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How did the all-purpose parenthesis come about?

Homage à Karine

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What is a parenthesis?

Oxford English Dictionary explains that it is

> An explanatory or qualifying word, clause, or sentence inserted into a passage with which it has not necessarily any grammatical connection, and from which it is usually marked off by round or square brackets, dashes, or commas.

This evidently concerns a parenthesis in a written text.

To this we may add that even spoken “text” may contain parentheses; they can be marked off by pauses or change of pitch. Both kinds we may refer to as “rhetorical parentheses”.

An *algebraic parenthesis*, however, is something different. It is used in arithmetic and algebra (and such mathematics that descend from them).

It is a mathematical expression that *as a whole* can be operated upon or taken as argument for a function.

The *parenthesis* (of either kind) is to be distinguished from the *brackets* that may mark it off.

The rhetorical and the algebraic parenthesis are thus utterly different things.

A rhetorical parenthesis can be left out without destroying the text from which is has been removed though impoverishing it.

«It is used in arithmetic and algebra ()» coming from «It is used in arithmetic and algebra (and such mathematics that descend from them)»

But what about $4\times()^2-3x = \log()$, produced by reduction of

$$4\times(x^2-1)-6x = \log(10x-19)$$

(solution $x = 2$) ?

It is utterly meaningless.

Some algebraic parentheses can serve specific purposes only – such as the parenthesis delimited by the long root sign $\sqrt{}$, or the parentheses defined by being the numerator and the denominator of "formal fractions" like $\frac{x^2+3x}{x^3+4x-5}$.

Such special-purpose parentheses had been in ample use in abbacus algebra since the earlier 14th century.

A few writers in this tradition (to which we may generously count Cardano and Bombelli, beyond such figures as Antonio de' Mazzinghi and Chuquet, *bona fide* members) had even used nested composite radicands.

The parenthesis notations used by Chuquet and Bombelli *could* have been used for other purposes *if such purposes had presented themselves*.

But they did not.

Round brackets (...) were in use as delimiters for *rhetorical* parentheses at least since 1494 (Pacioli uses them in his *Summa*).

The first to use them for mathematical purposes, but without any system, appears to have been Tartaglia; in 1608, Clavius used them systematically in his *Algebra*.

Both, however, only when delimiting composite radicands.

In consequence, even *potentially general* or “all-purpose” parenthesis delimitations remained “special-purpose”.

Viète and Descartes

Before 1600, abbacus algebra had thus made use of several special-purpose parentheses.

But before Viète, no attempt was ever made to use any of these special-purpose parentheses in a wider function than its original use.

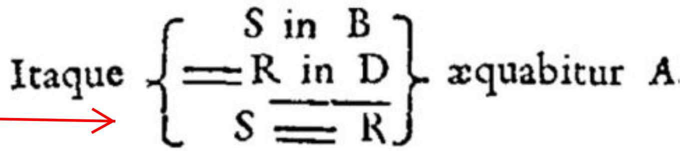
What then about Viète and Descartes, the creators of the new algebras?

Both offer something which at a first glance seems to be a possible candidate for the role of a general parenthesis. What we find in Viète, however, turns out not to *be* a parenthesis.

What Descartes offers reveals itself at closer inspection as yet another special-purpose parenthesis.

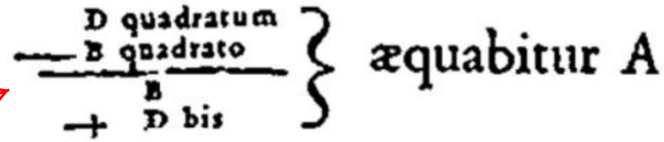
First Viète. In *Zeteticorum libri quinque* from 1591 we find copious use of braces (and much more copious use of formal fractions, with their inherent parentheses).

Mostly a pair of braces indicate that a formal fraction they contain is meant to be part of a single line.



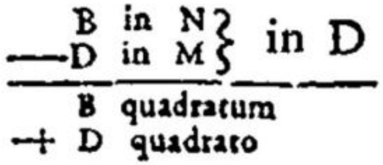
Fol. 2r

Regularly, however, a single brace to the right is used instead.

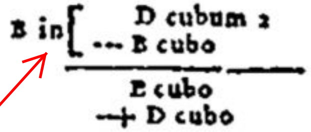


Fol. 17v

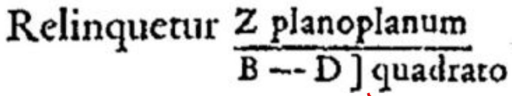
In some diagrams, the right brace is used to keep together a binomial spread over two lines; since this binomial is then multiplied by a factor, this has *the effect* to define a parenthesis.



Fol. 15r



Fol. 20v



Fol. 23r

A few times a left square bracket is used in a similar way.

Once, finally, we find the right square bracket after a sum, indicating that the sum as a whole has to be squared.

There are no indications in the text that Viète saw any difference between these diverse functions – his understanding may have been a general “keep together (for whatever purpose may be relevant)”.

We also find “*B*–*D* in *A* quadratum”, meaning in our notation $(B\text{–}D)\cdot A^2$, with no brace. In this place Viète sees no need to specify that *B*–*D* shall be treated as a parenthesis.

At most we may thus characterize Viète’s algebraic parenthesis as a vague intuition. *At most!*

Frans van Schooten certainly saw the difference when he published Viète's works in 1646.

Braces that do not serve to define a parenthesis are simply eliminated: van Schooten makes the fraction lines longer, which allows him to write numerator as well as denominator on a single line.

When a brace has the effect to create a parenthesis, van Schooten conserves it.

Alternatively, the parenthesis is instead kept together by a vinculum (a line over the expression), $\overline{B \text{ cubum} + D \text{ cubo } 2}$

Descartes similarly makes ample use of formal fractions in the *Géométrie* from 1637, and moreover of the extended root sign $\sqrt{}$.

But beyond these he introduces another parenthesis, marked off by a right brace – as here:

$$yy \infty \left.\begin{matrix}--dekzz\\+cfglz\end{matrix}\right\} y \frac{\left.\begin{matrix}--dezzx\\--cfgzx\\+bcgzx\end{matrix}\right\} y \begin{matrix}+bcfglx\\--bcfgxx\end{matrix}}{ezzz--cgzz.}$$

At times Descartes omits the brace but

$$y^6 \begin{matrix}+aa\\--2cc\end{matrix}\; y \;\begin{matrix}--a^4\\+c^4\end{matrix}\; yy \;\begin{matrix}--a^6\\--2a^4cc\\--aac^4\end{matrix} \infty 0$$

conserves the smaller type with the same meaning. Both notations are used for *composite coefficients* only, and thus provide yet another special-purpose parenthesis.

As such they are used consistently.

In either version, the notation is unfit to define a general parenthesis.

Wallis and Newton

Once the new algebra had been created, it was soon applied not only (as originally) to advanced geometric problems but also to infinites – first to infinite series.

Mathematical interest in infinite procedures was evidently not new.

Before the creation of the new algebra, however, there would have been no possibility to involve *algebra* in such inquiries.

That was done by John Wallis in his *Analysis infinitorum*, "Analysis of Infinites", in 1656.

We should not believe, however, that innovation in substance was accompanied by participation in any "progress" of notation.

In some respects Wallis takes over and unfolds what had been done by his predecessors. Quite significant is what he does to the exponent notation.

Descartes would write zz for the second power but mostly z^3 for the third (and similarly for higher powers).

Wallis broadens this notation to all rational exponents.

His parenthesis function, on the other hand, is weak.

He evidently uses a large number of formal fractions, and there he does as everybody else had done since three centuries.

For the root of a composite radicand, inspired by the 1648-edition of Oughtred's *Clavis mathematica*, he may use «√:*radicand*:», «√:*radicand*» alone, or «√*u*:» (*u* stands for *universal*, a term from the abbacus tradition).

Other polynomials are kept together in a similar way (still following Oughtred). The square on «*R*–*a*» may thus be written «Q:*R*–*a*:», and its cube «C:*R*–*a*:»

This double use of «:   :» is the closest Wallis gets to a "general parenthesis".

Oughtred had also made use of the vinculum, but Wallis does not adopt it.

In some letters to John Collins from 1673, Wallis produces nesting by means of round brackets – using «√(   )» for the inner root.

In *A Treatise of Algebra* from 1685, Wallis changes his ways more radically.

When it is adequate he adopts the Cartesian long root sign – yet without giving up his own notation.

For instance,

$$c = \sqrt{C. + ccc + \sqrt{:bbbbbb + cccccc}}$$

where we would have

$$c = \sqrt[3]{c^3 + \sqrt{b^6 + c^6}}$$

– that is, the «:» is conserved in the inner but not in the outer bound root.

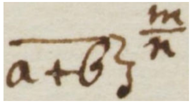
When coming to the “arithmetick of infinites”, Wallis first refers to his own work on the topic and then to Newton’s adoption of Wallis’s notation for broken and negative exponents and to the notation introduced by Newton:

for $Q:\overline{RR-aa}$. $\sqrt{}:\overline{RR-aa}$: &c.

he puts, $\overline{RR-aa}|^{2}$. $\overline{RR-aa}|^{\frac{1}{2}}$ &c.

As we see, Wallis presupposes the vinculum to be familiar; I have not noticed it earlier in the book, but in any case it was known from Oughtred, whom Wallis refers to repeatedly.

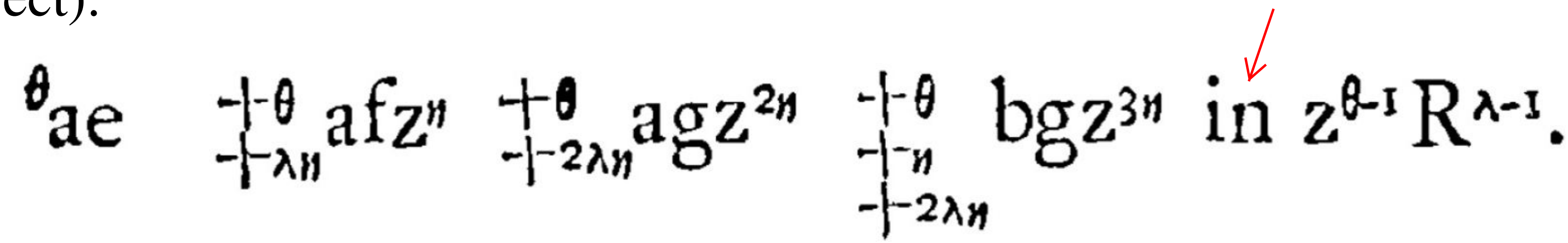
Newton (when building on Wallis's *Analysis of Infinites*) had used these notations already in his "Mathematical Manuscript" from *ca* 1664–1665 (with a right brace instead of the vertical stroke).

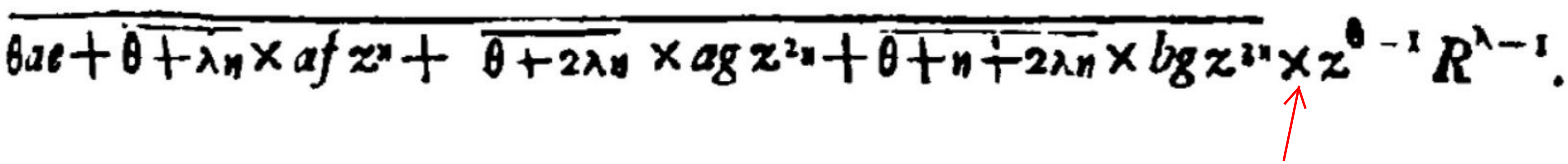
Here Newton makes use of vincula, at times combined with broken exponents – he thus develops  (i.e., $(a+b)^{\frac{m}{n}}$) as an infinite series.

Newton's *Tractatus de quadratura curvarum* was published in 1704, together with the *Opticks*, and then again in 1711 (edited then by John Collins).

The two editions are very close to each other; both make copious use of vincula, but the 1704-edition also of Cartesian braces.

In one formula, the 1711-version uses nested vincula but the 1704-version Cartesian braces for the inner parentheses while leaving the outer parenthesis unmarked (using a word for the multiplication with the same effect):





When dealing with rational powers of composite expressions, the two editions agree with each other as well as with what Wallis tells about Newton's notation,

All combine the vinculum with a vertical stroke instead of the brace used in the "Mathematical manuscript":

$\overline{\qquad}|^{\lambda}$ instead of $\overline{\qquad}\}^{\lambda}$.

Newton's *Arithmetica universalis* from 1707 contains an introduction to notations.

Here, the vinculum is explained (without being given a name, it is just shown).

It is only explained to keep together a parenthesis functioning as a factor, or two factors. Throughout the book, that is indeed its only use.

Occasionally we also encounter Cartesian composite coefficients (with or without a brace).

So, all in all, Newton treats the vinculum here not as an all-purpose parenthesis delimiter but as yet another special-purpose parenthesis.

Newton's use of $\overline{\quad\;}|^{\lambda}$ in other works (and similarly in the "Mathematical Manuscript") shows that he *could* go beyond this restricted use, but this was apparently an improvisation that never gave rise to a principle.

All in all, Newton as well as Wallis would use the new parentheses very sparingly outside infinitesimal calculus.

Basic algebra around 1700

What about the use of parentheses in basic algebra works published during the century around 1700? Their number is vast, so I shall sum up what is found in the sample that was at my immediate disposal.

– Carlo Renaldini's *Opus algebricum* from 1644;
– Johann Heinrich Rahn's *Teutsche Algebra oder algebraische Rechenkunst zusamt ihrem Gebrauch* from 1659;
– Gerard Kinckhuysen's *Stelkonst, Beschreven Tot diens van de Leerlinghen* from 1661, which Isaac Barrow wanted Newton to revise and comment upon in 1669;
– Erasmus Bartholin's *Principia matheseos universalis*, second edition from 1661;
– Jacob Brasser's *Regula cos, of algebra* from 1663;
– Andrés Puig's *Arithmetica especulativa, y practica, y arte de algebra* from 1672;
– Jean Prestet's *Élémens des mathématiques* from 1675;
– Jacques Ozanam's *Nouveax elemens d'algébre* from 1702;
– Jean-Pierre de Crouzas *Traite de l'algebre* from 1726;
– Nathaniel Hammond's *Elements of Algebra* from 1742.

Some continue the abbacus-/cossic tradition, and are unaffected by the new algebra.

Others take over Viète's notations, perhaps also Descartes' long root, and more rarely also use the vinculum.

All in all, these examples of elementary or less elementary algebra tell us that there was still *little use* for the vinculum (or any substitute we may imagine), and *no use* for it as a general determination of a parenthesis.

There was *certainly also no conceptualization* of the vinculum as having this function – not even what could be characterized as a “fuzzy” or intuitively understood concept.

When used, the vinculum created just another type of special-purpose parenthesis – keeping together a polynomial, mostly a binomial, as a factor.

Infinitesimal analysis

What then about the use of parentheses in infinitesimal analysis after Wallis?

In the early years, Newton kept his fluxional analysis as protected private property, and what he did privately is not easy to know.

The material collected by John Collins in 1711 shows nothing we do not already know from Newton's hand.

At the time, however, Leibniz and Johann Bernoulli were working in the field

– further, de l'Hospital published Bernoulli's work as if it had been his own (thereby diffusing it).

De l'Hospital uses vincula in 1696 not only to keep together composite factors but also for raising composite expressions to some power (as Wallis and Newton had done).

We may see this double use of the vinculum as a shared step toward a general parenthesis function.

Most significant, however, is de l'Hospital's *copious use* of the vinculum – it was obviously much more necessary in the context of infinitesimal analysis than in ordinary algebra.

Leibniz and Bernoulli show us more.

We may look, on one hand, at Leibniz's contributions to the *Acta eruditorum*, on the other at the Leibniz-Bernoulli correspondence.

First Leibniz in the *Acta*.

In the first pertinent article, from 1684, Leibniz uses a vinculum to keep together a composite expression (here *ax*) as argument for the taking of a differential, $d\,\overline{ax}$.

In 1686, and still in 1693, Leibniz employs a long «s» as symbol for integration (for instance, ſxdx) but replaces it by $\int$ in 1694.

The integrand when composite has to be kept together by a vinculum («dx» often following directly after $\int$).

This is already a manifestation of an understanding of the vinculum as an all-purpose generator of a parenthesis.

In 1695 we find $\overline{\quad} y\dagger dy \,\underline{\quad}^{\,x\dagger dx}$, where we would write $(y+dy)^{(x+dx)}$ – that is, a vinculum (written below, perhaps for typesetting reasons) holding together a composite exponent.

Yet another manifestation (now using round brackets) turns up in 1700: here, an infinite sum is composed from infinite groups:

$$0 = (01Y+02Y^2+03Y^3 \text{ \&c})+(10+11Y+12Y^2+13Y^3\text{\&c})Z^1+(20+21Y+22Y^2+23Y^3\text{\&c})Z^2+\text{\&c} .$$

$$0 = (01Y + 02Y + 03Y \text{ \&c}) + (10 + 11Y + 12Y^2 + 13Y^3 \text{ \&c})Z^1 + (20 + 21Y + 22Y^2 + 23Y^3 \text{ \&c})Z^2 + \text{\&c.}$$

The Leibniz-Bernoulli correspondence also exhibits ample use of parentheses, but initially Bernoulli's notation differs somewhat from that of Leibniz – and that of Leibniz differs from what he offers in the *Acta*.

First, however, a word about Bernoulli's *Lectiones mathematicae de methodo integralium*, written between 1691 and 1692 but only published in 1742.

I use an early manuscript now held in Paris, which we may expect to reflect the original notations.

Here, to delimit a parenthesis, Bernoulli normally (and often) draws a vinculum.

However, once when raising a binomial to a composite power, he delimits the binomial by means of a vinculum, while the exponent is in round brackets:

$$\overline{a+y}^{\left(\frac{1}{2}+1\right)}$$

Turning now to the correspondence, we notice in a letter from 1694 that Leibniz makes use of round brackets while he would still for long use vincula in the *Acta*.

In another letter from 1694, he writes

$$y = 1 + \frac{1}{1} xlx + \frac{1}{1.2} x^2 (lz)^2 + = \ldots$$

(l meaning *logarithm of*) and later

$$d(x^{\mathrm{h}} lx^2) \quad \text{and} \quad \int (x^{h-1} \mathrm{lxdx})$$

confirming Leibniz's preference for the round brackets in many functions.

In January 1695, Bernoulli uses I...dx (at that moment he uses «I» for "integral") to delimit a parenthesis (I$x^m lx^e dx$) (obviously another special-purpose parenthesis);

elsewhere, however, he lets «dx» follow immediately after «I», for which reason he needs the vinculum around the integrand.

In April, he returns to ∫ for the integral, and writes ∫(...), while still using vincula for other parentheses.

From June 1695 onward, he follows Leibniz in all respects, and even uses nested parentheses delimited by round brackets in June 1697.

After that, neither Leibniz nor Bernoulli ever use the vinculum in the correspondence.

All in all we see convergence toward what we still use today – round brackets serving to delimit the general parenthesis.

There is no trace, however, of a *conscious* decision taken at some moment to use for instance the round brackets to delimit a general parenthesis.

This habit grew out of a practice where parentheses were needed for ever-new purposes.

There the vinculum and the round brackets were at hand, and they were silently adopted.

And then?

Bernoulli (or his mathematically competent printer) changed his ways from 1691/92 when publishing his lectures on integration in 1742.

That made the treatise agree with the habits we have seen established during the Leibniz-Bernoulli correspondence.

In 1748, we see the tools fully unfolded in Euler's *Introductio in analysin infinitorum* – for instance in the development of the infinite fraction

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\,\&c.},$$

as the infinite sum of infinite sums

$$\begin{array}{l}
1+z\,(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\&c.) \\
+z^2(x^2+x^3+2x^4+2x^5+3x^6+3x^7+4x^8+4x^9+5x^{10}+\&c) \\
+z^3(x^3+x^4+2x^5+3x^6+4x^7+5x^8+7x^9+8x^{10}+10x^{11}+\&c.) \\
+z^4(x^4+x^5+2x^6+3x^7+5x^8+6x^9+9x^{10}+11x^{11}+15x^{12}+\&c.) \\
+z^5(x^5+x^6+2x^7+3x^8+5x^9+7x^{10}+10x^{11}+13x^{12}+18x^{13}+\&c.) \\
+z^6(x^6+x^7+2x^8+3x^9+5x^{10}+7x^{11}+11x^{12}+14x^{13}+20x^{14}+\&c.) \\
+z^7(x^7+x^8+2x^9+3x^{10}+5x^{11}+7x^{12}+11x^{13}+15x^{14}+21x^{15}+\&c) \\
+z^8(x^8+x^9+2x^{10}+3x^{11}+5x^{12}+7x^{13}+11x^{14}+15x^{15}+22x^{16}+\&c.) \\
\qquad\qquad\&c.,
\end{array}$$

Basic algebra once again

Infinitesimal analysis had needed the general parenthesis, as we have seen – at first modestly but increasingly as the field developed.

Basic algebra, as we have also seen, had had no need for it:

Along with the traditional special-purpose parentheses it adopted Descartes' composite coefficients and the vinculum keeping together a composite factor

but without using any of them very much nor for more general purposes.

Yet writers and teachers of basic algebra were aware of the developments created by the infinitesimal offspring of the new algebra, and gradually they felt it adequate to teach the tool.

I shall illustrate the process by two examples.

At the exam in 1799 at St. John's College, Cambridge, the students were expected to understand and solve this abstruse equation:

$$\frac{123+41\sqrt{x}}{5\sqrt{x}-x} = \frac{20\sqrt{x}+4x}{3-\sqrt{x}} - \frac{2x^2}{(5\sqrt{x}-x)(3-\sqrt{x})}\,.$$

The problem has obviously been constructed just in order to test the students' understanding of and ability to deal with parentheses

woe to the student who starts by performing the multiplication $(5\sqrt{x}-x)(3-\sqrt{x})$ and does not discover that $20\sqrt{x}+4x = 4(5\sqrt{x}+x)$ and $123+41\sqrt{x} = 41(3+\sqrt{x})$!

We do not know how the students of St. John's College were taught their algebra.

But we may see how the teaching was done in Paris at the same time in Lacroix's *Élémens d'algèbre* (I used the 1804 and 1808 editions).

Lacroix was very careful not to overload the students too early with conceptually difficult matters.

Parentheses (made by round brackets) only turn up from p. 66 onward, at first in the factorization of polynomials and then, very often, in the multiplication of polynomials.

The first time a parenthesis is *explicitly* spoken of (but presupposed to be known) is on pp. 153*f*, where the possibility to replace $\sqrt{4a^2b-2b^3+c^3}$ by $\sqrt{}(4a^2b–2b^3+c^3)$ is explained; the next (and last) time is on p. 195, where the development of $(4a^2–2ab+5b^2)^3$ is explained.

When discussing combinatorics on p. 204, Lacroix designates (ambiguously) by $P(m–(n–1))$ what we would write today $m!/(m–(n–1))!$.

> *n!* he writes *Qn*, showing that the brackets are not needed to contain a non-composite argument for combinatorial functions.

No doubt that Lacroix understood the round brackets as a an all-purpose delimitation of a parenthesis and expected his students to do so too.

The parenthesis has now been fully naturalized; even the great pedagogue Lacroix did not notice that an explanation was required before use

> (of course, Lacroix himself was fully at home in higher analysis and was thus accustomed).

So, basic algebra received the general parenthesis as a gift from its offspring – for long accepting it only with reluctance, but in the end becoming addicted.

Let us return to the observation that the habit to use either the vinculum or a pair of round brackets as *general* delimiters grew out “of a practice where parentheses were needed for ever-new purposes”.

And let us remember what was said in the beginning, that an algebraic parenthesis can be “taken as argument for a function”.

We may disregard the set-theoretical understanding of a function as a mapping from one set to another one, and restrict ourselves to the almost-modern idea that a function is something that can somehow be determined.

We shall also leave aside the intricate question, *in which sense* for instance a logarithm and a sine are supposed to be determined.

Mathematicians of the 18th century supposed to know what they meant, and only the nasty 19th-century mathematicians insisted that they did not.

So, accepting the earlier understanding, which were the mathematical functions known around 1650?

Polynomials and roots, evidently – also nested polynomials and roots.

Trigonometric functions.

Logarithms and their inverses.

More generally, what could be found by curves or their quadrature.

Polynomials and roots had been expressed *as such* for long.

Trigonometric functions were tabulated.

So were logarithms.

So, collecting these under a single heading, as constituting a *mathematical category*, is *our* idea.

However, as observed by Adolf P. Youschkévich, *our* idea seems to have dawned upon Johann Bernoulli in 1698, in an appendix to a letter to Leibniz.

In a previous letter (and also in earlier writings) Leibniz had used the word in a slightly different sense (corresponding to "performing some function").

Bernoulli spoke about finding a curve, the powers of the ordinates of which produced a certain result – but knowing from recent experience that this category of powers could be broadened, he spoke instead of these ordinates "raised to a given power or, in general, some functions of these ordinates".

One example of this recent experience was referred to a moment ago:

> In 1694, Leibniz had operated upon the expression
> $$y = 1 + \frac{1}{1} xlx + \frac{1}{1.2} x^2 (lz)^2 + = \ldots ,$$
> l meaning *logarithm of* ...

Within the practice of infinitesimal analysis, logarithms were thus no longer just entries in a table.

But there were many others.

Bernoulli's new use of the term could have been a soon forgotten linguistic accident – after all, most linguistic emergency solutions are happily forgotten a few minutes after they have conveyed their message.

However, some are not: this is how stable pidgin and contact languages come into existence.

In the present case, Leibniz was immediately pleased, as he said (*placet*).

– whether flattered by the use of a word he had used himself though in a different sense we cannot know; in any case Leibniz realized that this word had a niche waiting for its arrival.

That niche was also the one which had lead to the spontaneous development of a parenthesis notation which could serve to delimit a composite argument of any function or calculation that might be thought of.

Hereby a category was established – not quite, as the all-purpose parenthesis, the unplanned outcome of a practice:

> note was taken consciously perhaps by
> Bernoulli and in any case by Leibniz;

but still based on and called for by a practice that had developed spontaneously – *naturwüchsig*, in a word borrowed from a great German thinker.