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DESUSPENSION OF SPLITTING ELLIPTIC SYMBOLS II

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Abstract.

In the second part of our study we continue the development of the spectral calculus of elliptic operators which take the form $A = G_A(\partial/\partial t + B_t)$ near a submanifold Y of codimension 1 with self-adjoint B_t. The index of the general linear conjugation problem ("cutting and pasting" of elliptic operators) is determined. A thorough analysis of the geometry of Fredholm pairs of subspaces in Hilbert space and especially of the spaces of Cauchy data is undertaken. These methods lead to an alternative view of regular elliptic boundary value problems where main results (old and new ones) can be obtained through explicit transparent calculations.

DESUSPENSION OF SPLITTING ELLIPTIC SYMBOLS II

Bernhelm Booss, Krzysztof Wojciechowski

To Bogdan Bojarski

In the second part of our study we continue the development of the spectral calculus of elliptic operators which take the form $A = G_h(\partial/\partial t + B_t)$ near a submanifold Y of codimension 1 with self-adjoint B_t. The index of the general linear conjugation problem ("cutting and pasting" of elliptic operators) is determined. A thorough analysis of the geometry of Fredholm pairs of subspaces in Hilbert space and especially of the spaces of Cauchy data is undertaken. These methods lead to an alternative view of regular elliptic boundary value problems where main results (old and new ones) can be obtained through explicit transparent calculations.

0. Introduction and background

In the First Part of our paper [17, Theorem I.4.1] (here and in the following the Roman I refers always to Part I of our study) we obtained the following abstract scheme of "desuspension", i.e. reduction of index problems for elliptic differential operators to the calculation of the index of an explicitly constructed elliptic operator over a lower-dimensional manifold:

0.1. Theorem. If $\{B_t\}_{t\in I}$ is a family of elliptic self-adjoint operators of first order over a closed Riemannian manifold Y with $B_1 = g^{-1}B_0g$ for a suitable bundle automorphism g, then we get

$$sf\{B_t\}_{t\in\mathbb{Z}} = \text{index } P_+ - gP_- .$$

Here sf denotes the spectral flow, a certain integer valued invariant studied in § I.1, and $P_\pm$ are the spectral projections of B_0 such that $P_+ - gP_-$ becomes an elliptic pseudodifferential operator of 0-th order over Y.

In the present Part II we want to relate that abstract scheme to the concrete analysis of classical elliptic boundary value and transmission problems. It is partly our purpose to rewrite the "story" of elliptic boundary value problems in the spirit of one of the fundamental tenents of numerical analysis; that any work with partial differential equations should start with the analysis of the spaces of Cauchy data.
As with the investigation of pseudodifferential operators twenty years ago it turns out that the index problem - how restricted it may appear - is an effective pilot for the development of new constructive methods, perhaps because one always has something to "calculate" so that any empty building of definitions immediately can be recognized as inefficient.

Through this paper, X will denote a compact Riemannian manifold with smooth boundary Y or a closed Riemannian manifold which is divided into two parts X_+ and X_- by a smooth submanifold Y of codimension 1. As in Part I we restrict ourselves to elliptic operators of first order which take the form $A = G_A(y)(\partial/\partial t + B_t)$ near Y, where t is the normal coordinate, y the tangential coordinate, B_t an elliptic self-adjoint operator over Y, and the Green form G_A a bundle isomorphism over Y.
Our interest for that type of operators originated in working with the Cauchy-Riemann operator of complex analysis, the signature operator of the differential topology of 4k-dimensional manifolds, and the Dirac operator

of particle physics. However, if one starts with a system of arbitrary differential equations, our assumption is not really a restriction because any original problem incorporating higher order differential operators can be easily rewritten in that form. In fact, a large class of pseudodifferential elliptic operators can be reduced to that type, see e.g. [6, § 6] and below, § 4.

We build the analysis of elliptic problems upon two fundamental concepts, the "pasting" of elliptic operators from the two sides of a dividing submanifold and the projections onto the respective spaces of Cauchy data. This leads to the following structure of the present Part II:

Section 1 of Part II is devoted to the general linear conjugation problem ("cutting and pasting" of elliptic operators) where we get an explicit formula $\mu(g,A)$ = index $P_{+}-gP_{-}$ for the change of the index of elliptic operators under repasting with a bundle automorphism g over Y. We need not assume that A admits elliptic boundary value problems over Y. So roughly speaking, Section 1 treats the case of "simple" pasting and "intriguate" spectral projections. We consider the treatment of this case as a pilot study for elliptic boundary value problems, where in some sense the pasting is more complicated and the spectral projections are trivial because the spectral projections come from symbols which are then algebraically degenerated near Y.

Section 2 deals with spectral inequalities for operators with non-trivial symbols in $K^{-1}(TX)$ obtained originally by Vafa and Witten [35]. Their results provide an important example of the application of the spectral flow and show the full strength of the machinery we introduced in Part I.

Section 3 provides the necessary scheme for understanding the functional analysis of elliptic operators near submanifolds of codimension 1 and for the generalisations and relations of the transmission problem with the boundary value problem. Here the concept of Fredholm pairs of

subspaces and the analysis of the related projections play the same fundamental role in our approach as Fredholm operators and parametrices do in the usual approach. The power of the concept of Fredholm pairs of subspaces is perhaps most easily seen when we look for "dual" or "adjoint" problems. Then, it is much more elementary and constructive to investigate projections and orthogonal complements instead of building a whole machinery of parametrices. Of course, the greatest flexibility is obtained if one can easily switch between the space Fred(H) of Fredholm operators and the space $\mathrm{Fred}^2(H)$ of Fredholm pairs. This will be carried out in detail.
In that connection we also will show that $\mathrm{Fred}^2(H)$ is yet another classifying space for the functor K.

Section 4 is of independent interest and can be read without the preceding sections. It gives an introduction to the theory of local elliptic boundary value problems via the Fredholm pairs of Cauchy-data spaces and the Calderon projectors. This approach frees us both from the uncomfortable Gårding inequalities and from the elaborate parametrix machinery. Most of the results are certainly well-known for analysts working in that field. Our special aim is to write down explicitly and systematically the relations between the different concepts and notations and to show how elementary the most fundamental theorems really are. We close Section 4 with the construction of a boundary value problem which is equivalent to a given linear conjugation problem.

Section 4 has an Appendix which contains a new variant of the proof of the existence of the Calderon projector. We follow the exposition of Solomyak with some substantial short-cuts which are possible due to the results from the beginning of Section 4.

The results of Section 1 are extended in [39] to the more general case where the pasting admits a shift (diffeomorphism) of the base. There one also can find a more general version of the spectral flow (for families of operators with two rays of minimal growth in the terminology of [29] and [31]). The results of Section 3 were obtained after a further analysis of the notion of the spectral flow in the context of certain spaces of projections as carried out in [38]. A slightly different version of "repasting" was used in [37] for the construction of relative K-homology groups on $spin^c$-manifolds with boundary.

1. The General Linear Conjugation Problem

Let X be a smooth manifold and Y a smooth submanifold of codimension 1 which divides X into two submanifolds $X_\pm$ with boundary Y. We consider an elliptic operator

$$A: C^\infty(X;E) \rightarrow C^\infty(X;F)$$

acting between sections of smooth vector bundles E and F over X which splits near Y, i.e. which has over a tubular neighborhood N of Y in X the form $G_A(\partial/\partial t + B)$ where G_A: $E|Y \rightarrow F|Y$ is a fixed bundle isomorphism and B: $C^\infty(Y;E|Y) \rightarrow C^\infty(Y;E|Y)$ is a self-adjoint elliptic operator of first order. (By fixing a Riemannian structure on X and a Hermitian structure on E and F we provide the means for the necessary "parallel transport" of the (co)tangent vectors and sections over N).

1.1. Definition. Let g: $E|Y \rightarrow E|Y$ be a unitary automorphism of $E|Y$ (inducing the identity in the base space Y, i.e. mapping E_y onto E_y for each $y \in Y$) such that

(1.1) $$g\, \sigma_B\, g^{-1} = \sigma_B \;,$$

where σ_B denotes the principal symbol of B. Then we have

(1.2) $$g_F\, \sigma_A = \sigma_A\, g_E$$

where $g_E := g$ and $g_F := G_A\, g\, G_A^{-1}$ the corresponding automorphism on $F|Y$.

We define the *glued vector bundles*

$$E^g := E|X_- \cup_g E|X_+ \quad\text{and}\quad F^g := F|X_- \cup_{g_F} F|X_+ \;.$$

Then the principal symbol σ_A of A gives us a new symbol $\sigma_A{}^g$: $\pi^*(E^g) \rightarrow \pi^*(F^g)$ by

$$\sigma_A{}^g(x,\xi)v := \sigma_A(x,\xi)v \;, \quad x \in X,\; \xi \in (T_xX)^*,\; v \in E^g{}_x = E_x.$$

Here π: $SX \rightarrow X$ denotes the canonical projection. Note that $\sigma_A{}^g$ has the same values as σ_A but it operates on another bundle.

Now we take any operator A^g with the principal symbol $\sigma_A{}^g$ and investigate the value of the difference

$$\mu(g,A) := \text{index } A^g - \text{index } A.$$

We call this *The General Linear Conjugation Problem.*

Note. In our situation we can define A^g directly by

$$A^g := \begin{cases} A & \text{on } X \setminus N_- \\ G_A(\partial/\partial t + B_t) & \text{on } N_- \end{cases}$$

where $N_- := N \cap X_{-1}$ and

$$B_t := r(t)\, g^{-1} B g + (1-r(t))\, B$$

with a smooth real function equal to i near $i \in \{0,1\}$.

This problem was formulated by B. Bojarski in lectures given (after 1976) in Bielefeld, Darmstadt, and Tbilisi, cf. [9] and [10]. Its name is due to the strong interconnections between this problem and the classical Riemann-Hilbert problem which however might be more subtle from the analytical point of view since it deals with conjugated pairs of "local" solutions on the halves and hence with "serious" singularities over Y (cf. [30]) whereas our general linear conjugation problem deals with "truely global" solutions of A^g though not of A.

1.2. Examples.

(a) *The Classical Riemann-Hilbert Problem.* Let X be the 2-sphere $X := S^2 \simeq \mathbb{C} \cup \{\infty\}$, $Y := S^1$, hence

$$X_+ := \{z \mid |z| \geq 1\} \quad \text{and} \quad X_- := \{z \mid |z| \leq 1\}.$$

Let $g\colon Y \to \mathbb{C} \setminus \{0\}$ be a C^∞-map. We are looking for functions φ on $X \setminus Y$ sucht that

(i) $\partial\varphi/\partial z = 0$ on $\operatorname{int}(X_\pm)$,

(ii) $\varphi(\infty) = 0$,

(iii) $\varphi_\pm(z) := \lim_{z_\pm \to z} \varphi(z_\pm)$ exists for each $z \in Y$

where $z_\pm$ denotes a sequence of points in $\operatorname{int}(X_\pm)$ approaching z and $\varphi_\pm$ belongs to $L^2(S^1)$,

(iv) $\varphi_+(z) = g(z)\, \varphi_-(z)$ for almost all $z \in Y$.

This classical problem was posed by Hilbert (in modified form already by Riemann) and subsequently solved in whole generality by F. Noether, Vekua, Bojarski et al., cf. Muschelischwili [24]. The crucial step in all approaches to this problem is the analysis near the dividing contour Y; or more specifically, the reduction of the (differential) conjugation problem in two dimensions to an (integro-differential) problem over the contour Y, which is in one dimension. In our simplest case one has to consider the spaces $H_{\pm}$ of the functions on Y which can appear as limits of functions holomorphic in the outer (inner) region. One gets, see also [28, § 1]

$$H_+ = \{\Sigma_{k\geq 0}\ a_k z^k\} \quad \text{and} \quad H_- = \{\Sigma_{k<0}\ a_k z^k\}$$

and it turns out that the solutions φ are in one-one correspondence with the limit functions $\varphi_+ \in H_+ \cap gH_-$.
Let deg g > 0, then $H_+ \oplus gH_-$ span the whole $L^2(S^1)$ and

$$\dim H_+ \cap gH_- = \deg g .$$

Bojarski's aim was to understand the relations between the different integer valued invariants and indices involved.

(b) *The "Heat Equation" on the Torus.* Let X be the torus T^2 which is parametrized as $(I\times S^1)/(\{0,1\}\times S^1)$, see Fig. 1.

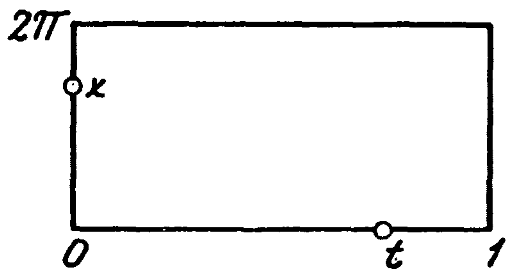


Fig. 1

Let $Y = \{(0,x) \mid x \in [0,2\pi]/\{0,2\pi\}\} \simeq S^1$ and let E be the trivial complex line bundle over T^2. Let g be the automorphism of E|Y given by $g(x) := e^{ix}$, $x \in S^1$. Then the bundle E^g is defined by

$$E^g = I\times S^1\times\mathbb{C}/\sim \quad \text{with} \quad (1,x,z) \sim (0,x,e^{-ix}z),$$

hence

$$C^\infty(T^2;E^g) \simeq \{f \in C^\infty(\mathbb{R}^2) \mid f(t+1,x) = e^{-ix} f(t,x) \text{ and } f(t,x+2\pi) = f(t,x)\}.$$

Let us analyze the situation of Example I.1.21(d) further:

$$A^g = \partial/\partial t - i\partial/\partial x + r(t) : C^\infty(T^2;E^g) \rightarrow C^\infty(T^2;E^g),$$

where $r(t)$ is again a smoothing function equal to i near $i \in \{0,1\}$ and $A := \partial/\partial t - i\partial/\partial x$. Obviously we have index $A = 0$ and, from the earlier-derived Theorem I.1.19,

$$\text{index } A^g = \text{sf}\{B_t\} = 1$$

where $B_t := -i\partial/\partial x + r(t)$.

Actually we are able to make the computation by hand for the operator $A^g = \partial/\partial t - i\partial/\partial x + t$. We can represent any section of the bundle E^g as a series $f(t,x) = \Sigma_{k\in\mathbb{Z}} f_k(t)e^{ikx}$ with $f_k(t+1)=f_{k+1}(t)$. Therefore the equation $A^g f = 0$ gives us the following equations for f_k

$$f_k'(t) + (k+t)f_k(t) = 0 , \quad k \in \mathbb{Z}$$

As a result we get $f_k(t) = C_k \exp(-1/2\ (k+t)^2)$. Moreover C_k doesn't depend on k, because of the equality $f_k(t+1) = f_{k+1}(t)$. Thus the solution of $A^g f = 0$ has the form

$$\Sigma_k\ C \exp(ikx) \exp(-1/2(k+t)^2)$$
$$= C \exp(-1/2\ t^2)\ \Sigma_k \exp(ikz - k^2/2)$$

where $z = x+it$. Thus the kernel of A^g is one-dimensional. The same calculation for the operator $-\partial/\partial t - i\partial/\partial x + t$ shows that there is no solution for $(A^g)^* f = 0$.

Now the spaces $H_\pm \subset L^2(S^1)$ of eigenfunctions of the operator $B = i\ \partial/\partial x$ with non-negative (negative) eigenvalues are of course equal to

$$H_\pm := \text{span } \{e^{ikx}\}_{k \underset{<}{\geq} 0}$$

and it is obvious that

$$\text{sf}\{B_t\} = \text{index } P_+(B) - gP_-(B) = 1$$

if $B_t := (1-t)B + tg^{-1}Bg$ and $P_\pm(B)$ are the spectral projections.

(c) *The Signature Operator over S^{2m} with Coefficients in an Auxiliary Bundle.* Recall the definition of the generalized signature operator D_V of a Hermitian bundle V over a closed oriented Riemannian manifold X of dimension $2m$, cf. [13, III.4.D] or [25, IV.9]:

$$D_V := (d_V + d_V^*)|\Omega_V^+ : \Omega_V^+ \rightarrow \Omega_V^- ,$$

where $\Omega_V^\pm$ denote the ± 1-eigenspaces of the involution

$$\tau := i^{p(p-1)+m}* : \Omega_V^p \rightarrow \Omega_V^p , \quad p \geq 0,$$

$$\Omega_V{}^p := C^\infty(X;\Lambda^p(TX)\otimes V),$$

$$d_V(u\otimes v) := du\otimes v + (-1)^p u\wedge\nabla_V v\ , \qquad u \in \Omega^p,\ v \in C^\infty(X;V)$$

and ∇_V the connection of V. If V is the trivial line bundle we are back in the situation of the standard signature operator $D_X\colon \Omega_X{}^+ \to \Omega_X{}^-$ and we obtain for the principal symbols

$$\sigma_{D_V} = \sigma_{D_X} \otimes Id_V\ .$$

Let Y be a closed submanifold dividing X into X_+ and X_- . We suppose that the bundle V is obtained by clutching the trivial bundles over the two components by a map g: Y $\to$ U(N), i.e.

$$V \simeq X_+\times\mathbb{C}^N \cup_g X_-\times\mathbb{C}^N \quad\text{and so}\quad D_V = A^g$$

where

$$A := D_{X\times\mathbb{C}^N} \simeq D_X \otimes Id_{\mathbb{C}^N} \simeq ND_X\ .$$

If we choose a Riemannian metric on X which is the product metric on $Y\times I \simeq N$, then the signature operator splits near Y, cf. [7, p. 63],

$$D_X = \varsigma\,(\partial/\partial t + B_0)$$

where B_0 is the boundary signature operator of Example I.2.5 (b). Hence we have the following integers to look at:

(i)	index A	=	$N\times$index D_X	=	N sign(X),
(ii)	index A^g	=	index D_V ,		
(iii)	sf$\{B_t\}$	=	index $P_+ - gP_-$		

where $\{B_t\}_{t\in I}$ is a family of self-adjoint elliptic operators connecting B_0 and $g^{-1}B_0g$ and $P_\pm$ are the spectral projections of B_0 .

From the definition it follows

$$\begin{aligned}\mu(g,A) &= \text{index } A^g - \text{index } A\\ &= \text{index}\{\partial/\partial t + B_t\} \quad (\text{operator on } Y\times S^1)\\ &= \text{sf}\{B_t\} \quad \text{by Theorem I.1.19}\\ &= \text{index } P_+ - gP_- \quad \text{by Theorem I.4.1}\\ &= \textstyle\int_{SY} \text{ch}[E_-;g]\tau(Y) \quad \text{by Corollary I.4.4}\end{aligned}$$

where E_- is the characteristic bundle of B_0, i.e. the range bundle of the principal symbol of P_- .

A simple exercise in K-theory shows $[E_-,g] = [\sigma'_{B_o}][Y\times\mathbb{C}^N,g]$

where the multiplication operates in

$$K^{-1}(TY) \otimes K^{-1}(Y) \rightarrow K(TY) \simeq K^{-1}(SY) ,$$

hence

$$\mu(g,A) = \int_{SY} ch[\sigma'_{B_o}]\, ch[Y\times\mathbb{C}^N,g]\, \tau(Y) .$$

Now let X be the 2m-sphere with the (2m-1)-sphere Y dividing X into two discs $X_\pm$. Then we have to evaluate the cup product $ch[\sigma'_{B_o}]\, ch[Y\times\mathbb{C}^N,g]\, \tau(Y)$ on the fundamental cycle $[TS^{2m-1}]$. Since $TS^{2m-1} \simeq S^{2m-1}\times\mathbb{R}^{2m-1}$ we get $K^{-1}(TS^{2m-1}) \simeq \mathbb{Z}$. Let α denote a generator of this group. It is shown in [25, XV.7] that

$$[\sigma'_{B_o}] = 2^{m-1}\alpha .$$

Since the Todd class is equal to the unity and $[Y\times\mathbb{C}^N,g] = k$ in $H^{2m-1}(S^{2m-1};\mathbb{R})$ where k is the winding number of g, we get in this example

$$\mu(g,A) = k\, 2^{m-1} .$$

After these examples one expects that the difference

$$\mu(g,A) = \text{index } A^g - \text{index } A$$

depends only on the principal symbol of B and on the automorphism g, i.e. only on objects living on Y. This is the case, which is also clear from topological reasoning.

<u>1.3. Theorem.</u> Let A be a first order elliptic operator acting between sections of Hermitian vector bundles E, F over a closed Riemannian manifold X which splits into

$$A = G_A(\partial/\partial t + B)$$

near a dividing submanifold Y, where B is a self-adjoint operator over Y and G_A a bundle isomorphism, and let g be a unitary automorphism of $E|Y$ compatible with B, i.e. satisfying the condition

$$g\, \sigma_L(B)\, g^{-1} = \sigma_L(B)$$

where $\sigma_L(B)$ is the principal symbol of B. Then we have

$$\mu(g,A) = \mathrm{sf}\ \{B_t\}_{t\in I}$$

where $\{B_t\}$ is a smooth family of elliptic self-adjoint operators over Y connecting $B_o := B$ and $B_1 := g^{-1}Bg$.

<u>Proof.</u> Recall the "Local Index Theorem", cf. [29] and [6],

$$\mathrm{index}\ A = \int_X \alpha(A)(x)\ dx$$

where $\alpha(A)$ is constructed from the full symbol of A. By the explicit definition of A^g given above in the Note after Definition 1.1 it is clear that

$$\alpha(A)(x) = \alpha(A^g)(x) \quad \text{for } x \in X \setminus N_- ,$$

hence

$$\mu(g,A) = \mathrm{index}\ A^g - \mathrm{index}\ A$$
$$= \int_{N_-} \alpha(A^g)(x)\ dx - \int_{N_-} \alpha(A)(x)\ dx .$$

The first integral gives us the index of the operator

$$G_A(\partial/\partial t - B_t): C^\infty(S^1 \times Y; E^{g'}) \to C^\infty(S^1 \times Y; F^{g'})$$

where

$E^{g'} = I \times E|Y / \sim$ with $(1,y,e)\sim(0,y,g^{-1}(y)e)$, $y\in Y$, $e\in E_y$.
The reason for that is that the full symbol of this operator is equal to the full symbol of A^g in each point of N_- (parametrized as $I\times Y$). So we have by Theorem I.1.19

$$\int_{N_-} \alpha(A^g)(x)\ dx = \mathrm{index}\ (\partial/\partial t - B_t) = \mathrm{sf}\ \{B_t\} .$$

The second integral does not contribute to $\mu(g,A)$ since it is equal to index $(\partial/\partial t - B) = \mathrm{sf}\ \{B'_t = B\} = 0$. ■

<u>1.4. Corollary.</u> Let A, B, and g be as in Theorem 1.3 and let $P_\pm(B)$ be the spectral projections of B. Then

$$\mu(g,A) = \mathrm{index}\ P_+(B) - gP_-(B) .$$

<u>Proof.</u> By Theorem I.4.1.

We could also express our result in the language of the Calderon projectors $P_\pm(A)$, cf. the Appendix below. Since the differences $P_\pm(B) - P_\pm(A)$ are compact, we obtain

1.5. Corollary. Let A and g be as in Theorem 1.3 and let $P_{\pm}(A)$ be the Calderon projectors belonging to A. Then

$$\mu(g,A) = \text{index } P_{+}(A) - gP_{-}(A).$$

Now we present several applications of our final formula given in Corollary 1.4. Our main purpose is to show how the integer $\mu(g,A)$ depends on the topology of Y, g, and B.

1.6. Proposition. Let A, Y, and g be as in Theorem 1.3. If g determines a torsion element in $K^{-1}(Y)$, then $\mu(g,A) = 0$.

Proof. Let g determine a torsion element in $K^{-1}(Y)$. Then there exists a natural number k such that

$$kg := \begin{pmatrix} g & 0 & \cdots & \cdots & 0 \\ 0 & g & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdots & \cdots & \cdots & g \end{pmatrix} : kE|Y \rightarrow kE|Y$$

is homotopically equivalent to the identity. More precisely, we have a path in the space of unitary automorphisms of $kE|Y$ joining kg with the identity, cf. Karoubi [22, p. 72]. Thus we are able to deform the family $\{kB_t\}$ into a family $\{C_t\}$ joining kB with itself without changing the spectral flow, hence

$$k\,sf\{B_t\} = sf\{kB_t\} = sf\{C_t\} = 0$$

since $\{C_t\}$ is contractible to the constant family. ∎

1.7. Proposition. Let A, Y, and g be as in Theorem 1.3. If A admits local elliptic boundary conditions on Y, then

$$\mu(g,A) = 0 .$$

Proof. Since A splits near Y it admits local elliptic boundary conditions on Y (in the sense of Shapiro Lopatinski, see also Definition 4.8 below), if and only if p_{+} , the principal symbol of the spectral projection $P_+(B)$, can be deformed into a projection onto a bundle over SY which is a pull back of a bundle over Y. Since $P_-(B) = Id - P_+(B)$ the symbol of $P_+(B)-gP_-(B)$ then becomes a matrix function of $y \in Y$ alone, and hence gives the trivial element of K(TY). ∎

Note. Let $\{p_t(y,\xi)\}_{t\in I}$, $y \in Y$, $\xi \in T^*Y_y$ be a continuous family of projection symbols such that

$$p_0(y,\xi) = p_+(y,\xi) \quad \text{and} \quad p_1(y,\xi) = p_1(y) .$$

Then we can not expect that g leads to an automorphism of the bundle $V := \text{Image } p_1$ since in general $p_1 g \neq g p_1$. Thus we must change g continuously by a continuous family $\{g_t\}$ of automorphisms of $V_t := \text{Image } p_t$ leading to a g_1 which commutes with p_1.

1.8. Proposition. Let A, Y, B, and g be as in Theorem 1.3 and let the principal symbol of B determine a torsion element in $K^{-1}(TY)$. Then we have $\mu(g,A) = 0$.

Proof. The assumption about B means that for some k, N the symbol

$$k\sigma_L(B) \otimes Id : \pi^*(k\ E|Y)\otimes\mathbb{C}^N \to \pi^*(k\ E|Y)\otimes\mathbb{C}^N$$

can be deformed into a symbol $v = q_+ - q_-$ where $q_\pm$ are projections onto some bundles lifted from Y. Thus we get the equalities

$$\begin{aligned} k\ sf\{B_t\} &= sf\ \{kB_t\otimes Id\} \\ &= \text{index } kP_+(B)\otimes Id - gkP_-(B)\otimes Id \\ &= t\text{-index } [q_-, g] = 0 \end{aligned}$$

where t-index : $K^{-1}(SY) \to \mathbb{Z}$ denotes the homomorphism given by

$$[\sigma] \mapsto \int_{SY} ch[\sigma]\ \pi_{SY}{}^*\tau(Y) . \quad ∎$$

Remarks.

So far we presented all results in detail only in the case when A splits near Y into the form

(*) $\quad A = G_A(\partial/\partial t + B)$, B self-adjoint

and g is a unitary automorphism of E|Y. Now we will indicate how one can weaken these assumptions.

(1) *Non self-adjoint* B. It is easy to see that in our calculations we can take for B any elliptic operator of which the principal symbol $\sigma_L(B)$ has no eigenvalues on the imaginary axis and is compatible with g, i.e.

$$g\ \sigma_L(B)\ g^{-1} = \sigma_L(B)\ .$$

Without changing the principal symbol class in $K^{-1}(TX)$ (hence the index neither) we can deform such a more general operator A into our form: Take the family

$$\{B_t\} := 1/2\{B+B^* + (1-r(t))(B-B^*)\}.$$

Thus we can deform A to an operator which on $N = I\times Y = N_-\cup N_+$ takes the form $A = G_A(\partial/\partial t + C_t)$ where

$$C_t := \begin{cases} B_{2t} & \text{for } 0 \le t \le 1/2 \\ B_{1-(2t-1)} & \text{for } 1/2 < t \le 1 . \end{cases}$$

Hence in a smaller collar neighbourhood A has the form (*).

In fact, we do not need this deformation to the self-adjoint case since one could define a spectral flow for all families of operators of which the principal symbol has no eigenvalues on the imaginary axis. We just count the number of eigenvalues whose real parts change the sign when t is going from 0 to 1.

(2) *Pasting with shift.* The preceding observation is essential to the more general situation where one considers, instead of g, a diffeomorphism Φ: $E|Y \to E|Y$ of the total spaces which is linear on the fibres though not inducing the identity in the base but an arbitrary diffeomorphism φ. In this case one can not reduce the problem to a self-adjoint family and the spectral calculus gets more advanced. Instead of $K^{-1}(TY)$ one must work with some suitable K-groups over the mapping torus $(Y\times I)^{\varphi}$. The details of this approach are worked out in Wojciechowski [39].

(3) *Non-splitting symbols.* Using the rather extensive machinery of parametrix calculus of the Vishik-Boutet-de-Monvel type one can get parts of the results of this section for operators which do not necessarily split near Y, cf. [14], [15].

(4) *Proof of the Index Theorem by Induction.* After Theorem 1.3 and Corollary 1.5 it is completely clear that the introduced machinery provides an alternative proof of the Atiyah-Singer Index Theorem. Using induction, we can decompose an arbitrary elliptic operator into simpler pieces in such a way that the decomposition coincides with a suitable handle decomposition of the underlying manifold; the changes of the indices can be followed arithmetically. We hope to complete all necessary details soon.

(5) *Relation with Boundary Value Problems.* We reformulate Theorem 1.3 and Corollary 1.4 in terms of elliptic boundary value problems at the end of Section 4. This provides a link to the work of [3] and [14], [15] and explains the architecture of our problem.

2. Spectral Flow and Spectral Properties of Elliptic Operators

Another set of corollaries of [17] and Section 1 is related to the spectral theory of elliptic operators. In what follows we will consider a self-adjoint elliptic operator $B: C^\infty(X;V) \to C^\infty(X;V)$ of order $m > 0$, acting on sections of a Hermitian vector bundle V over a closed Riemannian manifold X of dimension d.

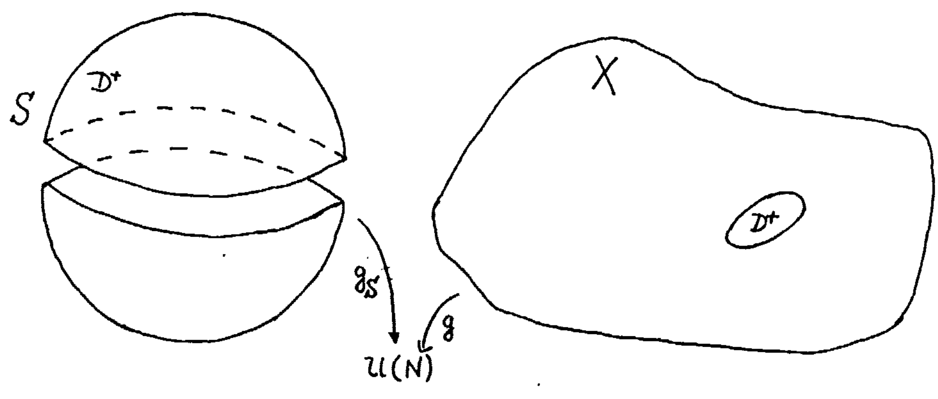
"Classical" spectral theory, as founded by H. Weyl (1911) and developed further by R. Courant (1920), B. M. Lewitan (1952, 1955), V. G. Avakumovic (1956), L. Hörmander (1968) and many other authors (see Shubin [31] for a recent survey), is concerned with the eigenvalues of the Laplace operator, of scalar operators or other half-bounded and therefore topologically trivial operators only. We, on the contrary, assume that $[\sigma_B] \in K^{-1}(TX)$ is a non-torsion element. As explained in Section 2 of [17], this implies that the spectrum of B consists of infinitely many positive and negative eigenvalues.

It is well known, that it is rather difficult to derive the asymptotic inequalities in the classical spectral analysis. It is therefore striking that one can obtain more precise results, namely exact inequalities, in the spectral analysis of operators with non-trivial symbol. This was first noticed by Vafa and Witten [35] in an attempt to find a mathematical explanation for the interrelation between the states and the masses of (Euclidean) bosons and fermions corresponding to the positive and negative eigenvalues of the Dirac operator of (Euclidean 3- and 4-dimensional) Quantum Field Theory. We want to show that the Vafa-Witten inequalities are properties of all self-adjoint elliptic differential operators of first order with a non-torsion stable symbol class, i.e. for all X and B there exists a constant $C' = C'(X,B)$ such that $|\lambda_r| \leq C' r^{1/d}$ for all r where the eigenvalues are indexed by increasing absolute value. In

fact, to a certain extent we can follow Vafa and Witten's presentation; see also Atiyah [4]. However, we have to be more careful in our arguments because we do not assume the existence of a spin structure on the manifold X.
We restrict ourselves to the case of an odd-dimensional, oriented X. The cases of an unoriented X and of an even-dimensional X need special treatment along the usual lines of index theory.

<u>2.1. Theorem.</u> Let X be an oriented, odd-dimensional, closed, Riemannian manifold and B an operator satisfying the conditions mentioned above: self-adjoint, elliptic, positive order, and topologically non-trivial. We assume that $[\sigma_B]$ is not a torsion element in $K^{-1}(TX)$, i.e. for any $k \in \mathbb{Z}$ we have $k[\sigma_B] \neq 0$. Then there exists a constant $C = C(X,B)$ depending only on X (and of course on the choice of the Riemannian structure) and B such that in any interval of the length C there is an eigenvalue of B.

<u>Proof.</u> Since X is odd-dimensional we can find a natural number N and a continuous map $g\colon X \to U(N)$ such that $\langle ch[X\times\mathbb{C}^N,g],[X]\rangle = 1$ where $[X\times\mathbb{C}^N,g] \in K^{-1}(X)$, $ch\colon K^{-1}(X) \to H^{odd}(X;\mathbb{Q})$ were defined in I.1.7 and $[X] \in H_d(X)$ is the fundamental class of the orientation of X. An explicit construction of such a mapping g could be derived from Example 1.2 (c) by proceeding as follows: First recall that on any odd-dimensional sphere S there exists $g_S\colon S \to U(N)$ with $\langle ch[S\times\mathbb{C}^N,g_S],[S]\rangle = 1$. We can assume that $g_S = Id$ outside of some disc D^+ in S. Then we proceed as in Fig. 2:



<u>Fig. 2</u>

We choose a disc D^+ in X with appropriate g_S: $D^+ \to U(N)$ and take as g the extension of g_S over the whole of X by the identity. Now we consider the operator $B_o := NB = B\otimes Id_N$. Its principal symbol commutes with the automorphism $h := Id(V)\otimes g$ of the bundle $V\otimes(X\times\mathbb{C}^N)$, hence $B_1 := h^{-1}B_o h$ has the same spectrum as B_o and we obtain a spectral flow for any family $\{B_t\}$ of self-adjoint elliptic operators which joins B_o with B_1. As explained in Section I.1 and Example 1.2 (c) above, the spectral flow is independant of the choice of the connecting family (in fact, we could take just the linear family) and can be expressed topologically by

$$sf\{B_t\} = \pm ch[b]\, ch[X\times\mathbb{C}^N, g]\, \tau(X)\, [TX],$$

where $\tau(X)$ is the Todd class of X, [TX] the fundamental class of the canonical (symplectic) orientation of TX, and the "boundary" Chern character ch: $K^{-1}(X) \to H^{odd}(X;\mathbb{Q})$ as defined in I.1.7. From the construction of the family it is clear that $sf\{B_t\} \neq 0$.

On the other hand we see that B_1-B_o is a pseudodifferential operator of order zero, hence it is bounded in $L^2(X;V\otimes\mathbb{C}^N)$. Let $t \mapsto j_r(t)$ be a parametrization of the eigenvalues of the family $\{B_t\}$. Then it is well known (see Lemma I.1.3), that

$$|j_r(0) - j_r(1)| \leq \|B_1-B_o\| =: C(X,B) \text{ for all } r.$$

Moreover we have $j_r(1) = j_{r+\ell}(0)$ where $\ell = sf\{B_t\}$, hence $|j_r(0) - j_{r+\ell}(0)| \leq C(X,B)$ for all r. Since $\ell \neq 0$, this proves that in any interval of length C there is an eigenvalue of B_o. However, the eigenvalues of $B_o = NB$ are the eigenvalues of B, just with N-times multiplicity. Therefore in any interval of the length C we have an eigenvalue of B. ■

Remark. This is the most general and most simple information about the spectrum of elliptic operators of our type (i.e. those with a non-vanishing stable symbol class) on odd-dimensional manifolds. One hardly gets more information without further assumptions, mostly because B_1-B_o is in general a self-adjoint pseudodifferential operator of order zero where infinitely many lower order terms in the symbol may be decisive for the spectral analysis. If we deal with differential operators, we can be more precise.

Before going further along these lines, we want to give the following (crude) estimate:

<u>2.2. Corollary.</u> Under the assumptions of the preceding Theorem, there exists a constant C such that

$$|\lambda_n| \leq Cn \quad \text{for all } n \in \mathbb{N}$$

where the eigenvalues of B are indexed by increasing absolute value: $|\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_n| \leq \ldots$

<u>Proof.</u> The estimate follows immediately from the preceding proof of Theorem 2.1 where it was shown that we have ℓ = $\mathrm{sf}\{B_t\}$ eigenvalues of B_0 and ℓ/N eigenvalues of B in any interval of length C. ∎

Now we prove the more precise inequality in the differential case.

<u>2.3. Theorem.</u> Let $B: C^\infty(X;V) \to C^\infty(X;V)$ be a self-adjoint elliptic differential operator of first order over a closed oriented Riemannian manifold X acting on sections of a Hermitian vector bundle V. If $d = \dim X$ is odd and if the stable symbol class of B is non-vanishing, then there exists a constant C such that

$$|\lambda_n| < C\, n^{1/d}$$

for all eigenvalues of B indexed by increasing absolute value.

<u>Proof.</u> We need to repeat and sharpen the arguments of the proof of Theorem 2.1. The plan of the proof consists of the following steps:

(i) We show that there exists a family $g_r: X \to U(N)$, $r \in \mathbb{Z}$ such that

$$\mathrm{ch}[X\times\mathbb{C}^N, g_r] = r^d\ \mathrm{ch}[X\times\mathbb{C}^N, g_1],$$

where $g_1 = g$ has "degree" 1 as in the proof above.

(ii) If $\{B(r)_t\}$ denotes the family $\{th(r)^{-1}B_0h(r) + (1-t)B_0\}$

where $h(r) = Id(V)\otimes g_r$ is an automorphisms of the bundle $V\otimes\times(X\times\mathbb{C}^N)$, then we will obtain

$$sf\{B(1)_t\} =: \ell \neq 0 \qquad \text{and}$$

$$sf\{B(r)_t\} = r^d \; sf\{B(1)_t\} \text{ for all } r \in \mathbb{Z}.$$

Here B_o again denotes the operator $NB = B\otimes Id_N$.

(iii) We show the inequalities

$$\|B(r)_1 - B(r)_o\| \leq r \; \|B(1)_1 - B(1)_o\| = rC(X,B) = rC.$$

In fact, these three properties are all we have to show. As a result we obtain that in any interval of length rC there are at least $r^d\ell$ eigenvalues of B_o. This means that in this interval we have $r^d\ell/N$ eigenvalues of B. Therefore the absolute value of the $r^d\ell/N$-th eigenvalue is less than rC, i.e.

$$|\lambda_{r^d\ell/N}| < rC .$$

This gives us the desired inequality for $n = r^d$ and hence for all n.

Now we prove the assertions (i)-(iii).

We start by looking at the extension NB of the operator B to sections of $V\otimes(X\times\mathbb{C}^N)$. From the construction of such an extension (see Example 1.2 (c) and [25, Chapter IV]) we get the following expression for NB in local coordinates $(x_1,\ldots,x_d)$

$$NB(v\otimes s) = (B\otimes_d Id_{\mathbb{C}^N})(v\otimes\Sigma_i f_i e_i)$$

$$= (Bv)\otimes s + \Sigma_{i,k} (b_k v)\otimes\left(\frac{\partial f_i}{\partial x_k}\right)e_i$$

$$= (Bv)\otimes s + \Sigma_k (b_k v)\otimes(\partial s/\partial x_k),$$

where $\{e_i\}$ is a local basis for $X\times\mathbb{C}^N$ such that $de_i = 0$ for all i (here the differential d is considered as a flat connection on $X\times\mathbb{C}^N$), $s = \Sigma_i f_i e_i$ locally where the f_i are smooth functions and $B = \Sigma_i b_i(x_1,\ldots x_d) \; \partial/\partial x_i + b_o(x_1,\ldots,x_d)$. Let us remark here that different choices of connections $d_1 = d + A$ for $X\times\mathbb{C}^N$ lead to different operators. Now, if we take any map $g: X \to U(N)$ then we get in the chosen coordinates for the operator $B_1 = (Id_V\otimes g)^{-1}(B\otimes Id_{\mathbb{C}^N})(Id_V\otimes g)$:

$$B_1(v\otimes s) = \{(Id_V\otimes g)^{-1}(B\otimes Id_{\mathbb{C}^N})(Id_V\otimes g)\}(v\otimes s)$$

$$= Bv \otimes s + \Sigma_k b_k v \otimes \partial s/\partial x_k + \Sigma_k b_k v \otimes (g^{-1} \partial g/\partial x_k \; s)$$

$= (B \otimes_d Id)(v\otimes s) \quad + \quad \{\Sigma_k \ (b_k v) \otimes (g^{-1} \ \partial g/\partial x_k \ s)\},$

$= B_o(v\otimes s) + \{...\}$

(i.e. $B_o = B \otimes_d Id_{\mathbb{C}^N}$ and $B_1 = B \otimes_{d+g^{-1}dh} Id_{\mathbb{C}^N}$).
Therefore the norm of the 0-th order operator $B_1 - B_o$ is bounded by a constant which depends only on the supremum of the norm of all matrices b_k , i.e. on the supremum of the norm of the principal symbol of B considered as a bundle isomorphism, and on the supremum of the matrices $g^{-1}\partial g/\partial x_k$.

The only thing left is choosing g. We are going to show that there exists a family of mappings g_r: $X \to U(N)$ such that $ch[X\times\mathbb{C}^N, g_r] = r^d$ and such that

$$\|g_r^{-1} \ \partial g_r/\partial x_k\| \ = \ k \ \|g_1^{-1} \ \partial g_1/\partial x_k\| \ .$$

First we construct such a family on the d-dimensional torus T^d with the standard (flat) metric. Let $(\varphi_1,..,\varphi_d)$ be the standard angular coordinates on T^d, $0 \le \varphi_i \le 2\pi$ and $\varphi_i + 2\pi = \varphi_i$. Let γ_1: $T^d \to SU(N)$ be a map of degree one chosen so that $\gamma_1(\varphi_1,..,\varphi_d) = Id$ if any of the coordinates φ_i is equal to 0 or to 2π; in particular

$$\gamma_1(\varphi_1+2k\pi,..,\varphi_d+2k\pi) \ = \ \gamma_1(\varphi_1,..,\varphi_d)$$

Now let α_r: $T^d \to T^d$ be given by the formula

$$\alpha_r(\varphi_1,..,\varphi_d) \ = \ (r\varphi_1,..,r\varphi_d)$$

Thus α_r is a map of degree r^d. Now we define

$$\gamma_r \ := \ \gamma_1 \circ \alpha_r$$

so

$$ch[T^d\times\mathbb{C}^N, \gamma_r][T^d] \ = \ r^d \ .$$

Moreover, the norm of the map $\gamma_r^{-1} \ \partial\gamma_r/\partial\varphi_i$ is precisely r times the norm of $\gamma_1^{-1} \ \partial\gamma_1/\partial\varphi_i$ since the derivatives of g_r are exactly r times the derivatives of γ_1 in absolute value. Thus the assertion is true for T^d.
We assume now that X is an arbitrary d-dimensional manifold. In such a manifold we embed the cube $I^d = I \times ... \times I$ and we define

$$g_r(x) := \begin{cases} \gamma_r & x \in I^d \\ & \text{for} \\ Id & x \in X \setminus I^d \ . \end{cases}$$

g_r is the desired map. ∎

We refer the reader to the papers [35] and [4] for the treatment of the special case of the Dirac operator with coefficients in an auxiliary vector bundle on a spin-manifold. In that case the related family of operators has a spectral flow just equal to 1. There one also finds a generalisation to the even-dimensional case.

3. Fredholm Pairs of Subspaces

In this Section we introduce one more concept which is useful in our consideration:

3.1. Definition. Let us assume that H is a complex separable Hilbert space. We define the space of *Fredholm pairs of subspaces* of H through

$$\mathrm{Fred}^2(H) := \{(H_1,H_2) \mid H_1, H_2 \text{ are closed infinite-dimensional subspaces of } H \text{ with } H_1\cap H_2 \text{ and } H/H_1+H_2 \text{ of finite dimension}\}$$

and we define

$$\mathrm{index}\ (H_1,H_2) = \dim H_1\cap H_2 - \dim H/H_1+H_2 .$$

We notice that the orthogonal complement $(H_1,H_2)^\perp = (H_1{}^\perp,H_2{}^\perp)$ of a Fredholm pair (H_1,H_2) is again a Fredholm pair with

$$\mathrm{index}\ (H_1,H_2)^\perp = -\ \mathrm{index}\ (H_1,H_2),$$

since $H_1{}^\perp\cap H_2{}^\perp = (H_1\oplus H_2)^\perp$.

This notion was introduced by Kato [23, IV.4.1] in an attempt to extend the stability properties of Fredholm operators from the case of bounded operators to the case of closed unbounded ones. See also the recent book by Cordes [19].

There are many examples of Fredholm pairs. In one sense, the rest of this paper is devoted to illuminating the fundamental role of Fredholm pairs in our direct approach to elliptic transmission and boundary value problems. Let us start with some simple examples:

3.2. Examples. (a) We have a natural mapping

$$\alpha: \mathrm{Fred}(H) \rightarrow \mathrm{Fred}^2(H\times H)$$

from the space of *Fredholm operators* on H into the space of Fredholm pairs of subspaces of $H\times H$ given by

$$F \mapsto (H\times 0,\ \mathrm{graph}\ F).$$

We get

$$(H\times 0) \cap \text{graph } F = \ker F \times \{0\} \simeq \ker F$$

and

$$H\times H = (H \times \text{image } F) \oplus (\{0\} \times \text{coker } F)$$
$$= ((H \times \{0\}) + \text{graph } F) \oplus \text{coker } F,$$

hence

$$\text{index } \alpha F = \text{index } F .$$

For the adjoint operator F^* we get

$$\text{index } \alpha F^* = \text{index } (H \times \{0\}, \text{graph } F^*)$$
$$= \text{index } (\{0\} \times H, \{(-F^*v,v) \mid v \in H\})$$
$$= -\text{ index } \alpha F$$

since $(\{0\} \times H, \{(-F^*v,v) \mid v \in H\}) = (H \times \{0\}, \text{graph } F)^{\perp}$.

(b) Another example is provided by the trivial Fredholm pair *of complementary eigenspaces (image* $P_+(B)$,*image* $P_-(B)$) *where* $P_{\pm}(B)$ are the *spectral projections* of an elliptic self-adjoint operator acting on sections of an Hermitian vector bundle E on a closed Riemannian manifold Y as in Part I, Section 2. Let $g \in GL_S$ with $S = P_+ - P_-$, i.e. g is an automorphism of L^2E which commutes with S modulo compact operators, cf. Part I, Section 3. Then (image P_+, g image P_-) is a Fredholm pair of subspaces of L^2E as well and we have

$$\text{index } P_+ - gP_- = \text{index } (\text{image } P_+, g \text{ image } P_-) .$$

To see this we recall from Lemma I.3.4

$$\ker P_+-gP_- \simeq \{u \in H_- \mid gu \in H_+\} = H_+ \cap gH_-$$

and

$$\text{coker } P_+-gP_- \simeq \{v \in H_+ \mid gv \in H_-\} \simeq \{u \in H_- \mid g^{-1}u \in H_+\}$$
$$\simeq \{u \in H_- \mid u \perp gH_-\} \simeq (H_+ \oplus gH_-)^{\perp}$$

where $H_{\pm} := \text{image } P_{\pm}$.

We will now describe a more important example:

3.3. Definition. Let A be an elliptic operator of first order over a closed manifold X divided by Y into two parts X_+ and X_- as assumed in Section 1. Let A take the form $G(y)(\partial/\partial t + B_t)$ near Y as assumed in the Introduction. We define the space of *Cauchy data* by

$$H_{\pm}(A) := \{u_{\pm}|Y \mid u_{\pm} \in C^{\infty}(X_{\pm};E|X_{\pm}) \text{ and } Au_{\pm} = 0 \text{ on } X_{\pm}\}.$$

Note. There exist naturally defined projections $P_{\pm}(A)$ of $C^{\infty}(Y;E|Y)$ onto the spaces $H_{\pm}(A)$ which are pseudodifferential operators of 0-th order. We describe these projections more carefully in the Appendix.

Bojarski [9] noticed the following

3.4. Lemma. Let X, Y, E and A be as assumed in Definition 3.3. Then the spaces of Cauchy data $H_{\pm}(A)$ are Fredholm pairs of subspaces of $L^2(Y;E|Y)$.

Proof. We consider the elliptic operator of 0-th order $P_+(A)-P_-(A)$. Since $\sigma_L(P_+(A)-P_-(A))^2 = Id$ we get

$$\text{index } P_+(A) - P_-(A) = 0 .$$

Actually it suffices to know that the operator $P_+(A)-P_-(A)$ is elliptic, hence $\dim \ker P_+(A)-P_-(A) < \infty$. Then we obtain

$$\dim H_+(A)+H_-(A) < \infty$$

and

$$\dim L^2(Y;E|Y) \;/\; (H_+(A)\cap H_-(A)) < \infty$$

since

$$\begin{aligned}\ker P_+(A)-P_-(A) \;=\; &\{f \in C^{\infty}(Y;E|Y) \mid P_+f = P_-f\}\\ &\cup \{f \in C^{\infty}(Y;E|Y) \mid f \text{ is orthogonal to}\\ &\quad \text{the space } H_+(A)+H_-(A) \text{ in } L^2(Y;E|Y)\}.\blacksquare\end{aligned}$$

Now we want to investigate the topological structure of the space $Fred^2(H)$. (By the way, we could have used a deformation argument to prove the preceding Lemma by reducing it to the situation described in Example 3.2 (b)).

$Fred^2(H)$ has a natural topology:

3.5. Definition. Let (H_1,H_2), (H_1',H_2') be two elements of $Fred^2(H)$ and let P_i, P_i' denote the orthogonal projections onto H_i and H_i' respectively. Then we can introduce a metric by the formula

$$\varrho((H_1,H_2),(H_1',H_2')) := \|P_1-P_1'\| + \|P_2-P_2'\|.$$

3.6. Lemma. A pair (H_1,H_2) of closed infinite-dimensional subspaces of H belongs to $Fred^2(H)$ if and only if the difference P_1-P_2 of the orthogonal projections is a Fredholm operator. In that case the operator

$$(Id-P_1)P_2 : H_2 \rightarrow H_1^{\perp}$$

is a Fredholm operator with

$$\text{index } (Id-P_1)P_2 = \text{index } (H_1,H_2).$$

Proof. The operator P_1-P_2 is self-adjoint, so

$$\begin{aligned} \text{coker } P_1-P_2 &\simeq \ker P_1-P_2 = \{f \mid (P_1-P_2)f = 0\} \\ &= \{f \in H_2 \mid P_1 f = f\} \oplus \{f \mid P_1 f = P_2 f = 0\} \\ &= (H_1 \cap H_2) \oplus (H_1 \oplus H_2)^{\perp}. \end{aligned}$$

Now let us consider the operator $(Id-P_1)P_2 : H_2 \rightarrow H_1^{\perp}$. We have

$$\ker (Id-P_1)P_2 = \{f \in H_2 \mid (Id-P_1)f = 0\} = H_1 \cap H_2$$

and

$$\begin{aligned} \text{coker } (Id-P_1)P_2 &= \{f \in H_1^{\perp} \mid P_2 f = 0\} = H_1^{\perp} \cap H_2^{\perp} \\ &= (H_1 \oplus H_2)^{\perp}. \quad \blacksquare \end{aligned}$$

3.7. Corollary. A pair (H_1,H_2) of closed infinite-dimensional subspaces is Fredholm if and only if $P_1-(Id-P_2)$ is a compact operator.

Proof. From Lemma 3.6 we know that $(H_1,H_2) \in Fred^2(H)$ if and only if $(Id-P_2)P_1$ is a Fredholm operator. We know also that there exists a unitary operator g such that $Id-P_2 = gP_1g^{-1}$ and in particular $H_2 = gH_1$ and

$$(Id-P_2)P_1 = gP_1g^{-1}P_1 : H_1 \rightarrow gH_1 .$$

Hence (H_1,H_2) is a Fredholm pair if and only if the operator $P_1g^{-1}P_1 : H_1 \rightarrow H_1$ is Fredholm. This is the case if and only if g is an element of the group GL_S for $S = P_1$. Hence

$$(Id-P_2) - P_1 = gP_1g^{-1} - P_1 = (gP_1-P_1g)g^{-1}$$

is a compact operator. $\blacksquare$

Now, let P be an orthogonal projection with infinite-dimensional range and kernel. In $GL_P = GL_{P-(Id-P)}$ we consider the subgroup U_P *of all unitary elements of* GL_P . *Any Fredholm* pair (H_1,H_2) with $H_1 = PH$ is of the form $(PH,(Id-gPg^{-1})H)$

for some $g \in U_P$. And we obtain the trivial Fredholm pair of complementary subspaces if and only if $gPg^{-1} = P$ or equivalently $g \in GL(PH) \oplus GL(PH^{\perp})$. This proves the following:

<u>3.8. Proposition.</u> The set of all Fredholm pairs (H_1, H_2) with $H_1 = PH$ for P as above can be identified with the space

$$U_P \;/\; U(PH) \oplus U(PH^{\perp}) .$$

We are going to show that the space $U_P \;/\; U(PH) \oplus U(PH^{\perp})$ is homotopically equivalent to GL_P . We start with the following

<u>3.9. Lemma.</u> U_P is homotopically equivalent to GL_P .

<u>Proof.</u> It is enough to show that we have a weak homotopy equivalence. Now, let us consider a map $R: S^n \to GL_P$. We have a polar decomposition $R(x) = U(x)\sqrt{R}(x)$ for all $x \in S^n$ and we know that $\sqrt{R}(x)$ is the limit of the sequence $\{h_n(x)\}_{n \in N}$ given by the formula

$h_1(x) = Id$,..., $h_{n+1}(x) = 1/2\ (h_n(x) + h_n^{-1}(x) R^*(x) R(x))$,..

So $\sqrt{R}(x)$ is a continuous family of positive operators from GL_P . We see that $U(x)$ is a continuous family of operators from U_P because

$$(U(x)P - PU(x))\sqrt{R}(x) = (R(x)P - PR(x)) - U(x)(\sqrt{R}(x)P - P\sqrt{R}(x))$$

is a compact operator and thus so is $U(x)P - PU(x)$.

We notice that the space $GL_P^{\ +}$ of positive operators from GL_P is a convex subspace, i.e. $tg + (1-t)Id \in GL_P^{\ +}$ for each $g \in GL_P^{\ +}$ and $t \in I$. So we can deform $\sqrt{R}(x)$ into the constant family equal to the identity on S^n through a family of positive operators from GL_P. ∎

Now we show that the natural projection

$$U_P \to U_P \;/\; U(PH) \oplus U(PH^{\perp})$$

is a principal fibre bundle with the structural group $U(PH) \oplus U(PH^{\perp})$. It is enough to construct local sections. The existence of such sections follows from the following

<u>3.10. Lemma.</u> Let P, P_1 be orthogonal projections such that $P - P_1$ is a compact operator and $\|P - P_1\| < 1$. Then there

exists $u \in U_P$ such that $uPu^{-1} = P_1$.

<u>Proof.</u> For the operator $v = (2P_1-Id)(2P-Id)+Id$ we have

$$\|v-2Id\| = \|(2P_1-Id)(2P-Id)-Id\| = \|(2P_1-Id)(2P-Id)-(2P_1-Id)^2\|$$
$$\leq \|2P_1-Id\|\|2(P-P_1)\| = 2\|P-P_1\| < 2,$$

so v is invertible. Moreover, we have $v(2P-Id) = (2P_1-Id)v$, so $vP = P_1 v$ or $vPv^{-1} = P_1$, hence $v \in GL_P$.
Similarly we obtain $(v^*)^{-1}Pv^* = P_1$, hence $vP = (v^*)^{-1}Pv^*v$ and $(v^*v)P(v^*v)^{-1} = P$.
From the construction of $\sqrt{v}$ given in the proof of Lemma 3.9 we see that $\sqrt{v}P = P\sqrt{v}$, hence

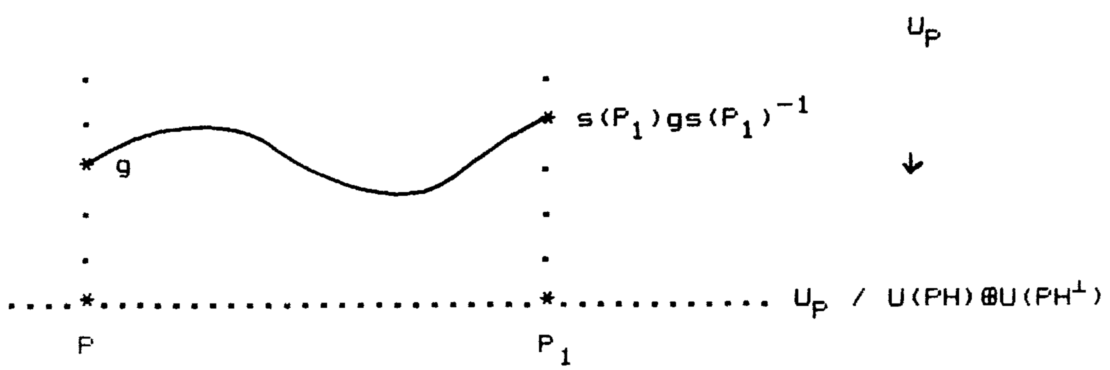
$$P_1 = vPv^{-1} = (v\sqrt{v}^{-1})\ P\ (v\sqrt{v}^{-1})^{-1} ,$$

so we can put $u := v\sqrt{v}^{-1}$. ∎

u is norm-continuous since * is norm-continuous. Moreover, $u = u(P_1,P) = s(P_1)$ has the following property: If $gP = Pg$ then

$$s(P_1)gs(P_1)^{-1}\ P_1 = P_1\ s(P_1)gs(P_1)^{-1} ,$$

i.e. starting from a projection P and a projection P_1 nearby, both representing elements of the base $U_P\ /\ U(PH)\oplus U(PH^\perp)$ we have shown that any element g in the fibre of U_P over P can be transformed through $g \mapsto s(P_1)gs(P_1)^{-1}$ into an element of the fibre of U_P over P_1 , see Figure 2. Hence we get a local trivialization of our bundle.



<u>Fig. 2</u>

<u>3.11. Proposition.</u> The homogeneous space $U_P\ /\ U(PH)\oplus U(PH^\perp)$

has the homotopy type of the space of Fredholm operators on H.

<u>Proof.</u> We have just shown that $U_P \to U_P \,/\, U(PH) \oplus U(PH^{\perp})$ is a principal fibre bundle with contractible fibre. So, the assertion follows from Lemma 3.9 and Theorem I.3.5. ■

The situation for the space $\mathrm{Fred}^2(H)$ is also clear. Let $\mathrm{Proj}^{\infty}(H)$ denote the space of all orthogonal projections with infinite-dimensional range and kernel. It is a contractible space by Kuiper's Theorem [, I.6]: This can easily be shown by the method described above and by using the fact that $\mathrm{Proj}^{\infty}(H) = U(H) \,/\, U(PH) \oplus U(PH^{\perp})$.

Now let us consider the map β: $\mathrm{Fred}^2(H) \to \mathrm{Proj}^{\infty}(H)$ given by the formula $\beta(H_1,H_2) := P_1$ where P_1 is the orthogonal projection onto H_1 . It turns out that $\beta^{-1}(P)$ can be identified with the space $U_P \,/\, U(PH) \oplus U(PH^{\perp})$, so we get a fibration. In fact it is a trivial fibre bundle. A global section is given by the formula

$$s(P) := (\text{image } P, \text{ image } (id-P)).$$

Thus we have proved the following

<u>3.12. Theorem.</u> The space $\mathrm{Fred}^2(H)$ is homeomorphic to the product space $\mathrm{Proj}^{\infty}(H) \times U_P/U(PH) \oplus U(PH^{\perp})$ where P is a projection with infinite-dimensional range and kernel. Hence $\mathrm{Fred}^2(H)$ is a classifying space for the K-functor.

<u>Remark.</u> Using the structure of an infinite-dimensional Lie group on U_P we can show that $\mathrm{Fred}^2(H)$ is a smooth manifold diffeomorphic to $\mathrm{Proj}^{\infty}(H) \times U_P/U(PH) \oplus U(PH^{\perp})$.

4. Local Elliptic Boundary Value Problems - Revisited

Essentially, there are no new results in this section. But one hardly finds them in standard text books devoted to the study of elliptic boundary value problems in the form we present here. We try to stress the usefulness of the language of projectors onto the spaces of Cauchy data. As a standard reference we use Chapter XX of the third volume of Hörmander's monograph "The Analysis of Linear Partial Differential Operators" [21]. However, to keep the paper readable by non-specialists, we restate the definition of elliptic boundary value problems and we also will give at least an explanation of the fact that any elliptic boundary value problem has a finite index. Furthermore, we only consider problems of first order of the type described below:

Let X be a compact smooth manifold with smooth boundary Y and let $A: C^{\infty}(X;E) \to C^{\infty}(X;F)$ be an elliptic pseudodifferential operator of first order acting on sections of smooth complex vector bundles over X. We assume that A satisfies the following two conditions. (We fix once and for all a Riemannian structure on X and Hermitian structures on E and F).

(a) In a local coordinate patch in X the complete symbol of A takes the form $\Sigma\, a_j(x,\xi)$. Assume then that each term in the sum is a rational function of ξ. Using the formulas of the symbolic calculus it can be shown that this condition is preserved under diffeomorphisms, transposition and composition of the operators, and passage to the parametrix in the elliptic case, c.f. [21, Chapter XVIII]. In particular, the condition is satisfied when A is an elliptic differential operator or its parametrix; and this is in fact all we need for the construction of the following pseudodifferential projectors.

Condition (a) gives us the regularity of the solutions of Au

= 0 up to the boundary, see e.g. [21, Chapter XVIII], which makes the definition of the space of Cauchy data below independent of the function spaces involved.

(b) In a collar neighbourhood $N = I \times Y$ of the boundary the operator A takes the form

$$A = G_A(t,y)(\partial/\partial t + B_t) .$$

Here $G_A(t,.): E|Y \to F|Y$ is a bundle isomorphism, t denotes the inward normal coordinate, $B_t: C^\infty(Y;E|Y) \to C^\infty(Y;F|Y)$ is an elliptic operator acting on sections of E restricted to Y depending smoothly on t. Without loss of generality we will assume that $G_A(t,y)$ is unitarian for all t and y.

We want to illustrate the technical advantages of the assumption (b) by investigating the problem of the continuation of a given elliptic operator A on X to an elliptic operator $\tilde{A}$ on the closed double $\tilde{X}$ of X. It is a standard procedure to obtain the smooth doubles for X and for the vector bundles E, F. However, there is no such natural construction for elliptic operators in general. Actually, there are topological obstructions which exclude the extension of the symbol of A to an elliptic symbol over the whole of $\tilde{X}$ in the double vector bundles $\tilde{E}$ and $\tilde{F}$. Take e.g. the Cauchy-Riemann operator over the 2-disc acting on sections of the trivial line bundle. If, on the contrary, A admits elliptic boundary conditions, one could reduce the order of A to zero and extend A onto the whole of $\tilde{X}$ by the identity after a homotopy near Y given by the boundary value conditions. The following construction has the advantage of being explicit and applying even if A does not admit elliptic boundary conditions.

<u>4.1. Lemma.</u> *Any elliptic operator A satisfying condition (b) extends to an elliptic operator on the closed double $\tilde{X}$ of X.*

<u>Proof.</u> First, we deform our operator on the collar in such a way that B_0 becomes a self-adjoint operator. Such a

deformation is standard and can always be done since $\sigma(B_t)$ has no purely imaginary eigenvalues. We change A a little on the collar N such that $B_t = B_o$, $G_A(t,y) = G_A(0,y)$ for $t \leq 1/2$ and all $y \in Y$. Then we paste $[-1,0]\times Y$ onto X. We extend E and F on this new manifold in an obvious way and we extend A to this collar by $G_A(0,.)(\partial/\partial t + B_t)$, where B_t for $t \in [-1,0[$ is given by the formula

$$B_t = 1/2\ (B_o + B_o^*) + 1/2\ r(t)\ (B_o - B_o^*)$$

where r is a smooth function on $[-1,0]$ equal to 1 near 0 and equal to 0 near -1. So, we can assume that $B_t = B_o$ is an elliptic self-adjoint operator for small t.

Now we repeat the construction from [37]. We define vector bundles $\tilde{E}$, $\tilde{F}$ over the double $\tilde{X}$ of X through

$$\tilde{E} = E \cup_{G_A} F \quad , \quad \tilde{F} = F \cup_{(G_A)^{-1}} E$$

where we identify $e \in E_{(o,y)}$ with $G_A(0,y)e \in F_{(o,y)}$ in the case of $\tilde{E}$ and similarily for $\tilde{F}$. Next we choose an elliptic operator

$$A \cup A^* : C^\infty(\tilde{X};\tilde{E}) \rightarrow C^\infty(\tilde{X};\tilde{F})$$

with principal symbol $a_1 \cup a_1^*$ equal to a_1 , the principal symbol of A, on one copy of X and equal to a_1^* , the principal symbol of A^*, on the other copy of X. (Since G_A is unitarian the clutching of the bundles fits with the pasting of the symbols. In the general case one would end with much more complicated formulas.) By smooth deformation we can finally obtain that $A \cup A^*$ is equal to A on X. ∎

Next we turn to the notion of Cauchy data spaces and of the Calderon projector. As noticed in Chapter 2, the space of *Cauchy data* for A is the space

$$H(A) = \{u \in C^\infty(Y;E|Y) \mid \text{there exists } v \in C^\infty(X;E) \text{ such that } Av = 0 \text{ and } v|Y = u\} .$$

<u>4.2. Proposition.</u> Let A satisfy the conditions (a) and (b). Then there exists a pseudodifferential operator, the *Calderon projector* belonging to A,

$$P(A): C^\infty(Y;E|Y) \rightarrow C^\infty(Y;E|Y)$$

such that Image $P(A) = H(A)$ and $P(A)^2 = P(A)$.

Remarks. (1) In the literature the term Calderon projector is usually used to describe a different pseudodifferential operator which is an approximate projection on $H(A)$, i.e. $P(A)^2 - P(A)$ is an operator with C^∞-kernel, c.f. Theorem 20.1.3 in [21].

(2) One can find the proof of Proposition 4.2 in [18]. In the Appendix to this paper we present another version of the proof given by Solomyak in [32] somewhat simplified thanks to Proposition 4.3 below.

(3) In the following we denote by $H(A)$ the closure of the space of Cauchy data in $L^2(Y;E|Y)$ instead of the space itself.

4.3. Proposition. Let A satisfy the conditions (a) and (b). (Actually, we will not use (a) nor the assumption of unitarian G). Then the orthogonal complement of $H(A)$ in $L^2(Y;E|Y)$ is the space $G_A*(H(A^*))$.

Proof. We define two operators acting on $L^2(Y;E|Y)$.

A_1 := A with the domain $\{u \mid Au \in L^2$ and $u|Y \in H(A)^\perp\}$

and

A_2 := A with the domain $\{u \mid Au \in L^2$ and $u|Y \in G_A*(H(A^*))\}$.

Both operators are closed, see [34, §V.3].

For operators of our type the Green formula

$$\langle Au,v\rangle - \langle u,A^*v\rangle = -(G_A(u|Y),v|Y)$$

is valid for all $u \in C^\infty(X;E)$ and $v \in C^\infty(X;F)$. This gives us at once that $G_A*(H(A^*))$ is contained in the orthogonal complement of $h(A)$, hence $\operatorname{dom}(A_2) \subset \operatorname{dom}(A_1)$. We have

$$\ker A_1 = \{u \mid Au = 0 \text{ and } u|Y = 0\} \subset \operatorname{dom}(A_2),$$

hence $\ker A_2 = \ker A_1$. Next we determine

$$\operatorname{coker} A_1 = \{v \mid \langle Au,v\rangle = 0 \text{ for each } u \in \operatorname{dom} A_1\}$$
$$= \{v \mid \langle u,A^*v\rangle - \langle u|Y,G_A*(v|Y)\rangle = 0,\ u \in \operatorname{dom} A_1\}.$$

The last equality is valid for any u with support in the interior of X, so in particular we get $A^*v = 0$ in the interior of X.

We notice

$$\operatorname{coker} A_1 = \{v \mid A^*v = 0 \text{ and } G_A*(v|Y) \in H(A)\}$$
$$= \{v \mid A^*v = 0 \text{ and } v|Y = 0\};$$

$$\operatorname{coker} A_2 = \{v \mid \langle Au,v\rangle = 0 \text{ for each } u \in \operatorname{dom} A_2\}$$

$$= \{v \mid A^*v = 0 \text{ and } G_{A^*}(v|Y) \in (G_{A^*}(H(A^*))^{\perp}\}$$
$$= \{v \mid A^*v = 0 \text{ and } v|Y = 0\}.$$

We summarize: $\mathrm{dom}(A_2) \subset \mathrm{dom}(A_1)$ and $A_1|\mathrm{dom}(A_2) = A_2$; the operators have finite dimensional kernels and cokernels; $\mathrm{coker}\, A_1 = \mathrm{coker}\, A_2$, hence $\mathrm{Image}(A_1) = \mathrm{Image}(A_2)$; $\ker A_1 = \ker A_2 \subset \mathrm{dom}\, A_2$.

From these properties it follows easily that $A_1 = A_2$: Let us assume on the contrary that there exists a $v \in \mathrm{dom}(A_1) \setminus \mathrm{dom}(A_2)$. Then there exists also a $u \in \mathrm{dom}(A_2)$ such that $A_1 v = A_2 u$. This gives $A_1(u-v) = 0$, so $(u-v) \in \ker A_1 \subset \mathrm{dom}\, A_2$, hence $v \in \mathrm{dom}(A_2)$ which gives us the contradiction. Thus we obtain $\mathrm{dom}(A_1) = \mathrm{dom}(A_2)$ which means that $G_{A^*}(H(A^*)) = (H(A))^{\perp}$. ∎

In the Appendix we use the preceding result in order to prove the existence of a projection onto the space of Cauchy data. Alternatively, we may reformulate the results in the language of the Calderon projector. We have to remark at this point, that we prefer to look at the Calderon projector as an (pseudodifferential) *orthogonal* projection onto $H(A)$. We may do that without loss of generality due to the following Lemma which is taken from [8].

<u>4.4. Lemma.</u> Let P be a projection in a separable Hilbert space (set of L^2 sections). Then

$$P_{ort} := PP^*(PP^* + (Id-P^*)(Id-P))^{-1}$$

is an orthogonal projection onto the range of P.

<u>Proof.</u> Let us first observe that

$$[PP^*;PP^*+(Id-P^*)(Id-P)] = [PP^*;PP^*+(Id-P^*)(Id-P)] = 0$$

and

$$PP^*+(Id-P^*)(Id-P) \geq 0$$

since

$$\langle (PP^* + (Id-P^*)(Id-P))u,u\rangle = \|P^*u\|^2 + \|(Id-P)u\|^2 \geq 0$$

with equality if and only if $Pu = u$ and $P^*u = 0$, hence

$$0 = \langle P^*u,u\rangle = \langle P^*Pu,u\rangle = \|Pu\|^2 = \|u\|^2$$

and so $u = 0$.

$$(P_{ort})^2 = PP^*(PP^*+(Id-P^*)(Id-P))^{-1}\, PP^*(PP^*+(Id-P^*)(Id-P))^{-1}$$

$$= \{(PP^*+(Id-P^*)(Id-P))-(Id-P^*)(Id-P)\}\{PP^*+(Id-P^*)(Id-P)\}^{-1}\cdot$$
$$PP^*\{PP^* + (Id-P^*)(Id-P)\}^{-1}$$
$$= P_{ort} - (Id-P^*)(Id-P)PP^*\{PP^*+(Id-P^*)(Id-P)\}^{-1} = P_{ort}$$

since $(Id-P)P = 0$.

Now we have $PP_{ort} = P_{ort}$, so we have only to show that $P_{ort}P = P$. Let $v = Pu$. Then

$$\begin{aligned} P_{ort}(v) &= PP^*\{PP^*+(Id-P^*)(Id-P)\}^{-1} (Pu) \\ &= Pu - (Id-P^*)(Id-P)\{PP^*+(Id-P^*)(Id-P)\}^{-1}(Pu) \\ &= Pu - (Id-P^*)(Id-P)P\{PP^*+(Id-P^*)(Id-P)\}^{-1}(u) \\ &= Pu \;. \;\blacksquare \end{aligned}$$

So, in what follows, $P(A)$ always denotes an orthogonal projection onto $H(A)$. Then Proposition 4.3 is equivalent to the following statement:

<u>4.5. Proposition.</u> Under the previous conditions we obtain

$$Id - P(A) = G_A{*}\, P(A^*)\ (G_A{*})^{-1}.$$

<u>Note.</u> If G_A is unitarian, then we can replace $(G_A{*})^{-1}$ by G_A since $G_A{*} = (G_A)^* = (G_A)^{-1}$.

Since this property of the Calderon projector has not been observed earlier in the literature, we present here some examples. (Actually, a different formula is well known, namely $Id - P_+(A) = P_-(A)$ for invertible A, see e.g. Seeley [28]. Roughly speaking, here $P_+(A) = P(A)$ and $P_-(A) = P(A_-)$ where A_- is an elliptic operator on a manifold X_- such that X_- is a "closing" of X, i.e. $X \cup X_-$ is a closed manifold and A_- is a continuation of A. However, this result is misleading in some sense and in general becomes wrong if A is not more invertible).

<u>4.6. Examples.</u>

(a) We begin with the investigation of the simple operator $A = \partial/\partial t + B$ on the *cylinder* $I\times Y$ where $B: C^\infty(Y;V) \to C^\infty(Y;V)$ is a self-adjoint first-order elliptic operator with spectral decomposition $\{\lambda_k,\varphi_k\}_{k\in\mathbb{Z}}$. Then any solution of Au

= 0 has the form

$$u = \Sigma a_k e^{-t\lambda_k} \varphi_k ,$$

so the space of Cauchy data H(A) is a subspace of $L^2(\{0\}\times Y)\oplus L^2(\{1\}\times Y)$ with the base

$$1/\sqrt{1+e^{-\lambda_k}} \;(\underset{\text{over } 0}{\varphi_k} \;;\; \underset{\text{over } 1}{e^{-\lambda_k} \varphi_k}) .$$

For $H(A^*)$ a suitable base is

$$1/\sqrt{1+e^{-\lambda_k}} \;(\underset{\text{over } 0}{\varphi_k} \;;\; \underset{\text{over } 1}{e^{\lambda_k} \varphi_k}) .$$

The relation between H(A) and $H(A^*)$ is given by the Green form. Since $A(1,y) = -(-\partial/\partial t - B)$ we get

$$G_A(0,y) = 1 = G_A{}^*(0,y) \text{ and } G_A(1,y) = -1 = G_A{}^*(1,y),$$

so $G_A{}^*(H(A^*))$ is the subspace with the base

$$1/\sqrt{1+e^{-\lambda_k}} \;(\underset{\text{over } 0}{\varphi_k} \;;\; \underset{\text{over } 1}{-e^{\lambda_k} \varphi_k}) ,$$

hence $H(A)^{\perp} = (G_A)^* H(A^*)$.

(b) Now we consider the *Cauchy-Riemann operator* on the 2-disk D^2, i.e.

$$\bar\partial = 1/2i \;(\partial/\partial x + i\, \partial/\partial y) \text{ on the set } \{z \in \mathbb{C} \mid |z|\le 1\}.$$

The Cauchy data space for $\bar\partial$ on S^1 is

$$H(\bar\partial) = \{\Sigma_{k\ge 0}\, a_k z^k \mid \Sigma |a_k|^2 < \infty\}.$$

The space of Cauchy data of its adjoint

$$\partial = 1/2i \;(\partial/\partial x - i\, \partial/\partial y)$$

is equal to

$$H(\partial) = \{\Sigma_{k\ge 0}\, a_k z^{-k} \mid \Sigma |a_k|^2 < \infty\},$$

so we get

$$H(\bar\partial)\cap H(\partial) \simeq \mathbb{C} .$$

We find an explanation for the non-vanishing intersection when we pass to polar coordinates and find for $\bar\partial$ the form

$$\bar\partial = e^{i\varphi}/2i \;(\partial/\partial r + ri\, \partial/\partial\varphi).$$

Let us notice that the Green form $e^{i\varphi}$ defines a non-trivial element of $K^1(S^1)$ as it was explained e.g. in Lemma I.1.8. This, of course gives us $H(\partial) = e^{-i\varphi}\,(H(\bar\partial))^{\perp}$ in accordance with Example (a).

(c) Finally let A be a self-adjoint operator. Then the equalities $G_A{}^2 = -\mathrm{Id}$, $G_A B = -BG_A$ hold. This is due to the fact that the formal adjoint to the operator A takes in a collar neighbourhood of Y the form

$$A^* = (G_A)^{-1}(-\partial/\partial t + G_A B(G_A)^{-1}).$$

(For simplicity we have assumed that G_A is unitarian). Moreover, G_A maps any section of $E|Y$ into an orthogonal section with respect to the Hermitian structure of $E|Y$ since

$$0 = \langle Au,u\rangle - \langle u,Au\rangle = \langle G_A(u|Y),u|Y\rangle .$$

We find that G_A maps $H(A)$ onto $H(A)^{\perp}$, hence $-G_A P(A) G_A = \mathrm{Id}-P(A)$.

Now we define elliptic boundary value conditions for an operator A satisfying conditions (a) and (b) from the beginning of this chapter.

<u>4.7. Definition.</u> Let $A\colon C^\infty(X;E) \to C^\infty(X;F)$ be an elliptic pseudodifferential operator of first order satisfying conditions (a) and (b) and let $R\colon C^\infty(Y;E|Y) \to C^\infty(Y;W)$ be a pseudodifferential operator of 0-th order with W a vector bundle over Y. We say that the map

$$(A,R)\colon C^\infty(X;E) \to C^\infty(X;F) \oplus C^\infty(Y;W)$$

given by the formula $(A,R)u := (Au,R(u|Y))$ is an *elliptic boundary value problem* if the mapping $\sigma_L(R)|E_+ : E_+ \to \pi^*(W)$ is an isomorphism. Here $\pi\colon SY \to Y$ denotes the natural projection of the cotangent sphere bundle and E_+ denotes the image of $\pi^* E$ under the symbol p_+ of the Calderon projector $P(A)$.

<u>Remarks.</u> (1) The bundle E_+ was introduced already in Lemma I.2.3 in the context of self-adjoint elliptic operators on Y. One easily sees that E_+ is really identical with the *indicator bundle* M^+ and $\sigma_L(R)|E_+$ can be interpreted as the *initial-value map* as introduced by Atiyah and Bott [5], cf. Booss and Bleecker [13, II.6] where the equivalence is shown of this "regular ellipticity" with the usual definition of ellipticity of boundary value problems in the sense of

Shapiro-Lopatinski.

(2) In general, one would like to consider problems of arbitrary order. As is shown in [21, § 20.3] (see also [34, § 5.1]), we can reduce the order of elliptic differential problems to 1 and this concern ourselves with the class of problems described in Definition 4.7.

(3) At this point it would be appropriate to investigate the regularity of the solutions of elliptic problems. However, the regularity is a very well known fact and we refer to [21] for the proof that the kernel and cokernel of (A,R) consist only of smooth sections; see also e.g. [18] and especially [34].

(4) Assuming regularity we are able at once to prove that (A,R) has a finite index. This is a classical result, too. But we want to show the reader how the machinery of the Calderon projectors works; in fact, in the proof of the following proposition we are mainly interested in the explicit description of the elements of the kernel and of the cokernel of (A,R). Of course, this description depends on the choice of the Riemannian structure on X and the choice of the collar of Y.

<u>4.8. Theorem.</u> Any elliptic boundary value problem (A,R) has a finite index.

<u>Proof.</u> The Calderon projector $P(A)$ provides a natural decomposition

$$\begin{aligned} \ker (A,R) &= \{u \mid Au = 0 \text{ and } R(u|Y) = 0\} \\ &\simeq \{u \mid Au = 0 \text{ and } u|Y = 0\} \oplus \ker RP(A). \end{aligned}$$

In fact, let $v \in \ker RP(A)$, i.e. $v \in C^{\infty}(Y;E|Y)$ and $RP(A)v = 0$. Then we can write $P(A)v = w|Y$ with $Aw = 0$, hence any pair (u,v) with $Au = 0$, $u|Y = 0$ and $RP(A)v = 0$ yields an element $u+w \in \ker (A,R)$ and vice versa. (Using the Calderon potential operator $K_+ : H(A) \to C^{\infty}(X;E)$ one could make the decomposition more explicit).

The first summand, which is in fact the space of "interior solutions" of $Au = 0$, is finite-dimensional, see also Calderon [18]. It can be shown at once: One can paste A over one copy X_+ of X with A^* over another copy X_- and get an elliptic operator $A \cup A^*$ over the closed double $\tilde{X}$ of X as in Lemma 4.1. Then

$$\{u \mid Au = 0 \text{ and } u|Y = 0\} \simeq \{\tilde{u} \mid A\tilde{u} = 0 \text{ on } X_+ \text{ and } \tilde{u}|X_- = 0\}$$

is contained in the kernel of $A \cup A^*$, hence it is finite-dimensional.

Our definition of ellipticity of boundary value problems allows us to construct a pseudodifferential operator

$$T: C^\infty(Y;W) \to C^\infty(Y;E|Y)$$

with principal symbol $\sigma_L(T) = (\sigma_L(R)\ p_+)^{-1}$, hence fulfilling the following conditions

$$\sigma_L(T)\ \sigma_L(R) = Id|E_+ \quad , \quad \sigma_L(R)\ p_+\ \sigma_L(T) = Id|W ,$$

where p_+ is the principal symbol of the projection $P(A)$.

It turns out that $RP(A): H(A) \to C^\infty(Y;W)$ is a Fredholm operator with parametrix $P(A)T: C^\infty(Y;W) \to H(A)$, in fact

$$RP(A)P(A)T = Id_W + \text{compact} \quad , \quad P(A)TRP(A) = P(A)(Id_{H(A)} + \text{compact})P(A) .$$

Hence the second summand of the decomposition of the kernel of (A,R) is also finite-dimensional.

Now, we have to investigate

$$\text{coker } (A,R) = \{(v,r) \in C^\infty(X;F) \oplus C^\infty(Y;W) \text{ such that } \langle Au,v\rangle + \langle R(u|Y),r\rangle = 0 \text{ for each } u \in C^\infty(X;E)\}.$$

From Green's Formula we get

$$\langle u,A^*v\rangle = \langle u|Y,\ G_A{}^*(v|Y) - R^*r\rangle ,$$

where $G_A : E|Y \to F|Y$ is the bundle isomorphism which appears when we write the first order differential operator A in the form $A = G_A(\partial/\partial t + B_t)$ near Y.

The formula is true for each u with support in $X \setminus Y$, so $A^*v = 0$, if (v,r) in coker (A,R), hence $P(A^*)(v|Y) = v|Y$, where $P(A^*)$ is the Calderon projector belonging to A^*.

We get some more information from Green's formula / Proposition 4.3, namely

$$(G_A)^*(H(A^*)) = H(A)^\perp .$$

In particular, this means that

$$(Id - P(A))R^*r = R^*r \quad , \quad R^*r = G_A{}^*P(A^*)z \quad \text{for some } z.$$

Hence $r \in \ker P(A)R^* = \text{coker } RP(A)$ which is finite dimensional. ∎

From the argumentation in the preceding proof we immediately obtain the following useful and well-known (cf. e.g. [21, p. 258]) observation:

4.9. Corollary. Let $A_R : L^2(X;E) \to L^2(X;F)$ denote the closed operator A with the domain

$$\operatorname{dom} A_R = \{u \in L^2(X;E) \mid Au \in L^2(X;F) \text{ and } R(u|Y) = 0\}.$$

Then

index A_R = *index* (A,R) *if and only if* R *is surjective.*

Proof.

$$\ker A_R = \{u \mid Au = 0 \text{ and } R(u|Y) = 0\} = \ker(A,R).$$

$$\operatorname{coker}(A,R) = \{(v,r) \in C^\infty(X;F)\oplus C^\infty(Y;W) \mid A^*v = 0 \text{ and } G_A(v|Y) = R^*r\}.$$

$$\begin{aligned}\operatorname{coker} A_R &= \{v \in C^\infty(X;F) \mid \langle Au,v\rangle = 0 \text{ for each } u \in \operatorname{dom} A_R\} \\ &= \{v \mid A^*v = 0 \text{ and } \textstyle\int_Y \langle u|Y, G_A{}^*(v|Y)\rangle = 0 \text{ for each } u \in \operatorname{dom} A_R\} \\ &= \{v \mid A^*v = 0 \text{ and } G_A{}^*(v|Y) \perp \ker R\} \\ &= \{v \mid A^*v = 0 \text{ and } G_A{}^*(v|Y) \in \operatorname{Im} R^*\}. \quad \blacksquare\end{aligned}$$

Now, we deform the boundary value problem (A,R) without changing the ellipticity condition through

$$t \mapsto (A,\ RP(A) + (1-t)R(Id-P(A))) \quad , t \in I .$$

Then index $(A,RP(A))$ = index (A,R). This proves the following

4.10. Corollary.

$$\operatorname{index}(A,R) = \operatorname{index}(A,RP(A)) = \operatorname{ind}_{X\setminus Y}A + \operatorname{index} RP(A) ,$$

where

$$\operatorname{ind}_{X\setminus Y}A := \dim\{u \mid Au = 0 \text{ and } u|Y = 0\} - \dim\{v \mid A^*v = 0 \text{ and } v|Y = 0\}.$$

Remarks. (1) As we have seen, the index of any elliptic boundary value problem (A,R) is described by three integers $\operatorname{ind}_{X\setminus Y}A$, $\dim\ker RP(A)$, and $\dim\{r \mid R^*r \in (G_A)^*(H(A^*))\}$, and for (A,RP(A)) by

$$\mathrm{ind}_{X\setminus Y}A \quad , \quad \dim \ker RP(A) \quad , \quad \text{and} \dim \operatorname{coker} RP(A) \; .$$

The first two integers do not change under our deformation; therefore neither does the third. The reason for this is in fact both complicated and intuitively surprising as explained in Proposition 4.3 above.

(2) Corollary 4.10 shows that one topological invariant – the index (A,R) of an elliptic boundary value problem – is equal to the sum of two indices, $\mathrm{ind}_{X\setminus Y}A$ and index RP(A), although neither is homotopy invariant. In the case of $\mathrm{ind}_{X\setminus Y}A$ this is due to the possible non-uniqueness of the solution of the Cauchy problem which has local character, cf. Alinhac [1] for a recent survey on this topic.
The second expression, index RP(A) ,looks like the index of an elliptic operator; but this is a misunderstanding: Let us consider the simplest situation, i.e. when the symbol p_+ of P(A) is a projection which does not depend on $\xi \in TY$ but only on y itself. Then it is very natural to consider the operator Rp_+ which in this case is an elliptic pseudodifferential operator over Y. In general, its index is not equal to index RP(A) . In fact

$$\text{index } Rp_+ = \text{index } RP(A) + \text{index } P(A)p_+ \; .$$

It is not hard to see that $P(A)p_+ : C^\infty(Y;E_+) \rightarrow H(A)$ is a Fredholm operator and that its index can take any value under compact deformations of P(A) , cf Wojciechowski [40]. It turns out that this index is generally not expressed as the index of an elliptic operator, see the discussion at the end of [38].

(3) If A is self-adjoint, index(A,R) is truely equal to index RP(A) . In this case we get a topological expression for its value through the Atiyah-Bott index formula. This has a nice methodological relevance: In Example 2.2 (b) we obtained a representation of the index of Fredholm pairs of subspaces by the index of an elliptic operator over a closed manifold. Now we obtain a topological formula for its index through representation by a boundary value problem:

$$\text{index } (H(A)^\perp, gH(A)) = \text{index } RP(A) = \text{index}(A,R),$$

if we put $g := RP(A)+h \in GL_{P(A)-(Id-P(A))}$, where
$RP(A): H(A) \leftarrow L^2(Y;W) \simeq H(A)$, $L^2(Y;E|Y) = H(A) \oplus H(A)^{\perp}$,
and $h: H(A)^{\perp} \oplus \ker RP(A) \to H(A)^{\perp} \oplus \operatorname{coker} RP(A)$ any isomorphism.

Now we would like to reprove the famous *Agranovic-Dynin formula.*

4.11. Theorem. Let (A,R_1) and (A,R_2) be two regular boundary value problems. Then
$\operatorname{index}(A,R_1) - \operatorname{index}(A,R_2) = \operatorname{index} R_1P(A)T_2 = \operatorname{index} R_1(R_2)^*$,
where T_2 is a right parametrix for R_2 constructed as in the proof of Theorem 4.8.

Proof. By Corollary 4.10 we have

$$\begin{aligned}\operatorname{index}(A,R_1) - \operatorname{index}(A,R_2) &= \operatorname{index} R_1P(A) - \operatorname{index} R_2P(A)\\ &= \operatorname{index} R_1P(A) + \operatorname{index} P(A)T_2\\ &= \operatorname{index} R_1P(A)T_2 = \operatorname{index} R_1R_2^*\end{aligned}$$

since there symbols are homotopically equivalent. ∎

Now we want to explain the relation of the General Linear Conjugation Problem (Section 1) with elliptic boundary value problems. We show that we can obtain $\mu(g,A)$ as the index of a suitable elliptic boundary value problem. Our presentation is highly inspired by Atiyah [3, §§ 7-8].

Let us consider the operator

$$A' = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} : \begin{matrix} C^\infty(X_+;E) \\ \oplus \\ C^\infty(X_+;F) \end{matrix} \to \begin{matrix} C^\infty(X_+;E) \\ \oplus \\ C^\infty(X_+;F) \end{matrix} .$$

Near the boundary it takes the form

$$A' = \begin{pmatrix} Id_E & 0 \\ 0 & G_A \end{pmatrix} \begin{pmatrix} 0 & -Id_E \\ Id_E & 0 \end{pmatrix} \left(\partial/\partial t + \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} \right) \begin{pmatrix} Id_E & 0 \\ 0 & G_A^* \end{pmatrix}$$

and it is clear that $H(A') = H(A) \oplus H(A^*) \subset C^\infty(Y;E) \oplus C^\infty(Y;F)$.
Now we define the elliptic boundary value problem

$$A'_g(u,v) := (A^*v\ ,\ Au\ ,\ P(A)(u|Y) - gG_A{}^*P(A^*)(v|Y)).$$

The boundary operator takes the form

$$R(r,z) = P(A)r - gG_A{}^*P(A^*)z.$$

Let $p_+(A)$ be the principal symbol of the Calderon projector $P(A)$. Then we know that

$$(Id - p_+(A))G_A{}^*p_+(A^*) = G_A{}^*p_+(A^*) : E_+(A^*) \to E_+(A)^\perp.$$

This gives the ellipticity condition by the isomorphy of

$$\sigma_L(R) : E_+(A) \oplus F_+(A^*) \to E\ ,$$

because

$$p_+(A) - gG_A{}^*p_+(A^*) = p_+(A) - (Id - p_+(A))g(Id - p_+(A))G_A{}^*p_+(A^*)$$

and g is an automorphism of $E_+(A)^\perp$.

<u>4.12. Theorem.</u>

$$\text{index } A'_g = \text{index } A^g - \text{index } A\ .$$

<u>Proof.</u> A' is a self-adjoint operator, so by Corollary 4.10

$$\begin{aligned}\text{index } A'_g\ (&= \text{index } RP_+)\\ &= \text{index}(P(A) - gG_A{}^*P(A^*))(P(A) + P(A^*))\\ &= \text{index}(P(A) - g(Id - P(A))(P(A) + G_A{}^*P(A^*)).\end{aligned}$$

The second factor

$$P(A) + G_A{}^*P(A^*) : H(A) \oplus H(A^*) \to H(A) \oplus H(A)^\perp$$

is an isomorphism as follows from Proposition 4.5. So

$$\begin{aligned}\text{index } A'_g &= \text{index } P(A) - g(Id - P(A))\\ &= \text{index } A^g - \text{index } A\ . \quad \blacksquare\end{aligned}$$

<u>Remark.</u> A shorter proof follows from the preceding Agranovic-Dynin Formula

$$\text{index } A'_g - \text{index } A'_{Id} = \text{index } (P(A) - g(Id - P(A)).$$

Appendix. The Calderon Projector

Our results on the "twisting" of the Cauchy data spaces, see above Proposition 4.3 and Proposition 4.5, permit a new presentation of the Calderon projector. From our construction it immediately follows that the Calderon projector is a true projection - in contrast to Hormander who only considers an approximate projection, see [20, Theorem 20.1.3]. Nevertheless, our construction is very much in the spirit of Hörmander's approach, especially in focusing on the general situation of not necessarily invertible first-order differential operators and in taking advantage of the related possibilities of symbolic calculus. This allows us some substantial short-cuts in the use of "elliptic towers". The elliptic tower concept was introduced by Birman and Solomyak [8], [34] in order to construct the Calderon projector and to prove its projection property.

Let X be a compact Riemannian manifold with boundary Y, E, F Hermitian vector bundles over X and $A: C^\infty(X;E) \to C^\infty(X;F)$ a pseudodifferential elliptic operator of first order. We make the same assumptions as in Section 4 above, i.e.

(a) In any local coordinate patch in X the complete symbol of A is a sum $\Sigma\, a_j(x,\xi)$ of rational functions of ξ .

(b) In a collar neighbourhood $N = I \times Y$ of the boundary the *operator A takes the form $A = G_A(\partial/\partial t + B)$ where B:* $C^\infty(Y;E|Y) \to C^\infty(Y;E|Y)$ is an elliptic operator acting on Y and G_A: $E|Y \to F|Y$ is a bundle isometry.

The main purpose of this Appendix is to give a new variant of the proof of the following theorem:

A.1 Theorem (Calderon [17]). *Under the preceding assumptions there exists a pseudodifferential operator $P(A)$: $C^\infty(Y;E|Y) \to$* $C^\infty(Y;E|Y)$ of order zero such that $P(A)P(A) = P(A)$ and Image $P(A) = H(A)$, where $H(A) := \{u|Y \mid u \in C^\infty(X;E) \text{ and } Au = 0\}$ is the space of Cauchy data of A.

Our proof is built upon the following results on "elliptic towers" which is taken from [8].

<u>A.2. Definition.</u> Let E, $E_1,..,E_n$ be Hermitian vector bundles over a closed Riemannian manifold M and let A_i: $C^\infty(M;E) \to C^\infty(M;E_i)$ be pseudodifferential operators of order zero with principal symbols α_i. We say that $(A_1,..,A_n)$ is an *elliptic tower* of pseudodifferential operators, if the following condiitions are satisfied:

(i) $A_i A_j^* = 0$ for $i \neq j$,

(ii) $\sum_{i=1}^{n} \operatorname{rank} \alpha_i = \operatorname{rank} E$ (= fiber dimension) .

The relation of this concept to usual elliptic theory is given by the following lemma.

<u>A.3. Lemma.</u> A system $(A_1,..,A_n)$ of pseudodifferential operators of order zero fulfilling the preceding condition (i) is an elliptic tower if and only if

$\Delta = \sum_{i=1}^{n} A_i^* A_i$ is an elliptic operator.

The proof is trivial.

<u>A.4. Proposition.</u> Any elliptic tower $(A_1,..,A_n)$ induces the following surjective pseudodifferential projections:

Image A_i Image A_i^*

$P_i = A_i(\Delta+K)^{-1}A_i^*$ ↖   ↗ $Q_i = A_i^*A_i(\Delta+K)^{-1}$

$C^\infty(M;E)$

$\mathrm{Id}-A_i^*A_i(\Delta+K)^{-1} = \mathrm{Id} - Q_i$ ↙   ↘ $\mathrm{Id}-A^i(\Delta+K)_{-1}A^i{}_* = \mathrm{Id} - P_i$

ker A_i ker A_i^* .

Here and subsequently K denotes the orthogonal projection onto the kernel of Δ .

Proof. Note that ker Δ is finite-dimensional; hence K is an operator with a smooth kernel. To see that $\Delta+K$ is invertible we notice $\ker(\Delta+K) = \{0\}$ after construction, so also $\operatorname{coker}(\Delta+K) = \{0\}$, since $\Delta+K$ is self-adjoint.
Now we observe that $A_i{}^*A_iK = KA_i{}^*A_i = 0$ from which we get

$$(\Delta+K)A_i{}^*A_i = A_i{}^*A_i(\Delta+K)$$

and as a result

$$\begin{aligned} A_i &= A_i(\Delta+K)(\Delta+K)^{-1} = A_i(K+\Sigma A_i{}^*A_i)(\Delta+K)^{-1} \\ &= A_iA_i{}^*A_i(\Delta+K)^{-1} = A_i(\Delta+K)^{-1}A_i{}^*A_i = P_iA_i , \end{aligned}$$

$$\begin{aligned} P_i{}^2 &= A_i(\Delta+K)A_i{}^*A_i(\Delta+K)A_i{}^* = P_iA_i(\Delta+K)^{-1}A_i \\ &= A_i(\Delta+K)^{-1}A_i = P . \end{aligned}$$

From $A_i = P_iA_i$ it is clear that Image $A_i \subset$ Image P_i , the reverse inclusion follows from the definition of P_i . The proof for Q_i and Image $A_i{}^*$ is similar. The desired results for ker A_i and ker $A_i{}^*$ follow from the fact that these spaces are orthogonally complementary to Image $A_i{}^*$, Image A_i.∎

Now we are able to prove Theorem A.1 by a simple construction of a suitable elliptic tower $(A_1,..)$ with Image $A_1 = H(A)$. We follow the scheme from [34] with the above mentioned modifications. The construction and the proof are presented as a series of lemmata.

Since the operator A^*A is strongly elliptic, i.e. $\sigma(A^*A)$ is positive definite, the Dirichlet problem $(A^*A, \mathrm{Id}|C^\infty(Y;E|Y))$ behaves like an elliptic boundary value problem (in the sense of our Definition 4.7) and we can apply all the classical results, see e.g. [36, Chapter V]. In particular we have

$$\dim\ker(A^*A,\mathrm{Id})) = \dim\{u \in C^\infty(X;E) \mid AA^*u = 0 \text{ and } u|Y = 0\} < \infty.$$

If $u|Y = 0$, we have $\langle A^*Au,u\rangle_E = \langle Au,Au\rangle_F$, hence

$$\ker(A^*A,\mathrm{Id}) = \ker_X A = \{u \in C^\infty(X;E) \mid A = 0 \text{ and } u|Y = 0\}.$$

Thus, we are permitted to interchange the spaces $\ker_X A^*A$ and $\ker_X A$ in the following.

Next we recall that for every $r \in C^\infty(Y;E|Y)$ satisfying a certain finite number of linear conditions, there is a solution u of

$$A^*Au = 0 \quad \text{and} \quad u|Y = r,$$

i.e. we have a subspace $W(A) = \{u|Y \mid A^*Au = 0\}$ in $C^\infty(Y;E|Y)$ of finite codimension. Let P_W denote the orthogonal projection onto $W(A)$. P_W is a pseudodifferential operator of order zero with principal symbol equal to the identity. We define an operator $\Gamma^0(A)\colon W(A) \to C^\infty(X;E)$ by the condition

$$u = \Gamma^0(A)r \quad \text{if and only if} \quad \begin{cases} u \in C^\infty(X;E)\ ,\ A^*Au = 0 \\ \qquad\text{and} \\ u|Y = r\ ,\ u \perp \ker_X A \end{cases}$$

Now we put

$$\Gamma(A) \quad := \quad \Gamma^0(A)P_W\ .$$

<u>A.5. Lemma (Theorem 1 from [34]).</u> We have the following equalities

$$H(A) = \text{Image } \gamma A^*\Gamma(A^*) \quad \text{and} \quad H(A^*) = \text{Image } \gamma A\Gamma(A),$$

where $\gamma A^*\Gamma(A^*)\colon C^\infty(Y;E|Y) \to C^\infty(Y;E|Y)$ and $\gamma u := u|Y$.

<u>Proof.</u> We prove the statement for $H(A)$. Let $r \in W(A^*)$ and $v := \Gamma(A^*)r$. Then $AA^*v = 0$ and for $u := A^*v$ we have $Au = 0$, so $\gamma u = u|Y \in H(A)$. This proves Image $\gamma A^*\Gamma(A^*) \subset H(A)$.
Now let $r \in H(A)$ and let $u \in (\ker_X A)^\perp$ with $\gamma u = r$ be a corresponding solution of the equation $Au = 0$. Then the classical theory of the Dirichlet problem, cf. [20, Chapter XVII and the beginning of Chapter XX], guarantees the existence of a solution w of the equations $A^*Aw = u$ and $\gamma w = r$. We can choose $w \in (\ker_X A)^\perp$.
We consider $v := Aw$. We have $A^*v = u$, hence $AA^*v = 0$ and for any $f \in \ker_X A^*$ we get from Green's formula

$$\langle v,f\rangle = \langle Aw,f\rangle = \langle w,A^*f\rangle - \langle G_A\gamma w,\gamma f\rangle = 0 - 0 = 0,$$

so $v \in (\ker_X A^*)^\perp$. Together with $AA^*v = 0$ this means $v = \Gamma(A^*)(\gamma v)$, hence $r = \gamma u = \gamma A^*v = \gamma A^*\Gamma(A^*)(\gamma v)$. ∎

<u>A.6. Lemma.</u> The system $(\gamma A^*\Gamma(A^*)\ ,\ G_A\gamma A\Gamma(A)(G_A)^{-1})$ is an

elliptic tower.

Proof. We get condition A.2.i from Proposition 4.3 and Lemma A.5. To prove condition A.2.ii we recall that $\gamma A^*\Gamma(A^*)$ and $\gamma A\Gamma(A)$ are classical pseudodifferential operators. Their principal symbols are mapings into suitable spaces of Cauchy data of ordinary differential equations:
Let a be the principal symbol of A and $(y,\xi) \in T_yY$. We consider the space

$$M_{(y,\xi)}(A) := \{w: \mathbb{R} \to E_y \mid a(0,y;\xi+\partial/\partial t)w = 0 \text{ and } w(t) \to 0 \text{ as } t \to 0\}.$$

It is well known, see e.g. [12, Chapter II.6], that this space has constant dimension (over connected components of the boundary) and that

$$\dim M_{(y,\xi)}(A) + \dim M_{(y,\xi)}(A^*) = \operatorname{rank} E.$$

Moreover it is clear that the principal symbol of $\gamma A^*\Gamma(A^*)$ provides a surjection of the "initial data" E_y onto $M_{(y,\xi)}(A)$ at each point (y,ξ) of SY; and the same is also true for the principal symbol of $\gamma A\Gamma(A)$ and $M_{(y,\xi)}(A^*)$, which completes the proof of A.2.ii. ∎

Lemma A.6 together with Proposition A.4 ends the proof of Theorem A.1.

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Projektrapport af Erik Gade, Hans Hedal, Henrik Lau og Finn Physant.
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Jens Højgaard Jensen.

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Projektrapport af Arne Jørgensen, Bruno Petersen og Jan Vedde.
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Vejleder: Bent C. Jørgensen.

45/82

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Projektrapport af: Mikael Wennerberg Johansen, Poul Kattler og Torben J. Andreasen
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