On holomorphic critical quasi circle maps

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On Holomorphic quasi circle maps

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The so-called Herman-Świątek Theorem, combined with a surgery argument described among others by Douady, implies the following theorem : The Siegel disk of a quadratic polynomial P whose rotation number is Diophantine with exponent 2, has a quasi circle boundary containing a critical point for the k-th iterate of P, where k is the period of the cycle of Siegel disks.
In this paper the Herman-Świątek Theorem is generalized to holomorphic selfhomeomorphisms of quasi circles. The generalized theorem implies a converse to the above theorem about Siegel disks. In fact it implies the following unpublished theorem of Michel Herman:
If a Siegel disk or Arnold-Herman ring for a rational map has a boundary component, which is a quasi circle containing a critical point for the k-th iterate, where k denotes the period of the cycle of disks or rings, then the associated rotation number is Diophantine of exponent 2.

On holomorphic critical quasi circle maps.

Carsten Lunde Petersen *

Dedicated to the memory of Michel Herman

Abstract

In this paper the so-called Herman-Światek Theorem is generalized to holomorphic selfhomeomorphisms of quasi circles. This result implies an unpublished theorem of Michel Herman: If a Siegel disk or Arnold-Herman ring for a rational map has a boundary component, which is a quasi circle containing a critical point, then the associated rotation number is Diophantine of exponent 2.

1 Introduction

Definition 1.1 *A* holomorphic quasi circle map *on a quasi circle* $\Gamma \subset \mathbb{C}$ *is a selfhomeomorphism* $f : \Gamma \longrightarrow \Gamma$, *which extends to a holomorphic map* $f : U \longrightarrow \mathbb{C}$ *defined in an open neighbourhood* U *of* Γ. *If in addition* Γ *contains a critical point for the holomorphic map* f, *then* f *is a* (holomorphic) critical quasi circle map.

The main result in this paper is the following generalization of the so called Herman-Światek Theorem:

Main Theorem 1 *Let* $f : \Gamma \longrightarrow \Gamma$ *be a critical quasi circle map with irrational rotation number* $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. *Then* θ *has bounded type (or equivalently is Diophantine of exponent* 2*) if and only if* f *is quasi conformally linearizable on* Γ, *i.e. iff there exists a qc. homeomorphsim* $\phi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ *with* $\phi(\Gamma) = \mathbb{S}^1$ *and such that*

$$\forall z \in \mathbb{S}^1 : \phi \circ f \circ \phi^{-1}(z) = R_\theta(z) = e^{i2\pi\theta} z.$$

*I would like to thank Université de Paris-Sud Mathématiques, Orsay, France for its hospitality while this paper was conceived and Saeed Zakeri for his careful comments to a preliminary version of this paper.

The above theorem contains as a special case the following theorem of Michel Herman, a theorem which remained unpublished until his death.

Theorem 1.2 (Herman 89) *Let $f : U \longrightarrow \overline{\mathbb{C}}$ be a holomorphic map with a k-periodic Siegel disk (or Arnold-Herman ring) Δ of rotation number θ. If $\partial\Delta$ (a component of the boundary of Δ) is a quasi-circle, contained in U and containing an f^k-critical point, then θ is of bounded type.*

The proof of the above Theorem is based on two main ingredients. One ingredient is the so called Herman-Światek Theorem, [Her], [Pet], [Swi2], which states that the above Theorem holds in the case where the quasi-circle Γ is a true euclidean circle. The other and new ingredient is a geometrization of a Theorem of Światek. The original Theorem of Światek states that under certain analytic conditions on a circle map a certain cross ratio distortion inequality holds, even in the presence of a finite number of critical points [Swi1](see also (2) on page 4). The inequality is the main ingredient in the proof of the Herman-Światek Theorem. What I propose here is to replace the analytic conditions on the circle map near the critical points with geometric conditions, which are more adequate for surgery applications.

In what follows we shall not destinguish a circle map $f : \mathbb{T} \longrightarrow \mathbb{T}$ from its lifts to $\mathbb{R}$ under the natural projection map $z \mapsto \exp(i2\pi z)$. Also Df will denote the derivative of f. A map $f : U \longrightarrow \mathbb{C}$, where $U \subseteq \mathbb{C}$ is some domain, is called quasi regular, if f has partial derivatives in the sense of distributions $\partial f/\partial z, \partial f/\overline{\partial z} \in H^1_{\text{loc}}$ and if $|\partial f/\overline{\partial z}| \leq k|\partial f/\partial z|$ for some $0 \leq k < 1$. By the Morrey-Ahlfors-Bers Integration Theorem for almost complex structures, the map f is quasi regular if and only if there exists a quasi-conformal homeomorphism $\phi : V \longrightarrow U$ such that $f \circ \phi : V \longrightarrow \mathbb{C}$ is holomorphic. A critical point w for a quasi-regular map f is a point where the local degree $\deg(f, w)$ is strictly bigger than one or equivalently, with ϕ as above, a point $w = \phi(c)$, where c is a critical point for $f \circ \phi$. As for holomorphic maps the natural number $\deg(f, w) - 1$ will be called the multiplicity of the critical point w.

Definition 1.3 *Let $f : U \longrightarrow \mathbb{C}$ be a quasi-regular map and let $\gamma \subset U$ be a quasi arc. A critical point $w \in \gamma$ is called* a nice critical point (relative to γ), *if its multiplicity is even, say $2l$, $l \in \mathbb{N}$ and if there exists two quasi conformal homeomorphisms $\phi : \mathbb{D} \longrightarrow \omega_i \subseteq U$, $\psi : \omega_f \subseteq f(U) \longrightarrow \mathbb{D}$ such that*

1. $\phi(0) = w$, $\phi(]-1, 1[) \subseteq \gamma$.
2. $\psi \circ f \circ \phi(z) = z^{2l+1}$.

3. *The map f is holomorphic on the two 'sectors' $\phi(\Sigma_\pm(1, \frac{1}{(2l+1)}))$, where*

$$\Sigma_\pm(r, \epsilon) = \{\pm s e^{i2\pi\eta} | 0 < s < r, |\eta| < \epsilon/2\}. \quad (1)$$

Remark that neither ϕ nor ψ is by any means unique, and that if $f(\gamma) \subset \mathbb{R}$, then we can choose ψ to be affine. In what follows γ will be clear from the context (usually $\mathbb{S}^1$ or $\mathbb{T}$) and will hence not be mentioned.

By a quasi-regular circle map we shall infer a circle homeomorphism $f : \mathbb{T} \longrightarrow \mathbb{T}$, which is the restriction to $\mathbb{T}$ of a quasi-regular map f defined on some neighbourhood $\omega(\mathbb{T})$.

Proposition 1.4 *Let $f : \Gamma \longrightarrow \Gamma$ be a holomorphic quasi-circle map. Then there exists a quasi regular circle map $\widehat{f} : \omega(\mathbb{S}^1) \longrightarrow \mathbb{C}$ and a quasiconformal homeomorphism $\psi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$, with $\psi(\Gamma) = \mathbb{S}^1$ such that :*

1. *The restriction $\psi_{|\Gamma}$ conjugates f to $\widehat{f}_{|\mathbb{S}^1}$.*

2. *The restriction $\widehat{f}_{|\mathbb{S}^1}$ is real analytic except possibly at finitely many critical points for $\widehat{f}$, all of which are nice.*

3. *If f is a critical quasi circle map, then $\widehat{f}$ can be chosen to have at least one nice critical point on $\mathbb{S}^1$.*

Proof : We shall suppose f is a critical quasi circle map, leaving the easier non critical case to the reader. Let $f : U \longrightarrow \mathbb{C}$ be a holomorphic extension of f and let $z_0 \in \Gamma$ be a critical point for the extension. Then at least one of the two complimentary components of Γ intersects $f^{-1}(\Gamma)$ in any neighbourhood of z_0. Denote such a component by V and let $\psi : \overline{V} \longrightarrow \overline{\mathbb{D}}$ be a homeomorphically extended Riemann map, say mapping z_0 to 1. Define $\omega_1 = \psi(U)$ and extend ψ to a quasiconformal homeomorphism $\psi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ by reflection in Γ and $\mathbb{S}^1$. Define a continuous map, quasi-regular in the interior, $\widetilde{f} : \omega_1 \longrightarrow \overline{\mathbb{C}}$ by $\widetilde{f} = \psi \circ f \circ \psi^{-1}$. Extend f by reflection in $\mathbb{S}^1$ to a quasi regular circle map defined on $\omega = \omega_1 \cup \tau(\omega_1)$, where $\tau(z) = 1/\overline{z}$ denotes the reflection through $\mathbb{S}^1$. The maps $\widehat{f}$ and ψ are easily seen to fulfill the conclusions of the Proposition, with 1 being a nice critical point. **q.e.d.**

A real four tuple (a, b, c, d) is called admissible if either $a < b < c < d$ or $d < c < b < a$. If additionally $a, b, c, d \in \mathbb{T}$ we shall also infer $|d - a| \leq 1$.

For such a four tuple we define the cross ratio $[a,b,c,d] = \frac{b-a}{c-a}\frac{d-c}{d-b}$ and given a homeomorphism $f : \mathbb{R} \longrightarrow \mathbb{R}$ we define the cross ratio distortion

$$D(a,b,c,d;f) = \frac{[f(a),f(b),f(c),f(d)]}{[a,b,c,d]}.$$

The main new technical result in this paper is the following geometrization of the Światek cross-ratio distortion inequality:

Theorem 1.5 (Geometric Światek cross ratio distortion inequality) *Let $f : \omega(\mathbb{T}) \longrightarrow \mathbb{C}$ be a quasi-regular circle map. Assume that f is analytic on $\mathbb{T}$ except possibly at finitely many critical points, all of which are nice. Then*

1. *There exists a constant $C = C(Df) > 1$ such that for all $N \in \mathbb{N}$ and for all families of admissible quadruples $\{(a_i, b_i, c_i, d_i)\}_{i\in I}$ for which the covering number $\sup_{z\in\mathbb{T}} \#\{i \in I | z \in]a_i, d_i[\}$ is at most N :*

$$\prod_{i\in I} D(a_i, b_i, c_i, d_i; f) \leq C^N. \tag{2}$$

2. *For any (nice) critical point $w \in \mathbb{T}$ there exists $C', \nu > 1$ (depending on Df only) such that for all $x, y \in \mathbb{T}$ with $|x - w| \leq |y - w|$:*

$$\left|\frac{f(x) - f(w)}{f(y) - f(w)}\right| \leq C' \left|\frac{x - w}{y - w}\right|^{\nu}. \tag{3}$$

We shall see (Proposition 1.10) that for f as above Df is continuous with value zero at the nice critical points. Combining this Theorem with the previous Proposition 1.4 and the following Theorem 1.6 of Herman our Main Theorem 1 follows as an immediate Corollary.

Theorem 1.6 (Herman) *Suppose $f : \mathbb{T} \longrightarrow \mathbb{T}$ is a circle homeomorphism of irrational rotation number θ. If f satisfies the cross ratio distortion inequality (2) and θ is of bounded type, then f is quasi-symmetrically conjugate to the rigid rotation R_θ. On the other hand if f is quasi symmetrically conjugate to R_θ, satisfies (2) and has a critical point w which satisfies (3) for some constants $C', \nu > 1$, then the rotation number θ is of bounded type.*

Proof : See any of the papers [Her],[Pet] and [Swi2]. **q.e.d.**

The rest of the paper is dedicated to the proof of Theorem 1.5. Any bounds (bounding constants) that will occur from here on will without explicit mention be functions of Df only.

Theorem 1.7 *Let $f : \omega(\mathbb{T}) \longrightarrow \mathbb{C}$ be a quasi-regular circlemap. Assume that f is analytic on $\mathbb{T}$ except possibly at finitely many critical points, all of which are nice. Then there exists $K > 1$ such that for any admissible quadruple (a, b, c, d)*

$$D(a, b, c, d; f) \leq K.$$

Theorem 1.8 *Let $f : \omega(\mathbb{T}) \longrightarrow \mathbb{C}$ be a quasi-regular circlemap. Assume that f is analytic on $\mathbb{T}$ except possibly at finitely many critical points, all of which are nice. Then there exists $l_0 > 0$ such that for any admissible quadruple $(a, b, c, d) \in \mathbb{T}$, for which $]a, d[$ does not contain a critical point*

$$D(a, b, c, d; f) \leq \exp(l_0 \cdot |f(d) - f(a)|).$$

Define $\mathbb{H}_+ = \{z = x + iy | x > 0\}$ and $\mathbb{D}_+ = \mathbb{D} \cap \mathbb{H}_+$. We need the following Julia-Wolf type Lemma.

Lemma 1.9 *Let $f : \mathbb{D}_+ \longrightarrow \mathbb{H}_+$ be a holomorphic function. Then*

$$\limsup_{x \to 0} \frac{x|f'(x)|}{|f(x)|} \leq 1 \tag{4}$$

Proof : Let $\lambda_1(z)|dz|, \lambda_2(z)|dz|$ denote the hyperbolic metrics on $\mathbb{D}_+$ and $\mathbb{H}_+$ respectively. Then $\lambda_2(x + iy) = 1/x$ and for all $z = x + iy \in \mathbb{D}_+$ $1 \geq \lambda_2(z)/\lambda_1(z) \to 1$ as $z \to 0$. By the Schwarz-Picks Lemma for hyperbolic domains

$$\forall x \in \mathbb{R} \cap \mathbb{D}_+ : 1 \geq \frac{\lambda_2(f(x))|f'(x)|}{\lambda_1(x)} \geq \frac{x|f'(x)|}{|f(x)|} \cdot \frac{\lambda_2(x)}{\lambda_1(x)}. \tag{5}$$

Thus (4) follows. **q.e.d.**

Proposition 1.10 *Let $f : U \longrightarrow \mathbb{C}$ be a quasi-regular map. Suppose w in $]u, v[\subseteq \mathbb{R} \cap U$ is a nice critical point and $f(]u, v[) \subseteq \mathbb{R}$. Then there exist $\delta > 0$, $C > 1$ and $\nu_+ > \nu_- > 1$ such that*

$$\forall x, |x| \leq \delta : \qquad \frac{1}{C} \leq \frac{f(w+x) - f(w)}{f(w) - f(w-x)} \leq C \tag{6}$$

$$\nu_- \leq \frac{x f'(w+x)}{f(w+x) - f(w)} \leq \nu_+ \tag{7}$$

It follows that for all x, y with $0 < |y| \leq \delta$ and $x \in]0, y[$:

$$\left(\frac{x}{y}\right)^{\nu_+} \leq \frac{f(w+x) - f(w)}{f(w+y) - f(w)} \leq \left(\frac{x}{y}\right)^{\nu_-}. \tag{8}$$

Consequently f is continuously differentiable (as a real function) at w with derivative 0.

Proof : Note at first that given $0 < x < y \leq \delta$ we obtain inequality (8) by integrating the logarithmic derivative $f'(t+w)/(f(t+w) - f(w))$ from x to y and bounding the integrand using (7).

To prove (6) and (7) note that both inequalities are invariant under both pre and post composition of f with affine maps. Thus we can start out assuming that f is order preserving, $w = f(w) = 0$ and that the map ψ in the definition of nice critical point is the identity map. Let $\phi : \mathbb{D} \longrightarrow \omega$ be the corresponding quasiconformal homeomorphism, so that $f \circ \phi(z) = z^{2l+1}$, $\phi(0) = 0$ and ϕ maps $] - 1, 1[$ increasingly into $\mathbb{R}$. Rescaling f in the domain and range if necessary, we can suppose $\phi(\overline{\mathbb{D}}) = \overline{\omega}$ is a quasidisk for which $\overline{\omega} \cap \mathbb{R}$ is a closed subinterval of $]u, v[$. Inequality (6) immediately follows, because ϕ is quasi symmetric on the real axis.

Using our fredom to pre compose f by linear maps we can assume that $\mathbb{D} \subseteq \omega \subseteq \mathbb{D}_R$ for some $R > 1$. Let $\Xi_\pm = \phi(\Sigma_\pm(1, \frac{1}{(2l+1)}))$ (where $\Sigma_\pm(\cdot, \cdot)$ were defined in Definition 1.3). I claim that there exists $\nu_+ > \nu_- > 1$ such that

$$\Sigma_\pm(1, 1/\nu_+) \subseteq \Xi_\pm \subseteq \Sigma_\pm(R, 1/\nu_-). \tag{9}$$

In fact as ω is a quasi disk with $\omega \cap \mathbb{R}$ connected we need only show that

$$\pi/\nu'_+ < \liminf_{\partial\Sigma_\pm \ni z \to 0} |\arg(\pm z)|, \quad \limsup_{\partial\Sigma_\pm \ni z \to 0} |\arg(\pm z)| < \pi/\nu'_- \tag{10}$$

for some constants $\nu'_+ > \nu'_- > 1$, which depends only on the degree $2l+1$ and the dilatation of ϕ. Moreover we need only prove the first of the two inequalitites in (10) because Ξ_+ and Ξ_- are disjoint. Finally by symmetry we need only prove the inequality for Ξ_+. By the assumptions on the geometry of ω we have a quasi conformal restriction $\phi : \mathbb{D} \setminus [0,1[\longrightarrow \mathbb{D}(R) \setminus [0,R[$. Such a map expands hyperbolic distances by at most a bounded amount depending only on the dilatation of ϕ and the distance. We have $\phi(]-1,0[) \subset]-R,0[$ and for $z \in \mathbb{D}(r) \setminus [0,r[:= B_r$ with $\arg(z) = \eta$ the hyperbolic distance in B_r $\mathrm{d}_{B_r}(z,]-r,0])$ converges to $\log\cot(|\eta|/4)$ as $z \to 0$. Thus the bounds (10) and hence (9) follows.

Suppose $x > 0$ (the case $x < 0$ is similar). Applying the Julia-Wolf Lemma 4 to the two functions $f(z^{\frac{2}{\nu_+}})^{\frac{1}{2}}$ and $(f^{-1}(z^2))^{\frac{\nu_-}{2}}$ we obtain

$$\nu_- \leq \liminf_{x \to 0} \frac{x \cdot f'(x)}{f(x)} \leq \limsup_{x \to 0} \frac{x \cdot f'(x)}{f(x)} \leq \nu_+.$$

Thus increasing ν_+ slightly and decreasing $\nu_- > 1$ slightly, if necessary there exists $\delta > 0$ such that the inequality (7) holds. Finally (7) and (8) imply that f is continuously differentiable (as a real function) at w with derivative 0. **q.e.d.**

Proof of Theorem 1.5: First, 1. of Theorem 1.5 follows immediately from the combination of the two Theorems 1.7 and 1.8. Simply take $C = K^n \cdot \mathrm{e}^{l_0}$, where n is the number of critical points. Secondly 2. of Theorem 1.5 follows immediately from the two inequalities (6) and (8) in Proposition 1.10, and the 1-periodicity of f. **q.e.d.**

Thus we are left with the task of proving Theorems 1.7 and 1.8.

To facilitate the proof of Theorem 1.7 we introduce, just as Światek did, the half cross ratio distortion : Given a real triple (a,b,c) with $b \in]a,c[$ we define

$$D_h(a,b,c;f) = \frac{f(b)-f(a)}{b-a} \frac{c-a}{f(c)-f(a)}.$$

The half cross ratio distortion is handy, because it is still invariant under pre and post composition by affine maps and it can be used to compute the cross ratio distortion:

$$D(a,b,c,d;f) = D_h(a,b,c;f) \cdot D_h(d,c,b;f).$$

Proposition 1.11 *Suppose $f :]w-\delta, w+\delta[\longrightarrow \mathbb{R}$, $\delta > 0$ is a continuous injection, satisfying (6) – (8) for some constants C, ν^+, ν^-, as in the conclusion of Proposition 1.10. Then there exists a constant $C_1 > 1$ depending only on C, ν^+, ν^- such that for all triples (a,b,c) with $b \in]a,c[\subseteq]w-\delta, w+\delta[$:*

$$D_h(a,b,c;f) \leq C_1. \tag{11}$$

Proof : We can suppose $w = f(w) = 0$ and we can further normalize so that given a point $x \neq 0$ we have say $x = f(x) = -1$. Let $x_- = \frac{1}{C}^{\frac{1}{\nu^+-1}}$ and $x_+ = C^{\frac{1}{\nu^--1}}$. If $f(-1) = -1$ then

$$\begin{array}{rl} \forall x < -1 & f(x) < x, \\ \forall x, -1 < x < 0 & f(x) > x, \\ \forall x, 0 < x < x_- & f(x) < x, \\ \forall x > x_+ & f(x) > x, \end{array}$$

moreover

$$\forall x, -1 \leq x \leq 0 \qquad 1 \leq \frac{f(x)+1}{x+1} \leq \nu_+.$$

For instance if $x > x_+$ then

$$f(x) \geq -\frac{1}{C} f(-x) \geq \frac{1}{C} x x^{\nu^- -1} > \frac{1}{C} x x_+^{\nu^- -1} = x.$$

By symmetry and using the freedom to normalize, it suffices to consider the following four different cases :

1. $c < -1 = b = f(b) < a \leq 0$, where we obtain
$$D_h(a,b,c,f) = \frac{a-c}{f(a)-f(c)} \cdot \frac{f(a)+1}{a+1} \leq \frac{f(a)+1}{a+1} \leq \nu^+.$$

2. $-1 = a = f(a) < b < c \leq 0$ yielding :
$$D_h(a,b,c,f) = \frac{c+1}{f(c)+1} \cdot \frac{f(b)+1}{b+1} \leq \frac{f(b)+1}{b+1} \leq \nu^+.$$

3. $c = f(c) = -1 < b \leq 0 < a$, where we obtain $D = D_h(a,b,c,f) = \frac{a+1}{f(a)+1} \cdot \frac{f(a)-f(b)}{a-b}$. If $f(a) \leq a$ then $a \leq x_+$ and $D \leq \frac{a+1}{f(a)+1} \leq x_+ + 1$. And if $f(a) \geq a$ then $a \geq x_-$ and $D \leq \frac{a+1}{a-b} \leq \frac{x_-+1}{x_--b} \leq \frac{x_-+1}{x_-}$, where the middle inequality stems from the fact that the Möbius transformation $a \mapsto \frac{a+1}{a-b}$ is decreasing on the interval $[b, \infty] \supset \mathbb{R}_+$.

4. $a = f(a) = -1 < b < 0 < c$, where we obtain $D = D_h(a,b,c,f) = \frac{c+1}{f(c)+1} \cdot \frac{f(b)+1}{b+1}$. If $f(c) \geq c$ then $D \leq \frac{f(b)+1}{b+1} \leq \nu^+$, And if $f(c) \leq c$ then $c \leq x_+$ and $D \leq \nu^+ \cdot (x_+ + 1)$.

This completes the proof. **q.e.d.**

Proposition 1.12 *Let f be a C^1 circle homeomorphism with at most finitely many critical points $w_1, \ldots w_n$ each satisfying the inequalitites (6) – (8) for some constants $\delta_i > 0, C_i > 1$, and $\nu_i^+ > \nu_i^- > 1$. Then there exists $K > 1$ such that for any admissible quadruple (a,b,c,d)*

$$D(a,b,c,d;f) \leq K.$$

Proof : It suffices to prove that there exists $K_1 > 1$ such that for every triple (a,b,c) with $b \in]a,c[$

$$D_h(a,b,c;f) \leq K_1,$$

because $D(a,b,c,d;f) = D_h(a,b,c;f) \cdot D_h(d,c,b;f)$. Write $2\delta_i$ in place of δ_i and let

$$M = \sup\{f'(z) | z \in \mathbb{T}\}, \qquad m = \inf\{f'(z) | z \in \mathbb{T} \setminus \cup_{i=1}^n]w_i - \delta_i, w_i + \delta_i[\}.$$

We consider three possible cases separately:

1. There exists i such that $]a,c[\subset [w_i - 2\delta_i, w_i - 2\delta_i]$. The conclusion then follows from Proposition 1.11.

2. For every i, $]a,c[\cap [w_i - \delta_i, w_i + \delta_i] = \emptyset$. Then by the Mean Value Theorem $D_h(a,b,c,f) \leq M/m$.

3. Neither 1. nor 2.. Then there exists i such that either $]w - 2\delta_i, w - \delta_i[\subseteq]a,c[$ or $]w + \delta_i, w + 2\delta_i[\subseteq]a,c[$. In either case $D_h(a,b,c;f)$ is bounded by

$$M \cdot \max_{1 \leq i \leq n} \left\{ \frac{1}{f(w_i - \delta_i) - f(w_i - 2\delta_i)}, \frac{1}{f(w_i + 2\delta_i) - f(w_i + \delta_i)} \right\}$$

q.e.d.

Proof of Theorem 1.7: By assumption f is analytic except possibly at the nice critical points w_i. By Proposition 1.10 it is continuously differentiable at those points, thus f is C^1. Moreover also by Proposition 1.10 each of those critical points satisfies the inequalitites (6) – (8) for some constants $\delta_i > 0, C_i > 1$, and $\nu_i^+ > \nu_i^- > 1$. Thus the hypotheses of Proposition 1.12 is satisfied. This completes the proof. **q.e.d.**

Finally we prove Theorem 1.8. To this end we shall transform the cross ratio distortion bound into a bound on contraction of hyperbolic length.

For a bounded open interval $J \subset \mathbb{R}$ we define the doubly slit plane with gap J by $\mathbb{C}_J = (\mathbb{C} \setminus \mathbb{R}) \cup J$. For $R > 0$ we define the slit R-neighbourhood as

$$\Omega_J(R) = \{z \in \mathbb{C}_J \,|\, \mathrm{dist}(z, J) < R\}$$

($\mathrm{dist}(\cdot,\cdot)$ being standard euclidean distance). For a hyperbolic domain $U \subset \mathbb{C}$ write the infinitessimal hyperbolic metric as $\lambda_U(z)|dz|$. Moreover let $\mathrm{d}_U(\cdot,\cdot)$ and $l(\cdot)$ denote the hyperbolic distance and arclength repsectively. However in the case of $U = \mathbb{C}_J$ we reduce the notation to $\mathrm{d}_J(\cdot,\cdot)$ and $l_J(\cdot)$ respectively.

Lemma 1.13 *There exists a strictly increasing function $L : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that for any $R > 0$ and any bounded nonempty open interval $J =]a, d[\subset \mathbb{R}$ and any pair of points $b, c \in J$*

$$l_{\Omega_J(R)}([b,c]) \leq l_J([b,c]) + L\left(\frac{|J|}{R}\right). \tag{12}$$

Moreover, the function L satisfies

$$L(r) = \mathcal{O}(r) \qquad \text{as } r \to 0^+. \tag{13}$$

Proof : Fix J and R as above and note that $\lambda_{\Omega_J(R)}(z) > \lambda_J(z)$ for any $z \in \Omega_J(R)$. Thus

$$\begin{aligned} l_{\Omega_J}([b,c]) - l_J([b,c]) &= \int_b^c (\lambda_{\Omega_J}(x) - \lambda_J(x))dx \\ &\leq \int_a^d (\lambda_{\Omega_J}(x) - \lambda_J(x))dx = \int_J (\lambda_{\Omega_J}(x) - \lambda_J(x))dx. \end{aligned}$$

We need to prove that the latter integral is finite and depends only on the ratio $|J|/R$. Let the open interval J' and $R' > 0$ satisfy $|J'|/R' = |J|/R$ or equivalently $|J'|/|J| = R'/R = \alpha > 0$. Then there exists $\beta \in \mathbb{R}$ such that the affine map $A(z) = \alpha z + \beta$ maps J onto J', $\mathbb{C}_J$ biholomorphically onto $\mathbb{C}_{J'}$ and $\Omega_J(R)$ biholomorphically onto $\Omega_{J'}(R')$. In particular A acts as a hypebolic isometry between both pairs of domains. Thus

$$\int_{J'} (\lambda_{\Omega_{J'}}(x) - \lambda_{J'}(x))dx = \int_J (\lambda_{\Omega_{J'}}(A(x)) - \lambda_{J'}(A(x))) \cdot A'(x)dx$$
$$= \int_J (\lambda_{\Omega_J}(x) - \lambda_J(x))dx. \tag{14}$$

Which proves that (14) depends on $|J|/R$ only. Denote the common value in (14) by $L(\frac{|J|}{R})$. By Schwartz Lemma L is increasing given that the integrals defining L are bounded. Thus it suffices to consider the case $J =]-1,1[$ with $|J| = 2$, which we shall do here after. Recall that, if $V \subseteq W$ are hyperbolic domains, then

$$\forall\, z \in V: \qquad 1 \leq \lambda_V(z)/\lambda_W(z) \leq \coth(\frac{1}{2}\,\mathrm{d}_W(z, \partial V)). \tag{15}$$

Suppose $\gamma : [0,l] \longrightarrow V$ is an arc which is parametrized by hypebolic W-length and which satisfies

$$\forall\, t \in [0,l]: \quad \mathrm{d}_W(\gamma(t), \partial V) \geq t + k \tag{16}$$

for some $k \geq 0$. Then by integration of (15) :

$$0 \leq l_V(\gamma) - l_W(\gamma) = \int_0^l (\lambda_V(\gamma(t))/\lambda_W(\gamma(t)) - 1)dt$$
$$\leq \int_0^l (\coth\frac{1}{2}(t+k) - 1)dt$$
$$\leq 2\log\frac{1 - \mathrm{e}^{-(k+l)}}{1 - \mathrm{e}^{-k}} \leq 2\log\frac{1}{1 - \mathrm{e}^{-k}} \tag{17}$$

To prove boundedness of $L(|J|/R) = L(2/R)$ it suffices to prove that for some $0 \leq x < 1$ the two geodesic segments $]-1,-x]$ and $[x,1[$ satisfy (16) for some $k > 0$ (here $\mathbb{C}_J = W$, $\Omega_J(R) = V$ and $l = \infty$). Moreover by symmetry it suffices to consider the arc $[x,1[$ only. Similarly to prove that $L(r) = \mathcal{O}(r)$ for $r = 2/R$ small, we shall prove that there exists $c > 0$ such that for R big enough $[0,1[$ satisfies (16) with $k \geq \log(R/c)$. From this (13) follows.

To produce these hyperbolic bounds on distances, it is convenient to change coordinates using $M(z) = \frac{z^2-1}{z^2+1}$. This degree two rational map restricts to an isomorphism $M : \mathbb{H}_+ \longrightarrow \mathbb{C}_J$, and it maps $\mathbb{R}_+$ increasingly onto J. For $R > 0$ let $\Delta_\pm(R)$ denote the two connected components of the preimage of $\overline{\mathbb{C}} \setminus \Omega_J(R)$ centered at $\pm i$. Let $\rho = \rho(R) > 0$ denote the smallest radius such that $\Delta_\pm(R) \subseteq \overline{\mathbb{D}(\pm i, \rho(R))}$. Set $k = 1 + \log \rho$. Then the map $\gamma(t) = \mathrm{e}^{t+k} : [0, \infty[\longrightarrow [\mathrm{e}^k, \infty[$ is a parametrization of the latter interval by hypebolic length in $\mathbb{H}_+$ and it satisfies (16). This proves boundedness of the integral in (14) and hence the existence of the function L. To prove the asymptotic behavior (13) of L we note that

$$M(z \pm i) = \frac{\pm i}{z} + \frac{1}{2} + \mathcal{O}(|z|) \qquad \text{for small } |z|.$$

It follows that $\rho(R) = \mathcal{O}(1/R)$. As above $\gamma(t) = \mathrm{e}^t : [0, \infty] \longrightarrow [1, \infty]$ is a parametrization of the latter interval by hypebolic length in $\mathbb{H}_+$ and it satisfies (16) with $k(R) = -\log \rho(R)$. That is there exists $R_0 > 0$ and $c > 0$ such that for $R \geq R_0$: $k(R) \geq \log(R/c)$. Thus completing the proof. **q.e.d.**

An elementary calculation shows that the cross ratio $[a, b, c, d]$ is related to the hyperbolic distance between b and c in the doubly slit plane $\mathbb{C}_{]a,d[}$ by

$$\mathrm{d}_{]a,d[}(b, c) = -\log[a, b, c, d].$$

Hence the cross ratio distortion $D(a, b, c, d; f)$ can be calculated as

$$D(a, b, c, d; f) = \exp\left(\mathrm{d}_{]a,d[}(b, c) - \mathrm{d}_{]f(a),f(d)[}(f(b), f(c))\right),$$

and the cross ratio distortion inequality (1) on page 4 becomes

$$\sum_{i \in I} \mathrm{d}_{]a_i,d_i[}(b_i, c_i) - \mathrm{d}_{]f(a_i),f(d_i)[}(f(b_i), f(c_i)) \leq N \log C.$$

Thus the cross ratio distortion inequality implies that the map f only shortens the hyperbolic length of $[b, c] \subset \mathbb{C}_{]a,d[}$ by a finite amount. In this perspective the Światek cross ratio distortion inequality can be viewed as a real pre decessor of the Carleson-Jones-Yoccoz bound for the distortion of proper analytic maps of simply connected domains [CJY].

Proposition 1.14 *Let $f : J' \longrightarrow J$ be a real analytic diffeomorphism and suppose that f extends to a univalent map $f : U \longrightarrow \Omega_J(R)$, where $R > 0$*

and $J' \subset U \subsetneq \mathbb{C}_{J'}$. Then there exists $l_0 > 0$ such that for any admissible quadruple (a, b, c, d) with $]a, d[\subseteq J'$

$$D(a, b, c, d; f) \leq \exp(l_0 \cdot |f(d) - f(a)|).$$

Proof : It follows from (13) that there exists $l_1 > 0$ such that for any l with $0 < l \leq |J| : L(l/R) \leq l_1 \cdot l/R$. Define $l_0 = l_1/R$ and let (a, b, c, d) be an admissible quadruple with $I' =]a, d[\subseteq J'$. Define moreover $I =]f(a), f(d)[$ and note that $\Omega_I(R) \subseteq \Omega_J(R)$. Define $U_{I'}$ to be the preimage of $\Omega_I(R)$ inside U. Then we have

$$\begin{aligned} \log\left(D(a, b, c, d; f)\right) &= \mathrm{d}_{I'}(b, c) - \mathrm{d}_I(f(b), f(c)) \\ &\leq \mathrm{d}_{U_{I'}}(b, c) - \mathrm{d}_I(f(b), f(c)) \\ &= \mathrm{d}_{\Omega_I(R)}(f(b), f(c)) - \mathrm{d}_I(f(b), f(c)) \\ &\leq L(|I|/R) \leq l_0|I| = l_0 \cdot |f(d) - f(a)|. \end{aligned}$$

q.e.d.

Proof of Theorem 1.8: Let $w, w' \in \mathbb{T}$ be a pair of neighbouring critical points bounding an open interval $J =]w, w'[$. Then there exist $R = R(J) > 0$ and an open neighbourhood U_J of J, such that $f : U \longrightarrow \Omega_{f(J)}(R)$ is biholomorphic. This is because f is real analytic except possibly at the critical endpoints, both of which are nice. Thus Proposition 1.14 proves Theorem 1.8 for J. As f is assumed to have only finitely many critical points, we are done.

q.e.d.

References

[Her] M. R. Herman. Conjugaison quasi symétrique des homéomorphismes analytiques du cercle à des rotations. Version très très préliminaire, 19 pages manuscript, 1987.

[CJY] P. Jones L. Carleson and J.C. Yoccoz. Julia and John. *Bol. Soc. Brasil. Mat. (N.S.)* **25** N. 1. (1994), 1–30.

[Pet] C .L. Petersen. The Herman-Swiatek Theorems with applications. In Tan Lei, editor, *The Mandelbrot set, Theme and Variation.* LMS Lecture Note Series **274** Cambridge university Press, 2000.

[Swi1] G. Swiatek. Rational Rotation Numbers for Maps of the Circle. *Comm. Math. Phys.* **119** (1988), 109–128.

[Swi2] G. Swiatek. On critical circle homeomorphisms. *Bol. Soc. Brasil. Mat. (N.S.)* **29** N. 2. (1998), 329–351.

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