopo's algebra must be presumed to have its direct roots in the Ibero-Provençal area, with further ancestry in al-Andalus and the Maghreb; there is absolutely no trace of inspiration from Fibonacci nor of direct influence of Arabic classics like al-Khwārizmī or Abū Kāmil (nor any Arabisms suggesting direct impact of other Arabic writings or settings). Jacopo offers no geometric proofs but only rules, and the very mixture of commercial and algebraic mathematics is characteristic of the Maghreb–al-Andalus tradition (as also reflected in the *Liber mahamaleth*). A particular multiplicative writing for Roman numerals (for example $\genfrac{}{}{0pt}{}{m}{cccc}$, used as explanation of the Hindu-Arabic number 400000) *could* also be inspired by the Maghreb algebraic notation (it may also have been an independent invention, Middle Kingdom Egyptian scribes and Diophantos sometimes put the "de-

[29] Comparing only lists of the equation types dealt with in various abbacus algebras and believing in a steady progress of their number within each family, Warren Van Egmond claims (2008, p. 313) that the algebra of the Vatican manuscript "falls entirely within the much later and securely dated Benedetto tradition and was undoubtedly added to a manuscript containing some sections from Jacopo's earlier work" (actually, it contains fewer types than the manuscript from c. 1390 which Van Egmond takes as the starting point for this tradition). If he had looked at the words used in the manuscripts he refers to he would have discovered that the Vatican algebra agrees verbatim with a section of an algebra manuscript from c. 1365, which however fills out a calculational lacuna left open in the Vatican manuscript and therefore represents a more developed form of the text (and combines it with other material – details in (Høyrup, 2007a, pp. 163*f*)). Van Egmond's dating can be safely dismissed.

[30] There is one instance of ℞ (fol. 44^r, ed. (Høyrup, 2007a, p. 326); as the single appearance of ℞ in Fibonacci's *Pratica geometrie* (see note 19), this is likely to be a copyist's *lapsus calami*.

nomination" above the "coefficient" in a similar way, and there is no reason to believe that these notations were connected to the Maghreb invention).

In 1328, also in Montpellier, a certain Paolo Gherardi (as Jacopo, unknown apart from the name) wrote a *Libro di ragioni*, known from a later copy (Florence, Bibl. Naz. Centr., Magl. XI, 87, ed. (Arrighi, 1987, p. 13–107)). Its final section is another presentation of algebra.[31] Part of this presentation is so close to Jacopo's algebra that it must descend either from that text (by reduction) or from a close source; but whereas Jacopo only deals (correctly) with 20 (of the possible 22) quadratic, cubic and quartic basic equations ("cases") that can be solved by reduction to quadratic equations or by simple root extraction,[32] Gherardi (omitting all quartics) introduces false rules for the solution of several cubics that cannot be solved in these ways (with examples that are "solved" by means of the false rules). Comparison with later sources show that they are unlikely to be of his own invention. A couple of the cases he shares with Jacopo also differ from the latter in their choice of examples, one of them agreeing at the same time with what can be found in a slightly later Provençal treatise (see imminently).

Gherardi's algebra is almost as rhetorical as Jacopo's, but not fully. Firstly, the abbreviation ℞ is used copiously though not systematically. This *may* be due to the copyist – the effort of Jacopo's and Fibonacci's copyists to conserve the features of the original was no general rule; but it could also correspond to Gherardi's own text. More important is the reference to a diagram in one example (100 is first divided by some number, next by five more, and the sum of the two quotients is given); this diagram is actually missing in the copy, but so clearly described in the text that it can be seen to correspond to the diagram found in a parallel text:[33]

$$\begin{array}{rl} 100 & 1 \text{ cosa} \\ 100 & 1 \text{ cosa piu } 5 \end{array}$$

The operations performed on the diagram ("cross-multiplication" and the other operations needed to add fractions) are described in a way that implies underlying operations with the "formal fractions" $\frac{100}{1\,cosa}$ and $\frac{100}{1\,cosa\;piu\;5}$. No abbreviations being used, we may speak of what goes on as a beginning of symbolic *syntax* without symbolic *vocabulary*.

Such formal fractions, we may observe, constitute an element of "symbolic algebra" that does not presuppose that "*cosa*" itself be replaced by a sym-

[31] Beyond Arrighi's complete edition of the treatise (1987, pp. 97–107), there is an edition of the algebra text with translation and mathematical commentary in (Van Egmond, 1978).
[32] The lacking equations are the two mixed biquadratics that correspond to al-Khwārizmī's (and Jacopo's) fifth and sixth case. Only the six simple cases (linear and quadratic) are provided with examples – ten in total, half of which are dressed as commercial problems. For the others, only rules are offered.
[33] Florence, Ricc. 2252, see (Van Egmond, 1978, p. 169).

bol – but certainly an isolated element only. It must be acknowledged, on the other hand, that this isolated element already made possible calculations that were impossible within a purely rhetorical framework. Jacopo, as already al-Khwārizmī, could get rid of one division by a binomial via multiplication. However, problems of the type where Gherardi and later abbacus algebra use two formal fractions were either solved geometrically by al-Khwārizmī, Abū Kāmil and Fibonacci, as I discuss in a forthcoming paper,[34] or they were replaced before being expressed algebraically *without explanation* by a different problem, namely the one resulting from multiplication by the denominators (al-Khwārizmī, ed. (Hughes, 1986, p. 51)).

A third abbacus book written in Provence (this one in Avignon) is the *Trattato di tutta l'arte dell'abbacho*. As shown by Jean Cassinet (2001), it must be dated to 1334. Cassinet also shows that the traditional ascription to Paolo dell'Abbaco is unfounded.[35] Exactly how much should be counted to the treatise is not clear. The codex Florence, Bibl. Naz. Centr., fond. princ. II.IX.57 (the author's own draft according to (Van Egmond, 1980, p. 140)) contains a part that is not found in the other copies[36] but which is informative about algebra and algebraic notation; however, since this extra part is in the same hand as the main treatise (Van Egmond, 1980, p. 140), it is unimportant whether it went into what the author eventually decided to put into the final version.

There is no systematic presentation of algebra nor listing of rules in this part,[37] only a number of problems solved by a rhetorical *censo-cosa* technique.[38] The author uses no abbreviations for *cosa*, *censo* and *radice* – but at one point (fol. 159^r) an astonishing notation turns up: $\frac{10}{cose}$, meaning "10 *cose*". The idea is the same as we encountered in the *Columbia Algorism* when it writes $\frac{1}{gran}\frac{1}{2}$ meaning "1 *gran* $\frac{1}{2}$": that what is written below the line is a denomination; indeed, many manuscripts write "il $\frac{1}{3}$" in the sense of "the

34 "'Proportions' in the Liber abbaci", to appear in the proceedings of the meeting "Proportions: Arts – Architecture – Musique – Mathématiques – Sciences", Centre d'études Supérieures de la Renaissance, Tours, 30 juin au 4 juillet 2008.

Al-Khwārizmī (ed. Hughes, 1986, p. 255) does not make the geometric argument explicit, but a division by 1 betrays his use of the same diagram as Abū Kāmil (ed. Sesiano, 1993, p. 370).

35 Arguments speaking against the ascription are given in (Høyrup, 2008, p. 11 n. 29).

36 I have compared with Rome, Acc. Naz. dei Lincei, Cors. 1875, from c. 1340. For other manuscripts, see (Cassinet, 2001) and (Van Egmond, 1980, passim).

37 The codex contains a list of four rules (fol. 171^v), three of which are followed by examples, written on paper from the same years (according to the watermark) but in a different hand than the recto of the sheet and thus apparently added by a user of the manuscript. It contains one of the examples which Gherardi does not share with Jacopo, confirming that his extra examples came from what circulated in the Provençal area. It contains no algebraic abbreviations nor anything else suggesting symbolism.

38 Jean Cassinet (2001, pp. 124–127) gives an almost complete list.

third" (as ordinal number as well as fraction) – that is, the notation for the fraction was understood as an *image of the spoken form*, not of the division procedure (cf. also the writing of *quinte* as $\frac{e}{5}$ in the *Liber mahamaleth*, see note 17).

The compiler of the *Trattato di tutta l'arte* was certainly not the first to use this algebraic notation – who introduces a new notation does not restrict himself to using it a single time in a passage well hidden in an odd corner of a text. He just happens to be our earliest witness of a notation which for long was in the way of the development of one that could serve symbolic calculation.

This compiler was, indeed, not only not the first but also not the last to use this writing of monomials as quasi-fractions. It is used profusely in Dardi of Pisa's *Aliabraa Argibra* from 1344,[39] better known for being the first Italian-vernacular treatise dedicated exclusively to algebra and for its presentation of rules for solving no less than 194+4 algebraic cases, 194 of which are solved according to generally valid rules (with two slips, explained by Van Egmond (1983, p. 417)), while the rules for the last four cases are pointed out by Dardi to hold only under particular (unspecified) circumstances.[40]

Dardi uses algebraic abbreviations systematically. *Radice* is always R̷, *meno* ("less") is *m̂*, *cosa* is *c*, *censo* is *ç*, *numero*/*numeri* are *nũo*/*nũi*. *Cubo* is unabridged, *censo de censo* (the fourth power) appears not as *çç* but in the expanded linguistic form *ç de ç*, which we may take as an indication that Dardi merely thinks in terms of abbreviation and nothing more. Roots of composite entities are written by a partially rhetorical expression, for instance (fol. 9^v) "R̷ *de zonto* $\frac{1}{4}$ *cō* R̷ *de* 12" (meaning $\sqrt{\frac{1}{4}+\sqrt{12}}$; *zonto* corresponds to Tuscan *gionto*, "joined").

As just mentioned, Dardi also employs the quasi-fraction notation for monomials, and does so quite systematically in the rules and the examples (but only here).[41] When coefficients are mixed numbers Dardi also uses the

[39] See (Van Egmond, 1983). The three principal manuscripts are Vatican, Chigi M.VIII.170 written in Venetian in c. 1395; Siena, Biblioteca Comunale I.VII.17 from c. 1470 (ed. Franci, 2001); and a manuscript from Mantua written in 1429 and actually held by Arizona State University Temple, which I am grateful to know from Van Egmond's personal transcription. In some of the details, the Arizona manuscript appears to be superior to the others, but at the level of overall structure the Chigi manuscript is demonstrably better – see (Høyrup, 2007a, pp. 169*f*). Considerations of consistency suggests it to be better also in its use of abbreviations and other quasi-symbolism, for which reason I shall build my presentation on this manuscript (cross-checking with the transcription of the Arizona-manuscript – differences on this account are minimal); for references I shall use the original foliation.

A fourth manuscript from c. 1495 (Florence, Bibl. Med.-Laur., Ash. 1199, partial ed. (Libri, 1838, III, pp. 349–356)) appears to be very close to the Siena manuscript.

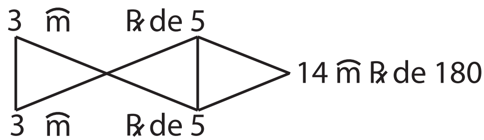
A critical edition of the treatise should be forthcoming from Van Egmond's hand.

[40] Dardi reaches this impressive number of resolvable cases by making ample use of radicals.

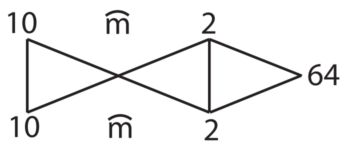
[41] This notation appears only to be present in the Chigi and Arizona manuscripts; Franci

formalism systematically in a way which suggests ascending continued fractions, writing for instance $2\frac{1}{2}c$ not quite as $\frac{2\;1}{c\;2}$ but as $\frac{2}{c}\frac{1}{2}$ (which however *could* also mean simply "2 *censi* and $\frac{1}{2}$". Often, a number term is written as a quasi-fraction, for example as $\frac{325}{n}$. How far this notation is from any operative symbolism is revealed by the way multiples of the *censo de censo* are sometimes written – namely for example as $\frac{81}{ç}$ *de ç* (fol. 46^v).

None the less, symbolic operations are not absent from Dardi's treatise. They turn up when he teaches the multiplication of binomials (either algebraic or containing numbers and square roots) – for instance, for $(3-\sqrt{5})\cdot(3-\sqrt{5})$,



Noteworthy is also Dardi's use of a similar scheme



as support for his proof of the sign rule "less times less makes plus" on fol. 5^v:

> Now I want to demonstrate by number how less times less makes plus, so that every times you have in a construction to multiply less times less you see with certainty that it makes plus, of which I shall give you an obvious example. 8 times 8 makes 64, and this 8 is 2 less than 10, and to multiply by the other 8, which is still 2 less than 10, it should similarly make 64. This is the proof. Multiply 10 by 10, it makes 100, and 10 times 2 less makes 20 less, and the other 10 times 2 less makes 40 less, which 40 less detract from 100, and there remains 60. Now it is left for the completion of the multiplication to multiply 2 less times 2 less, it amounts to 4 plus, which 4 plus join above 60, it amounts to 64. And if 2 less times two less had been 4 less, this 4 less should have been detracted from 60, and 56 would remain, and thus it would appear that 10 less 2 times 10 less two had been 56, which is not true. And so also if 2 less times 2 less had been nothing, then the multiplication of 10 less 2 times 10 less 2 would come to be 60, which is still false. Hence less times less by necessity comes to be plus.

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does not mention it in her edition of the much later Siena manuscript, and composite

Such schemes were no more Dardi's invention than the quasi-fraction notation (even though he may well have been more systematic in the use of both than his precursors). The clearest evidence for this is offered by an anonymous *Trattato dell'alcibra amuchabile* from c. 1365 (ed. Simi, 1994), contained in the codex Florence, Ricc. 2263. This is the treatise referred to in note 29, part of which agrees verbatim with Jacopo's algebra. It also has Gherardi's false rules. However, here the agreement is not verbatim, showing Gherardi not to be the immediate source (a compiler who follows one source verbatim will not use another one freely) – cf. (Høyrup, 2007a, p. 163).

The treatise consists of several parts. The first presents the arithmetic of monomials and binomials, the second contains rules and examples for 24 algebraic cases (mostly shared with Jacopo or Gherardi), the third a collection of 40 algebraic problems. All are purely rhetorical in formulation, except for using ℞ in the schemes of the first part (see imminently). However, the first and third part contain the same kinds of non-verbal operations as we have encountered in Gherardi and Dardi, and throws more light on the former.

In part 3, there are indeed a number of additions of formal fractions, for example (in problem #13) $\frac{100}{1\,cosa} + \frac{100}{1\,cosa+5}$. This is shown as

$$\frac{100 \qquad\qquad 100}{\textit{per una cosa} \qquad \textit{per una cosa e 5}}$$

and explained with reference to the parallel $\frac{24}{4} + \frac{24}{6}$ (cross-multiplication of denominators with numerators followed by addition, multiplication of the denominators, etc.). Gherardi's small scheme (see just after note 33) must build on the same insights (whether shared by Gherardi or not).

Part 1 explains the multiplication of binomials with schemes similar to those used by Dardi – for example

5	e	piu	℞	di	20
via					
5	e	meno	℞	di	20

As we see, the scheme is very similar to those of Dardi but more rudimentary. It also differs from Dardi in its use of the ungrammatical expressions *e più* and *e meno*, where Dardi uses the grammatical *e* for addition and the abbreviation $\hat{m}$ for subtraction.[42] There is thus no reason to suppose it should

expressions where their presence might be revealed show no trace of them. They are also absent from Guglielmo Libri's extract of the Florence manuscript.

[42] The expression *e meno n*, as we remember, corresponds to what was done by al-Karajī, see note 8. The appearance of the parallel expression *e più n* shows that the attribute "subtractivity" was seen to ask for the existence of a corresponding attribute "addivity" – another instance of "symbolic syntax" without "symbolic vocabulary" (or, in a different terminology but with the same meaning, the incipient shaping of the language of algebra

be borrowed from Dardi's earlier treatise – influence from which is on the whole totally absent. Schemes of this kind must hence have been around in the environment or in the source area for early abbacus algebra before 1340, just as the calculation with formal fractions must have been around before 1328, and the quasi-fractions for monomials before 1334.[43] On the whole, this tells us how far the development of algebraic symbolic operations had gone in abbacus algebra in the early fourteenth century – and that all that was taken over from the Maghreb symbolism was the calculation with formal fractions; a very dubious use of the ascending continued fractions; and possibly the idea of presenting *radice*, *cosa* and *censo* by single-letter abbreviations (implemented consistently by Dardi but not broadly, and not necessarily a borrowing).

1.5 The decades around 1400

The Venetian manuscript Vatican, Vat. lat. 10488 (*Alchune ragione*), written in 1424, connects the early phase of abbacus algebra with its own times. The manuscript is written by several hands, but clearly as a single project (hands may change in the middle of a page; we should perhaps think of an abbacus master and his assistants). From fol. 29^v to fol. 32^r it contains a short introduction to algebra, taken from a text written in 1339 by Giovanni di Davizzo, a member of a well-known Florentine abbacist family, see (Ulivi, 2002, pp. 39, 197, 200). At first come sign rules and rules for the multiplication of algebraic powers, next a strange section with rules for the division of algebraic powers where "roots" take the place of negative powers;[44] then a short section about the arithmetic of roots (including binomials containing roots)[45] somehow but indirectly pointing back to al-Karajī; and finally 20 rules for algebraic cases without examples, of which one is false and the rest parallel to those of Jacopo (not borrowed from him but sharing the same source tradition). Everywhere, *radice* is R, but "less", *cosa* and *censo* all appear unabbreviated (*censo* mostly as *zenso*, which cannot have been the Florentine Giovanni's spelling).

as an *artificial* language).

In the proof that "less times less makes plus" (see above), Dardi speaks of subtractive numbere, e.g., as "2 *meno*"/"2 less", etc., whereas additive numbers are not characterized explicitly as such.

[43] This latter presence leads naturally to the question whether the notation in the al-Khwārizmī–redaction from c. 1300 should belong to the same family. This cannot be completely excluded, but the absence of a fraction line from the notation of the redaction speaks against it. It remains more plausible that the latter notation is inspired from the Maghreb, or an independent invention.

[44] An edition, English translation and analysis of this initial part of the introduction can be found in (Høyrup, 2007b, pp. 479–484).

[45] Translation in (Høyrup, 2009, pp. 56f).

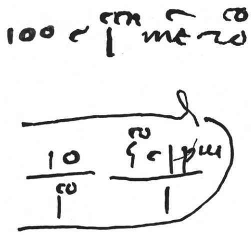

Fig. 1.6: The “equations” from VAT 10488 fol. 37^v (top) and fol. 39^v (bottom)

This introduction comes in the middle of a long section containing number problems mostly solved by means of algebra (many of them about numbers in continued proportion).[46] Here, abbreviations abound. *Radice* is always ℞, *meno* is often ⓜ, $\hat{m}$, or $\widehat{me}$ (different shapes may occur in the same line). More interesting, however, is the frequent use of *co*, □, (occasionally *ce*) and *no* written above the coefficient, precisely as in the Maghreb notation (and quite likely inspired by it). However, these notations are not used systematically, and only used once for formal calculation, namely in a marginal “equation” without equation sign[47] on fol. 39^v – see Figure 1.6, bottom.[48] In another place (fol. 37^r, Figure 1.6 top) the running text formulates a genuine equation, but this is merely an abbreviation for *100 è 1 censo meno 20 cose*. It serves within the rhetorical argument without being operated upon.

Later in the text comes another extensive collection of problems solved by means of algebra (some of them number problems, others dressed as business problems), and inside it another collection of rules for algebraic cases (17 in total, only 2 overlapping the first collection). In its use of abbreviations, this second cluster of problems and rules is quite similar to the first cluster, the only exception being a problem (fols. 95r–96v) where the use of coefficients

[46] Even these are borrowed en bloc, as revealed by a commentary within the running text on fol. 36^r, where the compiler tells how a certain problem should be made *al parere mio*, “in my opinion”. The several hands of the manuscripts are thus not professional scribes copying without following the argument.

[47] Two formal fractions are indicated to be equal; the hand seems to be the same as that of the main text and of marginal notes adding words that were omitted during copying.

[48] The treatment of the problem is quite interesting. The problem asks for a number which, when divided into 10 yields 5 times the same number and 1 more. Instead of writing “$\frac{10}{\overset{co}{1}} = \overset{co}{5}$ e 1 piu” it expresses the right-hand side as a fraction $\frac{\overset{co}{5}\ e\ 1\ piu}{1}$, thus opening the way to the usual cross-multiplication.

As in several cases below, I have had to redraw the extract from the manuscript in order to get clear contours, my scanned microfilm being too much grey in grey.

with superscript power is so dense (without being fully systematic) that it may possibly have facilitated understanding of the argument by making most of the multiples of *cosa* and *censo* stand out visually.

In the whole manuscript, addition is normally indicated by a simple *e*, "and". I have located three occurrences of *più*,[49] none of them abbreviated. The expressions *e più* and *e meno* appear to be wholly absent.

It is fairly obvious that this casual use of what could be a symbolism was not invented by the compilers of the manuscript, and certainly not something they were experimenting with. They used for convenience something which was familiar, without probing its possibilities. If anybody else in the abbacus environment had used the notation as a symbolism and not merely as a set of abbreviations (and the single case of an equation between formal fractions suggests that this may well have been the case), then the compilers of the present manuscript have not really discovered – or they reveal, which would be more significant, that the contents of abbacus algebra did not call for and justify the effort needed to implement a symbolism to which its practitioners were not accustomed.[50] They might *almost* as well have used Dardi's quasi-fractions – only in the equation between formal fractions would the left-hand side have collided with it by meaning simply "10 *cose*".

Though not using the notation as a symbolism, the compilers of Vat. lat. 10488 at least show that they knew it. However, this should not make us believe that every abbacus writer on algebra from the same period was familiar with the notation, or at least not that everybody adopted it. As an example we may look at two closely related manuscripts coming from Bologna, one (Palermo, Biblioteca Comunale 2 Qq E 13, *Libro merchatantesche*) written in 1398, the other (Vatican, Vat. lat. 4825, Tomaso de Jachomo Lione, *Libro da razioni*) in 1429.[51] They both contain a list of 27 algebraic cases with examples followed by a brief section about the arithmetic of roots (definition, multiplication, division, addition and subtraction). The former has a very fanciful abbreviation for *meno*, namely , which corresponds, however, to the way *che* and various other non-algebraic words are abbreviated, and is thus merely a personal style of the scribe; the other writes *meno* in full, and none of the two manuscripts have any other abbreviation whatsoever of algebraic terms – not even R̷ for *radice* which they are unlikely not to have known, which suggests but does not prove that the other abbreviations were also avoided consciously.

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[49] In a marginal scheme and the running text of a problem about combined works (fol. 90^r), and once in an algebra problem (fol. 94^r). There may be more instances, but they will be rare.

[50] The latter proviso is needed. For us, accustomed as we are to symbolic algebra, it is often much easier to follow a complex abbacus texts if we make symbolic notes on a sheet of paper.

[51] More precisely, 7 March 1429 – which with year change at Easter means 1430 according to our calendar, the date given in (Van Egmond, 1980, p. 223).

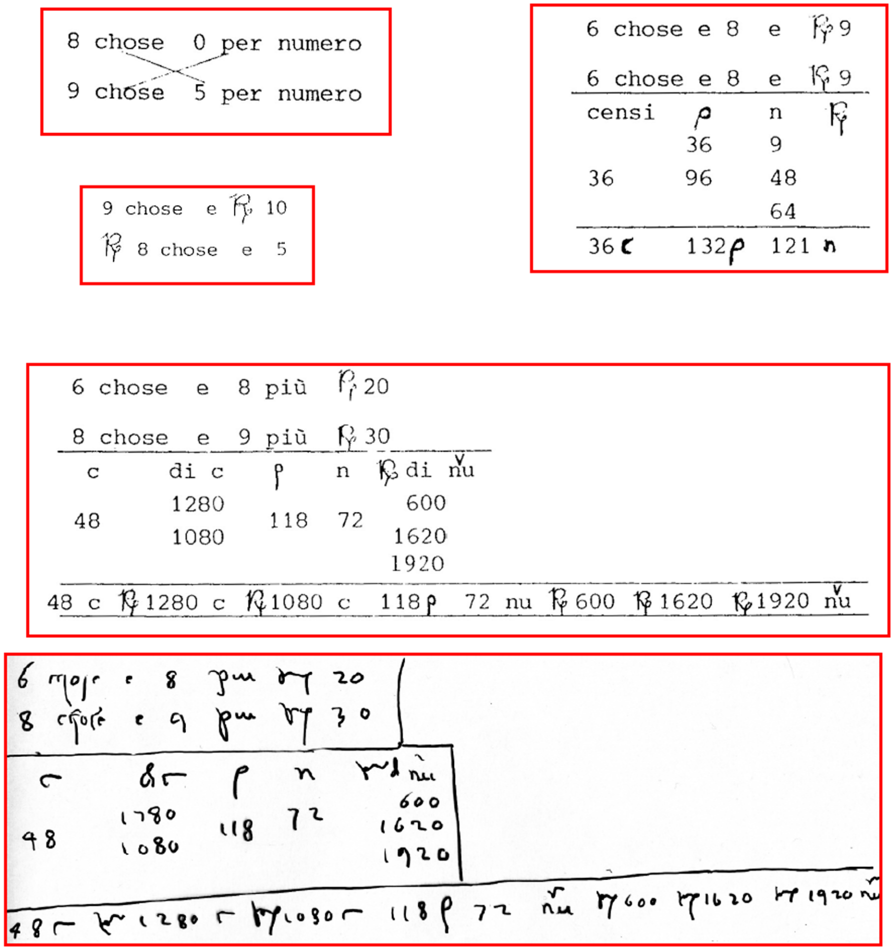

Fig. 1.7: Schemes for the multiplication of polynomials, from (Franci and Pancanti, 1988, pp. 812), and from the manuscript, fol. 146^v

Maybe we should not be surprised not to find any daring development in these two manuscripts. In general, they offer no evidence of deep mathematical insight. In this perspective, the manuscript Florence, Bibl. Naz. Centr., fondo princ. II.V.152 (*Tratato sopra l'arte della arismetricha*) is more illuminating. Its algebraic section was edited by Franci and Pancanti (1988).[52] It

[52] I have controlled on a scan of a microfilm, but since it is almost illegible my principal basis for discussing the treatise is this edition.

was written in Florence in c. 1390, and offers both a clear discussion of the sequence of algebraic powers as a geometric progression and sophisticated use of polynomial algebra in the transformation of equation types – see (Høyrup, 2008, pp. 30–34).

In the running text, there are no abbreviations nor anything else which foreshadows symbolism. However, inserted to the left we find a number of schemes explained by the text and showing multiplication of polynomials with two or three terms (numbers, roots and/or algebraic powers), of which Figure 1.7 shows some examples – four as rendered by Franci and Pancanti, the last of these also as appearing in the manuscript (redrawn for clarity).

Those involving only binomials are easily seen to be related to what we find in the *Trattato dell'alcibra amuchabile* and in Dardi's *Aliabraa Argibra* – but also to schemes used in non-algebraic sections of other treatises, for instance the Palermo-treatise discussed above, see Figure 1.8, which should warn us against seeing any direct connection.

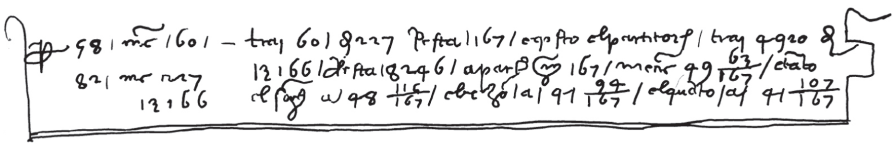

Fig. 1.8: Non-algebraic scheme from Palermo, Biblioteca Comunale 2 Qq E 13, fol. 38^v

The schemes for the multiplication of three-term polynomials are different. They emulate the scheme for multiplying multi-digit numbers, and the text itself justly refers to multiplication *a chasella* (ed. Franci and Pancanti, 1988, p. 9). The *a casella* version of the algorithm uses vertical columns, while the scheme for multiplying polynomials used in the Jerba manuscript (ed. Abdeljaouad, 2002, p. 47) follows the older algorithm *a scacchiera* with slanted columns; none the less inspiration from the Maghreb is plausible, in particular because another odd feature of the manuscript suggests a pipeline to the Arabic world. In a wage problem, an unknown amount of money is posited to be a *censo*, whereas Biagio *il vecchio* (ed. Pieraccini, 1983, p. 89*f*) posits it to be a *cosa* in the same problem in a treatise written at least 50 years earlier. But the present author does not understand that a *censo* can be an amount of money, and therefore feels obliged to find its square root – only to square it again to find the amount of money asked for. He thus uses the terminology without understanding it, and therefore cannot have shaped the solution himself; nor can the source be anything of what we have discussed so far.

Schemes of this kind (and other schemes for calculating with polynomials) turn up not only in later abbacus writings (for instance, in Raffaello Canacci, see below) but also in Stifel's *Arithmetica integra* (1544, fols. $238^r - 239^r$), in Jacques Peletier's *L'Algèbre* (1554, pp. 15–22) and in Petrus Ramus's *Algebra* (1560, fol. *Aiii*^{r}).

Returning to the schemes of the present treatise we observe that the *cosa* is represented (within the calculations, not in the statement lines) by a symbol looking like ρ, and the *censo* by *c*. *Radice* is R in statement as well as calculation. The writing of *meno* is not quite systematic – whether it is written in full, abbreviated *me* or as ⓜ (rendered "m." by Franci and Pancanti) seems mostly to depend on the space available in the line. Addition may be *e* or *più* (*più* being mostly but not always nor exclusively used before R); when space is insufficient, and only then, *più* may be abbreviated *p*.[53] All in all, the writer can be seen to have taken advantage of this incipient symbolism but not to have felt any need to use it systematically – it stays on the watershed, between facultative abbreviation and symbolic notation.

1.6 The mid-15th-century abbacus encyclopediæ

Around 1460, three extensive "abbacus encyclopediae" were written in Florence. Most famous among these is, and was, Benedetto da Firenze's *Trattato de praticha d'arismetrica* – it is the only one of them which is known from several manuscripts.[54]

Earliest of these is Siena, Biblioteca Comunale degli Intronati, L.IV.21, which I have used together with the editions of some of its books.[55] According to the colophon (fol. 1^r) it was "conpilato da B. a uno suo charo amicho negl'anni di Christo MCCCCLXIII". It consists of 495 folios, 106 of which deal with algebra.

The algebra part consists of the following books:

- XIII: Benedetto's own introduction to the field, starting with a 23-lines' excerpt from Guglielmo de Lunis's lost translation of al-Khwārizmī (cf. note 21). Then follows a presentation of the six fundamental cases with geomet-

[53] The phrases *e più* and *e meno* occur each around half a dozen times, but apparently in a processual meaning, "and (then) added" respectively "and (then) subtracted". Nothing suggest a use of *più* and *meno* as attributes of numbers, even though the author does operate with negative (not merely subtractive) numbers in his transformation of cubic equations – see (Høyrup, 2008, p. 33).

[54] On Benedetto and his historical setting, see the exhaustive study in (Ulivi, 2002).

[55] (Salomone, 1982); (Pieraccini, 1983); (Pancanti, 1982); (Arrighi, 1967). All of these editions were made from the same Siena manuscript, which is also described in detail with extensive extracts in (Arrighi, 2004/1965).

ric proofs, built on al-Khwārizmī; a second chapter on the multiplication and division of algebraic powers (*nomi*, "names") and the multiplication of binomials; and a third chapter containing rules and examples for 36 cases (none of them false);
- XIV: a problem collection going back to Biagio *il vecchio* († c. 1340 according to Benedetto);
- XV: containing a translation of the algebra chapter from the *Liber abbaci*, provided with "some clarifications, specification of the rules in relation to the cases presented in book XIII, and the completion of calculations, which the ancient master had often neglected, indicating only the result" (Franci and Toti Rigatelli, 1983, p. 309); a problem collection going back to Giovanni di Bartolo (fl. 1390–1430, a disciple of Antonio de' Mazzinghi); and Antonio de' Mazzinghi's *Fioretti* from 1373 or earlier (Ulivi, 1998, p. 122).

The basic problem in using this manuscript is to which extent we can rely on Benedetto as a faithful witness of the notations and possible symbolism of the earlier authors he cites. A secondary problem is whether we should ascribe to Benedetto himself or to a later user a number of marginal quasi-symbolic calculations.

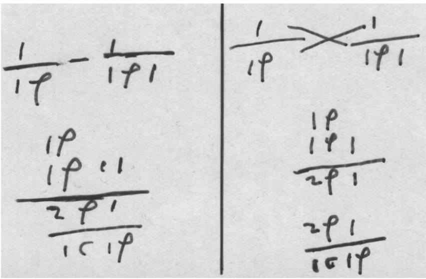

Fig. 1.9: A marginal calculation accompanying the same problem from Antonio's *Fioretti* in Siena L.IV.21, fol. 456^r and Ottobon. lat. 3307, fol. 338^v

Regarding the first problem we may observe that there are no abbreviations or any other hints of incipient symbolism in the chapters borrowed from Fibonacci and al-Khwārizmī. This suggests that Benedetto is a fairly faithful witness, at least as far as the presence or absence of such things is concerned. On the other hand it is striking that the symbols he uses are the same throughout;[56] this could mean that he employed his own notation when rendering the

[56] One partial exception to this rule is pointed out below, note 59.

notations of others, but could also be explained by the fact that all the abbacists he cites from Biagio onward belong to his own school tradition – as observed by Raffaella Franci and Laura Toti Rigatelli (1983, p. 307), the *Trattato* is not without "a certain parochialism".

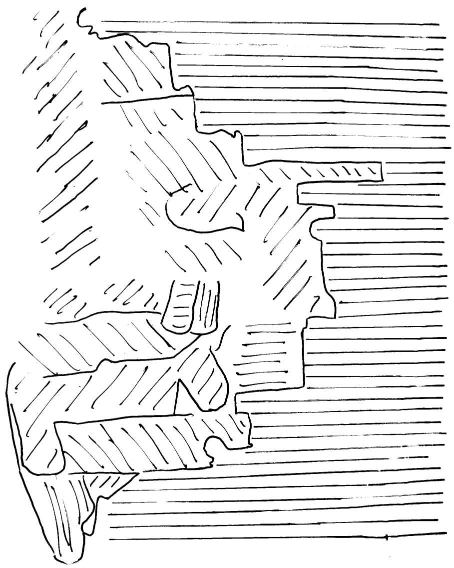

Fig. 1.10: The structure of Siena, L.IV.21, fol. 263^v. To the right, the orderly lines of the text proper. Left a variety of numerical calculations, separated by Benedetto by curved lines drawn ad hoc.

Marginal calculations along borrowed problems can obviously not be supposed a priori to be borrowed, and not even to have been written by the compiler. However, the marginal calculations in the algebraic chapters appear to be made in the same hand as marginal calculations and diagrams for which partial space is made in indentions in book XIII, chapter 2 as well as in earlier books of the treatise. Often, the irregular shape of the insertions shows these earlier calculations and diagrams to have been written before the main text, cf. fol. 263^v as shown in Figure 1.10.[57] This order of writing shows that the manuscript is Benedetto's original, and that he worked out the calculations

[57] This page presents a particularly striking case, and contains calculations for a very complicated problem dealing with two unknowns, a *borsa*, "[the unknown contents of] a purse", and a *quantità*, the share received by the first of those who divide its contents.

while making it – in particular because the marginal calculations are never indented in the algebra chapters copied from earlier authors.

Comparison of the marginal calculations accompanying a problem in the excerpt from Antonio's *Fioretti* and the same problem as contained in the manuscript Vatican, Ottobon. lat. 3307 from c. 1465 (on which below) show astonishing agreement, proving that these calculations were neither made by a later user nor invented by Benedetto and the compiler of the Vatican manuscript – see Figure 1.9. In principle, the calculations in the two manuscripts *could* have been added in a manuscript drawn from the *Fioretti* that had been written after Antonio's time and on which both encyclopedias build; given that the encyclopedias do not contain the same selection it seems reasonable, however, to assume that they reflect Antonio's own style – not least, as we shall see, because we are not far from what can be found in the equally Florentine *Tratato sopra l'arte della arismetricha* c. 1390, discussed around note 52.

What Benedetto does when he approaches symbolism can be summed up as follows: He uses ρ (often a shape more or less like φ) and (much less often) c and c^o for *cosa* respectively *censo* (and their plurals), but almost exclusively within formal fractions.[58] Even in formal fractions, *censo* may also be written in full. *Meno* is mostly abbreviated $\widehat{me}$ in formal fractions.[59] *Radice* may be abbreviated R in the running text, but often, and without system, it is left unabridged; within formal fractions, where there is little space for the usual abbreviation, it may become *r* or *ra*. Both when written in full and when appearing as R, it may be encircled if it is to be taken of a composite expression. In later times (e.g., in Pacioli's *Summa*, see below) this root was to be called *radice legata* or *radice universale*; the use of the circle to indicate it goes back at least to Gilio of Siena's *Questioni d'algebra* from 1384 (Franci, 1983, p. xxiii), and presumably to Antonio, since Gilio's is likely to have been taught by him or at least to have known his works well (*ibid.* pp. iv*f*). The concept itself, we remember, was expressed by Dardi as "R *de zonto ... con* ...", close in meaning to *radice legata*.

All of this suggests that the "symbolism" is only a set of facultative abbreviations, and not really an incipient symbolism. However, in a number of

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[58] Outside such fractions, I have noticed ρ three times in the main text of the *Fioretti*, *viz* on fols. 453r, 469r and 469v (of which the first occurrence seems to be explained by an initial omission of the word chosa leaving hardly space for the abbreviation), and c^o once, on fol. 458^r. Arrighi (1967, p. 22) claims another c^o on fol. 453^r, but the manuscript writes *chosa* in the corresponding place.

[59] Additively composite symbolic expressions are mostly constructed by juxtaposition (in running text as well as marginal computations); in rhetorical exposition, *e* or (when a root and a number are added) an unabbreviated *più* is used. A few marginal diagrams in the section copied from Bartolo mark additive contributions to a sum by *p*, and all subtractive contributions by *m*.

marginal calculations it does serve as carrier of the reasoning. One example was shown in Figure 1.9, another one (fol. 455^r, see Figure 1.11) performs a multiplication which, in slightly mixed notation, looks as follows:

$$(1\rho\,\overset{\frown}{me}\,\text{℞}[13\frac{1}{2}\,\overset{\frown}{me}\,1\text{ c}]) \times (1\rho\text{ p[iù] ℞}[13\frac{1}{2}\,\overset{\frown}{me}\,1\text{ c}])$$

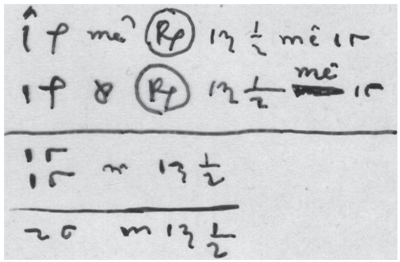

Fig. 1.11: The multiplication of $1\rho - \sqrt{13\frac{1}{2} - 1c}$ by $1\rho + \sqrt{13\frac{1}{2} - 1c}$

Formal fractions *without* abbreviation are used in the presentation of the arithmetic of algebraic powers in Book XIII (fols. 372r–373r). At first in this piece of text we find

> Partendo chose per censi ne viene rotto nominato da chose chome partendo 48 chose per 8 censi ne viene $\frac{6}{1\ chosa}$.

in translation

> Dividing *things* by *censi* results in a fraction denominated by *things*, as dividing 48 *things* by 8 *censi* results in $\frac{6}{1\ chosa}$.

Afterwards we find denominators "1 *censo*", "1 *cubo*", "1 *cubo di censo*", etc. When addition of such expressions and the division by a binomial are taught, we also find denominators like "3 *cubi* and 2 *cose*".[60]

Long before we come to the algebra, namely on fols. 259v–260v, there is an interesting appearance of formal fractions in problems of combined works, involving not a *cosa* or a *censo* but a *quantità* – such as $\frac{8}{1\ quantita}$ and

[60] This whole section looks as if it was inspired by al-Karajī or the tradition he inaugurated; but more or less independent invention is not to be excluded: once the notation for fractions is combined with interest in the arithmetic of algebraic monomials and binomials things should go by themselves.

$\frac{1\ quantita\ meno\ 8}{1\ chosa}$.[61] These fractions are written without any abbreviation.[62] Together with the explanation of the division of algebraic powers they demonstrate (as we already saw it in the *Trattato dell'alcibra amuchabile*) that the use of and the argumentation based on formal fractions do not depend on the presence of standard abbreviations for the unknown (even though calculations involving products of unknown quantities become heavy without standard abbreviations).

The manuscript Vatican, Ottobon. lat. 3307, was already mentioned above.[63] Like Benedetto's *Trattato*, it was written in Florence; it dates from c. 1465, and is also encyclopedic in character but somewhat less extensive than Benedetto's treatise, of which it is probably independent in substance.[64] It presents itself (fol. 1^r) as *Libro di praticha d'arismetrica, cioè fioretti tracti di più libri facti da Lionardo pisano* – which is to be taken *cum grano salis*, Fibonacci is certainly not the main source.

Judged as a mathematician (and as a Humanist digging in his historical tradition), the present compiler does not reach Benedetto's shoulders. However, from our present point of view he is very similar, and the manuscript even presents us with a couple of innovations (which are certainly not of the compiler's own invention).

Even in this text, margin calculations are often indented into the text in a way that shows them to have been written first, indicating that it is the compiler's autograph.[65] Already in an intricate problem about combined works (not the same as Benedetto's, but closely related) use is made of formal fractions involving an unknown (unabbreviated) *quantità*. Now, even the square of the *quantità* turns up, as *quantità di quantità*.

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[61] Benedetto would probably see these solutions not as applications of algebra but of the *regula recta* – which he speaks of as *modo retto/repto/recto* in the *Tractato d'abbaco*, ed. (Arrighi, 1974, pp. 153, 168, 181), everywhere using *quantità* for the unknown.

[62] However, in the slightly later problem about a *borsa* and a *quantità* mentioned in note 57, these are abbreviated in the marginal computations – perhaps not only in order to save space (already a valid consideration given how full the page is) but also because it makes it easier to schematize the calculations.

[63] Description with extracts in (Arrighi, 2004/1968).

[64] The *idea* of producing an encyclopedic presentation of abbacus mathematics may of course have been inspired by Benedetto's *Trattato* from 1463 – unless the inspiration goes the other way, the dating "c. 1465" is based on watermarks (Van Egmond, 1980, p. 213) and is therefore only approximate. If the present compiler had emulated Benedetto, one might perhaps expect that he would have indicated it in a heading, as does Benedetto when bringing a whole sequence of problems borrowed from Antonio. In consequence, I tend to suspect that the Ottoboniano manuscript precedes Benedetto's *Trattato*.

[65] This happens seven times from fol. 48^v to fol. 54^v. On fols. 176^v and 211^v there are empty indentions, but these are quite different in character, wedge-shaped and made in the beginning of problems, and thus expressions of visual artistry and not evidence that the earlier indentions were made as empty space while the text was written and then filled out afterwards by the compiler or a user.

When presenting the quotients between powers, the compiler writes the names of powers in full within the formal fractions, just as done by Benedetto. The details of the exposition show beyond doubt, however, that the compiler does not copy Benedetto but that both draw on a common background; it seems likely that the present author makes an attempt to be creative, with little success. In the present treatise, the first fractional power is introduced like this (fol. 304^v):

> Partendo dramme per chose ne viene un rocto denominato da chose, chome partendo 48 dramme per 6 chose ne viene questo rotto cioè $\frac{48\ dramme}{1\ chosa}$.

The second example makes the same numerical error. From the third example onward, it has disappeared. The fourth one looks as follows (fol. 305^r):

> Partendo chose per chubi ne viene rotto nominato da chubi, come partendo 48 chose per 6 chubi, ne viene questo rotto, cioè $\frac{8\ chose}{1\ chubo}$.

Only afterwards is the reduction of the ratio between powers (*schifare*) introduced, for instance, that $\frac{8\ chose}{1\ chubo}$ is $\frac{8\ dramme}{1\ censo}$.

Abbreviations for the powers are absent not only from this discussion but also from the presentation of the rules. When we come to the examples, however, marginal calculations with binomials expressed by means of abbreviations abound. That for *cosa* changes between ρ and φ, that for *censo* between c (written ⊂̄) and σ (actually σ̄); in both cases the difference is simply the length of the initial stroke; since all intermediate shapes are present, a single grapheme is certainly meant for *cosa* as well as *censo*. c^o appears to be absent. In the marginal computations, *più* may appear as *p*, whereas *meno* may be may be *m* or *mê*.[66] However, addition may also indicated by mere juxtaposition. The marginal calculations mostly have the same character as those of Benedetto, cf. Figure 1.9; in the running text abbreviations are reserved for formal fractions and otherwise as absent as from Benedetto's *Trattato*.

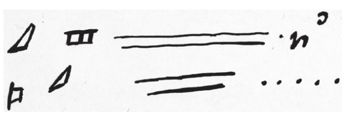

Fig. 1.12: The marginal note from Ottobon. lat. 3307 fol. 309^r

On two points the present manuscript goes slightly beyond Benedetto. Alongside a passage in the main text which introduces cases involving *cubi* and *censi di censi* (fol. 309^r), the margin contains the note shown in Figure 1.12.

[66] *m* and *mê* appear in the same calculation on fol. 31^v – by the way together with *p*.

n^o being *numero* and the superscript square being known (for instance from Vat. lat. 10488, cf. above) to be a possible representative for *censo*, it is a reasonable assumption (which we shall find fully confirmed below) that the triangle stands for the cube and the double square for *censo di censi*, the whole diagram thus being a pointer to the equation types "*cubi* and *censi di censi* equal number" and "*censi* and *cubi* equal number". We observe that equality is indicated by a double line.[67] As we shall see imminently, the compiler and several other fifteenth-century writers indicate equality by a single line. This, as well as the deviating symbols for the powers, suggests that this particular note was made by a later user of the manuscript.

The other innovation can be safely ascribed to the hand of the compiler if not (as an innovation) to his mind. It is a marginal calculation found on fol. 331^v, alongside a problem $\frac{100}{1\,\rho} + \frac{100}{1\,\rho+7} = 40$ (these formal fractions, without + and =, stand in the text). The solution follows from a transformation

$$\frac{100\rho + 100\cdot(\rho+7)}{(1\rho)\cdot(1\rho+7)} = \frac{100\rho + (100\rho+700)}{1\sigma + 7\rho} = 40$$

whence $200\rho + 700 = 40\sigma + 280\rho$. In the margin, the same solution is given schematically:

$$\begin{array}{lll}
100\rho & & \\
\underline{100\rho\ 700\qquad\quad} & & \\
\underline{200\rho\ 700\qquad\quad} & & \\
\quad 1\sigma\ \underline{7\rho\qquad\qquad} & & \\
 & 40 & \\
 & & \\
200\rho\ 700 \text{ ——— } & 40\sigma & \langle 280\rho\rangle
\end{array}$$

(the omitted $\langle 280\rho\rangle$ in the last line is present within the main text). The strokes before 40 and 40σ appear to be meant as equation signs. It might be better, however, to understand them as all-purpose "confrontation signs" – in the margin of fol. 338^r, ——— means that one commercial partner has $\frac{3000}{1\rho\,5000}$, the other $\frac{4000}{1\rho\,6000}$ (see Figure 1.13).[68]

[67] The double line is also used for equality in a Bologna manuscript from the mid-sixteenth century reproduced in (Cajori, 1928, I, p. 129); whether Recorde's introduction of the same symbol in 1557 was independent of this little known Italian tradition is difficult to decide. In any case, the combination with the geometric symbols indicates that the present example (and thus the Italian tradition) predates Recorde by at least half a century or so.

[68] As we shall see, Raffaello Canacci also uses the line both for equality and for confrontation. Even Widmann (1489) uses the long stroke for confrontation: fols. 12r, 21r–v, 23r, 27r, 38v when confronting the numbers 9 and 7 with the schemes for casting out nines and sevens, fol. 193^v (and elsewhere) when stakes and profits in a partnership are confronted.

This is one of Antonio's problems. In Benedetto's manuscript, we find the same problem and the same diagram on fol. 456^r – with the only difference that the line is replaced by an X indicating the cross-multiplication that is to be performed – see Figure 1.9. The "confrontation line" is thus not part of the inheritance from Antonio (nor, in general, of the inheritance shared with Benedetto). Though hardly due to the present compiler, it *is* an innovation.

The reason to doubt the innovative role of our compiler is one of Regiomontanus's notes for the Bianchini correspondence from c. 1460 (ed. Curtze, 1902, p. 278). For the problem $\frac{100}{1\rho}+\frac{100}{1\rho+8}$, he uses exactly the same scheme, including the "confrontation line":

$$
\begin{array}{rlll}
 & \frac{100}{1\rho} & \frac{100}{1\rho+8} & \\
 & & & \\
 & 100\rho \text{ et } 800 & & \\
 & \underline{100\rho} & & \\
 & \underline{200\rho \text{ et } 800} & & \\
 & 1\rho \text{ et } 8\ \sigma & \text{———} & 40 \\
40\ \sigma \text{ et } 320\rho \text{ ———} & & 200\rho \text{ et } 800 & \\
40\ \sigma \text{ et } 120\rho \text{ ———} & & 800 & \\
1\ \sigma \text{ et } 3\rho \text{ ———} & & 20 &
\end{array}
$$

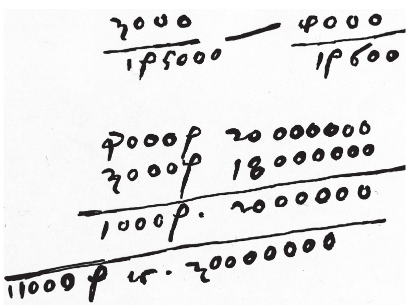

Fig. 1.13: The confrontation sign of Ottobon. lat. 3307 fol. 338^r

(Regiomontanus extends the initial stroke of ρ even more than our compiler, to ϼ; his variant of σ, *census*, is ℭ, possibly a different extension of c)[69].

A third Florentine encyclopedic abbacus treatise is Florence, Bibl. Naz. Centr., Palat. 573.[70] Van Egmond (1980, p. 124) dates it to c. 1460 on the

[69] Curtze does not show these shapes in his edition, but see (Cajori, 1928, I, p. 95).

[70] Described with sometimes extensive extracts from the beginnings of all chapters in

basis of dates contained in problems, but since the compiler refers (fol. 1^r) to Benedetto's *Trattato* (from 1463) as having been made "already some time ago" (*già è più tenpo*), a date around 1470 seems more plausible. This is confirmed by the watermarks referred to by Van Egmond – even this manuscript can be seen from marginal calculations made before the writing of the main text to be the compiler's original, whose date must therefore fit the watermarks.

As regards algebraic notations and incipient symbolism, this treatise teaches us nothing new. It does not copy Benedetto (in the passages I checked) but does not go beyond him in any respect; it uses the same abbreviations for algebraic powers, in marginal calculations and (sparingly) in formal fractions within the main text – including the encircled *radice* and ℞. In the chapter copying Fibonacci's algebra it has no marginal calculations (only indications of forgotten words), which confirms that the compilers of the three encyclopedic treatises copied the marginal calculations and did not add on their own when copying – at least not when copying venerated predecessors mentioned by name.

1.7 Late fifteenth-century Italy

The three encyclopediae confirm that no systematic effort to develop notations or to extend the range of symbolic calculation characterizes the mid-century Italian abbacus environment – not even among those masters who, like Benedetto and the compiler of Palat. 573, reveal scholarly and Humanist ambitions by including such matters as the Boethian names for ratios in their treatises and by basing their introduction of algebra on its oldest author (al-Khwārizmī).[71] The experiments and innovations of the fourteenth century – mostly, so it seems, vague reflections of Maghreb practices – had not been developed further.[72] In that respect, their attitude is not too far from that of mid-fifteenth–century mainstream Humanism.

(Arrighi, 2004/1967).

[71] Benedetto (ed. Salomone, 1982, p. 20) gives this argument explicitly; the compiler of Palat. 573 speaks of his wish that "the work of Maumetto the Arab which has been almost lost be renovated" (Arrighi, 2004/1967, p. 191).

[72] It is true that we have not seen the quotients between powers expressed as formal fractions in earlier manuscripts; however, the way they turn up independently in all three encyclopædiæ shows that they were already part of the heritage – *perhaps* from Antonio. The interest in such quotients is already documented in Giovanni di Davizzo in 1339, who however makes the unlucky choice to identify negative powers with roots – see (Høyrup, 2007c, pp. 478–484) (and cf. above, before note 44).

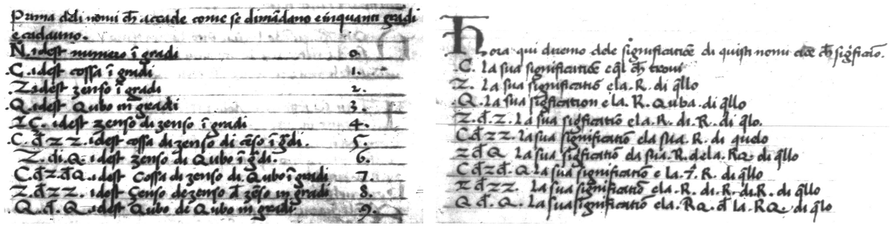

Fig. 1.14: The two presentations of the algebraic powers in Bibl. Estense, ital. 578

Towards the end of the century we have evidence of more conscious exploration of the potentialities of symbolic notations. A first manuscript to be mentioned here is Modena, Bibl. Estense, ital. 578 from c. 1485 (according to the orthography written in northern Italy – e.g., *zonzi* and *mazore* where Tuscan normal orthography would have *giongi* and *magiore*).[73] It contains (fols. 5r–20r) an algebra, starting with a presentation of symbols for the powers with a double explanation, first with symbols and corresponding "degrees", *gradi* (fol. 5^r), next by symbols and signification (fol. 5^v) – see Figure 1.14.

As we see, the symbol for the *cosa* is the habitual *c*. For the *censo*, *z* is used, in agreement with the usual northern orthography *zenso* – however, in a writing which is quite different from the *z* used in full writing of *zenso* (**z** respectively ʒ, see also Figure 1.15); the *cubo* is *Q*, the fourth power is *z di z*. The fifth power is *c di zz*, obviously meant as a multiplicative composition (as the traditional *cubo di censo*), the sixth instead *z di Q*, that is, composed by embedding. The seventh degree is *c di z di Q*, mixing the two principles, the eight again made with embedding as *z di zz*. So is the ninth, *QQ*.

Fig. 1.15: Three graphemes from Bibl. Estense, ital. 578. Left, *z* abbreviating *zenso* in the initial overview; centre, *z* as written as part of the running text; right, the digit 3

Then follow the significations. *c* is "that which you find", *z* "the root of that", *Q* "the cube root of that", and *z di z* "the root of the root of that". Already now we may wonder – why "roots"? I have no answer, but discuss possible

[73] (Van Egmond, 1986) is an edition of the manuscript. It has some discussion of its symbolism but does not go into details with the written shapes, for which reason I base my

hints in (Høyrup, 2008, p. 31), in connection with the *Tratato sopra l'arte della arismetricha* (see just before note 52), from where these "root-names" are known for the first time.[74] It is reasonable to assume a connection – this *Tratato* has the same mixture of multiplicative and embedding-based formation of the names for powers, though calling the fifth degree *cubo di censo*, and the sixth (like here) *censo di cubo*.[75]

The root names go on with "root of this" for the fifth power – which is probably meant as "5th root of this", since the seventh power is "the 7th root of this". The names for the sixth, eighth and ninth degree are made by embedding.

After explaining algebraic operations and the arithmetic of monomials and binomials the manuscript offers a list of algebraic cases followed by examples illustrating them. Here the same symbols are used within the text (there are no marginal calculations) – with one exception, instead of z a sign is used which is a transformed version of Dardi's *ç* – ʒ, with variations that sometimes make it look like a z provided with an initial and a final curlicue.[76]

The problems are grouped in *capitoli* asking for the same procedure in spite of involving different powers – chapter 14, for instance, combines "*zz* and *z di zz* equal to n^o" and "*c di zz* and *QQ* equal to *c*". The orderly presentation of the powers in a scheme and the concept of numerical *gradi*, "degrees", (our exponents) has facilitated this further ordering. This is clear from the presentation – in chapter 14, "When you find three names of which one is 4 degrees more than the other ...". Beyond this, the abbreviations seem to serve as nothing but abbreviations, though used consistently.

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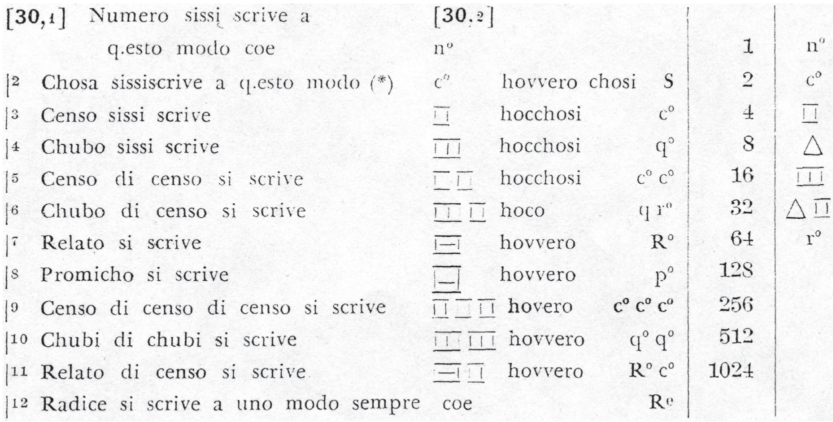
discussion on the manuscript.

[74] Van Egmond (1986, 20) "explains" them $Z = R$, $x^2 = n \rightarrow x\sqrt{n}$ etc., which however, while being an impeccable piece of mathematics, is completely at odds with the words of the text.

[75] This difference may tell us something about the spontaneous psychology of embedding: it seems to be easier to embed within a single than within a repeated multiplication – that is, to grasp *censo* of P as $(P)^2$ than to understand *cubo* of R as $(R)^3$.

[76] There are a few slips. In the initial list, a full *zenso* is once written *çenso* (written with ʒ), and ʒ itself appears once; within the list of cases and the examples a few instances of zenso abbreviated z (written ȝ, not ƶ) occur. Van Egmond (1986, p. 23) reads these as "3", and takes this as evidence that the manuscript was made by a copyist who did not really understand but had a tendency to replace a z used in the original by *ç*. However, even though the writings of z and 3 are similar, magnification shows them quite clearly to be different, and makes it clear that the copyist did not write 3 where he should have written z (see Figure 1.15). Other errors pointed out by Van Egmond demonstrate beyond doubt that the beautifully written manuscript *is* a copy. However, the almost systematic distinction between the abbreviations ƶ and ʒ, as well as the general idea of applying stylized shapes of letters when used as symbols, is likely to reflect the ways of the original – an unskilled copyist would hardly introduce them.



[30,1]	[30,2]				
Numero sissi scrive a q.esto modo coe	n°			1	n°
2 Chosa sississcrive a q.esto modo (*)	c°	hovvero chosi	S	2	c°
3 Censo sissi scrive		hocchosi	c°	4	
4 Chubo sissi scrive		hocchosi	q°	8	△
5 Censo di censo si scrive		hocchosi	c° c°	16	
6 Chubo di censo si scrive		hoco	q r°	32	△
7 Relato si scrive		hovvero	R°	64	r°
8 Promicho si scrive		hovvero	p°	128	
9 Censo di censo di censo si scrive		hovero	c° c° c°	256	
10 Chubi di chubi si scrive		hovvero	q° q°	512	
11 Relato di censo si scrive		hovvero	R° c°	1024	
12 Radice si scrive a uno modo sempre	coe		R°		

Fig. 1.16: Canaccis scheme with the naming of powers, after (Procissi, 1954, p. 432)

Raffaello Canacci's use of schemes for the calculation with polynomials (including multiplication *a casella*) in the *Ragionamenti d'algebra*[77] from c. 1495 (ed. Procissi 1954, pp. 316–323) was mentioned above. In a couple of these he employs geometric signs for the powers, but mostly he writes *s* for *cosa* and *censo* in full. Addition may be indicated by juxtaposition, by *e*, by *più* or by *p*, subtraction by *m̃* or *me*.[78] Later he presents an ordered list, with three different systems alongside each other – see Figure 1.16. To the right we find an extension of a different "geometric" system – namely the one which was found in a (secondary) marginal note in the Ottoboniano encyclopædia. Next toward the left we find powers of 2 corresponding to the algebraic powers (an explanatory stratagem also used by Pacioli in the *Summa*); then letter abbreviations; and then finally, just to the right of the column with Canacci's full names, his own "geometric" system (not necessarily invented by him, cf. imminently, but the one he uses in the schemes) – better planned for the economy of drawing than as a support for operations or algebraic thought. According to Cajori (1928, I, pp. 112*f*) the system turns up again in Ghaligai's *Pratica d'arithmetica* from 1552 (and probably in the first edition from 1521, entitled *Summa de arithmetica*), where their use is ascribed to Ghaligai's teacher Giovanni del Sodo.

[77] Florence, Bibl. Naz. Centr., Palat. 567. I have not seen the manuscript but only Angiolo Procissi's diplomatic transcriptions.

[78] However, *p n* and *p n°* stand for "per numero". In schemes showing the stepwise calculation of products (pp. 313f), *m* stands for multiplication. In one scheme p. 318), a first *p* stands for *più*, a second in this way for *per*.

Canacci uses these last geometric signs immediately afterwards in a brief exposition of the rules for multiplying powers – and then no more. In a couple of marginal notes to the long collection of problems (ed. Procissi, 1983, pp. 58, 62–64) he uses the letter abbreviations (only s and c^o) – but also the line as an indication, once of equality, twice of confrontation or correspondence not involving equality. The running text, including formal fractions, writes the powers unabridged (except *numero*, which once is n^o); even *più* and *meno* are mostly written in full, but *meno* sometimes (pp. 21–23) with a brief stroke "–" – the earliest occurrence of the minus sign in Italy I know of.[79]

Three works by Luca Pacioli are of interest: the Perugia manuscript from 1478, the *Summa de arithmetica* from 1494, and his translation of Piero della Francesca's *Libellus de quinque corporibus regularibus* as printed in (Pacioli, 1509).

Since there is only one brief observation to make on the latter work, I shall start by that. According to the manuscript Vatican, Urb. lat. 632 as edited by G. Mancini (1916, pp. 499–501), Piero uses the familiar superscript square for *censo* when performing algebraic calculations, or he writes words; for *res* he uses a horizontal stroke over the coefficient, but mostly also keeps the word.[80] Pacioli (1509, fols. 3v–26r, *passim*) instead uses a sign $\diamond$ for the *cosa* and $\square$ for the *censo* (or, in the old unsystematic way, words). *Censo di censi* is $\square\square$ on fol. 4^r and $\square$ *de* $\square$ on fols. 4r and 11v. These geometric signs are absent from Pacioli's other works, and they must rather be considered a typographic experiment – given that their use is not systematic, they can hardly be understood as an instance of mathematical exploration beyond what Pacioli had done before. It is difficult to agree with Paola Manni (2001, p. 146) that they should represent "progress of mathematical symbolism" with respect to the more systematic use of letter abbreviations in the Perugia manuscript and the *Summa* (see imminently; and cf. the quotation from Woepcke after note 12). Indeed, the *Libellus* is an appendix to Pacioli's *Divina proportione*, in which Pacioli (1509, fol. 3^v) explains that various professions, among whom *le mathematici per algebra*, use specific *caratheri e abreviature* "in order to avoid prolixity in writing and also of reading".[81]

The 1478 Perugia manuscript *Suis carissimis disciplis ...* (Vatican, Vat. lat. 3129) has lost the systematic algebra chapters listed in the initial table

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[79] As well known, "–" is already used in the *Deutsche algebra* from 1481 (ed. Vogel, 1981, p. 20). Whether this is part of the very mixed Italian heritage of this manuscript (see below, note 88 and surrounding text) or a German innovation eventually borrowed by Ghaligai is undecidable unless supplementary evidence should turn up.

[80] The same (lack of) system is found in his abbacus treatise, see (Arrighi, 1970, p. 12).

[81] That Pacioli really thinks in terms of abbreviations is confirmed by a list of examples given in the manuscript of the treatise (Milano, Biblioteca Ambrosiana, Ms. 170 Sup., written in 1498), see (Maia Bertato, 2008, 13): it mixes the abbreviations for *radice*, *più*, *meno*, *quadrato* (*cosa* and *censo* are absent) with others for, *inter alia*, *linea*, *geometria*

of contents,[82] but it does contain a large amount of algebraic calculation. Everywhere here – in the main text as well as in the margin, and in the neat original prepared in 1478 as well as in fols. 350r–360v, added at a later moment and obviously very private notes – we find the signs from Canacci's right-hand column (Figure 1.16) written superscript and to the right – on fol. 360^v extended until $\overset{\square}{\square\square}$, *censi di censi di censi*. *Meno* is ⓜ and *più* (both signifying addition and as a normal word) a corresponding encircled *p*. This is thus the system which Pacioli used when calculating for himself, at least at that moment.[83] He uses the equality line in the margin (but also the same line indicating confrontation/correspondence, e.g., fol. 130^r).

Most important (in the sense that it was immensely influential and the other two works not) is of course the *Summa* (Pacioli, 1494). Typographic constraints are likely to have caused Pacioli to give up his usual notation. In ordinary algebraic explanation and computation, he now uses *.co.* and *.ce.* written on the line, and *più* and *meno* have become $\tilde{p}$ and $\tilde{m}$ (*meno* sometimes *mē*) – both as operators and as indicators of positivity and negativity (not only additivity and subtractivity).[84] However, he also has more systematic presentations. The first, in the margin of fol. 67^v, shows how the sequence *.co.-.ce.* is to be continued, namely (third power) *cubo*, (4th) *censo de censo*, (5th) *primo relato*, (6th) *censo de cubo*/*cubo de senso*, (7th) *secundo relato*, (8th) *censo de censo de censo*, (9th) *cubo de cubo*, (10th) *censo de primo relato*, (11th) *terzo relato*, etc. until the 29th power. As we see, the embedding principle has taken over completely, creating problems for the naming of prime-number powers. For each power the "root name" is indicated, number being "℞ *prima*", *cosa* "℞ 2*a*", *censo* "℞ 3*a*", etc.[85] As we see, the "root number" is *not* the exponent, but the exponent augmented by 1. This diminishes the heuristic value of the concept: it still permits to see directly that "6th roots and 4th roots equal 2nd roots" must be equivalent to "5th roots and 3rd roots equal 1st roots", but it requires as much thinking as in Jacopo's days almost 200 years earlier to see that this is a biquadratic problem that must be solved in the same way as "3rd roots and 2nd roots equals 1st roots".

and *arithmetica*).

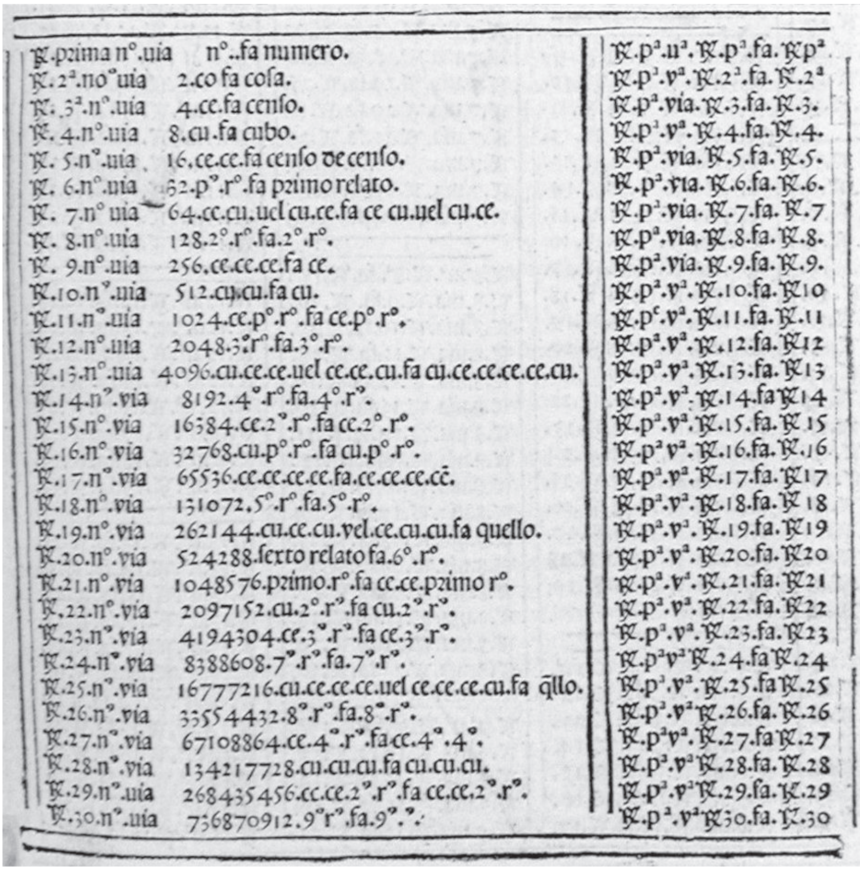
[82] See the meticulous description in (Derenzini, 1998), here p. 173. Since all abbreviations except the superscript symbols are expanded in the edition (Calzoni and Gavalzoni, 1996), I have used a scan of the manuscript.

[83] This restriction is probably unnecessary. At least the encircled *p* and *m* and the square are in the list offered by the 1498 manuscript, cf. note 81.

[84] E.g., on see fol. 114^r, "a partir .m̃.16.p̲.m̃.2. ne vene .p̃.8", and the proof that "meno via meno fa più" on fol. 11^3r, which is characterized as "absurda" and referred to the concept of a debt – if only subtractive numbers were involved, as in Dardi's corresponding proof, nothing would be absurd.

[85] Pacioli believes (or at least asserts) that these names go back to "the practice of algebra according to the Arabs, first inventors of this art". Could he have been led to this belief by the equivalence of "root" and *thing*/*cosa* in al-Khwārizmī's algebra?

After this list comes a list of symbols for "normal" roots: ℞ meaning *radici*; ℞℞ meaning *radici de radici*; ℞u. meaning *radici universale* or *radici legata*, that is, root of a composite expression following the root sign (encircled in Benedetto's *Trattato* and spoken of as "℞ *de zonzo*" by Dardi, we remember); and ℞ cu., cube root.



℞.prima n°.uia	n°..fa numero.	℞.pª.uª.℞.pª.fa.℞pª
℞.2ª.no°uia	2.co fa cosa.	℞.pª.vª.℞.2ª.fa.℞.2ª
℞.3ª.n°.uia	4.ce.fa censo.	℞.pª.via.℞.3.fa.℞.3.
℞.4.n°.uia	8.cu.fa cubo.	℞.pª.vª.℞.4.fa.℞.4.
℞.5.n°.uia	16.ce.ce.fa censo de censo.	℞.pª.via.℞.5.fa.℞.5.
℞.6.n°.uia	32.p°.r°.fa primo relato.	℞.pª.via.℞.6.fa.℞.6.
℞.7.n° uia	64.ce.cu.uel cu.ce.fa ce cu.uel cu.ce.	℞.pª.via.℞.7.fa.℞.7
℞.8.n°.uia	128.2°.r°.fa.2°.r°.	℞.pª.via.℞.8.fa.℞.8.
℞.9.n°.uia	256.ce.ce.ce.fa ce.	℞.pª.via.℞.9.fa.℞.9.
℞.10.n°.uia	512.cu.cu.fa cu.	℞.pª.vª.℞.10.fa.℞10
℞.11.n°.uia	1024.ce.p°.r°.fa ce.p°.r°.	℞.pª.vª.℞.11.fa.℞.11
℞.12.n°.uia	2048.3°.r°.fa.3°.r°.	℞.pª.vª.℞.12.fa.℞12
℞.13.n°.uia	4096.cu.ce.ce.uel ce.ce.cu.fa cu.ce.ce.ce.ce.cu.	℞.pª.vª.℞.13.fa.℞13
℞.14.n°.via	8192.4°.r°.fa.4°.r°.	℞.pª.vª.℞.14.fa℞14
℞.15.n°.via	16384.ce.2°.r°.fa ce.2°.r°.	℞.pª.vª.℞.15.fa.℞15
℞.16.n°.via	32768.cu.p°.r°.fa cu.p°.r°.	℞.pª vª.℞.16.fa.℞.16
℞.17.n°.via	65536.ce.ce.ce.ce.fa.ce.ce.ce.cẽ.	℞.pª.vª.℞.17.fa.℞17
℞.18.n°.via	131072.5°.r°.fa.5°.r°.	℞.pª.vª.℞.18.fa.℞18
℞.19.n°.via	262144.cu.ce.cu.vel.ce.cu.cu.fa quello.	℞.pª.vª.℞.19.fa.℞19
℞.20.n°.via	524288.sexto relato fa.6°.r°.	℞.pª.vª.℞.20.fa.℞20
℞.21.n°.via	1048576.primo.r°.fa ce.ce.primo r°.	℞.pª.vª.℞.21.fa.℞21
℞.22.n°.via	2097152.cu.2°.r°.fa cu.2°.r°.	℞.pª.vª.℞.22.fa.℞22
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℞.26.n°.via	33554432.8°.r°.fa.8°.r°.	℞.pª.vª℞.26.fa.℞.26
℞.27.n°.via	67108864.ce.4°.r°.fa ce.4°.4°.	℞.pªvª.℞.27.fa℞.27
℞.28.n°.via	134217728.cu.cu.cu.fa cu.cu.cu.	℞.pª.vª℞.28.fa.℞.28
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Fig. 1.17: Paciolis scheme (1494, fol. 143^r) showing the powers with root names

On fol. 143^r follows a scheme that deals with the first 30 powers (*dignità*), and with how they are brought forth as products (*li nascimenti pratici o li 30 gradi de li caratteri algebratici*). It runs in four tangled columns and 30 rows. The first column has the numbered "root name" of the power, the second formulates in Pacioli's normal language or in abbreviations that number times this power gives the same power. The third, written inside the second, indicates the corresponding power of 2. The fourth, finally, repeats the second column, now translated into root names – see Figure 1.17.

On the next page follow further schemes, expressed in roots names, for the products of the nth root with all roots from the nth to the $(31-n)$th (meaning that all products remain within the range defined by the 30th root), $2 \leq n \leq 15$.

All in all, we may say that Pacioli explored existing symbolic notations to a greater extent (and used them more consistently) than for example Benedetto, thus offering those of his readers who wanted it matters to chew; but he hardly gave them many solutions they could build on (and as we have seen, he thought of his notations as mere abbreviations serving to avoid prolixity). Even in this respect, subsequent authors could easily have found reasons to criticize him while standing on his shoulders (as they did regularly), if only their own understanding of the real progress they offered had been sufficient for that. Tartaglia, for instance, gives the list of *dignitates* until the 29th in *La sesta parte del general trattato* (Tartaglia, 1560, fol. 2^r), with names agreeing with Pacioli's *.co.-.ce.*-list and indication of the corresponding exponents (now *segni*), alongside a text that explains how multiplication of *dignitates* corresponds to addition of *segni*; that, however, was well after Stifel's *Arithmetica integra*, which Tartaglia knew well.

1.8 Summary observations about the German and French adoption

Regiomontanus shows familiarity with algebraic practice, not only in the notes for the Bianchini-correspondence (cf. above) but also elsewhere – several articles in (Folkerts, 2006) elucidate the topic in detail. Not only the calculation before note 69 but also some of his abbreviations (and the variability of these) are evident borrowings from Italian models (Høyrup, 2007c, p. 134). It might seem a not impossible assumption that Regiomontanus was the main channel for the adoption of Italian abbacus algebra into German areas, in spite of his purely ideological ascription of the algebraic domain to Diophantos and Jordanus (above, text before note 24).

An influence cannot be excluded, even though those of Regiomontanus' algebraic notes we know about may not have circulated widely. However, those of his symbolic notations or abbreviations which are not to be identified as Italian are already present in a section of a manuscript possessed by Regiomontanus but not written by him (Folkerts, 2006, V, pp. 201*f*), cf. (Høyrup, 2007c, pp. 136*f*).[86].

That Regiomontanus was at most one of several channels can also be seen from the so-called *Deutsche Algebra* from 1481 (ed. Vogel, 1981). Its symbols[87] for *number* (*denarius*, replaces earlier *dragma*), *thing* and *census* coincide with those of the Robert-Appendix,[88] that for the *cube* with the one Regiomontanus employs for *census* – hardly evidence for inspiration from the latter. A token of Italian inspiration certainly *not* passing through Regiomontanus is occasional use of the quasi-fraction notation for powers and of 1c for *cosa* (Vogel, 1981, p. 10) – all in all, as Kurt Vogel observes, evidence that a number of sources flow together in this manuscript.

I shall not consider in detail German algebraic writings from the sixteenth century (Rudolff, Ries, Stifel, Scheubel), only sum up that with time German algebra tends to be more systematic and coherent in its use of symbolism (for notation as well as calculation) than any single Italian treatise.[89] But what the German authors do is to combine and put into system ideas that are all present in *some* Italian work. They never really go beyond the Italian inspiration *seen as a whole*, and never attain the coherence which appears to have been reached by the Maghreb algebraists of the twelfth century.[90]

I shall also be brief on what happened in French area. Scrutiny of Nicolas Chuquet's daring exploration of the possibilities of symbolism in the *Triparty* from 1484 (ed. Marre, 1880) would be a task of its own; his parenthesis (an underlining[91]) and his complete arithmetization of the notation for powers

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[86] The *thing* symbol in the appendix to Robert of Chester's translation of al-Khwārizmī is the same as Regiomontanus's transformation of ρ ; the *census* symbol is a z provided with a final curlicue and which *could* be derived from the ç which we find in the Modena-manuscript but is much more likely to correspond to its initial use of z in this function.

[87] Listed in (Vogel, 1981, p. 11).

[88] With ∂ as an alternative for *thing*, standing probably for *dingk*.

[89] The use of schemes for polynomial arithmetical calculation by Stifel (1544) and Scheubel (1551) was mentioned above. They also appear in Rudolff's *Coss* (1525).

[90] Quite new, as far as I know, and awkwardly related to the drive toward more systematic use of notations (but maybe more closely to the teaching of Aristotelian logic), is the idea to represent persons appearing in commercial problems by letters A, B, C, ... I have noticed it in Magister Wolack's Erfurt lecture from 1467, apparently the earliest public presentation of abbacus mathematics in German land (ed. Wappler, 1900, pp. 53f), and again in Christoff Rudolff's *Behend und hübsch Rechnung durch die kunstreichen Regeln Algebra* #128 (1525, fol. Nv^{r-v}).

[91] The only parentheses Italian symbolic notation had made use of were those marked off by the fraction line and the R̸ *de zonzo/legata/universale*. The latter, furthermore, was ambiguous – how far does the expression go that it is meant to include? (Actually, I have not seen it go beyond two terms, which may indeed have been part of the concept.) A parenthesis as good and universal as that of Chuquet had to await Bombelli (1572), even though Pacioli (1494) uses brackets containing *textual* parentheses (e.g., on fol. 3^r). As we remember from note 12, even Descartes eschews general use of the parenthesis.

as well as roots certainly goes beyond what can be found in anything Italian until Bombelli, and (as far as the symbols for powers and roots are concerned) even beyond the Maghreb notation. However, his innovations were historical dead ends; Etienne de la Roche, while transmitting other aspects of Chuquet's mathematics in his *Larismetique* from 1520, returned to more familiar notations (Moss, 1988, pp. 120*f*). What later authors learned (or, like Buteo, refused to learn, *ibid.*, p. 123) from de la Roche could as well have been Italian.[92]

As a representative of the French mid-sixteenth century I shall choose Jacques Peletier's *L'algebre* from (1554) – interesting not least because his orthographic reform proposal (1555; 1554, final unpaged note) shows him to have reflected on notation. Peletier knows Stifel's *Arithmetica integra*, cites it often and learns from it. But he must be acquainted with the Italian abbacus tradition, and not only through Pacioli and Cardano, both of whom he cites on p. 2: he speaks of the powers as *nombres radicaus* (p. 5), and uses ℞ for the first power (this, as well as the *nombres radicaus*, *could* at a pinch be inspired by Pacioli) and the stylized ç (ʓ) which we know from the Modena-manuscript for the second power (following Stifel for higher powers). That certainly does not help him go beyond the combination of the most developed elements of Italian symbolism we know from the German authors – and like Stifel he does not get beyond.

1.9 Why should they?

As we have seen, Italian abbacus algebra makes use of a variety of elements that might have been (and in the main probably were) borrowed from the Maghreb, most of them already present in one or the other manuscript from the fourteenth century. But the abbacus masters do not seem to have been eager to use them consistently, to learn from each other or to surpass each other in this domain (to which extent they wanted to avoid to *teach* symbolism is difficult to know – it will not have had the same value in the competition for jobs and pupils as the ability to solve intricate questions); Benedetto and the compilers of the Ottoboniano and Palatino encyclopædiae were quite satisfied with repeating a heritage that may reach back to Antonio, and did not care about the schemes for polynomial arithmetic that had been in circulation at least since Dardi's times. Only with the Modena manuscript, with Canacci

[92] The question to which extent the Provençal tradition which Chuquet draws upon was independent of the Italian tradition (to some extent it certainly was) is immaterial for the present discussion; no surviving earlier or near-contemporary Provençal writings offer as much incipient symbolism as the Italian abbacus writers.

and with Pacioli's *Summa* do we find some effort to be encyclopedic (if not systematic) also in the presentation of notations.

Our meeting is about the "philosophical aspects of symbolic reasoning", and about "early modern science and mathematics". The philosophical question to raise to the material presented above is whether the abbacus masters of the fourteenth and fifteenth century, and even the algebraic writers of the early and mid-sixteenth century, had any *reason* to develop a coherent symbolic approach. The answer seems to be that they had none (cf. also note 50 and preceding text). The kind of mathematics they were engaged in (even when they applied their art to *Elements* X, as do for instance Fibonacci and Stifel) did not ask for that. They might sometimes extrapolate their technique further than their mathematical practice asked for – 29 algebraic powers is an example of that, as is of course the creation of never-used symbols for these powers. But without a genuine practice there was nothing which could force these extrapolations to merge into a consistent conceptual and operational framework. Even those abbacus authors that had scholarly ambitions – as Benedetto and his contemporary encyclopedists, Pacioli and Tartaglia – did not encounter anything within the practice of university or Humanist mathematics which asked for much more than they did. To the contrary, the aspiration to connect their mathematics to the Euclidean ideal made them re-attach geometric proofs to a tradition from which these had mostly been absent, barring thereby the insight that purely arithmetical reasoning could be made as rigorous as geometric proofs – barring it indeed to such an extent that Ries and Scheubel rejected Jordanus' arithmetical rigor and borrowed only his problems, as we have seen.

That changed in the outgoing sixteenth century. By then (if I may be allowed some concluding sweeping statements), Apollonios, Archimedes and Pappos were no longer mere names (or at most authors of difficult texts to be assimilated) but providers of problems to be worked on, and trigonometry had become an advanced topic. This was probably what created the pull on the development of symbolic reasoning and of those notations that symbolic reasoning presupposed if it was to go beyond simple formal fractions;[93] the reaction to this pull (which at first created a complex of new mathematical developments) was what ultimately transformed symbolic mathematics into

[93] It may perhaps be allowed to give a frivolous illustration of a sweeping statement: the problems which the 16–17 years old Huygens investigated by means of Cartesian algebra under the guidance of Frans van Schooten. Quite a few of them deal with matters from Archimedes or Apollonios (Huygens, 1908, 27–60). The problems he dealt with 4–5 years later (pp. 217–275 in the same volume) are derived from Pappos, and even they make extensive use of Descartes' technique. This is thus what a young but brilliant mathematical mind was training itself at a decade after the appearance of Descartes' *Geometrie*.

It is difficult to imagine that these problems could have been well served by *cossic* algebra, with or without the abbreviations that had been standardized in the mid-sixteenth century.

a factor that could (eventually) push the development of (some constituents of) early modern science.

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