A Note on Hybrid Modal Logic with Propositional Quantiers (Work in Progress)

Blackburn, Patrick Rowan; Braüner, Torben; Kofod, Julie Lundbak

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Preface

This volume is a collection of abstracts of presentations at **ICLA2021: 9th Indian Conference on Logic and its Applications** held online from March 4 to 7, 2021.

The *Indian Conference on Logic and its Applications* series aims to bring together researchers from a wide variety of fields that formal logic plays a significant rôle in, along with mathematicians, philosophers, computer scientists, linguists and logicians studying foundations of formal logic in itself. A special feature of this conference comprises of studies in systems of logic in the Indian tradition, and historical research on logic. The biennial conference is organized by the Association for Logic in India.

The papers in the volume span a wide range of themes. We have contributions to algebraic logic and model theory and philosophical logic. Modal logics, with applications to computer science and game theory, are discussed.

Like the previous conferences (the last one at IIT-Delhi in March 2019), the ninth conference also manifested the confluence of several disciplines of logic. As in the previous years, we were fortunate to have a number of highly eminent researchers giving plenary talks. It gives us great pleasure to thank Maria Aloni (University of Amsterdam), Nick Bezhanishvili (University of Amsterdam), Agata Ciabattoni (TU Wien), Adnan Darwiche (University of California, Los Angeles), Hans van Ditmarsch (LORIA), Julia Knight (University of Notre Dame), Marta Kwiatkowska (University of Oxford) and Thomas Schwentick (TU Dortmund) for agreeing to give invited talks at ICLA 2021.

In addition, the conference featured two very interesting **Panel Discussions**, one on Logic education, moderated by Serafina Lapenta (University of Salerno) and the other titled Logic and experimental studies: A new paradigm or contradiction in terms?, moderated by Torben Braüner (Roskilde University). The former panel discussed a range of issues in teaching logic at various levels, from high school to the university, and tools for teaching logic. The latter addressed the question of how logic fits together (or does not fit together) with empirical studies. We thank the moderators of these panels as well as the panelists Viviane Durand-Guerrier (Université de Montpellier), Nina Gierasimczuk (Technical University of Denmark), Paula Quinon (Warsaw University of Technology), François Schwarzentruber (IRISA, Rennes), John Slaney (Australian National University) and Niels Taatgen (University of Groningen) for their participation. The discussions were thoughtful, and presented varied perspectives on content and practice of logic.

Since this was an online event, in a departure from the past, ICLA decided to experiment with a different mode of contributed papers, inviting short **extended abstracts**, rather than full papers, including on-going research work. A selection of papers from the conference will constitute a special issue of the *Journal of Logic, Language and Information*, constituting formal post-proceedings of the conference.

We express our gratitude to all members of the Programme Committee for their help in putting together the entire programme as well as reading the submitted extended abstracts and selecting the ones included in this volume.

Despite the pandemic, we have managed to hold our discussions in the community. We are grateful to the Zoom platform for enabling this online conference in troubled times, and the social platform *gather.town* for supporting informal chats.

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Sujata Ghosh
(Indian Statistical Institute, Chennai)
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March 2021

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Invited talks

A logic for pragmatic intrusion

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Abstract

In a state-based modal logic, formulas are interpreted with respect to sets of possible worlds rather than individual worlds. In my talk I will present a bilateral version of a state-based modal logic motivated by linguistic phenomena at the semantics-pragmatics interface, including phenomena of free choice.

In free choice inferences conjunctive meanings are derived from disjunctive sentences contrary to the prescriptions of classical logic:

You may eat pizza or pasta $\implies$ You may eat pizza and you may eat pasta

Free choice inferences present a challenge to the canonical divide between semantics and pragmatics. They are not validities in classical modal logic and although they are derivable by conversational principles, they lack other characteristic properties of canonical pragmatic inferences, they are often non-cancellable, they are sometimes embeddable and their processing time has been shown to equal that of literal interpretations.

In my bilateral state-based modal logic free choice and related inferences are derived by allowing pragmatic principles intrude in the recursive process of meaning composition. Contrary to most existing accounts where free choice inferences are viewed as special cases of quantity implicatures, the relevant pragmatic principle in our logic-based approach will be a version of Grice's Maxim of Quality.

Polyhedral modal logic

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Abstract

Spatial logic studies various spatial structures though the prism of logic. The celebrated McKinsey-Tarski theorem states that if we interpret modal diamond as closure and hence modal box as interior, then $S4$ is the modal logic of any dense-in-itself metric space. In particular, this implies that the modal logic of each Euclidean space is $S4$. However, we can distinguish the logics of Euclidean spaces of different dimensions by restricting the valuation to polyhedral subsets, resulting in polyhedral semantics of modal logic.

In this talk I will discuss this semantics in detail including some recent axiomatization and completeness results. I will also review some applications of this approach.

Normative reasoning in Mīmāṃsā: A deontic logic approach

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Abstract

Normative statements, which involve concepts such as obligation and prohibition, are enormously important in a variety of fields—from law and ethics to artificial intelligence. Reasoning with and about them requires deontic logic, which is a quite recent area of research. By contrast, for more than two millennia, one of the most important systems of Indian philosophy focused on analyzing normative statements. Mīmāṃsā, as it is called, looks at these statements found in the Vedas, the sacred texts of Hinduism, and interprets them by explaining precisely what course of action they require. In my talk I will discuss connections and synergies between Mīmāṃsā and deontic logic. The results I am going to present arise from a collaboration between logicians, sanskritists and computer scientists.

Reasoning about what was learned

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Abstract

The traditional role of logic in AI has been to represent and reason with categorical knowledge about the world, putting it in competition with other formalisms such as probabilistic reasoning that deals with uncertainty and machine learning that relies on learning from data instead of explicit knowledge. The landscape has been changing recently with logic playing a broader role that complements these formalisms. For example, logic and symbolic manipulations have been providing a computational foundation for probabilistic reasoning and new approaches have been emerging for learning from a combination of categorical knowledge and data. In this talk, I will briefly review these additional roles and then focus on the use of logic for meta reasoning about what was learned. This role of logic has been facilitated by new advances that compile the input-output behavior of some machine learning systems into logical representations, allowing one to explain the decisions made by these systems, reason about their robustness and validate some of their properties (e.g., whether they are biased or monotonic).

Dynamic epistemic logic for distributed computing: asynchrony and concurrency

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Abstract

We will present some recent work on asynchrony and concurrency in dynamic epistemic logics (DEL), building on foundations in distributed computing and combining threads in dynamic logics and temporal epistemic logics. One way to model asynchrony in DEL is to reasoning over histories of actions of different length. Within DEL there has been work on asynchronous protocol-generated forests along different depths of trees (older work by Witzel and others, recent work by Pinchinat and others), and hovering in between DEL and PDL there is more recent work applied to gossip protocols by Apt and others, and van Ditmarsch and others.

An entirely different emerging thread involving histories of different length is within combinatorial topology, namely DEL-motivated modellings of algorithms in distributed computing such as the immediate snapshot algorithm, by Rajsbaum and others.

A different kind of DEL asynchrony involves separating in the logical language the sending and receiving of messages, such as announcements. We present recent proposals by Knight and others, and Balbiani and others. Topics of interest there are axiomatization, reduction (elimination of dynamic modalities) and complexity, and also generalizations to communications between intersecting subgroups of agents building on older work involving Mukund and Ramanujam.

A modelling solution wherein knowing a proposition is identified with acknowledging receipt can be demonstrated by, what else, the *Muddy Children problem*. Finally, the Muddy Children problem also perfectly illustrates issues with concurrency in DEL: the action of n muddy children not stepping forward because none of them know whether they are muddy, is always modelled as the public announcement of a conjunction with n conjuncts. Should this not be a concurrent action with n components?

Describing structures and classes of structures

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Abstract

The talk will provide an overview of notions and results related to the following:

- describing a specific countable structure, up to isomorphism,

- describing a class of countable structures, closed under isomorphism,

- classifying the members of a given class, giving invariants that distinguish the members from each other, up to isomorphism.

The notions and results will all be illustrated with examples involving familiar kinds of mathematical structures: groups, fields, graphs.

Probabilistic model checking for strategic equilibria-based decision making

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Abstract

Software faults have plagued computing systems since the early days, leading to the development of methods based on mathematical logic, such as proof assistants or model checking, to ensure their correctness. The rise of AI calls for automated decision making that incorporates strategic reasoning and coordination of behaviour of multiple autonomous agents acting concurrently and in presence of uncertainty. Traditionally, game-theoretic solutions such as Nash equilibria are employed to analyse strategic interactions between multiple independent entities, but model checking tools for scenarios exhibiting concurrency, stochasticity and equilibria have been lacking. This lecture will focus on a recent extension of probabilistic model checker PRISM-games, which supports quantitative reasoning and strategy synthesis for concurrent multiplayer stochastic games against temporal logic that can express coalitional, zero-sum and equilibria-based properties. Game-theoretic models arise naturally in the context of autonomous computing infrastructure, including user-centric networks, robotics and security. Using illustrative examples, this lecture will give an overview of recent progress in probabilistic model checking for stochastic games and highlight challenges and opportunities for the future.

Dynamic Complexity: Basics and recent directions

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Abstract

In most real-life databases data changes frequently and thus makes efficient query answering challenging. Auxiliary data might help to avoid computing query answers from scratch all the time. One way to study this incremental maintenance scenario is from the perspective of dynamic algorithms with the goal to reduce (re-) computation time.

Another option is to investigate it from the perspective of logic. Since first-order logic corresponds to the core of the standard database query language SQL, one naturally arrives at the question which queries can be answered/maintained dynamically with first-order predicate logic ($DynFO$) [1,2]. This approach has been termed dynamic complexity in the literature [1].

The talk gives an introduction into dynamic complexity and presents and discusses old and recent methods and results [3,4,5,6,7,8].

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Panels

Panel on logic education

Viviane Durand-Guerrier (Université de Montpellier)
Serafina Lapenta (University of Salerno), **Moderator**
François Schwarzentruber (IRISA, Rennes)
John Slaney (Australian National University)

Abstract

The discussions in the logic education panel will centre on the place of logic in the university. This will include several themes:

- familiarising students from mathematics, philosophy, computer science and AI to topics and techniques in logic, attracting them to research in logic.

- identifying areas of relevance of logic to mathematics, the sciences and the social sciences.

- the use of tools in teaching logic (theorem provers, sat solvers etc).

- the place of logic in school curricula and advocacy for logic at school.

The presentations

1. Viviane Durand-Guerrier
   The place of logic in school curricula and advocacy for logic at school

   A common view is that the learning of mathematics at school is sufficient to develop the logical skills required by advanced mathematics studies. However there is research based-evidence that it is not the case. On another side, several authors acknowledge that teaching logic in separate module does not necessarily improve students' mathematical reasoning. This lead to introduce elements of logic inside the mathematics curriculum and to propose activities aiming to make students experience the intertwining between mathematics contents and logical skills. I will provide some examples of such activities, for class and for teacher training.

2. Francois Schwarzentruber
   Use of tools in teaching logic

   Logic is the root of computer science and has many direct applications: database, program design and verification, architecture, constraint solving, theorem proving, machine learning, etc. In this talk, we will make a tour on existing tools that shows the applicability of logics, and thus could motivate students to study logic. We then show how tools can be used for students to grasp theoretical foundations of logic. We will then discuss some guidelines for the future of tools for teaching logic.

3. John Slaney
   Introductory logic: opportunity and challenge

   This is a talk from my personal perspective, drawing on the experience of teaching elementary logic to university students. One of the main pedagogic difficulties we face is the diversity of backgrounds of our students: it is hard to pitch the course simultaneously

to philosophy, mathematics and computing majors, and with the increasing popularity of logic this diversity of backgrounds is also increasing. The online tool 'Logic for Fun' is intended to appeal to students across the range of disciplinary backgrounds, introducing them to the art of using the resources of first order logic to express and solve problems. I describe the tool and report briefly on the experience of using it over ten years of introductory logic courses.

Panel on logic and experimental studies: A new paradigm or contradiction in terms?

Torben Braüner (Roskilde University), **Moderator**
Nina Gierasimczuk (Technical University of Denmark)
Paula Quinon (Warsaw University of Technology)
Niels Taatgen (University of Groningen)

Abstract

The main idea is to discuss how logic fits together (or doesn't fit together) with empirical studies. By empirical studies we mean scientific studies modeled after experimental natural sciences like physics (not natural sciences of a more historical nature, like geology). Standard textbooks on the philosophy of science give demarcation criteria like verifiability and falsifiability. Cognitive psychology and cognitive science clearly tries to live up to such ideals inherited from natural science.

When we say *paradigm* we are not only referring to Thomas Kuhn (1962) in its original grandeur, but also to weaker notions of paradigm that might be more descriptive. The relation between logic and psychology has been discussed extensively since Frege, and is still continuing to play a role in our understanding of normative vis-a-vis descriptive enterprises. The natural questions one can ask: what would be an empirically informed logical formalism, or what would be a formalism informed experiment. In recent days, attempts have been made towards answering these questions: to this end, one could ask for a systematic study of these attempts in order to answer the query we put forward.

The presentations

1. Nina Gierasimczuk
   Approximate number sense and semantic universals: an experimental simulation study

   Languages vary in their systems of categories but the variation is constrained by several well-defined properties called universals. Explanations thereof have been sought in universal constraints of human cognition, communication, complexity, and pragmatics. In this walk we examine whether the perceptual constraints of approximate number sense (ANS) contribute to the development of two universals in the semantic domain of quantities: monotonicity and convexity. Using a state-of-the-art multi-agent language-coordination model, with the perceptual layer of agents substituted by the ANS, we evolve communicatively usable quantity terminologies. We compare the degrees of convexity and monotonicity of languages evolving with and without ANS. The results suggest that ANS supports the development of monotonicity and, to a lesser extent, convexity.

   This is a joint work with Dariusz Kalociński, Franciszek Rakowski and Jakub Uszyński. It was generously supported by N.G.'s National Science Centre Poland grant no. 2015/19/B/HS1/03292.

2. Paula Quinon
   The core cognitive paradigm and foundations of logic, arithmetic, and computation

Spelke in *Core knowledge* (2000) put forward the idea that complex cognitive skills and concepts can be based on a set of "building block" systems (called also "core knowledge" or "core cognition" systems) that emerge early in human ontogeny, play an important role in phylogeny of concepts, and are detected in non-human animals. Humans combine representations from these systems and build new complex concepts. I call the "core cognition paradigm" a framework that commits to the view that the conceptual content of a concept is scaffolded on core cognition.

The concept of natural number has been studied extensively within this paradigm and it has been established that the ability to subitize up to four elements, the ability to approximate quantities without counting, and the counting routine, play important roles in the formation and acquisition of the concept of natural number. A similar endeavor is currently being undertaken with respect to logically complex representations (such as logical connectives, logical rules, or quantifiers) and to computational representations (such as computable functions).

Regardless of which building blocks will be disclosed as playing a role in the formation of mature complex representations, both in logic and in computation, it will be interesting to answer the question of whether the core cognition paradigm can in any way help in understanding the nuances of formal concepts.

3. Niels Taatgen
   Cognitive architectures and predictive models

Cognitive architectures are formal systems that allow building simulations of human intelligent behavior. They encompass constraints derived from human cognition, such as memory, attention and decision making. Cognitive architectures are used to build cognitive models of particular cognitive phenomena, either to explain existing empirical data, or to make predictions about new data. I will illustrate this with an example in which we successfully predicted a bottleneck in human multitasking.

Contributed talks

On Computability-Theoretic Properties
of Heyting Algebras[*]

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Abstract

We prove that the class K containing all Heyting algebras with distinguished atoms and coatoms is complete with respect to effective dimensions. This implies that there is no simple syntactic characterization of computably categorical members of K (i.e. structures from K possessing a unique computable copy, up to computable isomorphisms). This contrasts with Remmel's result that gives a simple algebraic characterization of computably categorical Boolean algebras with distinguished atoms.

1 Introduction

The paper studies algorithmic properties of countable Heyting algebras. Working within the framework of computable structure theory [2, 6], we consider computable copies of Heyting algebras. A structure $\mathcal{S}$ is *computable* if its domain $\mathrm{dom}(\mathcal{S})$ is a (Turing) computable subset of the set of natural numbers ω, and the signature operations and predicates of $\mathcal{S}$ are uniformly computable.

A lot of familiar countable algebraic structures have computable isomorphic copies. A standard example of a computable structure is the semiring of natural numbers $(\omega; +, \cdot)$. Another typical example is provided by the ordered field of rationals $(\mathbb{Q}; +, \cdot, \leq)$ — one can build a computable copy of the field by effectively encoding irreducible fractions via natural numbers. Another class of examples comes from group theory: a finitely presented group G has a computable copy if and only if the word problem for G is decidable.

From the point of view of the classical algebra, isomorphic copies of the same structure $\mathcal{S}$ have the same algebraic properties. Nevertheless, in computable structure theory, two isomorphic computable copies of $\mathcal{S}$ can exhibit strikingly different algorithmic behavior. One of the first examples of this phenomenon was witnessed by Fröhlich and Shepherdson [8]: they proved that there exist isomorphic computable fields F and K such that F has a splitting algorithm, but G has no splitting algorithm.

Motivated by these kinds of phenomena, Mal'tsev [13] introduced the notion of *computable categoricity*. A computable structure $\mathcal{S}$ is *computably categorical* (or autostable) if for every computable copy $\mathcal{A}$ of $\mathcal{S}$, there is a computable isomorphism f (i.e. an isomorphism, which is also a computable function) from $\mathcal{A}$ onto $\mathcal{S}$. Roughly speaking, if a structure $\mathcal{S}$ is computably categorical, then *all* computable copies of $\mathcal{S}$ have similar algorithmic behavior. More formally, one can say that $\mathcal{S}$ possesses a unique computable copy, up to computable isomorphisms.

Since the works of Mal'tsev [12, 13], the following classification problem has been one of the main driving forces behind the developments of computable structure theory: Obtain a description of computably categorical structures from familiar algebraic classes.

We give several examples of known results in this line of research. A computable linear order is computably categorical if and only if it has only finitely many pairs of adjacent elements [10, 17]. A computable Boolean algebra is computably categorical if and only if its set of atoms is finite [10, 16]. In general, it is impossible to provide a nice algebraic characterization of computably categorical structures: the paper [5] proves that the index set of computably categorical graphs is m-complete Π^1_1. Informally speaking, this means that there is no simpler way to syntactically describe computable categoricity than the original definition given by Mal'tsev.

A *Boolean algebra with distinguished atoms* is a structure of the form $(\mathcal{B}, \mathrm{At})$, where $\mathcal{B}$ is a Boolean algebra, and the unary predicate At is precisely the set of atoms of $\mathcal{B}$. Remmel [15] (see also Theorem 5.16 of [18]) proved that a computable Boolean algebra with distinguished atoms $(\mathcal{B}, \mathrm{At})$ is computably categorical if and only if $\mathcal{B}$ is isomorphic to a finite product of the following Boolean algebras: the countable atomless algebra, the algebra of finite and cofinite subsets of ω, and finite algebras.

In this extended abstract, we consider the class of Heyting algebras with distinguished atoms and coatoms, denoted by HA_{AtCoat}. We sketch the proof of the following result (to be explained in Section 2):

Theorem 1. *The class HA_{AtCoat} is complete with respect to effective dimensions.*

The proof of this theorem implies that the result of [5] mentioned above can be transferred into our setting: the index set of computably categorical members of HA_{AtCoat} is m-complete Π^1_1. This contrasts with the result of Remmel [15] above (to be elaborated in Remark 2.1). Informally speaking, Theorem 1 shows that the class of Heyting algebras provides a richer computability-theoretic environment than the class of Boolean algebras.

2　Preliminaries and Discussion

We refer the reader to [2, 6] for the background on computable structure theory. Let $\mathcal{S}$ be a computable structure, and let $\mathbf{d}$ be a Turing degree. The $\mathbf{d}$-*computable dimension* of $\mathcal{S}$, denoted by $\dim_{\mathbf{d}}(\mathcal{S})$, is the number of computable copies of $\mathcal{S}$, up to $\mathbf{d}$-computable isomorphisms. If $\mathbf{d} = \mathbf{0}$, then we omit the prefix $\mathbf{d}$- and talk about computable dimension.

A class of structures K is *complete with respect to effective dimensions* if for every computable structure $\mathcal{S}$, there exists a computable structure $\mathcal{A}_{\mathcal{S}} \in K$ such that for any Turing degree $\mathbf{d}$, we have $\dim_{\mathbf{d}}(\mathcal{A}_{\mathcal{S}}) = \dim_{\mathbf{d}}(\mathcal{S})$.

Heyting algebras are treated as algebras in the signature $\sigma_{HA} = \{\vee, \wedge, \rightarrow, 0, 1\}$. The signature of Boolean algebras is equal to $\sigma_{BA} = \{\vee, \wedge, \overline{(\ )}, 0, 1\}$. For the preliminaries on Heyting algebras, we refer to the book [7]. Structures from the class HA_{AtCoat} are considered in the signature $\sigma_0 = \sigma_{HA} \cup \{\mathrm{At}, \mathrm{Coat}\}$.

Remark 2.1. Note that any nontrivial Boolean algebra $\mathcal{B}$ satisfies the following: an element a is a coatom of $\mathcal{B}$ if and only if its complement $\overline{(a)}$ is an atom of $\mathcal{B}$. This implies that the structure $(\mathcal{B}, \mathrm{At}, \mathrm{Coat})$ is computably categorical if and only if $(\mathcal{B}, \mathrm{At})$ is computably categorical. Using the result of Remmel [15], one can deduce that the index set I of computably categorical Boolean algebras with distinguished atoms and coatoms is arithmetical. Clearly, I cannot be Π^1_1-complete.

One can recover another computability-theoretic difference between Boolean algebras and Heyting algebras. Recall that Goncharov [9] proved that for any natural number $n \geq 2$, there is a computable structure having computable dimension n. Therefore, Theorem 1 implies that

computable structures from the class HA_{AtCoat} can have arbitrary finite computable dimension. On the other hand, Alaev (Corollary 2 of [1]) showed the following. Consider a computable structure $\mathcal{C}$, which is obtained by enriching a Boolean algebra $\mathcal{B}$ by unary predicates for some ideals — $I_1, I_2, \ldots, I_k$, and for the sets of atoms with respect to these ideals — $\mathrm{At}_1, \mathrm{At}_2, \ldots, \mathrm{At}_k$. Then the computable dimension of $\mathcal{C}$ is either one, or ∞.

We note that a modification of the proof of Theorem 1 provides a *stronger* computability-theoretic result: one can show that the class HA_{AtCoat} is HKSS-complete (see [11] and Section 4.3 of [14] for the formal definition of HKSS-completeness). Nevertheless, for the sake of clarity of exposition, here we focus only on effective dimensions. More technical details can be recovered in a way similar to the papers [3, 4], which consider some natural classes of enrichments of Boolean algebras.

3 Proof Sketch for Theorem 1

Here we concentrate on a more "algebraic" part of the proof. The omitted "computability-theoretic" details are somewhat technical, but still pretty straightforward.

Let G be a computable, undirected (i.e. symmetric and irreflexive) graph with domain ω. By Edge we denote the edge relation inside G. We build a Heyting algebra $\mathcal{H}_G$, our construction proceeds in three stages.

<u>Stage 1.</u> We define a poset $R[G]$ as follows. The domain of $R[G]$ equals:

$$\mathrm{dom}(R[G]) = \{a_i : i \in \omega\} \cup \{b_{i,j}, c_{i,j}, d_{i,j}, e_{i,j} : i < j, \ i, j \in \omega\}.$$

The ordering $\preceq$ on $\mathrm{dom}(R[G])$ is obtained as the least partial order, which contains the following relations: for all $i < j$,

- we have $a_i \prec b_{i,j}$, $a_j \prec b_{i,j}$, $b_{i,j} \prec c_{i,j} \prec e_{i,j}$, $b_{i,j} \prec d_{i,j} \prec e_{i,j}$;

- if $G \models \mathrm{Edge}(i, j)$, then we additionally set $c_{i,j} \prec d_{i,j}$.

Recall that a *closure algebra* is a structure $(\mathcal{B}, \mathrm{Cl})$, where $\mathcal{B}$ is a Boolean algebra, and the closure operator $\mathrm{Cl}\colon \mathcal{B} \to \mathcal{B}$ satisfies the following axioms: $\mathrm{Cl}(0) = 0$, and for all $a, b \in \mathcal{B}$, we have $a \leq \mathrm{Cl}(a)$, $\mathrm{Cl}(\mathrm{Cl}(a)) \leq \mathrm{Cl}(a)$, and $\mathrm{Cl}(a \vee b) = \mathrm{Cl}(a) \vee \mathrm{Cl}(b)$.

For a closure algebra $(\mathcal{B}, \mathrm{Cl})$, the *interior operator* $\mathrm{Int}\colon \mathcal{B} \to \mathcal{B}$ is defined as follows: $\mathrm{Int}(a) := \overline{(\mathrm{Cl}(\overline{a}))}$. An element $a \in \mathcal{B}$ is *open* if $\mathrm{Int}(a) = a$.

For a set X, by $P(X)$ we denote the set of all subsets of X.

<u>Stage 2.</u> Given the poset $R[G]$, we define a closure algebra $\mathcal{C}[G]$.

It is known that the algebra $\mathcal{C}_{\mathrm{Cl}} = (P(\mathrm{dom}(R[G])), \cup, \cap, \overline{(\ )}, \emptyset, \mathrm{dom}(R[G]), \mathrm{Cl}_{\preceq})$, where

$$\text{for a set } X \subseteq \mathrm{dom}(R[G]), \ \mathrm{Cl}_{\preceq}(X) = \{y : y \preceq x \text{ for some } x \in X\},$$

is a closure algebra (see, e.g., Proposition 2.2.8 of [7]). Note that $\mathcal{C}_{\mathrm{Cl}}$ has cardinality continuum.

Let $\mathcal{B}$ be the Boolean algebra containing all finite and cofinite subsets of $\mathrm{dom}(R[G])$. We prove that $\mathcal{B}$ is closed with respect to the operation $\mathrm{Cl}_{\preceq}$. In order to establish this, it is sufficient to show the following: for any element $x \in R[G]$, the set $\mathrm{Cl}_{\preceq}(\{x\})$ is finite. If $x = a_i$ for some i, then clearly $\mathrm{Cl}_{\preceq}(\{x\}) = \{x\}$. Otherwise, x belongs to $\{b_{i,j}, c_{i,j}, d_{i,j}, e_{i,j}\}$ for some $i < j$. Then $\mathrm{Cl}_{\preceq}(\{x\}) \subseteq \{a_i, a_j, b_{i,j}, c_{i,j}, d_{i,j}, e_{i,j}\}$, and $\mathrm{Cl}_{\preceq}(\{x\})$ is finite.

Hence, we deduce that $(\mathcal{B}, \mathrm{Cl}_{\preceq})$ is a closure subalgebra of $\mathcal{C}_{\mathrm{Cl}}$. We put $\mathcal{C}[G] := (\mathcal{B}, \mathrm{Cl}_{\preceq})$.

<u>Stage 3.</u> Now we are ready to define the desired Heyting algebra $\mathcal{H}_G$. This is the algebra of all open elements of the closure algebra $\mathcal{C}[G]$ (see, e.g., Proposition 2.2.4 of [7]). Its Heyting

operations are the standard set-theoretic union $\cup$ and intersection $\cap$, and the implication is defined as follows: $a \to b = \overline{\mathrm{Cl}(a \cap \overline{(b)})}$.

A straightforward computability-theoretic analysis of the construction shows that there is a uniform procedure, which given a computable graph G, produces a computable copy of the Heyting algebra $\mathcal{H}_G$. Thus, without loss of generality, we may assume that the algebra $\mathcal{H}_G$ itself is computable.

Recall that every element of $\mathcal{H}_G$ is either finite or cofinite subset of $\mathrm{dom}(R[G])$. It would be useful to keep this intuition in mind. Let D denote the set $\mathrm{dom}(R[G])$.

Lemma 3.1. *Suppose that v is either finite or cofinite subset of D.*

1. *$v \in \mathcal{H}_G$ if and only if v is upwards $\preceq$-closed, i.e.*

$$v = \{y \in R[G] : x \preceq y \text{ for some } x \in v\}. \tag{1}$$

2. *An element $v \in \mathcal{H}_G$ is an atom if and only if $v = \{e_{i,j}\}$ for some $i < j$.*

3. *An element $v \in \mathcal{H}_G$ is a coatom if and only if $v = D \setminus \{a_i\}$ for some $i \in \omega$.*

For reasons of space, the proof of Lemma 3.1 is omitted.

Lemma 3.1 implies that the structure $\mathcal{H}_G^* := (\mathcal{H}_G, \mathrm{At}, \mathrm{Coat})$ is *computable*. Note that item (3) of Lemma 3.1 says the following: the set $\{D \setminus \{a_i\} : i \in \omega\}$ is definable inside $\mathcal{H}_G^*$ via the quantifier-free formula $\mathrm{Coat}(z)$. Hence, it is natural to consider the following (informal) encoding: a node i from the original graph G is encoded by the coatom $D \setminus \{a_i\}$ of Heyting algebra $\mathcal{H}_G$.

The next lemma clarifies the properties of this encoding. It shows that inside the structure $\mathcal{H}_G^*$, our encoding induces an isomorphic copy of the graph G. The copy is definable via both existential and universal formulas (without parameters).

Lemma 3.2. *There are existential σ_0-formulas $\psi_0(u, v)$ and $\psi_1(u, v)$ (without parameters) such that for all $i < j$, we have:*

$$G \models \mathrm{Edge}(i, j) \iff \mathcal{H}_G^* \models \psi_0(D \setminus \{a_i\}, D \setminus \{a_j\}) \iff \mathcal{H}_G^* \models \neg\psi_1(D \setminus \{a_i\}, D \setminus \{a_j\}). \tag{2}$$

Proof. First, we note the following: for an atom $m = \{e_{i,j}\}$, its pseudocomplement $m \to \emptyset$ is the $\subseteq$-greatest among upwards $\preceq$-closed sets that do not contain $e_{i,j}$. Hence, $(m \to \emptyset) = D \setminus \{a_i, a_j, b_{i,j}, c_{i,j}, d_{i,j}, e_{i,j}\}$.

The argument above implies that for any $i < j$, there is a unique atom m such that $D \setminus \{a_i, a_j\} \supseteq (m \to \emptyset)$. This m equals $D \setminus \{a_i, a_j, b_{i,j}, c_{i,j}, d_{i,j}, e_{i,j}\}$. Consider a half-open interval $I := \,](m \to \emptyset); D \setminus \{a_i, a_j\}]$ inside our Heyting algebra. It is not hard to see the following:

- If $G \models \mathrm{Edge}(i, j)$, then I is a chain containing four elements.

- If $G \not\models \mathrm{Edge}(i, j)$, then I is isomorphic to the finite Boolean lattice with two atoms.

Hence, we define:

$$\psi_0(u, v) = \exists m \exists w_1 \exists w_2 \exists w_3 [\mathrm{At}(m) \,\&\, (m \to \emptyset) \subset w_1 \subset w_2 \subset w_3 \subset u \cap v],$$
$$\psi_1(u, v) = \exists m \exists w_1 \exists w_2 \exists w_3 [\mathrm{At}(m) \,\&\, (m \to \emptyset) \subset w_1 \subset w_2 \subset u \cap v \,\&\, w_1 \subset w_3 \subset u \cap v$$
$$\&\, (w_2 \text{ and } w_3 \text{ are incomparable w.r.t. } \subseteq)].$$

It is not difficult to show that these formulas satisfy (2). $\qquad \square$

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Lemma 3.2 provides a nice first-order interpretation of G inside $\mathcal{H}_G^*$. After that, a technical argument shows the following: for any Turing degree $\mathbf{d}$, we have $\dim_{\mathbf{d}}(\mathcal{H}_G^*) = \dim_{\mathbf{d}}(G)$. Since the class of undirected graphs is complete with respect to effective dimensions (see, e.g., Theorem 3.1 of [11]), the proof of Theorem 1 is complete. $\square$

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Aggregating Relational Structures

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1   Introduction

How to "fairly" aggregate information/preferences from multiple sources into a collective opinion/preference? How to define fairness, and are some definitions of it unattainable?

With the seminal result in [1], Arrow introduced the problem of aggregating voters' preferences into a collective preference ordering over a set of alternatives (candidates) and showed that no such aggregation could satisfy a reasonable sounding definition of fairness, laying the foundations for social choice theory. There have been numerous extensions of the result, all fundamentally attempting to (re)define "fairness" over a new/broader problem. Results in this domain are typically either of:

- Negative results that prove a certain definition of fairness is unattainable.

- Positive results that propose new aggregation algorithms and prove their properties.

As computational tools make it easy to survey and assess data, summarisation and aggregation tools will be required to make inferences or decisions based on observed relationships such as planning city routes given access to graphs corresponding to each pedestrian's activity. As examples of applied research taking note from social choice theory, in [2] and [8] we see aggregation arising as a key concern in group recommender systems and parsing user review data. [3] gives historical context on the aggregation problem and summarises some current applications.

Some important negative results for aggregating partial orders and equivalence relations are given in [7] and [5] respectively. In [4], the impossibility result is established for the general binary relation/graph setting - significant progress from results for specific relations such as linear orders or equivalence relations. However, not all information worth aggregating can be adequately represented with binary relations. Consider the relationship of "knowing" someone on a social network inferred from being tagged in the same picture, a group of three friends in the same photo carries more information than all combinations of pairs and can be best expressed by a 2-simplex. It is thus valuable to extend the result to mathematical domains that allow representing more social choice scenarios. Attempts to extend the result to the setting of aggregating committees, i.e., subsets as preferences are made in [6]. With a similar motivation of addressing problems beyond binary relations, we attempt to prove the result for relations of arbitrary arity, which differs from the subset setting by being sensitive to permutation of tuples of "candidates" to be aggregated.

2 Aggregating k-ary Relations

Consider the predicate language $\mathcal{L}$ consisting of a single predicate symbol R of arity $k \geq 3$. Also fix a non-empty finite set A of candidates. We will deal with models of some $\mathcal{L}$-theory T, to be specified later, with universe A. We denote the collection of models of the theory T by $\mathcal{M}(A)$.

A social choice situation over the pair $(\mathcal{L}, A)$ consists of the following. Let $\mathcal{I}$ denote a set of voters/individuals. Each voter chooses $\mathcal{A}_i \in \mathcal{M}(A)$. An aggregation rule (also known as a social welfare function) is a map $\sigma : \mathcal{D} \subseteq \mathcal{M}(A)^{\mathcal{I}} \to \mathcal{M}(A)$ that satisfies some desirable properties, where $\mathcal{D}$ is the set of allowed ballots or profiles. By appropriately choosing T and the properties of the aggregation rule, we will prove that the only legitimate choice of the aggregated structure is either a filtered product or an ultraproduct of $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$; the latter case corresponds to a "dictatorship" when $\mathcal{I}$ is finite.

In order to describe the properties of k-ary relations we begin by setting up some notations.

- We refer to arbitrary tuples of length k as $\bar{a}_k = (a_1, ..., a_k)$.

- $(\bar{a}_k)_r^{+\bar{i}, -\bar{j}}$ for $\bar{i} = (i_1, ..., i_v)$ and $\bar{j} = (j_1, ..., j_w)$, refers to the set of subsequences of $\bar{a}_k$ of length r with elements $a_{i_1}, ..., a_{i_v}$ necessarily present and elements $a_{j_1}, ..., a_{j_w}$ necessarily absent. In a slight abuse of notation, when i or j are single-membered they will simply be referred to as $(\bar{a}_k)_r^{+i, -j}$.

- $(\bar{a}_{k+1})^{-j} \mid b$ refers to the singleton element in $(a_1, ..., a_j, b, a_{j+1}, ..., a_k)_k^{-j}$, essentially replacing the jth element of the tuple by b.

- For a permutation τ (τ on $\bar{a}_k$ is just a bijective map from $\{1, ..., k\}$ to itself), $\bar{a}_k^{\tau}$ refers to the tuple $(a_{\tau(1)}, ..., a_{\tau(k)})$.

Now we define some desirable properties of k-ary relations that will constitute theory T. A k-ary relation R is called

- **connected** if, for each pairwise-distinct $\bar{a}_k \in A$, there is a permutation τ of $\{1, ..., k\}$ such that $\bar{a}_k^{\tau} \in R$.

- **exclusive** if for each pairwise-distinct $\bar{a}_k \in A$, there is a permutation τ of $\{1, ..., k\}$ such that $\bar{a}_k^{\tau} \notin R$.

- **simplicial transitive** if for each sequence of pairwise-distinct elements $\bar{a}_{k+1}$ for each $j \in \{1, ..., k+1\}$ if $(\bar{a}_{k+1})_k^{+j} \subseteq R$ then $(\bar{a}_{k+1})_k^{-j} \in R$.

Below we present natural aggregation scenarios to justify that our choice of properties is not arbitrary.

- **Moderate Voters** Consider a collection of voters and a group of electoral candidates. Each voter interprets the political inclination of each candidate as left or right leaning compared to the others resulting in a total order over the set of candidates. If each voter prefers the moderate candidate in a group of 3 candidates, then this voting behaviour can be captured by a "betweenness" relation, with $(a, b, c) \in R_i \leftrightarrow (c, b, a) \in R_i$ representing the ith voters preference for b over a and c. This relation is connected, exclusive as well as simplicial transitive.

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- **Seating along a circular table** Consider a party of dinner guests to be seated on circular table in groups of 4 (any cyclic arrangement is fine) and preferences over how every subsets of 4 people should be seated. This quaternary relation R on the party of dinner guests is such that if $(a, b, c, d) \in R$ then all cyclic permutations $(b, c, d, a), (c, d, a, b), (d, a, b, c) \in R$ and no other permutation of $\{a, b, c, d\}$ is in R. This relation also is connected, exclusive and simplicial transitive.

With the motivation and notation in place, we can define relevant properties for the aggregation rule and state our first result.

(**UD**) $\forall a_1, ..., a_k \in A. \forall p \in \mathcal{M}(\{a_1, ..., a_{k+1}\})^{\mathcal{I}}.\exists q \in \mathcal{D}.q|_{a_1,...,a_k} = p$

(**P**) $\forall \bar{a}_k \in A^k. \forall p \in \mathcal{D}.(\forall i \in \mathcal{I}.p_i \models R[\bar{a}_k]) \Rightarrow \sigma(p) \models R[\bar{a}_k]$

(**IIA**) $\forall \bar{a}_k \in A^k. \forall p, q \in \mathcal{D}.(\forall i \in \mathcal{I}.p_i \models R[\bar{a}_k] \Leftrightarrow q_i \models R[\bar{a}_k]) \Rightarrow (\sigma(p) \models R[\bar{a}_k] \Leftrightarrow \sigma(q) \models R[\bar{a}_k])$

(**D**) $\exists i \in \mathcal{I}.\forall \bar{a}_k \in A^k. \forall p \in \mathcal{D}.(p_i \models R[\bar{a}_k] \Leftrightarrow \sigma(p) \models R[\bar{a}_k])$

Theorem 1. *Let* $(A, \mathcal{I}, \mathcal{D}, \sigma)$ *be a social choice situation over k-ary relation R language $\mathcal{L}$ with* $|A| \geq k + 1$, *satisfying* **UD**, **P**, *and* **IIA** *where R is simplicial/path transitive, exclusive, and connected. Then for finite $\mathcal{I}$, it also satisfies* **D**.

3   Metaproperties for k-ary relations

The above result relies on the specific definitions of properties that we started with, restricting its usefulness in applications. Two ways this has been previously addressed are:

- In [6], the authors establish a framework using a SAT solver to assist with the proof, making it easier to establish the result for variations of the original definitions.

- In [4], the authors attempt to characterize sufficient conditions on the definitions for the proof to go through, reducing the burden of proof to establishing that the problem definitions satisfy certain "metaproperties".

We rely on the second approach, adapting [4]'s definitions to our setting and getting inspiration on the definition of metaproperties.

On the fixed set A, we talk about k-ary relations $R \subseteq A^k$. Since such relations can be aptly described as uniform directed k-hypergraphs, we interchangeably call them U_k-graphs, for short. Any property P of k-ary relations can be identified with the collection of all relations satisfying the property, i.e., a subset P of $\mathcal{P}(A^k)$.

The generalizations to U_k-graphs of properties of aggregation rules described in [4], namely unanimity, groundedness, independecne of irrelevant alternatives (IIE), and collective rationality with respect to a property P of k-ary relations are straightforward.

It is also natural to extend the definitions of contagious, implicative, and disjunctive properties to U_k-graph properties P. We also show that simplicial transitivity is contagious and implicative and that connectedness is disjunctive.

This set-up allows us to prove the following k-ary analogue of the **dictatorship theorem** [4, Theorem 16].

Theorem 2. *For $|A| \geq k + 1$, any unanimous, grounded, and IIE aggregation rule for k-ary relations that is collectively rational with respect to a contagious, implicative, and disjunctive property must be dictatorial on pairwise distinct tuples.*

4 Future directions

A lot of applications model relationships as simplicial complexes with bounded dimension. Apart from aggregating social relationships, simplicial complexes are also useful in diverse areas including rendering $3D$-graphics. Such aggregation problems could arise naturally in decentralized computing setups when each unit produces a simplicial complex built on a predetermined grid of points.

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A Note on Hybrid Modal Logic
with Propositional Quantifiers
(Work in progress)

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1 Introduction

Hybrid logic extends standard modal logic so it can refer to worlds. It does so using *nominals*, a second kind of propositional symbol, usually written i, j, and k, to distinguish them from the p, q, and r used for ordinary propositional symbols. Nominals are true at one and only one world in any model, so a nominal is an atomic 'propositional term' that names a world (or time, or state, or...); here we call such symbols *standard nominals*. Arthur Prior introduced early forms of hybrid logic in the 1950s and 60s; see [4] for background.

Sometimes, rather than introducing nominals as a second kind of propositional symbol, Arthur Prior would create them using the Q operator: Qp, the result of prefixing the ordinary propositional symbol by Q operator, converted p to a nominal. As Prior put it in [6], page 237:

> For 'p is an individual' (or an instant, or a possible total world-state) we write Qp.
> If we have propositional quantifiers, we can define Qp thus:
> $$Qp = \diamond p \wedge \forall q(\square(p \to q) \vee \square(p \to \neg q))$$

Here the $\square$ and $\diamond$ are the box and diamond forms of the *universal modality*: $\square$ means *true at all worlds* and $\diamond$ means *true at some world*. Thus one way of reading this definition is that Qp says that p is both *possible* and *maximal*: p is true *somewhere* and in addition p strictly implies *every* proposition q or its negation. Read this way — taking the quantifier as ranging across all sets of world — Qp is a standard nominal.

But Prior's definition is ambiguous: there are *two* mathematically well understood ways of interpreting a propositional quantifier like $\forall q$. The first is to interpret it as quantifying across all subsets of the set of possible worlds (that is: as quantifying across all propositions). This is called the *standard* interpretation, and it is the standard interpretation that gives rise to standard nominals. But we can also interpret propositional quantifiers as ranging over a pre-selected set of subsets of worlds. These are usually called the *admissible subsets* or the *admissible propositions*. This interpretation traces back to Leon Henkin's pioneering work on higher-order logic in the 1950s, and is often called the *general* semantics; this alternative interpretation became important in modal logic in in 1972, when S. K. Thomason introduced *general frames* as part of his work on incomplete modal logics.

The distinction between the standard and the general semantics is of direct relevance to Prior's definition of the Q operator: when interpreted standardly we get standard nominals, but when interpreted according to the general semantics, we get something interestingly different; here we call them *non-standard nominals*. In this note we explore this distinction by working with a basic hybrid language enriched with propositional quantifiers. Thus we will have standard nominals and — because of the propositional quantifiers — we will be able to define Prior's Q operator and hence non-standard nominals too.

2 Syntax and semantics

The language we shall use in this paper is called $\mathcal{L}_{BHPQ}$. Choose (disjoint) countably infinite sets $PROP = \{p, q, r, \ldots\}$ and $NOM = \{i, j, k, \ldots\}$, the propositional symbols and standard nominals respectively. The formulas of $\mathcal{L}_{BHPQ}$ are:

$$\varphi ::= p \mid i \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid @_i\varphi \mid \forall p\varphi$$

where $p \in PROP$ and $i \in NOM$. Other booleans are defined as usual, and $\Diamond\varphi$ and $\exists p\varphi$ are defined as $\neg\Box\neg\varphi$ and $\neg\forall p\neg\varphi$ respectively. Note that standard nominals are used in two syntactically distinct ways: if i appears as a subscript to @, then we say it occurs in *operator position* and if it occurs as an atomic symbol, then we say it occurs in *formula position*.[1] We assume the usual distinction between free and bound symbols. When we later discuss universal instantiation we shall write $\phi[\chi/q]$ for the result of substituting a formula χ for free occurrences of a propositional symbol q in a formula ϕ we assume that all such substitutions are carried out sensibly (that is: avoiding accidental binding).

So much for the syntax; let us turn to the formal semantics. We shall interpret $\mathcal{L}_{BHPQ}$ with a Henkin-style semantics where the propositional quantifiers need not range over all subsets of the set of worlds. Accordingly, the semantics is given using S.K. Thomason's *general semantics*, which makes use of *general frames* and *general models*.

Definition 1. *A general frame is a triple $\langle W, R, \Pi \rangle$ where W is a non-empty set (worlds), R is a binary relation on W (the accessibility relation) and Π is a non-empty collection of subsets of W (the admissible subsets) closed under the following operations:*

- *relative complement: if $X \in \Pi$, then $W - X \in \Pi$*
- *intersection: if $X, Y \in \Pi$, then $X \cap Y \in \Pi$*
- *modal projection: if $X \in \Pi$, then $\{w \in W : \forall v(wRv \rightarrow v \in X)\} \in \Pi$*

Note that when we work with a general frame with $\Pi = \mathscr{P}(W)$, then we are in effect working with a standard frame; general semantics contains the standard semantics as a special case. However, we do not need to work with all possible subsets, just with collections of subsets of worlds with 'enough logical structure'. This is what the closure conditions ensure.

Definition 2. *A general model $\mathfrak{M}$ based on a general frame $\langle W, R, \Pi \rangle$ is a tuple $\langle W, R, \Pi, N, V \rangle$ where $N : NOM \rightarrow W$ and $V : PROP \rightarrow \Pi$. The truth conditions are as follows:*

$$
\begin{array}{lll}
\mathfrak{M}, w \models p & \text{iff} & w \in V(p) \text{ where } p \in PROP \\
\mathfrak{M}, w \models i & \text{iff} & w = N(i) \\
\mathfrak{M}, w \models \neg\varphi & \text{iff} & \text{it is not the case that } \mathfrak{M}, w \models \varphi \\
\mathfrak{M}, w \models \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models \Box\varphi & \text{iff} & \text{for all } v \in W \text{ such that } wRv, \text{ we have } \mathfrak{M}, v \models \varphi \\
\mathfrak{M}, w \models @_i\varphi & \text{iff} & \mathfrak{M}, N(i) \models \varphi \\
\mathfrak{M}, w \models \forall p\varphi & \text{iff} & \text{for all } \mathfrak{M}' = \langle W, R, \Pi, N, V' \rangle \text{ such that } V'(q) = V(q) \\
& & \text{whenever } q \neq p, \text{ we have } \mathfrak{M}', w \models \varphi
\end{array}
$$

Apart from occasional side remarks, in what follows we shall confine our attention to universally related general frames, that is, to general frames $\langle W, R, \Pi \rangle$ where $R = W \times W$. We will say that a formula φ is a *general validity* on a general (universally related) frame $\langle W, R, \Pi \rangle$

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[1] A formula of the form $@_i\varphi$ says that φ is true at one particular world, namely the world the standard nominal i refers to, and similarly for j, k and so on.

iff for any model $\langle W, R, \Pi, N, V \rangle$ and any $w \in W$ we have that $\langle W, R, \Pi, N, V \rangle, w \models \varphi$, and we will say it is a *general validity* iff it is valid on all such frames. Similarly, we will say that a formula φ is a *standard validity* on a standard (universally related) frame $\langle W, R, \Pi \rangle$ iff for any model $\langle W, R, \Pi, N, V \rangle$ and any $w \in W$ we have that $\langle W, R, \Pi, N, V \rangle, w \models \varphi$, and we will say it is a *standard validity* iff it is valid on all such frames.

3 Fine's formula

In this section we shall make some semantic remarks centered around a formula noted by Kit Fine in an important early paper on modal logic with propositional quantifiers [5].
$$(*) \quad \exists p (p \wedge \forall q (q \rightarrow \Box(p \rightarrow q))).$$

This is relevant here because (1) it is a standard validity, (2) it is not a general validity, (3) it is a general validity on any general frame satisfying a *discreteness* property, (4) is also general validity on any general frame satisfying an *atomicity* property and (5) it is closely related to the formula Prior used to define Q. Let's look a little closer at all five claims.

First, Fine's formula (*) is true at any world in any standard model at all (whether universally related or not). For let w be the world of evaluation. Let the outermost existential quantifier pick out the singleton set $\{w\}$; the truth of (*) follows immediately. As w was arbitrary, we can never falsify this formula on (any) standard model.

Second, Fine's formula *can* be falsified on some general models (so it is not a general validity). We won't prove this, but point the reader to the second general frame in Example 5.67, page 307 in [2], which falsifies (*) on a general frame when we interpret $\Box$ and $\Diamond$ using $W \times W$. This is actually an instructive example of a general frame with an element that does not belong to an atom, namely the element ∞, because any admissible set containing ∞ is co-finite, meaning that a smaller admissible set containing ∞ can always be found.[2]

Third, Fine's formula is valid on any *discrete* general frame, that is, on general frames where all singleton sets of worlds are admissible (that is, where for any $w \in W$, we have $\{w\} \in \Pi$).[3] This is easy to see — just use the argument used in our first observation above.

Fourth — as Fine points out in his paper — (*) says that the set of propositions must be *atomic* over the set of worlds; we won't prove this here, but we will explain the terminology. A general frame is *atomic* when every $w \in W$ belongs to some minimal non-empty element of Π. Clearly standard frames and discrete frames are atomic as $\{w\}$ is the required minimal non-empty element of Π in both. The more interesting point is that in some atomic frames *atoms may contain multiple worlds*, and such atomic frames are *not* discrete.[4]

Lastly, Fine's formula is closely related to the formula Prior used to define Qp:
$$(**) \quad \Diamond p \wedge \forall q (\Box(p \rightarrow q) \vee \Box(p \rightarrow \neg q))$$

but whereas Fine's formula (*) says that the world of evaluation belongs to an atom in Π, Prior's formula (**) says that the denotation of p is an atom in Π. If we are working with a standard

---

[2]So the inclusion ATOMIC $\subset$ GENERAL is proper.

[3] Obviously any standard frame is discrete — but there are also plenty of discrete frames that are *not* standard. This is not so obvious — any discrete non-standard frame must be infinite. This is because, if we have all singleton sets of a finite set, then we can generate *all* of its subsets simply by taking intersections and complements, which means that finite discrete frames are standard. However if we try to generate all subsets of an infinite set in this way, we end up with a general frame in which Π contains just the *finite* and *co-finite* sets, rather than all elements of $\mathscr{P}(W)$; see page 30 of [2] for the relevant definitions.

[4]So to sum up: STANDARD $\subset$ DISCRETE $\subset$ ATOMIC. The class of standard frames is included in the class of discrete frames, which in turn is included in the class of atomic frames. Both inclusions are proper: there are discrete frames which are not standard (see Footnote 3) and there are also atomic frames which are not discrete (we will see a simple example in Section 4).

model (or a discrete model) then condition $(**)$ obviously boils down to the denotation of p being a singleton set, and we are have standard nominals. And in fact, Kit Fine on pages 339–340 of [5], considers the Q operator in connection with the formula

$$(***) \qquad \exists p(p \wedge Qp).$$

It is straightforward to show that the formulas $(***)$ and $(*)$ are equivalent in any (universally related) general frame. That is: Fine's formula can be defined using Prior's Q; both say something about atomicity.

4 Universal instantiation

To make the differences between our two species of nominals concrete, it will help to have a proof system. There are several we could have used for this purpose (see [3] for a general introduction to the proof-theory of hybrid logic) and we are currently focussing on tableau and Hilbert-style systems. Here we will make some comments on the universal instantiation rules used in our tableaux system (the full system can be found in [1]).

$$
\begin{array}{cc}
@_i \forall p \varphi & \neg @_i \forall p \varphi \\
\downarrow & \downarrow \\
@_i \varphi[\psi/p]\dagger & \neg @_i \varphi[q/p]*
\end{array}
$$

$\dagger$: *where ψ is free for p in φ and ψ does not contain any standard nominal in formula position.*

$*$: *where q is a new propositional symbol.*

The $\neg\forall$ rule is easy: suppose we have $\neg @_i \forall p \varphi$. This says it is *false* at the i-world i that $\forall p \varphi$. But then there is *some* proposition — call it q — which witnesses this falsehood. So, throwing away the quantifier, and substituting the new symbol q for the newly-freed occurrence of p, we deduce that it is false at the i-world that $\varphi[q/p]$, or to put it another way, we deduce $\neg @_i \varphi[q/p]$.

And now for the crucial $\forall$ rule, universal instantiation. The basic idea is straightforward: suppose we are given $@_i \forall p \varphi$. This is a universal claim: it says that at the i-world, $\forall p \varphi$ is true. Hence we should be able to pick any formula ψ, throw away the $\forall p$, and substitute ψ inside φ. Now, the first part of the $\dagger$ side-condition simply prevents accidental symbol binding (defined in the usuak way). But what about the second part of the restriction $\dagger$ side-condition? This is where the distinction between the standard and non-standard nominals becomes important.

First we observe that (unrestricted) universal instantiation is sound with respect to the standard semantics — this is easy to check, and we leave it to the reader. On the other hand, (unrestricted) universal instantiation is *not* sound with respect to the general semantics is less clear. We can see this as follows. Consider the general frame where $W = \{a, b\}$, $R = W \times W$ and $\Pi = \{\emptyset, W\}$. It is straightforward to check that Π satisfies the closure properties: the relative complement and intersection properties are immediate, and as for the modal projection condition, we have that:

$$
\begin{array}{l}
\{w \in W : \forall v(wRv \rightarrow v \in W)\} = W \\
\{w \in W : \forall v(wRv \rightarrow v \in \emptyset)\} = \{w \in W : \neg\exists v(wRv)\} = \emptyset
\end{array}
$$

So: Π has propositional structure and thus we have a genuine general frame. Now, extend the general frame to a general model by choosing some valuation V, letting $N(i) = a$. Then the formula $@_i \forall q(q \rightarrow \Box q)$ is *true*: if $V'(q) = \emptyset$, then the implication $q \rightarrow \Box q$ is trivially true at a,

and if $V'(q) = W$, then the implication is true at a since q is true at both a and b. On the other hand, if the rule $\forall$ is applied to $@_i\forall q(q \to \Box q)$ with the substitution $[i/q]$, then the resulting formula $@_i(i \to \Box i)$ is *false* at a since i is not true at b. So we have falsified an instance of unrestricted universal instantiation on a general model, and shown that the (unrestricted) rule is *not* sound for the general semantics.[5]

We next observe that the restricted universal instantiation rule is sound with respect to the general semantics: if there are no nominals in formula position, then all formulas are either propositional symbols or of the form $\neg\varphi$, $\varphi \wedge \psi$, $\Box\varphi$, $\forall p\varphi$ or $@_i\varphi$.The denotation of all such formulas (with the exception of those of the form $@_i\varphi$) are guaranteed to be in Π because of the way valuations are defined and the three properties imposed on Π. What about formulas of the form $@_i\varphi$? Well — first note that all such formulas are either true everywhere (that is, have denotation W), or false everywhere (that is, have denotation $\emptyset$). But both $\emptyset$ and W are admissible sets in any general model, so these are fine too. We are currently working on a completeness proof for the proof-system.

Lastly we observe that the unrestricted universal instantiation rule is sound with respect to *discrete* general frames. So: discreteness guarantees the soundness of (unrestricted) universal instantiation. This is easy to check, and we leave it to the reader.[6]

In the table below we map out the different classes of frames considered in the present note: there is a column for each class and results associated with it.

STANDARD	DISCRETE	ATOMIC	GENERAL
			Restricted UI sound
		Formula ($*$) valid	Formula ($*$) invalid
	Unrestricted UI sound	Unrestricted UI unsound	
?	?		

Formula ($*$) is Kit Fine's formula displayed in Section 3. As indicated earlier, the frame classes in the top row are strictly included in each other, read from left to right. Note that the observation regarding the unrestricted UI rule can easily be turned into a formula valid on discrete frames, but invalid on atomic ones, namely the formula $@_i\forall q(q \to \Box q) \to @_i(i \to \Box i)$, cf. the counterexample to soundness we gave in Section 4.[7] The two question marks indicate future work, where we are considering formulas/proof-rules distinguishing the two classes of frames.

Acknowledgements

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[5]Indeed, we have done something a little stronger. The model just defined is *atomic* — note that W is a minimal non-empty set for both worlds a and b, thus W itself is a (rather unusual!) atom. Thus we have also shown that the (unrestricted) rule is unsound on the class of atomic general frames. Incidentally, it's worth explicitly noting that while this frame is atomic, it is *not* discrete, as neither $\{a\}$ nor $\{b\}$ is a proposition. Thus this example also shows that the class inclusion DISCRETE $\subset$ ATOMIC is proper, as claimed in Footnote 4.

[6]But we cannot loosen the discreteness requirement: Any non-discrete general frame can be extended to a general model in which the formula $\forall p\exists q\Box(q \leftrightarrow p)$ is true, but the instance $\exists q\Box(q \leftrightarrow i)$ is false.

[7]Note by the way that it is hybrid-logical machinery that enables building a formula that distinguishes between discrete and atomic frames: As indicated in [5], page 338, one can identify indistinguishable worlds in an atomic model, that is, the worlds that belong to the same propositions in Π, such that a discrete model is obtained. However, in the context of hybrid logic, such an identification cannot be carried out without possibly identifying the denotation of different nominals.

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Equivalance of Pointwise and Continuous Semantics of FO with Linear Constraints

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Abstract

We consider a first-order logic with linear constraints which can be interpreted naturally in both a pointwise or continuous way over timed words. We show that the two interpretations of this logic coincide in terms of expressiveness. As a consequence it follows that the pointwise and continuous semantics of the logic TPTL with the since operator also coincide. We also exhibit a useful normal form for formulas in these logics.

1 Introduction

Several real-time logics proposed in the literature have been interpreted over timed behaviours in two natural ways which have come to be known as the "pointwise" and "continuous" interpretations. In the pointwise semantics, formulas may be asserted only at points where an action occurs (the so called "action points"), while in the continuous semantics formulas may be asserted at arbitrary time points. To illustrate these semantics, consider the popular timed temporal logic Metric Temporal Logic (MTL) [8, 1, 9], which extends the U operator of classical LTL with an interval index, to allow formulas of the form $\theta U_I \eta$ which says that, with respect to the current time point, there is a future time point where η is satisfied and which lies at a distance that falls within the interval I, and at all time points in between θ is satisfied. Consider a timed word σ in which the first action is an a at time 2, followed subsequently by only b's. Then the MTL formula $\Diamond(\Diamond_{[1,1]}a)$ is satisfied in σ in the continuous semantics, but not in the pointwise semantics since there is no action point at time 1.

The Timed Propositional Temporal Logic (TPTL) of Alur and Henzinger [2] is a well-known timed temporal logic for specifying real-time behaviors. The logic is interpreted over timed words and extends classical LTL with the "freeze" quantifier $x.\theta$ which binds x to the value of the current time point, along with the ability to constrain these time points using linear constraints of the form $x \sim y + c$. For example the formula $x.(\Diamond y.(a \wedge y = x + 2))$ says that with respect to the current time point, an action a occurs exactly two time units later. Then the TPTL formula $\Diamond x.(\Diamond y.(a \wedge y = x + 1))$ is satisfied in the example timed word σ in the continuous semantics, but not in the pointwise semantics, since there is no action point at time 1. It is not difficult to see that for a typical timed temporal logic the continuous semantics is at least as expressive as the pointwise one, since one can ask for a time point to be an action point by asserting $\bigvee_{a \in \Sigma} a$ at each quantified time point.

There have been several results in the literature which show that for the logic MTL and its variants, the continuous semantics is in fact strictly more expressive than the pointwise one

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[*]Work done while at Indian Institute of Science, Bangalore

[3, 5, 10]. Thus the logics MTL, MTL$_S$ (MTL with the "since" operator S), MTL$_{S_I}$ (MTL with the S_I operator), and MITL (MTL restricted to non-singular intervals), are all strictly more expressive in the continuous semantics than their pointwise counterparts. Ho and others [6] further point out differences in the pointwise and continuous versions of MTL and also show an expressive completeness result, a la Kamp [7], for a variant of MTL in the pointwise semantics.

In this paper we consider the expressiveness of the pointwise and continuous interpretations of a natural first-order logic with linear constraints which is similar in flavour to TPTL. Thus the logic allows atomic predicates of the form $a(x)$ which says that an a-event occurs at time point x, and constraints of the form $x < y + c$. The interpretation of the quantifier $\exists x$ depends on the pointwise or continuous semantics: in the pointwise it is interpreted as "there exists an action point x", while in the continuous semantics it is interpreted as "there exists a time point x." As an example the FO($<_+$) formula $\exists x.\exists y.(a(y) \wedge y = x + 1)$ is satisfied in the example timed word σ in the continuous interpretation while it is not satisfied in the pointwise semantics.

Our main result in this paper is that the expressiveness of the logic in these two semantics coincides.

The main proof idea is to show that we can go from an arbitrary sentence in FO($<_+$) to a sentence in FO($<_+$) which uses only "active" quantifiers. We say a subformula of the form $\exists x \varphi$ is an actively quantified formula if φ is of the form $a(x) \wedge \psi$ for some action a and formula ψ; and say it is "passively" quantified otherwise. A sentence in which all quantifiers are active, is clearly equivalent to a pointwise formula. Thus, we show how to eliminate passively quantified variables using only actively quantified ones. As an example, the formula $\exists x(1 \leq x \wedge \exists y(b(y) \wedge x + 1 \leq y))$ has x passively quantified. We can eliminate x from it, by giving the equivalent actively quantified formula $\exists y(b(y) \wedge 2 \leq y)$.

As a corollary, we easily obtain that the pointwise and continuous interpretations of the logic TPTL with the Since operator have the same expressive power. A detailed version of this paper is available online in [4] at www.csa.iisc.ac.in/~deepakd/papers/icla-long.pdf.

2 Preliminaries

We begin with some preliminary definitions. Let $\mathbb{R}_{\geq 0}$ denote the set of non-negative real numbers, $\mathbb{Q}$ denote the set of rational numbers, and $\mathbb{N}$ denote the set of non-negative integers. We use the standard notation to represent intervals, which are convex subsets of $\mathbb{R}$. For example $[1, \infty)$ denotes the set $\{t \in \mathbb{R} \mid 1 \leq t\}$.

For an alphabet A we denote by A^ω the set of infinite words over A. Let Σ be a finite alphabet of actions, which we fix for the rest of this paper. An (infinite) timed word σ over Σ is an element of $(\Sigma \times \mathbb{R}_{\geq 0})^\omega$ of the form $(a_0, t_0)(a_1, t_1) \cdots$, satisfying the conditions that: for each $i \in \mathbb{N}$, $t_i < t_{i+1}$ (*monotonicity*), and for each $t \in \mathbb{R}_{\geq 0}$ there exists an $i \in \mathbb{N}$ such that $t < t_i$ (*progressiveness*). For convenience, we will also assume in this paper that $t_0 = 0$. We will sometimes represent the timed word σ above as a pair (α, τ), where $\alpha = a_0 a_1 \cdots$ and $\tau = t_0 t_1 \cdots$. Thus $\alpha(i)$ and $\tau(i)$ denote the the action and the time stamp respectively, in σ at position i. We write $T\Sigma^\omega$ to denote the set of all timed words over Σ.

We now introduce the linear constraints we use in this paper, and some notations for manipulating them. We assume a supply of variables $Var = \{x, y, \ldots\}$ which we will use in constraints as well as later in our logics. We use restricted linear constraints of the form $x \sim y + c$ or $x \sim c$, where x and y are variables in Var, $\sim$ is one of the relations $\{<, \leq, =, \geq, >\}$, and c is in $\mathbb{Q}$. We call these constraints *simple constraints*. In general, we will allow constraints to be boolean combinations of simple constraints.

An *interpretation* for variables is a map $\mathbb{I} : Var \to \mathbb{R}$. For $t \in \mathbb{R}$ and $x \in Var$ we will use $\mathbb{I}[t/x]$ to represent the interpretation which sends x to t, and agrees with $\mathbb{I}$ on all other variables. For an interpretation $\mathbb{I}$ and a constraint δ, we write $\mathbb{I} \models \delta$ to mean that the constraint δ is satisfied in the interpretation $\mathbb{I}$, and defined in the expected way.

Consider the conjunction and/or disjunction of simple constraints of the form $e \prec x$ with $\prec \in \{<, \leq\}$. For an interpretation $\mathbb{I}$, they define a left boundary of all the possible values of x. Similarly, the conjunction and/or disjunction of simple constraints of the form $x \prec e$ define a right boundary of all the possible values of x for a given interpretation. Consider the conjunction of left boundary of x and right boundary of x. For an interpretation it defines an interval in the real line. For any interval π, $Lt_x(\pi)$ denote the left boundary and $Rt_x(\pi)$ denote the right boundary. So $\pi(x)$ can be written as $Lt_x(\pi) \wedge Rt_x(\pi)$.

Given two intervals $\pi_1(x)$ and $\pi_2(x)$, it is possible to identify an interval which is a combination of left and right boundaries of these intervals and nonemptyness of these intervals can specify some property of the intervals $\pi_1(x)$ and $\pi_2(x)$. To illustrate, if left boundary of $\pi_2(x)$ begins after the right boundary of $\pi_1(x)$, then the interval $\neg Rt_x(\pi_1) \wedge \neg Lt_x(\pi_2)$ will not be empty. Similarly, if $\pi_1(x)$ begins before $\pi_2(x)$, then the interval $Lt_x(\pi_1) \wedge \neg Lt_x(\pi_2)$ will not be empty.

3   The $FO(<_+)$ logic

We now define our first order logic with simple constraints $FO(<_+)$, which is interpreted over timed words over the alphabet Σ. The formulas of $FO(<_+)$ are given by:

$$\varphi ::= a(x) \mid g \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \exists x \varphi,$$

where $a \in \Sigma$, $x \in Var$, and g is a simple constraint.

We now define the continuous semantics for $FO(<_+)$. Let φ be a formula in $FO(<_+)$. Let $\sigma = (\alpha, \tau)$ be the timed word over Σ, and let $\mathbb{I}$ be an interpretation for variables. Then the satisfaction relation $\sigma, \mathbb{I} \models_c \varphi$ (read "σ satisfies φ in the interpretation $\mathbb{I}$ in the continuous semantics") is inductively defined as:

$$
\begin{array}{lll}
\sigma, \mathbb{I} \models_c a(x) & \text{iff} & \exists i : \alpha(i) = a \text{ and } \tau(i) = \mathbb{I}(x) \\
\sigma, \mathbb{I} \models_c g & \text{iff} & \mathbb{I} \models g \\
\sigma, \mathbb{I} \models_c \neg\nu & \text{iff} & \sigma, \mathbb{I} \not\models_c \nu \\
\sigma, \mathbb{I} \models_c \nu \wedge \psi & \text{iff} & \sigma, \mathbb{I} \models_c \nu \text{ and } \sigma, \mathbb{I} \models_c \psi \\
\sigma, \mathbb{I} \models_c \nu \vee \psi & \text{iff} & \sigma, \mathbb{I} \models_c \nu \text{ or } \sigma, \mathbb{I} \models_c \psi \\
\sigma, \mathbb{I} \models_c \exists x \nu & \text{iff} & \exists t : t \in \mathbb{R}_{\geq 0} \text{ such that } \sigma, \mathbb{I}[t/x] \models_c \nu.
\end{array}
$$

A variable x is said to occur *free* in a formula φ if it is occurs outside the scope of a quantifier $\exists x$. A *sentence* is a formula in which there are no free occurrences of variables. Again, the satisfaction of a sentence is independent of an interpretation for variables.

The timed language defined by an $FO(<_+)$ sentence φ in the continuous semantics is given by $L^c(\varphi) = \{\sigma \in T\Sigma^\omega \mid \sigma \models_c \varphi\}$. We denote this continuous version of the logic by $FO^c(<_+)$.

We can similarly define the pointwise version of the logic $FO(<_+)$, where the quantification is over action points in the timed word. The satisfaction relation $\sigma, \mathbb{I} \models_{pw} \varphi$, read "$\sigma$ satisfies φ in the interpretation $\mathbb{I}$ in the pointwise semantics", is defined as above, except for the $\exists$ clause which is interpreted as follows:

$$\sigma, \mathbb{I} \models_{pw} \exists x \nu \quad \text{iff} \quad \exists i \in \mathbb{N} : \sigma, \mathbb{I}[\tau(i)/x] \models_{pw} \nu.$$

The timed language defined by a sentence φ in the pointwise semantics is given by $L^{pw}(\varphi) = \{\sigma \in T\Sigma^{\omega} \mid \sigma \models_{pw} \varphi\}$. We denote the pointwise version of the logic by $\text{FO}^{pw}(<_+)$.

4 A normal form for FO sentences

In this section we exhibit a normal form for $\text{FO}(<_+)$ sentences which will be useful in our proofs. We begin with a normal form for formulas of the form $\exists x\varphi$. An $\text{FO}(<_+)$ formula is said to be in $\exists$-*normal form* if it is of the form

$$\exists x(a(x) \wedge \pi(x) \wedge \nu),$$

where $a \in \Sigma$, $\pi(x)$ is a conjunction of simple constraints each containing x, and ν is a conjunction of formulas of the form ψ or $\neg\psi$, where each ψ is again in $\exists$-normal form. In addition, we allow any of the components $a(x)$ and ν to be absent. The figure alongside depicts a formula in normal form.
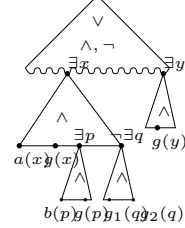
Theorem 4.1. *Any* $\text{FO}(<_+)$ *sentence can be written as a boolean combination of sentences which are in* $\exists$-*normal form.* □

5 Equivalence of FO^c and FO^{pw} semantics

In this section our aim is to show that the logics $\text{FO}^{pw}(<_+)$ and $\text{FO}^c(<_+)$ are expressively equivalent. It is easy to translate an $\text{FO}^{pw}(<_+)$ sentence φ to an equivalent $\text{FO}^c(<_+)$ sentence by simply replacing every $\exists x\varphi'$ subformula, by $\exists x(\bigvee_{a\in\Sigma} a(x) \wedge \varphi'')$, where φ'' is obtained by similarly replacing $\exists$ subformulas in φ'.

In the converse direction, let us call an $\text{FO}^c(<_+)$ formula φ *actively quantified* (or simply *active*) if every $\exists$ subformula is of the form $\exists x(a(x) \wedge \varphi')$, for some action $a \in \Sigma$ and formula φ'. Then, an active $\text{FO}^c(<_+)$ formula clearly defines the same language of timed words, regardless of the semantics being pointwise or continuous. Hence, our aim in the rest of this section is to show how we can go from an arbitrary formula in $\text{FO}^c(<_+)$ to an equivalent active formula.

An arbitrary formula in the continuous semantics has the obvious advantage of being able to associate any value in $\mathbb{R}_{\geq 0}$ to its variables, whereas an actively quantified variable can be asserted only at the action points in a timed word. For e.g., consider the language of all timed strings over a and b, where for every b in the interval $[1,2]$, there is an a in $[0,1]$ which is exactly at a distance of one time unit from that of the b. This can be written easily in FO^c, as shown below:

$$\neg\exists x((\neg a(x) \wedge 0 \leq x \wedge x \leq 1 \wedge \exists y(b(y) \wedge y = x+1)) \tag{1}$$

But if we restrict x to be actively quantified, then the above formula does not recognize the same language. This formula is not in the $\exists$-normal form and the normalization of this formula yields a disjunction of four formulas $\psi_1, \psi_2, \psi_3, \psi_4$, where x is the only variable which is passively quantified. If we can eliminate x from $\psi_1, \psi_2, \psi_3, \psi_4$, without introducing any new passively quantified variables, the disjunction of these actively quantified formulas recognizes the required language. The subformula involving x in each of ψ_i-s looks like $\exists x(\pi(x) \wedge \exists y(b(y) \wedge y = x+1))$. It is possible to remove x from such formulas using the Fourier-Motzkin elimination method.

Consider another example where x is passively quantified.

$$\exists x(0 \leq x \wedge x \leq 1 \wedge \exists y(a(y) \wedge x+1 \leq y \wedge y \leq x+1.2)). \tag{2}$$

The above formula is true, iff there is a point in [0,1], from which there is an action a at a distance which lies in [1,1.2]. Equivalently, (2) is true iff there is an action a in the interval [1,2.2]. So the equivalent active formula is:

$$\exists y(a(y) \wedge 1 \leq y \wedge y \leq 2.2). \tag{3}$$

As an another example, consider the following modified formula:

$$\exists x(0 \leq x \wedge x \leq 1 \wedge \neg \exists y(a(y) \wedge x + 1 \leq y \wedge y \leq x + 1.2)) \tag{4}$$
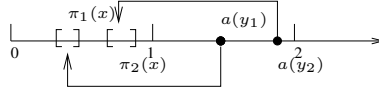
Figure 1:

To eliminate the passively quantified variable x from this formula, first consider "all" intervals of x in [0,1] which satisfy (2) in the given model (the bracketed region in Fig. 1.). (4) is true iff there are some "gaps" where formula (2) is not satisfied. There can be the following four types of gaps.

- No x satisfies (2), so the whole of [0,1] is a gap.

- Gap from 0 to the beginning of the first interval of x satisfying (2).

- Gap between two consecutive intervals of x satisfying formula (2).

- Gap from the end of the last interval of x satisfying (2) to 1.

The formula (4) can be satisfied iff any of the above gaps exist. We show that it is possible to identify those gaps using the syntax of $\mathrm{FO}^c(<_+)$.

Theorem 5.1. *Let $\widehat{\nu}$ be an active formula which is the conjunction of formulas in the $\exists$-normal form or in negated $\exists$-normal form. Consider the formula $\psi = \exists x(\pi \wedge \widehat{\nu})$ where x is the only passive variable. It is possible to get an equivalent formula $\psi' = \bigvee_i(\widehat{\mu}_i \barwedge \exists x \pi_i(x))$, where each $\widehat{\mu}_i$ is active and independent of x and $\pi_i(x)$ is an interval function of free variables and positively quantified variables of $\widehat{\mu}_i$. Further, ψ is satisfied at $x = t$ if and only if ψ' is satisfied for some i at $x = t$.* □

Corollary 5.2. *Let $\widehat{\nu}$ be an active formula which is the conjunction of formulas in the $\exists$-normal form or in negated $\exists$-normal form. Consider the formula $\psi = \exists x(\pi \wedge \widehat{\nu})$ where x is the only passive variable. It is possible to get an equivalent formula $\psi' = \bigvee_i(\widehat{\mu}_i)$, where each $\widehat{\mu}_i$ is active and independent of x.*

Proof. By Theorem 5.1, it is possible to get an equivalent formula of $\widehat{\nu}$ which is of the form $\bigvee_i(\widehat{\mu}_i' \barwedge \exists x \pi_i(x))$ where $\widehat{\mu}_i'$ is active and independent of x. But we can use the Fourier-Motzkin elimination method to get equivalent formulas of the form $\widehat{\mu}_i = \widehat{\mu}_i' \barwedge \mathrm{FME}_x(\pi_i)$ for each i. The new formula is $\widehat{\nu} = \bigvee_i \widehat{\mu}_i$ where each $\widehat{\mu}_i$ is independent of x. □

Theorem 5.3. *It is possible to translate any FO^c formula to an equivalent formula which defines the same language in the pointwise semantics.* □

As a corollary it follows that the logic TPTL with the Since modality [2] has the same expressive power in the pointwise and continuous semantics.

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Dualities and logical aspects of Baire functions

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Abstract

After introducing the infinitary logic $\mathcal{IRL}$ defined in [2], we characterize its Lindenbaum-Tarski algebra (build upon a set of variables of cardinality κ) as the algebras of Baire-measurable functions in $[0,1]^{[0,1]^{\kappa}}$. Furthermore, working in the infinitary variety $\mathbf{RMV}_\sigma$ – that provides the algebraic semantics of $\mathcal{IRL}$ – we discuss the so-called the Marra-Spada duality in this framework. Some of the results presented are included in [3], which is a work submitted for publication.

One of the most investigated many-valued logics is certainly Łukasiewicz logic, with the main reason being found in the fact that its algebraic semantics, the variety of MV-algebras, proved to be a very malleable class of algebras that carries many underlying relations with other areas of mathematics or theoretical computer science. One can see the appendixes of [6] to grasp the broad collection of tools and techniques from different fields that one can use to carry out state-of-art research on the topic.

A major contribution to the widespread of MV-algebras is undoubtedly their celebrated categorical equivalence with lattice-ordered groups with a strong unit, D. Mundici's equivalence, that dates back to 1986. This result led the way for an investigation of MV-algebras inspired by the theory of groups, and one of the outcomes of this point of view has been the definition of the class of *Riesz MV-algebras*. These algebras stand to MV-algebras like vector spaces over $\mathbb{R}$ stand to groups, see [4].

Riesz MV-algebras have proven to be a quite interesting and useful class of algebras in themselves. They provide the semantics for a conservative expansion of Łukasiewicz logic, but more importantly they are successful in overcoming some of the limitations of MV-algebras.

In this presentation we will not deal with the totality of Riesz MV-algebras. Indeed, our starting point will be the class of algebras considered in [2]: the infinitary variety of σ-complete Riesz MV-algebras, denoted by $\mathbf{RMV}_\sigma$. These algebras have a quite concrete representation, since they can be thought as intervals of some Banach lattices. A Banach lattice is a lattice-ordered vector space over $\mathbb{R}$ which is norm-complete. In the special case of a vector space with a strong order unit, one can consider the class of those Banach lattices that are complete with respect to a norm that is induced by the strong unit. These spaces turned out to be all $\mathbb{R}$-valued algebras of continuous functions over compact and Hausdorff topological spaces. Any Dedekind σ-complete lattice-ordered vector space with a strong order unit is norm-complete, and therefore, when in addition one requires suitable topological properties, these algebras of functions turn out to be equivalent "à la Mundici" to σ-complete Riesz MV-algebras.

The focus of this presentation is to discuss the infinitary variety $\mathbf{RMV}_\sigma$ from two points of view: description of its free objects and categorical dualities.

After introducing the needed preliminary notions, we dive into characterizing the free objects of $\mathbf{RMV}_\sigma$ in the case of an arbitrary set of generators. We denote by $IRL(X)$ the algebra of term functions $p : [0,1]^X \to [0,1]$ in the language of σ-complete Riesz MV-algebras. By the results of [2] (mainly the fact that $\mathbf{RMV}_\sigma$ is the infinitary variety generated by $[0,1]$) and standard results in universal algebra, it is easily seen that $IRL(X)$ is the Lindenbaum-Tarski

algebra of $\mathcal{IRL}$ build upon $|X|$-propositional variables, and it is also isomorphic with the free $|X|$-generated σ-complete Riesz MV-algebra. We call its elements IRL-polynomials.

Moreover, if we consider the hypercube $[0,1]^X$ endowed with its natural product topology, by Baire set we mean a subset of the σ-algebra generated by the zeroset of the continuous functions $f : [0,1]^X \to [0,1]$. In symbols, B is a Baire set if, and only if, there exists a continuous $f : [0,1]^X \to [0,1]$ such that $B = f^{-1}(\{0\})$. A function $p : [0,1]^X \to [0,1]$ is Baire-measurable if the preimage of a Baire set of $[0,1]$ is a Baire set of $[0,1]^X$.

With these definitions, the following holds.

Proposition 1. *For every set X, Baire sets and zerosets of IRL-polynomials in $[0,1]^X$ coincide.*

Theorem 2. *For an arbitrary non-empty set X, $IRL(X)$ is the algebra of all $[0,1]$-valued and Baire-measurable functions defined over $[0,1]^X$.*

Furthermore, applying the well-known Lebesgue-Hausdorff theorem (see e.g. [5, Appendix 4A3K]), if X is countable we recover the results of [2].

Corollary 3. *For a countable set X, $IRL(X)$ is the algebra of all $[0,1]$-valued and Borel-measurable functions defined over $[0,1]^X$*

We recall that the Borel σ-algebra of $[0,1]^X$ is the *sigma*-algebra generated by the open sets, and that Borel-measurable functions are defined analogously to Baire functions.

After having obtained this characterization of the free algebra in $\mathbf{RMV}_\sigma$, we will focus of obtaining a categorical duality of special classes of σ-complete Riesz MV-algebras.

In [1] the authors prove a very general dual adjunction between the category of (presented) algebras (in an arbitrary variety, even infinitary) and subsets of powers of a fixed algebra A in the variety at hand, with term functions as main characters. We briefly describe the adjunction in our setting, with $A = [0,1]$, and we obtain dualities for presented and finitely presented σ-complete Riesz MV-algebras.

For any subset $S \subseteq [0,1]^X$, we denote by $\mathbb{I}(S)$ the ideal of IRL-polynomials vanishing on S, that is

$$\mathbb{I}(S) = \{p \in IRL(X) \mid p(\mathbf{x}) = 0 \text{ for any } \mathbf{x} \in S\}.$$

Given a set J of IRL-polynomials in $IRL(X)$, we denote by $\mathbb{V}(J)$ the zeroset of J, that is

$$\mathbb{V}(J) = \{\mathbf{x} \in [0,1]^X \mid p(\mathbf{x}) = 0 \text{ for any } p \in J\} = \bigcap_{p \in J} \mathbb{V}(\{p\}).$$

These operators give a Galois connection between subsets of points and subsets of polynomials. Moreover, following [1, Section 4], we get the following categories and functors.

1. The category $\mathbf{RMV}_\sigma^\mathbf{p}$ whose objects are presented σ-complete Riesz MV-algebras and whose arrows are σ-homomorphisms of Riesz MV-algebras. More precisely, an object is a pair $(IRL(X), I)$, where I is an ideal in the free algebra $IRL(X)$. Intuitively, this pair represents the quotient algebra $IRL(X)/I$. Consequently, each morphism $h : (IRL(X), I) \to (IRL(Y), J)$ between pairs is induced by a unique homomorphism $h^p : IRL(X) \to IRL(Y)$ such that $h^p(I) \subseteq J$.

2. The category $\mathbf{Hyper}$, whose objects are subsets of hypercubes of type $[0,1]^X$, for an arbitrary X, and arrows are tuples of IRL-polynomials, that is, an arrow in $\mathbf{Hyper}$ is a map $\eta = (\eta_{y|_S})_{y \in Y} : S \subseteq [0,1]^X \to T \subseteq [0,1]^Y$, where each η_y belongs to $IRL(X)$. We remark that this definition implies that each $\eta : S \to T$ is the restriction of a tuple of IRL-polynomials $\tilde{\eta} : [0,1]^X \to [0,1]^Y$.

3. The functor $\mathcal{V} \colon \mathbf{RMV}_\sigma^{\mathbf{p}} \to \mathbf{Hyper}$, defined by

- $\mathcal{V}(IRL(X), J) = \mathbb{V}(J)$;
- for $h \colon (IRL(X), J) \to (IRL(Y), K)$, $\mathcal{V}(h) \colon \mathbb{V}(K) \to \mathbb{V}(J)$ is defined as follows. For any $x \in X$, take $p_x \in h^p(\pi_x)$, and note that $p_x \in IRL(Y)$. Now, for $(v_y)_{y \in Y} \in \mathbb{V}(K)$, $\mathcal{V}(h)((v_y)_{y \in Y}) = (p_x((v_y)_{y \in Y}))_{x \in X} \in \mathbb{V}(J)$.

4. The functor $\mathcal{I} \colon \mathbf{Hyper} \to \mathbf{RMV}_\sigma^{\mathbf{p}}$, defined by

- for $S \subseteq [0,1]^X$, $\mathcal{I}(S) = (IRL(X), \mathbb{I}(S))$,
- for $\eta = (\eta_y)_{y \in Y} \colon S \subseteq [0,1]^X \to T \subseteq [0,1]^Y$, $\mathcal{I}(\eta) \colon \mathcal{I}(T) \to \mathcal{I}(S)$ is the map given by $f \in IRL(Y) \mapsto f \circ \eta \in IRL(X)$.

Proposition 4. *[1, Corollary 4.8] The above defined functors $\mathcal{I}$ and $\mathcal{V}$ are an adjoint pair between the categories $(\mathbf{RMV}_\sigma^{\mathbf{p}})^{op}$ and $\mathbf{Hyper}$.*

Thus, after sketching these preliminary notions, our goal will be to describe the fixed points on both sides of the adjunction and obtain dualities. To do this, we start by recalling that each algebra in $A \in \mathbf{RMV}_\sigma$ is semisimple as an MV-algebra, which implies the intersection of all maximal ideals of A is 0, in symbols $\bigcap Max(A) = \{0\}$. Adapting the notion to our framework, we denote by $Id_\sigma(A)$ the set of all σ-complete ideals of A and we get the following.

Definition 5. An algebra $A \in \mathbf{RMV}_\sigma$ is called σ-semisimple if

$$\bigcap \{M \mid Max(A) \cap Id_\sigma(A)\} = \{0\}.$$

Thus, on one side of the adjunction, we decribe fixed point via the following proposition.

Proposition 6. *An algebra $IRL(X)/J \in \mathbf{RMV}_\sigma$ is σ-semisimple if, and only if, $J = \mathbb{I}(\mathbb{V}(J))$.*

On the other side, using Proposition 1, we infer the following.

Theorem 7. *Let S be a subset of $[0,1]^X$. Then $\mathbb{V}(\mathbb{I}(S))$ is the intersection of all Baire subsets of $[0,1]^X$ containing S. Thus, $\mathbb{V}(\mathbb{I}(S)) = S$ if, and only if, S is an intersection of Baire sets.*

Calling such intersections of Baire sets *IRL-algebraic varieties*, the duality is described as follows.

Corollary 8. *The adjunction $(\mathcal{I}, \mathcal{V})$ restricts to a duality between the full subcategory $\mathbf{ssRMV}_\sigma^{\mathbf{p}}$ of $\mathbf{RMV}_\sigma^{\mathbf{p}}$ whose objects are σ-semisimple algebras, and the full subcategory $\mathbf{IRL}$ of $\mathbf{Hyper}$ whose objects are IRL-algebraic varieties.*

Furthermore, the duality can be restricted to finitely presented objects, as follows.

Corollary 9. *The adjunction $(\mathcal{I}, \mathcal{V})$ restricts to a duality between the category $\mathbf{RMV}_\sigma^{\mathbf{fp}}$ of finitely presented σ-complete RMV-algebras and full subcategory $\mathbf{Baire}$ of $\mathbf{Hyper}$ whose objects are Baire subsets of the hypercubes.*

Summarizing, in this presentation we shall start from the infinitary logic $\mathcal{IRL}$ and describe its Lindenbaum-Tarski algebra in terms of Baire measurable functions. Subsequently we will build on this description and the work in [1] to obtain categorical dualities for special classes of σ-complete Riesz MV-algebras.

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An Epistemic Separation Logic with Action Models

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1 Introduction

The aim of the paper is to present Epistemic Separation Logic with Action Models (ESLAM), that can be seen as a generalization of Public Announcement Separation Logic (PASL [3]). The general context of this work is the study of extensions of Separation logics with modalities in order to manage various dynamic aspects and mainly here from Dynamic Epistemic Logic (with factual change) [1, 9, 8]. Let us remind that Separation Logics (SL) refer to a class of logics based on (intuitionistic) Logic of Bunched Implications (BI) or its classical counterpart Boolean BI (BBI) [6]. They combine both additive ($\wedge$, $\rightarrow$, $\vee$) and multiplicative ($*$, $-\!*$) connectives in the language, the latter expressing the concept of resource separation (or composition), and resource update [6]. Among extensions of Separation Logics with dynamics we mention Dynamic Modal BI (DMBI [2]) and Epistemic Resource Logic (ERL [4]). The first one is a BBI extension with the modalities $\square$, $\diamond$, and a dynamic modality $\langle a \rangle$, that allows us to investigate how resource properties change over dynamic processes taking place, with an emphasis on concurrent processes. The second one is a BBI extension with epistemic modalities, as well as a differentiation between ambient resource and local resources (assigned to each agent), and their compositions. A recent work on resource semantics to model updates and/or epistemic reasoning, lead to PASL, which extends BBI with a knowledge operator K_a, and a public announcement modality. The strenghts of this logic lie in its ability to model knowledge acquisition and information change over the course of truthful public communication [5].

In this paper we generalize the dynamic aspects of PASL, by defining Epistemic Separation Logic with Action Models (ESLAM), in which we replace public announcements with action models [1], motivated by their ability to model factual change, and instances of a more nuanced, private communication. A keypoint about integrating dynamic logics with BBI is that the available resources and the resource composition operator are based on the monoidal structure, which entails inclusion of a neutral element (neutral, or unit resource). As dynamic processes are carried out it is vital that – in any case – the structure of our updated model still contains the neutral element, so the monoidal structure is preserved. In PASL, possible worlds are considered resources and hence the issue was solved by a refinement semantics [7], that ensures that after an announcement of φ, all relations are severed between the states where φ is true, and the states where φ is false, but no state is ever removed from the model. In ESLAM, the relationship between states and resources is more implicit, as we define a resource function r, mapping every state (or several states) to a resource. Moreover, the updated epistemic resource model – obtained after action model execution – ensures that all state-to-resource mappings are preserved. We also require that an action model is covering, so a state assigned to a neutral resource is always part of the updated model domain.

In Section 2, we define ESLAM syntax, semantics, and associated structures. In Section 3, we propose a set of ESLAM reductions for elimination of the action model modality. In Section 4, a modelling example is presented, in which we can compare PASL and ESLAM with regard to their abilities to model public and private communications. In Section 5, we mention future works, as well as possible modifications of ESLAM to be investigated.

2 Epistemic Separation Logic with Action Models

The logic ESLAM is based on BBI, extended with a knowledge modality K_a and a dynamic modality $[\mathcal{E}_e]$ for action execution. Given a set of agents A and a set of propositional variables P, the language of ESLAM, $\mathcal{L}_{K*}$, is defined as follows, where $a \in A$ and $p \in P$:

$$\varphi ::= \quad p \mid \bot \mid I \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid K_a\varphi \mid \varphi * \varphi \mid \varphi \mathbin{-\!\!*} \varphi \mid [\mathcal{E}_e]\varphi$$

Expression $K_a\varphi$ means that *agent a knows that φ*. There are two multiplicative connectives ($*$ and $\mathbin{-\!\!*}$) referring to the separation respectively composition of resources. Expression $[\mathcal{E}_e]\varphi$ stands for 'after execution of action $\mathcal{E}_e$, φ is true. Such actions $\mathcal{E}_e$ will be defined below.

Definition 1 (Resource monoid). *A* partial resource monoid *(or* resource monoid*) is a structure $\mathcal{R} = (R, \bullet, n)$ where R is a set of* resources *containing a* neutral element $n \in R$, $\bullet : R \times R \to R$ is a resource composition operator *that is associative and commutative, that may be partial, and such for all $r \in R$, $r \bullet n = n \bullet r = r$. If $r \bullet r'$ is defined we write $r \bullet r' \downarrow$ and if $r \bullet r'$ is undefined we write $r \bullet r' \uparrow$. Whenever writing $r \bullet r' = r''$ we assume that $r \bullet r' \downarrow$.*

Definition 2 (Epistemic resource model). *An* epistemic frame *(frame) is a structure $(S, \sim)$ such that S is a set of* states *and $\sim : A \to \mathcal{P}(S \times S)$ is a function that maps each agent a to an equivalence relation $\sim(a)$ denoted as $\sim_a$. Given a resource monoid $\mathcal{R} = (R, \bullet, n)$, an* epistemic resource model *is a structure $\mathcal{M} = (S, \sim, r, V)$ such that $(S, \sim)$ is an epistemic frame, surjection $r : S \to R$ is a* resource function*, that maps each state to a resource and where we write r_s for $r(s)$, and $V : P \to \mathcal{P}(S)$ is a* valuation function*, where $V(p)$ denotes the set of states where variable p is true. Given $s \in S$, the pair $(\mathcal{M}, s)$ is a* pointed epistemic resource model*, also denoted $\mathcal{M}_s$.*

Definition 3 (Action model). *Given a logical language $\mathcal{L}$, an* action model $\mathcal{E}$ *is a structure $\mathcal{E} = (E, \approx, pre, post)$, such that E is a finite domain of* actions*, $\approx_a$ an equivalence relation on E for all $a \in A$, $pre : E \to \mathcal{L}$ is a precondition function, and $post : E \to P \nrightarrow \mathcal{L}$ is a postcondition function that is a partial function: its domain is a finite set of variables $Q \subseteq P$. Given $e \in E$, a* pointed action model *(or* epistemic action*) is a pair $(\mathcal{E}, e)$, denoted $\mathcal{E}_e$. An action model is* covering *if $\bigvee_{e \in E} pre(e)$ is a validity of the logic of $\mathcal{L}$.*

Definition 4. *Given an epistemic resource model $\mathcal{M} = (S, \sim, r, V)$ and a covering action model $\mathcal{E} = (E, \approx, pre, post)$, the* updated epistemic resource model *$\mathcal{M} \otimes \mathcal{E} = (S', \sim', r', V')$ is defined as*

$$
\begin{array}{lll}
S' & = & \{(s,e) \mid \mathcal{M}_s \models pre(e)\} \\
(s,e) \sim'_a (t,f) & \textit{iff} & s \sim_a t \textit{ and } e \approx_a f \\
(s,e) \in V'(p) & \textit{iff} & \mathcal{M}_s \models post(e)(p) \\
r'_{(s,e)} & = & r_s
\end{array}
$$

Definition 5 (Satisfaction relation). *Let $s \in S$. The* satisfaction relation $\models$ *between pointed epistemic resource models $\mathcal{M}_s$, where $\mathcal{M} = (S, \sim, r, V)$, for resources $\mathcal{R} = (R, \bullet, n)$, and formulas in $\mathcal{L}_{K*\otimes}(A, P)$, is defined by structural induction as follows:*

$$
\begin{array}{lll}
\mathcal{M}_s \models p & \textit{iff} & s \in V(p) \\
\mathcal{M}_s \models \bot & \textit{iff} & \textit{false} \\
\mathcal{M}_s \models I & \textit{iff} & r_s = n \\
\mathcal{M}_s \models \neg\varphi & \textit{iff} & \mathcal{M}_s \not\models \varphi \\
\mathcal{M}_s \models \varphi \wedge \psi & \textit{iff} & \mathcal{M}_s \models \varphi \textit{ and } \mathcal{M}_s \models \psi \\
\mathcal{M}_s \models \varphi \rightarrow \psi & \textit{iff} & \mathcal{M}_s \not\models \varphi \textit{ or } \mathcal{M}_s \models \psi
\end{array}
$$

$$\mathcal{M}_s \models \varphi * \psi \quad \textit{iff} \quad \textit{there are } t, u \in S \textit{ such that } r_s = r_t \bullet r_u, \mathcal{M}_t \models \varphi \textit{ and } \mathcal{M}_u \models \psi$$
$$\mathcal{M}_s \models \varphi \mathbin{-\!\!*} \psi \quad \textit{iff} \quad \textit{for all } t \in S \textit{ such that } r_s \bullet r_t \downarrow \textit{ and } \mathcal{M}_t \models \varphi$$
$$\textit{there is } u \in S \textit{ such that } r_u = r_s \bullet r_t \textit{ and } \mathcal{M}_u \models \psi$$
$$\mathcal{M}_s \models K_a \varphi \quad \textit{iff} \quad \mathcal{M}_t \models \varphi \textit{ for all } t \in S \textit{ such that } s \sim_a t$$
$$\mathcal{M}_s \models [\mathcal{E}_e]\varphi \quad \textit{iff} \quad \mathcal{M}_s \models \textit{pre}(e) \textit{ implies } (\mathcal{M} \otimes \mathcal{E})_{(s,e)} \models \varphi$$

A formula φ is valid on model $\mathcal{M}$ (notation: $\mathcal{M} \models \varphi$) iff for all $s \in S$, $\mathcal{M}_s \models \varphi$, and φ is valid (notation: $\models \varphi$) iff φ is valid on all models $\mathcal{M}$.

Note that $\mathcal{M} \otimes \mathcal{E}$ is again an epistemic resource model for the monoid $(\mathcal{R}, \bullet, n)$. For each resource r in $\mathcal{R}$ there is a state s in S such that $r_s = r$ (r was required to be a surjection). As the action model is covering, for each state s there is an action e such that $\mathcal{M}, s \models \textit{pre}(e)$, so that $(s, e) \in S'$. As $r_{(s,e)} = r_s = r$, $\mathcal{M} \otimes \mathcal{E}$ is again a resource model for $(\mathcal{R}, \bullet, n)$.

The semantics for $*$ and $\mathbin{-\!\!*}$ are different from their standard semantics in BBI. This is because they are not formulated directly in terms of resource but only indirectly by way of states mapped to resources. As different states can be mapped to the same resource, this obliges us to choose where the semantics for $*$ and $\mathbin{-\!\!*}$ is defined in terms of all such states or some such states (there is an extra quantifier that can be universal or existential). We also consider other semantics, for example, for $\mathbin{-\!\!*}$:

$$\mathcal{M}_s \models \varphi \mathbin{-\!\!*} \psi \quad \textit{iff} \quad \textit{for all } t, u \in S \textit{ such that } r_u = r_s \bullet r_t, \mathcal{M}_t \models \varphi \textit{ implies } \mathcal{M}_u \models \psi$$

We are exploring such alternatives in view of their theoretical properties (does a reduction exist?) and their applicability (which typical BBI modelling challenges or benchmarks under partial observation are best described by which version of the semantics?).

3 Eliminating Dynamic Modalities

We now define a set of ESLAM validities for action model modality elimination. To the well-known reduction axioms for Action Model Logic with factual change [8] we add two novel reductions for $*$ and $\mathbin{-\!\!*}$. At this stage we have proved such reduction for $*$ and $\mathbin{-\!\!*}$ for the diamond version of the action model modality but not yet for the box version. Therefore the system is given with the diamond modality as the primitive.

1. $\langle \mathcal{E}_e \rangle p \leftrightarrow (\textit{pre}(e) \wedge \textit{post}(e)(p))$
2. $\langle \mathcal{E}_e \rangle (\psi \wedge \varphi) \leftrightarrow \langle \mathcal{E}_e \rangle \psi \wedge \langle \mathcal{E}_e \rangle \varphi$
3. $\langle \mathcal{E}_e \rangle \neg \psi \leftrightarrow (\textit{pre}(e) \wedge \neg \langle \mathcal{E}_e \rangle \psi)$
4. $\langle \mathcal{E}_e \rangle K_a \psi \leftrightarrow (\textit{pre}(e) \wedge \bigwedge_{e \sim_a f} K_a \langle \mathcal{E}_f \rangle \psi)$
5. $\langle \mathcal{E}_e \rangle (\varphi * \psi) \leftrightarrow (\textit{pre}(e) \wedge \bigvee_{f,g \in E} (\langle \mathcal{E}_f \rangle \varphi * \langle \mathcal{E}_g \rangle \psi))$
6. $\langle \mathcal{E}_e \rangle (\varphi \mathbin{-\!\!*} \psi) \leftrightarrow (\textit{pre}(e) \wedge \bigwedge_{f \in E} (\langle \mathcal{E}_f \rangle \varphi \mathbin{-\!\!*} \bigvee_{g \in E} \langle \mathcal{E}_g \rangle \psi))$

We have shown that the above validities are reduction rules using a complexity measure taken from [9], where it was used to show the reduction for public announcement logic. In [9], the complexity of a formula with public announcement is calculated as follows: $c([\varphi]\psi) = (4 + c(\varphi)) \cdot c(\psi)$. For our current purposes this is generalized to: $c([\mathcal{E}_e]\psi) = (4 + c(\mathcal{E})) \cdot c(\psi)$. The complexity of a pointed action $\mathcal{E}_e$ does not depend on the point, which is why we see $c(\mathcal{E})$ on the right-hand side and not $c(\mathcal{E}_e)$, as maybe expected. Then, $c(\mathcal{E}) = |E|^2 + \max\{c(\textit{pre}(f)), c(\textit{post}(f)(p)) \mid f \in \mathcal{E}, p \in P, p \in \mathcal{D}(\textit{post}(f))\}$ (where E is the domain of $\mathcal{E}$).

4 Library example revisited

In this section we reconsider the modelling example of PASL [3] and illustrate what ESLAM allows us to express. The set of agents A is $\{A_1, A_2\}$ and the set of atoms (variables) P is $\{P_1, P_2, C\}$. The epistemic model $\mathcal{M} = (S, \sim, r, V)$ is now such that: $S = \{(i, j) \mid i, j \in \{0, 1, 2\}\}$; $(i_1, j_1) \sim_{A_1} (i_2, j_2)$ iff $i_1 = i_2$ and $(i_1, j_1) \sim_{A_2} (i_2, j_2)$ iff $j_1 = j_2$; $r_{(i,j)} = (i, j)$; and $V(C) = \{(i, j) \mid i + j \leq 2\}$, $V(P_1) = \{(1, 0)\}$, $V(P_2) = \{(0, 1)\}$. The atoms P_1 and P_2 express agents A_1 and A_2 (respectively) requesting one book each from a librarian, whereas C expresses that the librarian is capable of carrying the requested books. The partial resource monoid $\mathcal{R} = (S, \bullet, n)$, considered has as neutral element $n = (0, 0)$, and a composition operator $\bullet$ defined as:

$$(i_1, j_1) \bullet (i_2, j_2) = \begin{cases} \uparrow & \text{if } i_1 + i_2 \geq 2 \text{ or } j_1 + j_2 \geq 2 \\ (i_1 + i_2, j_1 + j_2) & \text{otherwise} \end{cases}$$

We present two modelling examples for the library setting. The action model $\mathcal{E}'$ emulates public announcement (equivalent to one defined for PASL [3]) and the action model $\mathcal{E}$, defined with ESLAM which, compared to PASL, enables private communication. In both cases we model an action of the librarian telling either: both agents (by means of $\mathcal{E}'$), agent A_1 only (in $\mathcal{E}$) that they can carry the books.

Public announcement action model:	Private announcement action model:
$\mathcal{E}' = \{E', \approx'_a, pre', post'\}$, where:	$\mathcal{E} = \{E, \approx_a, pre, post\}$, where:
$E' = \{e, f\}$	$E = \{e, f\}$
$\approx'_{A_1} = \{(e, e), (f, f)\}$	$\approx_{A1} = \{(e, e), (f, f)\}$
$\approx'_{A_2} = \{(e, e), (f, f)\}$	$\approx_{A2} = \{(e, f), (f, e), (e, e), (f, f)\}$
$pre'(e) = C$	$pre(e) = C$
$pre'(f) = \neg C$	$pre(f) = \neg C$
$post'(e)$ and $post'(f)$ have empty domain	$post(e)$ and $post(f)$ have empty domain

As presented above, the difference between the two lies in the definition of $\approx_a$. In $\mathcal{E}'$ all librarian's announcements are heard by both agents and their uncertainty is equally reduced as action model is executed. This is represented by the identity relation. In $\mathcal{E}$, the librarian addresses A_1 privately, that is why A_1 can tell $\mathcal{E}_e$ and $\mathcal{E}_f$ apart, but as A_2 is excluded from this communication, although A_2 can observe the communication taking place, A_2 cannot make that distinction.
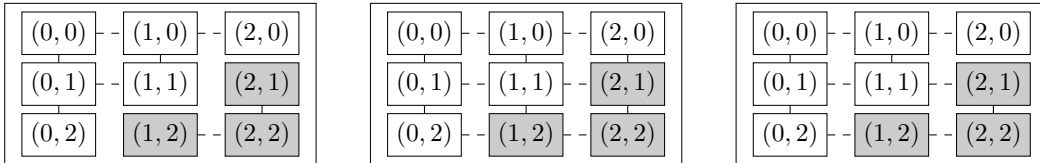
Figure 1: In the center, the initial model. On the left, the result a public announcement $\mathcal{E}'_e$. On the right, the result of a private announcement $\mathcal{E}'_e$. More explanations are found in the text.

Assume each agent wants one book, which corresponds to state $(1, 1)$. Let us compare two model updates: the librarian telling both agents they can carry the books ($\mathcal{E}'_e$), and the librarian telling just A_1 that they can carry the books ($\mathcal{E}_e$). (In ESLAM, as in PASL, a public announcement is a two-event action model, because of the requirement that the action is covering.)

$$(1,1) \models_{\mathcal{M}} \langle \mathcal{E}'_e \rangle (K_{A1}C \wedge K_{A2}C)$$
$$\Leftrightarrow$$
$$(1,1) \models_{\mathcal{M}} C$$
and
$$((1,1),e) \models_{(\mathcal{M} \otimes \mathcal{E}')} K_{A1}C \wedge K_{A2}C$$

$$(1,1) \models_{\mathcal{M}} \langle \mathcal{E}_e \rangle (K_{A1}C \wedge \neg K_{A2}C)$$
$$\Leftrightarrow$$
$$(1,1) \models_{\mathcal{M}} C$$
and
$$((1,1),e) \models_{(\mathcal{M} \otimes \mathcal{E})} K_{A1}C \wedge \neg K_{A2}C$$

In the center of Figure 4 we see the initial model of knowledge. Dashed links - - - represent the relation $\sim_{A_2}$. Solid links — represent the relation $\sim_{A_1}$. We assume reflexivity and transitivity. Grey means "cannot be carried". On the left in the figure we see the update of the model with $\mathcal{E}'_e$. On the right in the figure we see the update of the model with $\mathcal{E}_e$. After the public announcement is made, both agents stopped considering the scenarios where the number of books requested exceeds the librarian's limit. This is illustrated by all links between gray and white areas in the graph disappearing. After the private announcement this is the case only for A_2. This example shows that with ESLAM, compared to PASL, we can define instances of not only public announcement, but also private, more nuanced announcements as well as other forms of partial observation.

5 Conclusions

We have presented Epistemic Separation Logic with Action Models, where the relationship between resources and Kripke semantics is based on the resource function, mapping each state to a resource. Future works will be developed in different directions: defining an additional action resource model monoid, allowing composition and separation of action points, as well as modelling sequential action point execution, achieved by means of action composition operator. Moreover we will investigate the optimal semantics for $*$ and $-\!\!*$, taking into account the duality between these operations.

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Commonly Knowing Whether [*]

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1 Introduction

Common knowledge has been studied extensively in various areas [9, 4, 7, 3]. Intuitively, a proposition is common knowledge among a group, if the proposition is true, everyone knows that, everyone knows everyone knows that, everyone knows everyone knows everyone knows that, and so on, ad infinitum.[1] As the definition suggests, common knowledge is defined based on the notion of 'knowing that'.

Beyond 'knowing that', recent years have witnessed a growing interest in other types of knowledge, such as 'knowing whether', 'knowing how', 'knowing why', 'knowing who', see [13] for an overview. Among these notions, 'knowing whether' is the one most closely related to 'knowing that' and it is used frequently to specify knowledge goals and preconditions for actions [8, 11, 10]. It corresponds to the philosophical notion of non-contingency [2]. An agent *knows whether* a proposition φ, if the agent knows that φ is true, or the agent knows that φ is false; otherwise, the agent is ignorant about φ [5].

It is therefore natural to ask what the notion of 'commonly knowing whether' is in contrast to the notion of common knowledge based on 'knowing that'. These two notions are not equivalent. For instance, suppose you see two people chatting beside a window but you cannot look outside yourself. Then you know that they *commonly know whether* it is sunny outside, but you do not know that they *commonly know that* it is sunny since you do not see the weather. There has been no unanimous agreement yet on the formal definition of 'commonly knowing whether'. As we will show, there are at least five definitions for this notion, which are not logically equivalent over various frame classes.

In this extended abstract, Section 2 gives five formal definitions for 'commonly knowing whether' and the logical relations among them. Section 3 provides an axiomatization and sketches its soundness and completeness. Section 4 explores the relative expressivity of one of the five definitions, by distinguishing two classes of models. All proofs can be found in https://arxiv.org/pdf/2001.03945.pdf.

2 Definitions of 'Commonly Knowing Whether'

This section presents five definitions of 'commonly knowing whether'. We fix a denumerable set of propositional atoms P and a nonempty finite set of agents G. Let G^+ denote the set of finite nonempty sequences consisting of only agents from G. The language involved in this section is defined by the following BNF:

$$\textbf{CCw} \quad \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid Kw_i\varphi \mid Ew\varphi \mid C\varphi \mid Cw\varphi$$

where $p \in \mathsf{P}$ and $i \in \mathsf{G}$. Intuitively, $Kw_i\varphi$ is read "agent i knows whether φ", $Ew\varphi$ is read "everyone knows whether φ", $C\varphi$ is read "the group G commonly knows that φ", and $Cw\varphi$ is read "the group G

[1]This so-called 'iterative approach', attributed to Lewis [7], is a common view of common knowledge. It is compared to the 'fixed-point approach' and the 'shared-environment approach' in Barwise's paper [1].

commonly knows whether φ". As a basis for 'commonly knowing whether', we can give two possible definitions of 'everyone knowing whether'.

Definition 1. $Ew_1\varphi := E\varphi \vee E\neg\varphi$ $\qquad$ $Ew_2\varphi := \bigwedge_{i \in G} Kw_i\varphi$

It is easy to check that Ew_1 is stronger than Ew_2 over $\mathcal{K}$-frames.

Let us now give five possible definitions of 'commonly knowing whether'. The first one, Cw_1, is neither based on 'everyone knowing whether' nor on 'knowing whether', but others are.

Definition 2. $Cw_1\varphi := C\varphi \vee C\neg\varphi$

The definition of Cw_1 is structurally similar to that of 'knowing whether'. It says that a group *commonly knows whether* φ iff the group has common knowledge of φ or common knowledge of $\neg\varphi$. This definition fits well to the type of 'commonly knowing whether' in the sunny weather example in the introduction. Cw_1 also fits to question-answer contexts. Suppose that Sue attends a lecture and asks the speaker: "Is p true or false?" No matter whether the speaker says 'Yes' or 'No', all attendees will *commonly know whether* p after hearing the answer since the answer amounts to a public announcement whether p [12] (depending on the truth value of p), which leads to common knowledge of p or of $\neg p$.

Definition 3. $Cw_2\varphi := CEw\varphi$

According to the above definition, a group *commonly knows whether* φ, if it is common knowledge that everyone knows whether φ. Since there are two different definitions of Ew, we also have two different definitions of Cw_2, that is, $Cw_{21}\varphi := CEw_1\varphi$ and $Cw_{22}\varphi := CEw_2\varphi$.

Definition 4. $Cw_3\varphi := \bigwedge_{k \geq 1}(Ew)^k\varphi$

The above definition follows the iterative approach: a group *commonly knows whether* φ iff everyone knows whether φ, everyone knows whether everyone knows whether φ, everyone knows whether everyone knows whether everyone knows whether φ, and so on, ad infinitum. Again, we have two different definitions of Cw_3, namely, $Cw_{31}\varphi := \bigwedge_{k \geq 1}(Ew_1)^k\varphi$ and $Cw_{32}\varphi := \bigwedge_{k \geq 1}(Ew_2)^k\varphi$.

Definition 5. $Cw_4\varphi := \bigwedge_{i \in G} Cw_1 Kw_i\varphi$, that is, $Cw_4\varphi := \bigwedge_{i \in G}(CKw_i\varphi \vee C\neg Kw_i\varphi)$

According to the above definition, a group *commonly knows whether* φ iff for every member it is common knowledge that they know whether φ or it is common knowledge that they do not know whether φ.

Definition 6. $Cw_5\varphi := \bigwedge_{s \in G^+} Kw_s\varphi$, where $Kw_s\varphi := Kw_{s_1} \ldots Kw_{s_n}\varphi$ if $s = s_1 \ldots s_n$ *is a nonempty sequence of agents.*

The above definition is inspired by the hierarchy of inter-knowledge of a group given in [9]: 'commonly knowing whether' amounts to listing all the possible inter-'knowing whether' states for every nonempty sequence of group members.

2.1 Relations among the different definitions of 'commonly knowing whether'

Over different classes of frames, the logical relations among the above five types of 'commonly knowing whether' $Cw_1, \ldots, Cw_5$ are distinct. In Figures 1 and 2, a one-way arrow from one operator O to another O' means that for all φ, $O\varphi$ strictly logically implies $O'\varphi$. The arrow relation is transitive.

Theorem 1. *Over $\mathcal{K}$ and $\mathcal{KD}45$-frames, the logical relationships among Cw_1, Cw_{21}, Cw_{22}, Cw_{31}, Cw_{32}, Cw_4, and Cw_5 are as in Fig. 1. Over $\mathcal{T}$ and $\mathcal{S}5$-frames, the logical relationships are as in Fig. 2.*

The fact that Cw_1, Cw_2, Cw_3 and Cw_5 boil down to the same thing once the frame is reflexive can be attributed to agents' agreements on the value of φ. In contrast, in case $\mathcal{M}$ is not reflexive, it is possible that $\mathcal{M}, s \models K_i\varphi \wedge K_j\neg\varphi$.
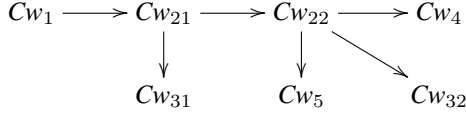
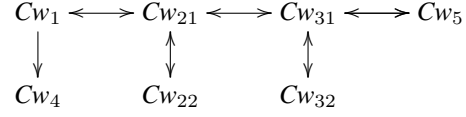Figure 1: Over $\mathcal{K}$ and $\mathcal{KD}45$-frames

Figure 2: Over $\mathcal{T}$ and $\mathcal{S}5$-frames

3 Axiomatization

As mentioned above, over $\mathcal{S}5$-frames, the five definitions of 'commonly knowing whether' are logically equivalent except for Cw_4. One can also verify that Ew_1 and Ew_2 are logically equivalent over $\mathcal{S}5$-frames. In this section, we axiomatize **Cw** over $\mathcal{S}5$-frames based on Cw_1. The language **Cw** can be defined by the following BNF:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid Kw_i\varphi \mid Cw\varphi,$$

The semantics of **Cw** is defined over *Kripke models* of the form $\mathcal{M} = \langle W, \{R_i \mid i \in \mathsf{G}\}, V \rangle$. We use $\to$ to denote the union of R_i for all $i \in \mathsf{G}$, and $\twoheadrightarrow$ to denote the reflexive-transitive closure of $\to$. The truth conditions of **Cw**-formulas are as follows (only nontrivial clauses are shown).

$\mathcal{M}, w \vDash Kw_i\varphi \iff$ for all u and v, if wR_iu and wR_iv, then $(\mathcal{M}, u \vDash \varphi \iff \mathcal{M}, v \vDash \varphi)$.
$\mathcal{M}, w \vDash Cw\varphi \iff$ for all u and v, if $w \twoheadrightarrow u$ and $w \twoheadrightarrow v$, then $(\mathcal{M}, u \vDash \varphi \iff \mathcal{M}, v \vDash \varphi)$.

3.1 Proof system $\mathbb{C}w\mathbb{S}5$

The proof system $\mathbb{C}w\mathbb{S}5$ is an extension of the axiom system of the logic of 'knowing whether' $\mathbb{CLS}5$ from [5] with axioms and rules concerning Cw.

Definition 7. $\mathbb{C}w\mathbb{S}5$ *consists of the following axiom schemas and inference rules:*

(TAUT)	*All instances of tautologies*	*(Kw-T)*	$Kw_i\varphi \wedge Kw_i(\varphi \to \psi) \wedge \varphi \to Kw_i\psi$
(Kw-$\leftrightarrow$)	$Kw_i\varphi \leftrightarrow Kw_i\neg\varphi$	*(Kw-DIS)*	$Kw_i\varphi \to Kw_i(\varphi \to \psi) \vee Kw_i(\neg\varphi \to \chi)$
(wKw-5)	$\neg Kw_i\varphi \to Kw_i\neg Kw_i\varphi$	*(Kw-CON)*	$Kw_i(\chi \to \varphi) \wedge Kw_i(\neg\chi \to \varphi) \to Kw_i\varphi$
(Cw-Mix)	$Cw\varphi \to Ew\varphi \wedge EwCw\varphi$	*(Cw-DIS)*	$Cw\varphi \to Cw(\varphi \to \psi) \vee Cw(\neg\varphi \to \chi)$
(Cw-$\leftrightarrow$)	$Cw\varphi \leftrightarrow Cw\neg\varphi$	*(Cw-CON)*	$Cw(\chi \to \varphi) \wedge Cw(\neg\chi \to \varphi) \to Cw\varphi$
(Cw-Ind)	$Cw(\varphi \to Ew\varphi) \to (\varphi \to Cw\varphi)$	*(Cw-T)*	$Cw\varphi \wedge Cw(\varphi \to \psi) \wedge \varphi \to Cw\psi$
(Kw-NEC)	*from φ infer $Kw_i\varphi$*	*(C-NEC)*	*from φ infer $Cw\varphi$*
(Kw-RE)	*from $\varphi \leftrightarrow \psi$ infer $Kw_i\varphi \leftrightarrow Kw_i\psi$*	*(Cw-RE)*	*from $\varphi \leftrightarrow \psi$ infer $Cw\varphi \leftrightarrow Cw\psi$*
(MP)	*from φ and $\varphi \to \psi$ infer ψ*		

Theorem 2. $\mathbb{C}w\mathbb{S}5$ *is sound with respect to $\mathcal{S}5$.*

3.2 Completeness of $\mathbb{C}w\mathbb{S}5$

It is easy to observe that the logic of **Cw** over $\mathcal{S}5$-frames is not compact. We therefore set out to prove the weak completeness of $\mathbb{C}w\mathbb{S}5$ over $\mathcal{S}5$-frames. Given a formula φ consistent with $\mathbb{C}w\mathbb{S}5$, we will construct a finite model of φ, within the closure of φ (see Definition 8). The definition of the canonical model, including the definition of the canonical relation, is inspired by the completeness proof for the contingency logic $\mathbb{CLS}5$ in [5]. Observe that Lemma 1 makes a finite version of the truth lemma (Lemma 2) go through for formulas of the form $Cw\psi$.

Definition 8. *The* closure *of φ, denoted as $cl(\varphi)$, is the smallest set satisfying the following conditions:*

1. $\varphi \in cl(\varphi)$.
2. *if $\psi \in cl(\varphi)$, then $sub(\psi) \subseteq cl(\varphi)$.*
3. *if $\psi \in cl(\varphi)$ and ψ is not itself of the form $\neg\chi$, then $\neg\psi \in cl(\varphi)$.*
4. *if $Kw_i\psi$, $Kw_i\chi \in cl(\varphi)$, and ψ, χ are not themselves conditionals, then $Kw_i(\chi \to \psi) \in cl(\varphi)$.*
5. *if $Cw\psi \in cl(\varphi)$, then $\{Kw_i\psi \mid i \in \mathbf{G}\} \subseteq cl(\varphi)$.*
6. *if $Cw\psi \in cl(\varphi)$, then $\{Kw_iCw\psi \mid i \in \mathbf{G}\} \subseteq cl(\varphi)$.*

Definition 9. *Let Φ be $cl(\varphi)$. We define the* canonical model based on Φ *as:*
$\mathcal{M}^c = \langle W^c, \{R_i^T \mid i \in \mathbf{G}\}, V^c \rangle$, *where:*

1. $W^c = \{\Sigma \mid \Sigma$ *is maximal consistent in* $\Phi\}$.
2. *For each $i \in \mathbf{G}$, let R_i^T be the reflexive closure of R_i^c, where $\Sigma R_i^c \Delta$ iff there exists a χ such that :*
   (a) χ is not a conditionals and $\neg Kw_i\chi \in \Sigma$;
   (b) for all $Kw_i\psi \in cl(\varphi)$: if $Kw_i\psi$, $Kw_i(\chi \to \psi) \in \Sigma$, then $\psi \in \Delta$.
3. $V^c(p) = \{\Sigma \in W^c \mid p \in \Sigma\}$

Lemma 1. *In the canonical model $\mathcal{M}^c$, $l = \langle \Sigma, \Gamma_1, \cdots, \Gamma_n, \Delta \rangle$ is a ψ-path if $\Sigma \twoheadrightarrow \Delta$ and for each $\Gamma_i \in l$, $\Sigma \twoheadrightarrow \Gamma_i$ and $\psi \in \Gamma_i$. If $Cw\psi \in \Phi$, then $Cw\psi \in \Sigma$ iff every path from Σ is a ψ-path or every path from Σ is a $\neg\psi$-path.*

Lemma 2. *(Finite Truth Lemma) For any $\mathbf{Cw}$-formula $\psi \in \Phi$, for all $\Sigma \in W^c$, $\mathcal{M}^c, \Sigma \models \psi$ iff $\psi \in \Sigma$.*

Theorem 3. *The logic $\mathbb{C}w\mathbb{S}5$ is weakly complete with respect to $\mathcal{S}5$.*

4 Expressivity

In this section, we will compare the expressivity of Cw_5 with that of common knowledge, since both notions are inspired by the hierarchy of inter-knowledge of a group given in [9]. The two languages are:

$$\mathbf{Cw_5} \quad \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid Kw_i\varphi \mid Cw_5\varphi \qquad \mathbf{C} \quad \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid C\varphi$$

Although Kw_i can be defined using the classical operator K_i, surprisingly, Cw_5 is not expressible in **C**. We prove this by constructing two series of models between which *no* **C**-formula can distinguish, whereas some **Cw$_5$**-formula can. To construct these two classes of models, we first define two series of sets of possible worlds.

Definition 10. *For every $n \geq 1$, we inductively define two sets of possible worlds T_n and Z_n as follows:*

- $T_0 = \{t_{00}\}$ *and* $Z_0 = \{z_0\}$;
- *If $t_i \in T_n$, then $t_{i0} \in T_{n+1}$ and $t_{i1} \in T_{n+1}$; if $z_i \in Z_n$, then $z_{i0} \in Z_{n+1}$ and $z_{i1} \in Z_{n+1}$;*
- T_n *and Z_n have no other possible worlds,*

where $|j|$ denotes the length of the subscript sequence j in each t_j and z_j.
   Then we define two series of models : $\mathcal{M} = \{\mathcal{M}_n = \langle W_n, R_n, V_n \rangle \mid n \in \mathbb{N}^+\}$ and $\mathcal{N} = \{\mathcal{N}_n = \langle W_n', R_n', V_n' \rangle \mid n \in \mathbb{N}^+\}$, where

- $W_n = T_n \cup \{r, t_0\}$,
- $R_n = \{(t_i, t_{i0}), (t_i, t_{i1}) \mid t_i \in T_n\} \cup \{(r, t_0), (t_0, t_{00})\}$,
- $V_n(p) = W_n \setminus \{t_{0i}\}$, *where $|i| = n + 1$ and i is a finite sequence of 0s,*
- $W_n' = W_n \cup Z_n$,
- $R_n' = R_n \cup \{(z_i, z_{i0}), (z_i, z_{i1}) \mid z_i \in Z_n\} \cup \{(r, z_0)\}$,
- $V_n'(p) = V_n(p) \cup (Z_n \setminus \{z_{0i}\})$, *where $|i| = 1$ and i is a finite sequence of 0s.*

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Since any **C**-formula φ has a finite modal depth n, we can prove that φ cannot distinguish between $\mathcal{M}_n \in \mathcal{M}$ and $\mathcal{N}_n \in \mathcal{N}$. Therefore, no **C**-formula can distinguish between $\mathcal{M}$ and $\mathcal{N}$. But there exists a $\mathbf{Cw}_5$-formula, namely, Kw_iCw_5p, which is satisfied on all models in $\mathcal{M}$, and falsified on all models in $\mathcal{N}$. In other words, Kw_iCw_5p distinguishes between $\mathcal{M}$ and $\mathcal{N}$.

On the other hand, **C** is not expressively weaker than $\mathbf{Cw}_5$. Consider two models $\mathcal{M}_1$ and $\mathcal{M}_2$, where $(\mathcal{M}_1, s_1)$ can only reach a single p-world by R_1, and $(\mathcal{M}_2, s_2)$ can only reach a single $\neg p$-world by R_1. So $\mathcal{M}_1, s_1 \vDash K_1p$ but $\mathcal{M}_2, s_2 \nvDash K_1p$, which implies that **C** can distinguish the pair of pointed models. However, these two pointed models cannot be distinguished by any $\mathbf{Cw}_5$-formula.

Theorem 4. *Over $\mathcal{K}$, $\mathbf{Cw}_5$ and $\mathbf{C}$ are incomparable in expressivity.*

5 Conclusion and Related Literature

We defined five possible notions of 'commonly knowing whether' and studied how they are related to one another. On $\mathcal{S}5$-frames four of the five notions collapse. We prove the soundness and weak completeness of a 'commonly knowing whether' logic on that class of frames. Finally, we study the expressivity of one of the proposed languages with respect to the standard common knowledge modal language on $\mathcal{K}$-frames. Herzig and Perrotin [6] also recently established an alternative axiomatization of the same logic for 'commonly knowing whether' on $\mathcal{S}5$-frames, based on a novel proof system for common knowledge.

There remains a lot of future work to be done. The axiomatizations over $\mathcal{KD}45$-frames would induce the logic of so-called *commonly believing whether*. And the operator Cw_5 sheds light on a possible formalization of the notion of *ultimate ignorance*, i.e., $I_\mathsf{G}\varphi := \bigwedge_{s \in \mathsf{G}^+} \neg Kw_s\varphi$.

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Double Boolean Algebras with Operators

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Formal concept analysis (FCA) [13, 4] has been a subject of intense study for the past several decades. The central objects of FCA are contexts and concepts. A *context* is a triple $\mathbb{K} := (G, M, I)$, where G, M are sets of *objects* and *properties* respectively, and $I \subseteq G \times M$. For $g \in G$, $I(g) := \{m \in M \ : \ gIm\}$ and for $m \in M$, $I^{-1}(m) := \{g \in G \ : \ gIm\}$. Based on the relation I, two operators, with the same notation $'$, are defined on G and M as follows. For any $A \subseteq G$ and $B \subseteq M$,

$A' := \{m \in M \mid \text{for all } g \in G (g \in A \implies gIm)\}$, and
$B' := \{g \in G \mid \text{for all } m \in M (m \in B \implies gIm)\}$.

A *concept* (A, B) of a context $\mathbb{K} := (G, M, I)$ is a pair of sets satisfying the conditions $A' = B$ and $B' = A$. The set of all concepts of a context $\mathbb{K}$ is denoted by $\mathfrak{B}(\mathbb{K})$. A relation $\leq$ is defined on $\mathfrak{B}(\mathbb{K})$ as follows: for all $(A, B), (C, D) \in \mathfrak{B}(\mathbb{K})$, $(A, B) \leq (C, D)$, if and only if $A \subseteq C$ (equivalently, $D \subseteq B$). $\leq$ is a partial order on $\mathfrak{B}(\mathbb{K})$, and $(\mathfrak{B}(\mathbb{K}), \leq)$ forms a complete lattice.

To develop a mathematical model for conceptual knowledge, Wille introduced negation in FCA in [14]. Simply considering the complements of the sets involved does not work in defining the negation of a concept, and he defined the notions of *semiconcepts* and *protoconcepts*. Let $A, C \subseteq G$ and $B, D \subseteq M$.

Definition 1. [15] A *semiconcept* of the context $\mathbb{K}$ is a pair (A, B) such that $A' = B$ or $A = B'$. A *protoconcept* of the context $\mathbb{K}$ is a pair (A, B) such that $A'' = B'$.

The set of all protoconcepts of the context $\mathbb{K}$ is denoted by $\mathfrak{P}(\mathbb{K})$ and that of all semiconcepts is denoted by $\mathfrak{H}(\mathbb{K})$. Note that, $\mathfrak{H}(\mathbb{K}) \subseteq \mathfrak{P}(\mathbb{K})$. A relation $\sqsubseteq$ is defined on the set $\mathfrak{P}(\mathbb{K})$ as follows. For $(A, B), (C, D) \in \mathfrak{P}(\mathbb{K})$, $(A, B) \sqsubseteq (C, D)$ if and only if $A \subseteq C$ and $D \subseteq B$. $\sqsubseteq$ is a partial order on $\mathfrak{P}(\mathbb{K})$. Further, the following operations are defined on $\mathfrak{P}(\mathbb{K})$. For any $(A_1, B_1), (A_2, B_2), (A, B) \in \mathfrak{P}(\mathbb{K})$,

$(A_1, B_1) \sqcap (A_2, B_2) := (A_1 \cap A_2, (A_1 \cap A_2)')$, $(A_1, B_1) \sqcup (A_2, B_2) := ((B_1 \cap B_2)', B_1 \cap B_2)$,
$\neg(A, B) := (A^c, A^{c'})$, $\lrcorner(A, B) := (B^{c'}, B^c)$, $\top := (G, \emptyset)$ and $\bot := (\emptyset, M)$.

The set $\mathfrak{P}(\mathbb{K})$ of protoconcepts of the context $\mathbb{K}$ fails to form a lattice but leads to a rich algebraic structure, $\underline{\mathfrak{P}}(\mathbb{K}) := (\mathfrak{P}(\mathbb{K}), \sqcap, \sqcup, \neg, \lrcorner, \top, \bot)$, called the *algebra of protoconcepts*. The set $\mathfrak{H}(\mathbb{K})$ of semiconcepts forms a subalgebra $\underline{\mathfrak{H}}(\mathbb{K})$ of the algebra $\underline{\mathfrak{P}}(\mathbb{K})$ of protoconcepts. On abstraction of properties of $\underline{\mathfrak{P}}(\mathbb{K})$ and $\underline{\mathfrak{H}}(\mathbb{K})$, Wille defined *double Boolean algebras* and *pure double Boolean algebras* respectively.

Definition 2. [15] A *double Boolean algebra (dBa)* is an abstract algebra $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ which satisfies the following properties. For any $x, y, z \in D$.

(1a) $(x \sqcap x) \sqcap y = x \sqcap y$ (1b) $(x \sqcup x) \sqcup y = x \sqcup y$

(2a) $x \sqcap y = y \sqcap x$ (2b) $x \sqcup y = y \sqcup x$

(3a) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$ (3b) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$

(4a) $\neg(x \sqcap x) = \neg x$ (4b) $\lrcorner(x \sqcup x) = \lrcorner x$

(5a) $x \sqcap (x \sqcup y) = x \sqcap x$ (5b) $x \sqcup (x \sqcap y) = x \sqcup x$

(6a) $x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$ (6b) $x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$

(7a) $x \sqcap (x \vee y) = x \sqcap x$ (7b) $x \sqcup (x \wedge y) = x \sqcup x$

(8a) $\neg\neg(x \sqcap y) = x \sqcap y$ (8b) $\lrcorner\lrcorner(x \sqcup y) = x \sqcup y$

(9a) $x \sqcap \neg x = \bot$ (9b) $x \sqcup \lrcorner x = \top$

(10a) $\neg\bot = \top \sqcap \top$ (10b) $\lrcorner\top = \bot \sqcup \bot$

(11a) $\neg\top = \bot$ (11b) $\lrcorner\bot = \top$

(12) $(x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$

where $x \vee y := \neg(\neg x \sqcap \neg y)$, and $x \wedge y := \lrcorner(\lrcorner x \sqcup \lrcorner y)$.

A dBa $\mathbf{D}$ is called *pure* if for all $x \in D$, either $x \sqcap x = x$ or $x \sqcup x = x$.

A relation $\sqsubseteq$ is defined on D by

$$x \sqsubseteq y : \iff x \sqcap y = x \sqcap x \text{ and } x \sqcup y = y \sqcup y.$$

$\sqsubseteq$ is a quasi-order on D. As intended, for any context $\mathbb{K} := (G, M, I)$, we get

Theorem 1. [15]
    (i) $\underline{\mathfrak{P}}(\mathbb{K}) := (\mathfrak{P}(\mathbb{K}), \sqcap, \sqcup, \neg, \lrcorner, \top, \bot)$ is a dBa.
    (ii) $\underline{\mathfrak{H}}(\mathbb{K}) := (\mathfrak{H}(\mathbb{K}), \sqcap, \sqcup, \neg, \lrcorner, \top, \bot)$ is a pure dBa.

It is clear that any Boolean algebra is also a pure dBa.
    Let us recall the sets $D_\sqcap := \{x \in D : x \sqcap x = x\}$ and $D_\sqcup := \{x \in D : x \sqcup x = x\}$. When $D = \mathfrak{P}(\mathbb{K})$, one has the following.

Observation 1. [15] Let $A \subseteq G$ and $B \subseteq M$.
    (i) $(A, B) \in \mathfrak{P}(\mathbb{K})_\sqcap$ if and only if $(A, B) = (A, A')$.
    (ii) $(A, B) \in \mathfrak{P}(\mathbb{K})_\sqcup$ if and only if $(A, B) = (B', B)$.

It is known that the sets $D_\sqcap$ and $D_\sqcup$ form Boolean algebras $\mathbf{D}_\sqcap := (D_\sqcap, \sqcap, \vee, \neg, \bot, \neg\bot)$ and $\mathbf{D}_\sqcup := (D_\sqcup, \sqcup, \wedge, \lrcorner, \top, \lrcorner\top)$. In particular, for the Boolean algebras formed by $\mathfrak{P}(\mathbb{K})_\sqcap$ and $\mathfrak{P}(\mathbb{K})_\sqcup$, it is shown that

Theorem 2. [15]
    (i) The power set Boolean algebra $(\mathcal{P}(G), \cap, \cup, ^c, G, \emptyset)$ is isomorphic to the Boolean algebra $\underline{\mathfrak{P}}(\mathbb{K})_\sqcap := (\mathfrak{P}(\mathbb{K})_\sqcap, \sqcap, \vee, \neg, \bot, \neg\bot)$, where any $A(\subseteq G)$ is mapped to $(A, A') \in \mathfrak{P}(\mathbb{K})_\sqcap$.
    (ii) The power set Boolean algebra $(\mathcal{P}(M), \cup, \cap, ^c, M, \emptyset)$ is anti-isomorphic to the Boolean algebra $\underline{\mathfrak{P}}(\mathbb{K})_\sqcup := (\mathfrak{P}(\mathbb{K})_\sqcup, \sqcup, \wedge, \lrcorner, \top, \lrcorner\top)$, where any $B(\subseteq M)$ is mapped to $(B', B) \in \mathfrak{P}(\mathbb{K})_\sqcup$.

The above features of the Boolean algebras $\underline{\mathfrak{P}}(\mathbb{K})_{\sqcap}$ and $\underline{\mathfrak{P}}(\mathbb{K})_{\sqcup}$ play a role in our work, as we shall see in the sequel.

FCA has been established as a useful tool for data analysis. On the other hand, another well-established area of study that has been extensively used in data analysis is Rough set theory, founded by Pawlak [11]. There have been several studies linking and comparing the two theories, e.g. in [16, 9, 7, 3, 17, 8, 5, 6, 12]. The basic notions in rough set theory [10] are *approximation spaces* and the *approximation operators* defined on them. In this paper, we work with *generalized* rough set models [18], instead of the classical Pawlakian ones. An approximation space in a generalized rough set model is a pair (W, E), where W is a set and E a binary relation on W. For $x \in W$, $E(x) := \{y \in W \ : \ xRy\}$. The *lower* and *upper* *approximations* of any $A(\subseteq W)$ are defined respectively as follows.

(i) $\underline{A}_E := \{x \in W \ : \ E(x) \subseteq A\}$.

(ii) $\overline{A}^E := \{x \in W \ : \ E(x) \cap A \neq \emptyset\}$.

If the relation is clear from the context, we shall omit the subscript and denote $\underline{A}_E$ by $\underline{A}$, $\overline{A}^E$ by $\overline{A}$. The following are easy consequences.

Proposition 1.

    (i) $\overline{A} = ((\underline{A^c}))^c, \underline{A} = (\overline{(A^c)})^c$.

    (ii) $\underline{W} = W$.

    (iii) $\underline{A \cap B} = \underline{A} \cap \underline{B}, \overline{A \cup B} = \overline{A} \cup \overline{B}$.

    (iv) $A \subseteq B$ implies that $\underline{A} \subseteq \underline{B}, \overline{A} \subseteq \overline{B}$.

Düntsch and Gediga [2] defined *property oriented concept lattices* based on modal style operators determined by a context. *Object oriented concept lattices* have been defined by Yao [16], using the same operators. The present authors introduced negation into the study by defining the notions of *object oriented semiconcepts* and *object oriented protoconcepts* [6]. Furthermore, in [5], *semitopological dBas* were proposed. The present work is a continuation of this direction of study. We modify the algebraic structure of a semitopological dBa and propose a *double Boolean algebra with operators*.

Definition 3. A *double Boolean algebra with operators (dBao)* is a structure $\mathfrak{O} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, I, C)$ such that

    (i) $(D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ is a dBa.

    (ii) I, C are monotonic operators on D such that for all $x, y \in D$ the following are satisfied.

1a $I(x \sqcap y) = I(x) \sqcap I(y)$	1b $C(x \sqcup y) = C(x) \sqcup C(y)$
2a $I(\neg\bot) = \neg\bot$	2b $C(\lrcorner\top) = \lrcorner\top$
3a $I(x \sqcap x) = I(x)$	3b $C(x \sqcup x) = C(x)$

Observe that any Boolean algebra with operators [1] is a dBao. We shall give another class of examples of dBao. For that, we have been motivated by the notions of *lower and upper concepts approximations* of any pair of sets in a context, defined by Saquer and Deogun in [12]. The definitions are based on rough set theory. It may be remarked here that approximations of concepts based on rough set theory were first introduced by Kent [8]. The work in [12] differs from that of Kent in choosing the "indiscernibility" relations. Kent considers an indiscernibility relation on the set G of objects which is externally given by some agent, whereas Saquer and

Deogun consider a relation that is determined by the given context. The latter defined relations E_1, E_2 on the set G of objects and the set M of properties of a given context $\mathbb{K} := (G, M, I)$, as follows.

(a) For $g_1, g_2 \in G$, $g_1 E_1 g_2$ if and only if $I(g_1) = I(g_2)$.

(b) For $m_1, m_2 \in M$, $m_1 E_2 m_2$ if and only if $I^{-1}(m_1) = I^{-1}(m_2)$.

For $A \subseteq G$, the *lower concept approximation* of A has been defined by the formal concept $((\underline{A}_{E_1})'', (\underline{A}_{E_1})')$ and the *upper concept approximation* of A by $((\overline{A}^{E_1})'', (\overline{A}^{E_1})')$.

Similarly for $B \subseteq M$, the lower and upper concept approximations of B have been defined by $((\underline{B}_{E_2})', (\underline{B}_{E_2})'')$ and $((\overline{B}^{E_2})', (\overline{B}^{E_2})'')$ respectively. Let us illustrate these notions by an example. The following context is a subcontext of a context given by Wille [4] with some modifications.

A Context

	a	b	c	g
Leech	*	*		*
Bream	*	*		*
Frog	*	*	*	*
Dog	*		*	*
Cat	*		*	*

where a:= needs water to live, b:= lives in water, c:= lives on land, g:=can move around.

Now observe that the properties a and g are indiscernible by objects, where as Leech and Bream are indiscernible by properties. Therefore the induced approximation spaces are, $(G, \{\{Leech, Bream\}, \{Frog\}, \{Dog, Cat\}\})$ and $(M, \{\{a, g\}, \{b\}, \{c\}\})$, where $G := \{Leech, Bream, Frog, Dog, Cat\}$ and $M := \{a, b, c, g\}$.

Let $A := \{Leech, Bream, Dog\}$. Then the lower concept approximation of A is $(\{Leech, Bream, Frog\}, \{a, b, g\})$. If $B := \{a, c\}$, the upper concept approximation of B is $(\{Frog, Dog, Cat\}, \{a, g, c\})$.

We now introduce the following.

Definition 4. A *Kripke context based on a context* $\mathbb{K} := (G, M, I)$ is a triple $\mathbb{KC} := ((G, R), (M, S), I)$, where R, S are relations on G and M respectively.

We would like to expand the algebra of protoconcepts of $\mathbb{K}$ to a dBao, for which we need to find two "interior and closure type" operators I, C on $\mathfrak{P}(\mathbb{K})$ satisfying 1a-3a and 1b-3b respectively of Definition 3. In particular, one focusses on 1a and 1b. For the Kripke frame (G, R), an interior-type operator is the map $-_R : \mathcal{P}(G) \to \mathcal{P}(G)$ defined as $-_R(A) := \underline{A}_R$ for all $A \in \mathcal{P}(G)$. This makes $(\mathcal{P}(G), \cap, \cup, ^c, G, \emptyset, -_R)$ a Boolean algebra with operators with respect to $-_R$. Similarly for the Kripke frame (M, S), one has the operator $-_S : \mathcal{P}(M) \to \mathcal{P}(M)$ defined by $-_M(B) := \underline{B}_S$ for all $B \in \mathcal{P}(M)$. Now from Theorem 2, we get the isomorphism $f : \mathcal{P}(G) \to \mathfrak{P}(\mathbb{K})_\sqcap$ given by $f(A) := (A, A')$ for all $A \in \mathcal{P}(G)$ and the anti-isomorphism $g : \mathcal{P}(M) \to \mathfrak{P}(\mathbb{K})_\sqcup$ given by $g(B) := (B', B)$ for all $B \in \mathcal{P}(M)$. Utilizing $f, -_R$ on the one hand, one gets a candidate for I in the form of the unary operator f_R given below. On the other hand, using the anti-isomorphism g and the operator $-_S$, we get a candidate for C as the unary operator f_S. For any $(A, B) \in \mathfrak{P}(\mathbb{K})$,

- $f_R((A, B)) := (\underline{A}_R, (\underline{A}_R)')$,

- $f_S((A, B)) := ((\underline{B}_S)', \underline{B}_S)$.

f_R, f_S are well-defined, as $(\underline{A}_R, (\underline{A}_R)')$ and $((\underline{B}_S)', \underline{B}_S)$ are both semiconcepts and hence protoconcepts. In Theorem 3 below, we show that the above choices for f_R, f_S indeed work for our purpose.

Notation 1. Let f_R^δ, f_S^δ denote the operators on $\mathfrak{P}(\mathbb{K})$ that are *dual* to f_R, f_S respectively. In other words, for each $x := (A, B) \in \mathfrak{P}(\mathbb{K})$,

$f_R^\delta(x) := \neg f_R(\neg x) = \neg f_R((A^c, A^{c\prime})) = \neg(\underline{A}_R^c, (\underline{A}_R^c)') = ((\underline{A}_R^c)^c, (\underline{A}_R^c)^{c\prime}) = (\overline{A}^R, (\overline{A}^R)')$, by Proposition 1(i).

Similarly $f_S^\delta(x) := \lrcorner f_S(\lrcorner x) = (\overline{B}^{S\prime}, \overline{B}^S)$.

Again, note that $f_R^\delta(x) = (\overline{A}^R, (\overline{A}^R)')$ and $f_S^\delta(x) = ((\overline{B}^S)', \overline{B}^S)$ are semiconcepts.

An immediate example of a Kripke context based on $\mathbb{K} := (G, M, I)$ is $\mathbb{KC} := ((G, E_1), (M, E_2), I)$, where E_1 and E_2 are the equivalence relations defined in (a) and (b) above. Moreover, for any $A \subseteq G$ and $B \subseteq M$, we observe that the lower concept approximations of A and B are expressible using the operators f_{E_1} and f_{E_2} respectively. Indeed, $f_{E_1}((A, A')) \sqcup f_{E_1}((A, A')) = (\underline{A}_{E_1}, (\underline{A}_{E_1})') \sqcup (\underline{A}_{E_1}, (\underline{A}_{E_1})') = ((\underline{A}_{E_1})'', (\underline{A}_{E_1})')$.
On the other hand, $f_{E_2}((B', B)) \sqcap f_{E_2}((B', B)) = ((\underline{B}_{E_2})', (\underline{B}_{E_2})'')$.
Similarly the upper concept approximations of A and B can be expressed using $f_{E_1}^\delta$ and $f_{E_2}^\delta$ respectively.

Now we arrive at a class of examples of dBao.

Theorem 3. Let $\mathbb{KC} := ((G, R), (M, S), I)$ be a Kripke context based on the context $\mathbb{K} := (G, M, I)$. Then $\underline{\mathfrak{P}}^+(\mathbb{KC}) := (\mathfrak{P}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, f_R, f_S)$ is a dBao.

Proof. It was mentioned earlier that $(\mathfrak{P}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ is a dBa. To show monotonicity of f_R, f_S, let $(A, B) \sqsubseteq (C, D)$. Then, by definition of $\sqsubseteq$, $A \subseteq C$ and $D \subseteq B$, and by using Proposition 1(iv), $\underline{A}_R \subseteq \underline{C}_R$ which will imply $(\underline{C}_R)' \subseteq (\underline{A}_R)'$. Hence $f_R((A, B)) \sqsubseteq f_R((C, D))$. Similarly, one can show monotonicity of f_S.
Let (A, B) and (C, D) belong to $\mathfrak{P}(\mathbb{K})$. Using Proposition 1(iii),
$f_R((A, B) \sqcap (C, D)) = f_R(A \cap C, (A \cap C)') = ((\underline{A \cap C})_R, ((\underline{A \cap C})_R)') = (\underline{A}_R \cap \underline{C}_R, (\underline{A}_R \cap \underline{C}_R)') = (\underline{A}_R, (\underline{A}_R)') \sqcap (\underline{C}_R, (\underline{C}_R)') = f_R((A, B)) \sqcap f_R((C, D))$ and
$f_S((A, B) \sqcup (C, D)) = f_S((B \cap D)', B \cap D) = (((\overline{B \cap D})_S)', (\overline{B \cap D})_S) = f_S((A, B)) \sqcup f_S((C, D))$.
Let $(A, B) \in \mathfrak{P}(\mathbb{K})$. Then $f_R((A, B) \sqcap (A, B)) = f_R((A, A')) = (\underline{A}_R, (\underline{A}_R)') = f_R((A, B))$. Now $f_R(\neg \bot) = f_R((G, G')) = (\underline{G}_R, (\underline{G}_R)') = (G, G') = \neg \bot$, by Proposition 1(ii). Similarly, one can show that $f_S((A, B) \sqcup (A, B)) = ((\underline{B}_S)', \underline{B}_S)$ and $f_S(\lrcorner \top) = \lrcorner \top$. Therefore, by Definition 3, $(\mathfrak{P}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, f_R, f_S)$ is a dBao. $\square$

The definitions of f_R, f_S could also be given on the set $\mathfrak{H}(\mathbb{K})$ of semiconcepts. As $\mathfrak{H}(\mathbb{K})$ forms a pure dBa, Theorem 3 would then yield an example $\underline{\mathfrak{H}}^+(\mathbb{KC})$ of a *pure dBao* (a dBao based on a pure dBa) corresponding to any Kripke context. Following the nomenclature used in modal logic [1], we call $\underline{\mathfrak{P}}^+(\mathbb{KC})$ the *full complex algebra* of the Kripke context $\mathbb{KC}$. Any subalgebra of $\underline{\mathfrak{P}}^+(\mathbb{KC})$ is called a *complex algebra* of $\mathbb{KC}$, and $\underline{\mathfrak{H}}^+(\mathbb{KC})$ is an instance of such an algebra.

Directions of ongoing study

Representation results related to dBas have been well-studied in literature. It has been shown that any dBa is quasi-embeddable into the algebra of protoconcepts of some context $\mathbb{K}$, while any pure dBa is embeddable into the algebra of semiconcepts of $\mathbb{K}$. We expect that, in the lines of the proof of the representation theorem for Boolean algebras with operators, one

will obtain a representation of dBao in terms of the dBao $\underline{\mathfrak{P}}^+(\mathbb{KC})$ and for a pure dBao, a representation in terms of the pure dBao $\underline{\mathfrak{H}}^+(\mathbb{KC})$, for some Kripke context $\mathbb{KC}$.

We have formulated modal systems corresponding to dBao and pure dBao – the last, a system with hypersequents. We expect that, apart from the algebraic semantics, a semantics based on the class of Kripke contexts can be imparted to these systems.

The Kripke context $\mathbb{KC} := ((G, R), (M, S), I)$ was defined with arbitrary binary relations R, S and I. As in modal correspondence theory, one may impose various conditions on the relations. We are exploring the resulting properties of the complex algebras $\underline{\mathfrak{P}}^+(\mathbb{KC})$, $\underline{\mathfrak{H}}^+(\mathbb{KC})$, the corresponding classes of dBao and modal systems.

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L-Topology via Generalised Geometric Logic[*]

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Abstract

This work introduces a notion of generalised geometric logic. Connections of generalised geometric logic with L-topological systems and L-topological spaces are established.

1 Introduction

This work is motivated by S. Vickers's work on topology via logic [17]. To show the connection of topology with geometric logic, the notion of topological system [17] played a crucial role. Topological system is an interesting mathematical structure, which unifies the concepts of topology, algebra, logic in a single framework. In our earlier work [1], we had introduced a notion of fuzzy geometric logic to answer the question viz. "From which logic can fuzzy topology be studied?". For this purpose first of all we introduced the notion of fuzzy topological system [6]. J. Denniston et al. introduced the notion of lattice valued topological system (L-topological system) by considering frame valued relation between X and A. In [3], a categorical relationship of Lattice valued topological space (L-topological space) with frame was established using the categorical relationships of them with L-topological systems. Moreover categorical equivalence between spatial L-topological systems with L-topological space was shown. In this paper the main focus is to answer the question viz. "From which logic can L-topology be studied?". From [1], it is clear that the satisfaction relation $\models$ of fuzzy topological systems reflects the notion of satisfiability (sat) of a geometric formula by a sequence over the domain of interpretation of the corresponding logic. Hence we considered the grade of satisfiability from $[0,1]$. As for the L-topological system the satisfaction relation is an L (frame)-valued relation, the natural tendency is to consider the grade of satisfiability from L. Keeping this in mind, generalised geometric logic is proposed to provide the answer of the raised question successfully.

2 Generalised Geometric Logic

Generalised geometric logic may be considered as a generalisation of fuzzy geometric logic and consequently of so called geometric logic. Detailed studies on fuzzy logic, geometric logic and fuzzy geometric logic may be found in [1, 4, 9, 10, 11, 12, 14, 15, 16, 17].

The **alphabet** of the language $\mathcal{L}$ of generalised geometric logic comprises of the connectives $\wedge$, $\bigvee$, the existential quantifier $\exists$, parentheses) and (as well as: countably many individual constants $c_1, c_2, \ldots$; denumerably many individual variables $x_1, x_2, \ldots$; propositional constants $\top$, $\bot$; for each $i > 0$, countably many i-place predicate symbols p_j^i's, including at least the 2-place symbol "=" for identity; for each $i > 0$, countably many i-place function symbols f_j^i's.

Definition 2.1 (Term). **Terms** are recursively defined in the usual way. Every constant symbol c_i is a term; every variable x_i is a term; if f_j is an i-place function symbol, and $t_1, t_2, \ldots, t_i$ are terms then $f_j^i t_1 t_2 \ldots t_i$ is a term; nothing else is a term.

Definition 2.2 (Geometric formula). **Geometric formulae** are recursively defined as follows: $\top$, $\bot$ are geometric formulae; if p_j is an i-place predicate symbol, and $t_1, t_2, \ldots, t_i$ are terms then $p_j^i t_1 t_2 \ldots t_i$ is a geometric formula; if t_i, t_j are terms then $(t_i = t_j)$ is a geometric formula; if ϕ and ψ are geometric formulae then $(\phi \wedge \psi)$ is a geometric formula; if ϕ_i's $(i \in I)$ are geometric formulae then $\bigvee \{\phi_i\}_{i \in I}$ is a geometric formula, when $I = \{1, 2\}$ then the above formula is written as $\phi_1 \vee \phi_2$; if ϕ is a geometric formula and x_i is a variable then $\exists x_i \phi$ is a geometric formula; nothing else is a geometric formula.

Definition 2.3 (Interpretation). An **interpretation** I consists of

- a set D, called the domain of interpretation;

- an element $I(c_i) \in D$ for each constant c_i;

- a function $I(f_j^i) : D^i \longrightarrow D$ for each function symbol f_j^i;

- an L-fuzzy relation $I(p_j^i) : D^i \longrightarrow L$, where L is a frame, for each predicate symbol p_j^i i.e. it is an L-fuzzy subset of D^i.

Definition 2.4 (Graded Satisfiability). Let s be a sequence over D. Let $s = (s_1, s_2, \ldots)$ be a sequence over D where $s_1, s_2, \ldots$ are all elements of D. Let d be an element of D. Then $s(d/x_i)$ is the result of replacing i'th coordinate of s by d i.e., $s(d/x_i) = (s_1, s_2, \ldots, s_{i-1}, d, s_{i+1}, \ldots)$. Let t be a term. Then s assigns an element $s(t)$ of D as follows:

- if t is the constant symbol c_i then $s(c_i) = I(c_i)$;

- if t is the variable x_i then $s(x_i) = s_i$;

- if t is the function symbol $f_j^i t_1 t_2 \ldots t_i$ then
  $s(f_j^i t_1 t_2 \ldots t_i) = I(f_j^i)(s(t_1), s(t_2), \ldots, s(t_i))$.

Now we define grade of satisfiability of ϕ by s written as $gr(s\ sat\ \phi)$, where ϕ is a geometric formula, as follows:

- $gr(s\ sat\ p_j^i t_1 t_2 \ldots t_i) = I(p_j^i)(s(t_1), s(t_2), \ldots, s(t_i))$;

- $gr(s\ sat\ \top) = 1_L$;

- $gr(s\ sat\ \bot) = 0_L$;

- $gr(s\ sat\ t_i = t_j) = \begin{cases} 1_L & if\ s(t_i) = s(t_j) \\ 0_L & otherwise; \end{cases}$

- $gr(s\ sat\ \phi_1 \wedge \phi_2) = gr(s\ sat\ \phi_1) \wedge gr(s\ sat\ \phi_2)$;

- $gr(s\ sat\ \phi_1 \vee \phi_2) = gr(s\ sat\ \phi_1) \vee gr(s\ sat\ \phi_2)$;

- $gr(s\ sat\ \bigvee \{\phi_i\}_{i \in I}) = sup\{gr(s\ sat\ \phi_i) \mid i \in I\}$;

- $gr(s\ sat\ \exists x_i \phi) = sup\{gr(s(d/x_i)\ sat\ \phi) \mid d \in D\}$.

Throughout this article $\wedge$ and $\vee$ in L will stand for the meet and join of the frame L respectively. The expression $\phi \vdash \psi$, where ϕ, ψ are wffs, is called a sequent. We now define satisfiability of a sequent.

Definition 2.5. 1. s sat $\phi \vdash \psi$ iff $gr(s$ sat $\phi) \leq gr(s$ sat $\psi)$.
2. $\phi \vdash \psi$ is valid in I iff s sat $\phi \vdash \psi$ for all s in the domain of I.
3. $\phi \vdash \psi$ is universally valid iff it is valid in all interpretations.

Theorem 2.6. *Let I be an interpretation and t be a term. If the sequences s and s' are such that they agree on the variables occurring in the term t then $s(t) = s'(t)$.*

Proof. By induction on t. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 2.7. *Let I be an interpretation and ϕ be a geometric formula. If the sequences s and s' are such that they agree on the free variables occurring in ϕ then $gr(s$ sat $\phi) = gr(s'$ sat $\phi)$.*

Proof. By induction on ϕ. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

A rule of inference for generalised geometric logic is of the form $\dfrac{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_i}{\mathcal{S}}$, where each of the $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_i$ and $\mathcal{S}$ is a sequent. The sequents $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_i$ are known as premises and the sequent $\mathcal{S}$ is called the conclusion. It should be noted that for a rule of inference the set of premises can be empty also.
The rules of inference for generalised geometric logic are as follows.

1. $\phi \vdash \phi$,

2. $\dfrac{\phi \vdash \psi \qquad \psi \vdash \chi}{\phi \vdash \chi}$,

3. (i) $\phi \vdash \top$, (ii) $\phi \wedge \psi \vdash \phi$, (iii) $\phi \wedge \psi \vdash \psi$, (iv) $\dfrac{\phi \vdash \psi \qquad \phi \vdash \chi}{\phi \vdash \psi \wedge \chi}$,

4. (i) $\phi \vdash \bigvee S$ $(\phi \in S)$, (ii) $\dfrac{\phi \vdash \psi \qquad \text{all } \phi \in S}{\bigvee S \vdash \psi}$,

5. $\phi \wedge \bigvee S \vdash \bigvee \{\phi \wedge \psi \mid \psi \in S\}$,

6. $\top \vdash (x = x)$,

7. $((x_1, \ldots, x_n) = (y_1, \ldots, y_n)) \wedge \phi \vdash \phi[(y_1, \ldots, y_n) \mid (x_1, \ldots, x_n)]$,

8. (i) $\dfrac{\phi \vdash \psi[x \mid y]}{\phi \vdash \exists y \psi}$, (ii) $\dfrac{\exists y \phi \vdash \psi}{\phi[x \mid y] \vdash \psi}$,

9. $\phi \wedge (\exists y)\psi \vdash (\exists y)(\phi \wedge \psi)$.

The soundness of a rule means that if all the premises are valid in an interpretation I then the conclusion must also be valid in the same interpretation I. Satisfaction relation being many-valued, the validity of a sequent has a meaning different from that in the classical geometric logic. In this subsection we will show the soundness of the above rules of inference.

Theorem 2.8. *The rules of inference for generalised geometric logic are universally valid.*

Let us consider the triplet $(X, \models, A)$ where X is the non empty set of assignments s, A is the set of geometric formulae and $\models$ defined as $gr(s \models \phi) = gr(s$ sat $\phi)$.

Theorem 2.9. *(i)* $gr(s \models \phi \wedge \psi) = gr(s \models \phi) \wedge gr(s \models \psi)$.
(ii) $gr(s \models \bigvee \{\phi_i\}_{i \in I}) = sup_{i \in I} \{gr(x \models \phi_i)\}$.

Definition 2.10. $\phi \approx \psi$ iff $gr(s \models \phi) = gr(s \models \psi)$ for any $s \in X$ and $\phi, \psi \in A$.

The above defined "$\approx$" is an equivalence relation. Thus we get $A/_\approx$.

Definition 2.11 (*L*-topological space)**.** Let X be a set, and τ be a collection of *L*-fuzzy subsets of X i.e., $\tilde{A} : X \to L$, where L is a frame, s.t.

1. $\tilde{\emptyset}$, $\tilde{X} \in \tau$, where $\tilde{\emptyset}(x) = 0_L$, for all $x \in X$ and $\tilde{X}(x) = 1_L$, for all $x \in X$;

2. $\tilde{A}_i \in \tau$ for $i \in I$ implies $\bigcup_{i \in I} \tilde{A}_i \in \tau$, where $\bigcup_{i \in I} \tilde{A}_i(x) = sup_{i \in I}(\tilde{A}_i(x))$;

3. $\tilde{A}_1$, $\tilde{A}_2 \in \tau$ implies $\tilde{A}_1 \cap \tilde{A}_2 \in \tau$, where $(\tilde{A}_1 \cap \tilde{A}_2)(x) = \tilde{A}_1(x) \wedge \tilde{A}_2(x)$.

Then (X, τ) is called an **L-topological space**. τ is called an **L-topology** over X.

Elements of τ are called **L-open sets** of *L*-topological space (X, τ).

Definition 2.12 (*L*-topological system)**.** [3] An **L-topological system** is a triple $(X, \models, A)$, where X is a non-empty set, A is a frame and $\models$ is an *L*-valued relation from X to A ($\models$: $X \times A \to L$) such that

1. if S is a finite subset of A, then $gr(x \models \bigwedge S) = inf\{gr(x \models s) \mid s \in S\}$;

2. if S is any subset of A, then $gr(x \models \bigvee S) = sup\{gr(x \models s) \mid s \in S\}$.

For our convenience $\models (x, a)$ will be expressed as $gr(x \models a)$ throughout this article. It is to be noted that $\bigwedge S$ is either $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ if $S = \{a_1, a_2, \ldots, a_n\}$ and is $\top$ if $S = \emptyset$. Note that if $L = [0, 1]$ then the triple is known as fuzzy topological system [1].

Definition 2.13 (Spatial)**.** An *L*-topological system $(X, \models, A)$ is said to be **spatial** if and only if (for any $x \in X$, $gr(x \models a) = gr(x \models b)$) imply $(a = b)$, for any $a, b \in A$.

Theorem 2.14. *[3] Category of spatial L-topological systems, for a fixed L, is equivalent to the category of L-topological spaces.*

Theorem 2.15. *$(X, \models', A/_\approx)$ is an L-topological system, where $\models'$ is defined by $gr(s \models' [\phi]) = gr(s \models \phi)$.*

Proposition 2.16. *In the L-topological system $(X, \models', A/_\approx)$, defined above, for all $s \in X$, $(gr(s \models' [\phi]) = gr(s \models' [\psi]))$ implies $([\phi] = [\psi])$.*

Proof. As $gr(s \models' [\phi]) = gr(s \models' [\psi])$, for any s, we have $gr(s \models \phi) = gr(s \models \psi)$, for any s. Hence $\phi \approx \psi$ and consequently $[\phi] = [\psi]$. $\square$

Theorem 2.17. *Let $(X, \models, A)$ be an L-topological system. Then $(X, ext(A))$, where $ext(A) = \{ext(a)\}_{a \in A}$ such that $ext(a)(x) = gr(x \models a)$ for each $x \in X$, is an L-topological space.*

From Theorem 2.15 and Theorem 2.17 we arrive at the conclusion that $(X, ext(A/_\approx))$ is an *L*-topological space. Proposition 2.16 indicates that $(X, \models', A/_\approx)$ is a spatial *L*-topological system and hence from Theorem 2.14 we arrive at the conclusion that $(X, \models', A/_\approx)$, $(A, ext(A/_\approx))$ are equivalent to each other. That is, $(X, \models', A/_\approx)$ and $(X, \in, ext(A/_\approx))$ represent the same *L*-topological system.

Let X be an *L*-topological space, τ is its *L*-topology. Then the corresponding generalised geometric theory can be defined as follows:

- for each L-open set $\tilde{T}$, a proposition $P_{\tilde{T}}$

- if $\tilde{T}_1 \subseteq \tilde{T}_2$, then an axiom
$$P_{\tilde{T}_1} \vdash P_{\tilde{T}_2}$$

- if S is a family of L-open sets, then an axiom
$$P_{\bigcup S} \vdash \bigvee_{\tilde{T} \in S} P_{\tilde{T}}$$

- if S is finite collection of L-open sets, then an axiom
$$\bigwedge_{\tilde{T} \in S} P_{\tilde{T}} \vdash P_{\bigcap S}$$

All other axioms for the (propositional) generalised geometric logic will follow from the above clauses.

If $x \in X$, then x gives a model of the theory in which the truth value of the interpretation of $P_{\tilde{T}}$ will be $\tilde{T}(x)$.

Hence one may study L-topology via generalised geometric logic.

3 Conclusion

In summary, the notion of generalised geometric logic is introduced and studied in details. Using the connection between L-topological system and L-topological space, the strong connection between the proposed logic and L-topological space is established. The interpretation of the predicate symbols for the generalised geometric logic are L (frame)-valued relations and so the proposed logic is more expressible. That is, the proposed logic has the capacity to interpret the situation where the truth values are incomparable. Generalising the proposed logic considering graded consequence relation is in future agenda.

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Quotient models for a class of non-classical set theories

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In this article, we explore the construction of models for non-classical set theories. In particular, we build (1) algebra-valued models that validate a paraconsistent logic and a set theoretic axiom system which is classically equivalent to ZF. Moreover, (2), we develop the respective quotient models, by defining equivalence classes on the algebra-valued models. The procedure behind (1) follows the presentation of [1] and its extension to non-classical contexts as in [4], [2], [5] etc. On the other hand, the methodology corresponding to (2) is new and it extends the classical construction of Tarskian models from Boolean-valued models as described in detail in [6]. We plan to extend this line of work to algebra-valued models of paraconsistent set theory and thus present here for the first time a class of quotient models for paraconsistent set theory.

The novelty of these results consists in being able to build proper models of paraconsistent set theory and not only algebra-valued models, which, however, are a proper definable class in the universe of all sets. The key ingredient of this construction is represented by the construction of equivalence classes, which offer a well-behaved semantics for set theory, because of the validity of the Leibniz's law of indiscernibility of identicals. The results of this paper thus are an improvement of those of [3], where for the first time it was shown that there are algebra-valued models of paraconsistent set theory that validate a classically equivalent form of ZF and Leibniz's law of indiscernibility of identicals. Morevoer, this paper also generalizes the class of algebras from [3], for which these constructions can be performed.

Definition 1. *Let $\mathbb{A} = (A, \wedge, \vee, \mathbf{1}, \mathbf{0})$ be a bounded lattice, where $\mathbf{1}$ and $\mathbf{0}$ are the top and bottom elements, respectively. A set $D \subset A$ is said to be a designated set on A if it is a filter on A, i.e. 1) $\mathbf{1} \in D$ and $\mathbf{0} \notin D$; 2) if $x \in D$ and $x \leq y$, then $y \in D$; and 3) for any $x, y \in D$, $x \wedge y \in D$.*

Definition 2. *Let $\mathbb{A} = (A, \wedge, \vee, \Rightarrow, {}^*, \mathbf{1}, \mathbf{0})$ be an algebra and D be a designated set on A. We say that $(\mathbb{A}, D)$ is an MTV-algebra if*

(i) *$(A, \wedge, \vee, \mathbf{1}, \mathbf{0})$ is a complete distributive lattice,*

(ii) *$\mathbb{A}$ has a unique atom and a unique co-atom[1],*

(iii) *the two algebraic operators $\Rightarrow$ and * are defined as follows:*

$$a \Rightarrow b = \begin{cases} \mathbf{0}, & \text{if } a \neq \mathbf{0} \text{ and } b = \mathbf{0}; \\ \mathbf{1}, & \text{otherwise;} \end{cases}$$

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[1]The necessity of having a unique atom and a unique co-atom in the definition of MTV-algebra will be discussed at the time of presentation.

$$a^* = \begin{cases} \mathbf{0}, & \text{if } a = \mathbf{1}; \\ a, & \text{if } a \in D \setminus \{\mathbf{1}\}; \\ \mathbf{1}, & \text{if } a \notin D. \end{cases}$$

Notice that in the structure of any complete distributive lattice $\mathbb{L}$, if we add one top element and one bottom element, associated with a designated set D together with the two operators $\Rightarrow$ and * defined as in Definition 2, then the new structure becomes an MTV-algebra. For example, MTV-algebra include all totally ordered algebras from [2], whose negation and implication are defined in terms of Hyting and co-Hyting negations[2].

Theorem 3. *The logic of any* MTV*-algebra* $(\mathbb{A}, D)$*, is a paraconsistent logic, i.e., there exist formulas* φ *and* ψ *such that* ψ *cannot be derived from* $\{\varphi, \neg\varphi\}$*.*

Now, we are in a position to define the algebra-valued models. Let us consider a model $\mathbf{V}$ of ZF and an algebra $\mathbb{A}$, which is a complete distributive lattice associated with two operators $\Rightarrow$ and *. Then, by transfinite recursion, we define the following set-theoretic universe:

$$\mathbf{V}_\alpha^{(\mathbb{A})} = \{x \; : \; x \text{ is a function and } \mathrm{ran}(x) \subseteq \mathbf{A}$$
$$\text{and there is } \xi < \alpha \text{ with } \mathrm{dom}(x) \subseteq \mathbf{V}_\xi^{(\mathbb{A})}\} \text{ and}$$
$$\mathbf{V}^{(\mathbb{A})} = \{x \; : \; \exists\alpha(x \in \mathbf{V}_\alpha^{(\mathbb{A})})\}.$$

Let $\mathcal{L}$ be the language of ZF and $\mathcal{L}_\mathbb{A}$ be its extension, constructed by adding constant symbols corresponding to every element in $\mathbf{V}^{(\mathbb{A})}$. A mapping $[\![\cdot]\!]_{\mathrm{PA}}$ is recursively defined from the collection of all closed well-formed formulas in $\mathcal{L}_\mathbb{A}$ into $\mathbb{A}$ as follows.

Definition 4. *For any pair of elements* $u, v \in \mathbf{V}^{(\mathbb{A})}$*,*

$$[\![u \in v]\!]_{\mathrm{PA}} = \bigvee_{x \in \mathrm{dom}(v)} \big(v(x) \wedge [\![x = u]\!]_{\mathrm{PA}}\big),$$
$$[\![u = v]\!]_{\mathrm{PA}} = \bigwedge_{x \in \mathrm{dom}(u)} \big((u(x) \Rightarrow [\![x \in v]\!]_{\mathrm{PA}}) \wedge ([\![x \in v]\!]_{\mathrm{PA}}^* \Rightarrow u(x)^*)\big)$$
$$\wedge \bigwedge_{y \in \mathrm{dom}(v)} \big((v(y) \Rightarrow [\![y \in u]\!]_{\mathrm{PA}}) \wedge ([\![y \in u]\!]_{\mathrm{PA}}^* \Rightarrow v(y)^*)\big);$$

then, the map $[\![\cdot]\!]_{\mathrm{PA}}$ *is extended to the non-atomic formulas: for any two closed well-formed formulas* φ *and* ψ*,*

$$[\![\varphi \wedge \psi]\!]_{\mathrm{PA}} = [\![\varphi]\!]_{\mathrm{PA}} \wedge [\![\psi]\!]_{\mathrm{PA}},$$
$$[\![\varphi \vee \psi]\!]_{\mathrm{PA}} = [\![\varphi]\!]_{\mathrm{PA}} \vee [\![\psi]\!]_{\mathrm{PA}},$$
$$[\![\varphi \rightarrow \psi]\!]_{\mathrm{PA}} = [\![\varphi]\!]_{\mathrm{PA}} \Rightarrow [\![\psi]\!]_{\mathrm{PA}},$$
$$[\![\neg\varphi]\!] = [\![\varphi]\!]_{\mathrm{PA}}^*,$$
$$[\![\forall x \varphi(x)]\!]_{\mathrm{PA}} = \bigwedge_{u \in \mathbf{V}^{(\mathbb{A})}} [\![\varphi(u)]\!]_{\mathrm{PA}}, \text{ and}$$

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[2]For this reason it could be interesting to investigate the connection between topologically defined algebras and MTV-algebras. We thank an anonymous referee for raising this issue.

$$\llbracket \exists x \varphi(x) \rrbracket_{\mathrm{PA}} = \bigvee_{u \in \mathbf{V}^{(\mathbb{A})}} \llbracket \varphi(u) \rrbracket_{\mathrm{PA}}.$$

Definition 5. *A sentence φ of $\mathcal{L}_{\mathbb{A}}$ is said to be valid in $\mathbf{V}^{(\mathbb{A})}$ with respect to the assignment function $\llbracket \cdot \rrbracket_{\mathrm{PA}}$ and a designated set D, whenever $\llbracket \varphi \rrbracket_{\mathrm{PA}} \in D$. We denote this fact by $\mathbf{V}^{(\mathbb{A},\, \llbracket \cdot \rrbracket_{\mathrm{PA}})} \models_D \varphi$.*

The main difference between the assignment functions used in this paper (viz. $\llbracket \cdot \rrbracket_{\mathrm{PA}}$) and the assignment functions $\llbracket \cdot \rrbracket$ used in the study of Boolean-valued models, Heyting-valued models, generalized algebra-valued models etc. (cf. [1], [4], [2]) is the algebraic expression of $\llbracket u = v \rrbracket$: in the latter cases, for any two elements $u, v \in \mathbf{V}^{(\mathbb{A})}$, we define

$$\llbracket u = v \rrbracket = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow \llbracket x \in v \rrbracket) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow \llbracket y \in u \rrbracket).$$

If $\mathbb{A}$ is a Boolean algebra, then $\llbracket \cdot \rrbracket = \llbracket \cdot \rrbracket_{\mathrm{PA}}$. However, these two assignment functions may not be identical if $\mathbb{A}$ is not a Boolean algebra. For any non-Boolean and non-Heyting *reasonable implication algebra* $\mathbb{A}$, defined in [4], which were used to build algebra-valued models of non-classical set theories, the relation $=$ becomes an equivalence relation in $\mathbf{V}^{(\mathbb{A})}$ with respect to the assignment functions $\llbracket \cdot \rrbracket$. Hence, we can have the quotient space $\mathbf{V}^{(\mathbb{A})}/=$. But the major problem in using the assignment function $\llbracket \cdot \rrbracket$ is that in $\mathbf{V}^{(\mathbb{A})}$, it does not satisfy the *Leibniz's law of indiscernibility of identicals*:

$$\forall x \forall y \big((x = y \wedge \varphi(x)) \to \varphi(y) \big) \tag{LL_φ}$$

for all formulas $\varphi(x)$. On the other hand, we can overcome this problem by using the assignment function $\llbracket \cdot \rrbracket_{\mathrm{PA}}$.

Theorem 6. *Let $(\mathbb{A}, D)$ be an MTV-algebra. Then, the Leibniz's law of indiscernibility of identicals holds in $\mathbf{V}^{(\mathbb{A},\, \llbracket \cdot \rrbracket_{\mathrm{PA}})}$, i.e., $\mathbf{V}^{(\mathbb{A},\, \llbracket \cdot \rrbracket_{\mathrm{PA}})} \models \mathsf{LL}_\varphi$, for all formulas φ in the language $\mathcal{L}_{\mathbb{A}}$.* $\square$

Notice that the Axiom of Extensionality is not valid in $\mathbf{V}^{(\mathbb{A},\, \llbracket \cdot \rrbracket_{\mathrm{PA}})}$, where $(\mathbb{A}, D)$ is an MTV-algebra. Let us consider the following sentence:

$$\forall x \forall y \forall z \big(((z \in x \leftrightarrow z \in y) \wedge (\neg z \in x \leftrightarrow \neg z \in y)) \to x = y \big). \tag{Extensionality^P}$$

Let ZF^P be the axiom system consisting of the axioms of ZF, where Extensionality^P is taken instead of the Axiom of Extensionality, i.e.,

$$\mathsf{ZF}^P := \mathsf{ZF} - \mathsf{Axiom\ of\ Extensionality} + \mathsf{Extensionality}^P.$$

It can be observed that ZF^P and ZF are classically equivalent.

Theorem 7. *For any MTV-algebra $(\mathbb{A}, D)$, $\mathbf{V}^{(\mathbb{A},\, \llbracket \cdot \rrbracket_{\mathrm{PA}})} \models_D \mathsf{ZF}^P$.* $\square$

Hence, we have found algebra-valued models of a class of paraconsistent set theories, which satisfy the Leibniz's law of indiscernibility of identicals. Let $(\mathbb{A}, D)$ be an MTV-algebra. We define a class relation $\sim$ in $\mathbf{V}^{(\mathbb{A})}$ as $u \sim v$ iff $\mathbf{V}^{(\mathbb{A},\, \llbracket \cdot \rrbracket_{\mathrm{PA}})} \models_D u = v$, for all $u, v \in \mathbf{V}^{(\mathbb{A})}$. It can be proved that $\sim$ is an equivalence relation and hence we have the quotient space $\mathbf{V}^{(\mathbb{A})}/\sim$. Then we go on and define the validity of formulas in the quotient model based on the validity of those formulas in the respective algebra-valued model. This ensures us that the resulting quotient model does validate the ZF^P axioms.

Theorem 8. *For any* MTV*-algebra* $(\mathbb{A}, D)$, $\mathbf{V}^{(\mathbb{A})}/\sim \models \mathsf{ZF}^P$. $\hfill \square$

Theorem 9. *Let* $(\mathbb{A}, D)$ *be an* MTV*-algebra and let* $\mathbf{V}^{(\mathbb{A})}/\sim$ *be the respective quotient space. Then for a sequence* $[\vec{u}] = ([u_1], [u_2], ...)$ *in* $\mathbf{V}^{(\mathbb{A})}/\sim$ *we have the following.*

(i) $\mathbf{V}^{(\mathbb{A})}/\sim \models (\varphi \to \psi)([\vec{u}])$ *iff* $\mathbf{V}^{(\mathbb{A})}/\sim \nvDash \varphi([\vec{u}])$ *or* $\mathbf{V}^{(\mathbb{A})}/\sim \models \psi([\vec{u}])$.

(ii) $\mathbf{V}^{(\mathbb{A})}/\sim \models (\varphi \wedge \psi)([\vec{u}])$ *iff* $\mathbf{V}^{(\mathbb{A})}/\sim \models \varphi([\vec{u}])$ *and* $\mathbf{V}^{(\mathbb{A})}/\sim \models \psi([\vec{u}])$.

(iii) $\mathbf{V}^{(\mathbb{A})}/\sim \models (\varphi \vee \psi)([\vec{u}])$ *iff* $\mathbf{V}^{(\mathbb{A})}/\sim \models \varphi([\vec{u}])$ *or* $\mathbf{V}^{(\mathbb{A})}/\sim \models \psi([\vec{u}])$.

(iv) if $\mathbf{V}^{(\mathbb{A})}/\sim \nvDash \varphi([\vec{u}])$, *then* $\mathbf{V}^{(\mathbb{A})}/\sim \models \neg\varphi([\vec{u}])$.

(v) $\mathbf{V}^{(\mathbb{A})}/\sim \models \forall x_k \varphi([\vec{u}])$ *iff* $\mathbf{V}^{(\mathbb{A})}/\sim \models \varphi([\vec{u}], d/k)$, *where* $[\vec{u}](d/k)$ *is identified with the sequence* $([u_1], ..., [u_{k-1}], d, [u_{k+1}])$, *for any* $d \in \mathbf{V}^{(\mathbb{A})}/\sim$.

(vi) $\mathbf{V}^{(\mathbb{A})}/\sim \models \exists x_k \varphi([\vec{u}])$ *iff* $\mathbf{V}^{(\mathbb{A})}/\sim \models \varphi([\vec{u}], d/k)$ *for some* $d \in \mathbf{V}^{(\mathbb{A})}/\sim$. $\hfill \square$

There are quotient models which are paraconsistent, i.e., there are MTV-algebra $(\mathbb{A}, D)$ and there exist sentences φ and ψ in the language of set theory such that $\mathbf{V}^{(\mathbb{A})}/\sim \models (\varphi \wedge \neg\varphi) \to \psi$. In conclusion, we have found quotient models for a class of paraconsistent set theory which validate ZF^P. These models seem intriguing because they extend the class of Boolean quotient models and so open up the possibility of carrying out forcing constructions beyond the limits of classicality.

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Reasoning about the Robustness of Self-organizing Multi-agent Systems (WIP)

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Abstract

Self-organization has been introduced to multi-agent systems as an internal control process or mechanism to solve difficult problems spontaneously. When the system is deployed in an open environment, agents can enter or exit the system as they want, which might bring the system to an undesired state. We propose a framework to reason about the robustness of self-organizing multi-agent systems under the change of participating agents.

1 Introduction

Self-organization is a process where a stable pattern is formed by the cooperative behavior between parts of an initially disordered system without external control or influence. It has been introduced to multi-agent systems as an internal control process or mechanism to solve difficult problems spontaneously, especially if the system is operated in an open environment thereby having no perfect and a priori design to be guaranteed [10][8][9]. When the system is deployed in an open environment, agents can enter or exit the system as they want, which might bring the system to an undesired state. Because in a self-organizing multi-agent system local rules work as guidance for agents to behave thus leading to specific outcomes, we can see the set of local rules as a mechanism that we implement in the system. A mechanism is a procedure, protocol or game for generating desired outcomes. If we want to know whether we can design a set of local rules to ensure desired outcomes, we then enter the field of mechanism design [7]. In the area of model checking, some work has been done to verify a multi-agent system from the perspective of mechanism design [6][2]. In [6], Max Knobbout, et al assume agents to have some preferences, which might be unknown to the system designers, and use a solution concept of Nash equilibrium for decision-making. A formal framework is developed to verify whether a normative system implements desired outcomes no matter what preferences agents have. In [2], Nils Bulling and Mehdi Dastani formally analyze and verify whether a specific behavior can be enforced by norms and sanctions if agents follow their subjective preferences and whether a set of norms and sanctions that realize the effect exists at all. In this paper, we verify the correctness of a mechanism from the perspective of robustness. In our framework of self-organizing multi-agent systems, agents follow their local rules to communicate with other agents in order to make their next moves, giving rise to the dependence relation between agents. Based on the dependence relation, we provide formal properties that can be used to verify whether a self-organizing multi-agent system is robust under the change of participating agents.

2 Framework

The semantic structure of this paper is concurrent game structures (CGSs). It is basically a model where agents can simultaneously choose actions that collectively bring the system from the current state to a successor state. Compared to other kripke models of transaction systems, each transition in a CGS is labeled with collective actions and the agents who perform those actions.Formally,

Definition 2.1. *A concurrent game structure is a tuple $\mathcal{S} = (k, Q, \pi, \Pi, ACT, d, \delta)$ such that:*

- *A natural number $k \geq 1$ of agents, and the set of all agents is $\Sigma = \{1, \ldots, k\}$; we use A to denote a coalition of agents $A \subseteq \Sigma$;*

- *A finite set Q of states;*

- *A finite set Π of propositions;*

- *A labeling function π which maps each state $q \in Q$ to a subset of propositions which are true at q; thus, for each $q \in Q$ we have $\pi(q) \subseteq \Pi$;*

- *A finite set ACT of actions;*

- *For each agent $i \in \Sigma$ and a state $q \in Q$, $d_i(q) \subseteq ACT$ is the non-empty set of actions available to agent i in q; $D(q) = d_1(q) \times \ldots \times d_k(q)$ is the set of joint actions in q; given a state $q \in Q$, an action vector is a tuple $\langle \alpha_1, \ldots, \alpha_k \rangle$ such that $\alpha_i \in d_i(q)$;*

- *A function δ which maps each state $q \in Q$ and a joint action $\langle \alpha_1, \ldots, \alpha_k \rangle \in D(q)$ to another state that results from state q if each agent adopted the action in the action vector, thus for each $q \in Q$ and each $\langle \alpha_1, \ldots, \alpha_k \rangle \in D(q)$ we have $\delta(q, \langle \alpha_1, \ldots, \alpha_k \rangle) \in Q$.*

Note that the model is deterministic: the same update function adopted in the same state will always result in the same resulting state. A computation over $\mathcal{S}$ is an infinite sequence $\lambda = q_0, q_1, q_2, \ldots$ of states such that for all positions $i \geq 0$, there is a joint action $\langle \alpha_1, \ldots, \alpha_k \rangle \in D(q_i)$ such that $\delta(q_i, \langle \alpha_1, \ldots, \alpha_k \rangle) = q_{i+1}$. For a computation λ and a position $i \geq 0$, we use $\lambda[i]$ to denote the ith state of λ. More elaboration of concurrent game structures can be found in [1].

Self-organization has been introduced into multi-agent systems for a long time to solve various problems in multi-agent systems [12][4]. It is a mechanism or a process which enables a system to finish a difficult task by the cooperative behavior between agents spontaneously [3]. In this paper, we argue that the cooperative behavior is defined by prescribed local rules that agents are supposed to follow with communication between agents as a prerequisite. Therefore, we can define a self-organizing multi-agent system as a concurrent game structure together with a set of local rules for agents to follow. For example, in ant colony optimization algorithms, ants are required to record their positions and the quality of their solutions (lay down pheromones) so that in later simulation iterations more ants locate better solutions. An internal function returns the information that is provided by participating agents themselves for communication at a particular state, which is referred to as an agent type and thus might be different from agent to agent. We thus define it as a function $m_i : Q \to \mathcal{L}_{prop}$ that maps a state $q \in Q$ to a propositional formula. We use $M = \{m_i \mid i \in \Sigma\}$ to denote the set of internal functions for all the participating agents. Depending on the application, we might have different interpretation on $m_i(q)$. For example, vehicles in a busy traffic situation are required to communicate their urgencies, and robots sensors in a self-deploy sensing network are required to communicate their sensing areas. Given an agent a, an abstract local rule for agent a is defined as a tuple $\langle \tau_a, \gamma_a \rangle$: it consists of two parts: the first part $\tau_a(q)$ states the agents with whom agent a is supposed to communicate in state q, and the second part $\gamma_a(M_a(q))$ states the action that agent a is supposed to take given the communication with agents in $\tau_a(q)$ for their internals. We see local rules not only as constraints but also guidance on agents' behavior, namely an agent does not know what to do if he does not communicate with other agents. Therefore, we exclude the case where agents get no constraint from their respective local rules. We use notation $out(q, \Gamma_A)$ is the set of computations starting from state q where agents in coalition A follow their respective local rules in Γ_A to behave. Now we are ready to define a self-organizing multi-agent system. Formally,

Definition 2.2 (Self-organizing Multi-agent Systems)**.** *A self-organizing multi-agent system (SOMAS) is a tuple $(\mathcal{S}, M, \Gamma)$, where $\mathcal{S}$ is a concurrent game structure, M is the set of internal functions and Γ is a set of local rules for agents in the system to follow.*

Example 1. *Consider a CGS scenario in [5][11] as Figure.1 where there are two trains, each controlled by an agent, going through a tunnel from the opposite side. The tunnel has only room for one train,*

and the agents can either wait or go. Starting from state q_0, if the agents choose to go simultaneously, the trains will crash, which is state q_4; if one agent goes and the other waits, they can both successfully go through the tunnel without crashing, which is q_3. Local rules are prescribed for both trains to follow: both agents communicate with each other for their urgencies u_1 and u_2 in state q_0; the one who is more urgent can go through the tunnel first and the other one has to wait for it; after that the agent who waits before can go. Therefore, if a_1 is more urgent than a_2 with respect to u_1 and u_2, the desired state q_3 is reached along with computation $q_0, q_2, q_3 \ldots$; if a_1 is less urgent than a_2 with respect to u_1 and u_2, the desired state q_3 is reached along with computation $q_0, q_1, q_3 \ldots$.
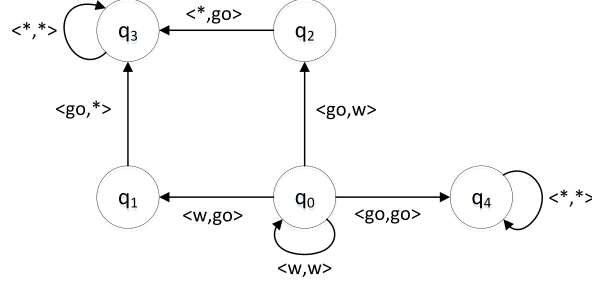

Figure 1: A CGS example.

In order to study how a self-organizing multi-agent system behaves under the change of participating agents, we first need to characterize the independence between agents in terms of their contributions to the system behavior. In this paper, it is characterized from two perspectives: a semantic perspective given by the underlying game structure and a structural perspective derived from abstract local rules. Similar to ATL [1], our language is interpreted over a concurrent game structure $\mathcal{S}$ that has the same propositions and agents as our language. It is an extension of classical propositional logic with path quantifiers and temporal cooperation modalities. A formula of the form $\langle A \rangle \psi$ means that coalition of agents A will bring about the subformula ψ by following their respective local rules in Γ_A, no matter what agents in $\Sigma \backslash A$ do. Our formulas are evaluated over a self-organizing multi-agent system $(\mathcal{S}, M, \Gamma)$ and a state $q \in Q$ in the form of $\mathcal{S}, M, \Gamma, q \models \langle A \rangle \psi$. Agents in the system follow their respective local rules to communicate with other agents and behave based on the communication results, which means that a local rule in this paper has agents' communication its prerequisite. Different participating agents might have different internal functions, making the communication results and thus the actions that are required to take different. Hence, even though we have $\mathcal{S}, M, \Gamma, q \models \langle A \rangle \psi$, the computations $out(q, \Gamma_A)$ that coalition A ensures to bring about ψ might be different, which shows coalition A self-organizes to bring about ψ in state q.

Example 2. *According to the local rules in the two-train example, the train who is more urgent can go through the tunnel first and the other one has to wait for him. We have that one train by itself cannot bring about the result of passing through the tunnel without crash through following the local rule, which can be expressed:*

$$\mathcal{S}, M, \Gamma, q_0 \not\models \langle a_1 \rangle \Diamond no\_crash,$$
$$\mathcal{S}, M, \Gamma, q_0 \not\models \langle a_2 \rangle \Diamond no\_crash.$$

Instead, both trains have to cooperate to bring about the result. Thus, we have that both agents by themselves can bring about the result of passing through the tunnel without crash through following the local rules, which can be expressed:

$$\mathcal{S}, M, \Gamma, q_0 \models \langle a_1, a_2 \rangle \Diamond no\_crash.$$

Notice that we have this formula when both trains follow their local rules to cooperate for the cases where a_1 is more urgent than a_2 and a_2 is more urgent than a_1. Interestingly, $\Diamond no\_crash$ is ensured along

the computation $q_0, q_2, q_3 \ldots$ for the former case, while $\diamond no\_crash$ is ensured along the computation $q_0, q_1, q_3 \ldots$ for the latter case.

3 Changing Participating Agents

As we have already highlighted in the introduction, when a self-organizing multi-agent system is deployed in an open environment, different types of agents participate in the system and their internals might be unknown to system designers. Therefore, it is important to verify whether the set of local rules still generates desired outcomes, more generally how the system behaves, under the change of participating agents. As in the two-train example, if both trains have the same urgency, they will wait and thus get stuck in state q_0 forever and cannot go through the tunnel by following their local rules, showing that the local rules are not robust to handle this case. Because in this paper an internal function $m_i(\cdot)$ is interpreted as an agent type, changing participating agents can be done by simply replacing internal functions.

Definition 3.1 (Change of Participating Agents)**.** *Given a SOMAS $(\mathcal{S}, M, \Gamma)$, a new SOMAS under the change of participating agents is denoted as $(\mathcal{S}, M', \Gamma)$, where there exists an agent $a \in \Sigma$ and a state $q \in Q$ such that $m_a(q) \neq m'_a(q)$.*

As we can see from the definition, what we meant by replacing internal functions is that in the new system there exists at least an agent's internal function such that given the same state its output is different from the one in the original system. Except the internal functions, the underlying concurrent game structure and the set of local rules remain the same. When we change participating agents in the system, some properties that hold in the original system might become false in the new system. The reason is that the change of agents' internal function might change the actions that agents are allowed to take by their local rules thus causing the new system to run along a computation that might be different from the original system. Accordingly, we use $out'(q, \Gamma_A)$ to denote the set of computations where agents in coalition A follow their respective local rules to behave in the new system. Therefore, local rules have to be well designed in order to ensure that desired properties remain unchanged under the change of participating agents. That gives rise to the notion of robustness.

Definition 3.2 (Robustness of Local Rules)**.** *Given a CCG $\mathcal{S}$, a set of local rules Γ is robust w.r.t. a temporal formula ψ under the change of participating agents M' iff for $q \in Q$ and $A \subseteq \Sigma$ if ψ is ensured by A in SOMAS $(\mathcal{S}, M, \Gamma)$ then ψ is also ensured by A in SOMAS $(\mathcal{S}, M', \Gamma)$.*

In words, we define the robustness of a set of local rules with respect to a temporal formula in the sense that the satisfaction of the temporal formula is persevered under the change of participating agents M'. Intuitively, we can verify the robustness of local rules w.r.t. ψ through checking $\mathcal{S}, M, \Gamma, q \models \langle A \rangle \psi$ and $\mathcal{S}, M', \Gamma, q \models \langle A \rangle \psi$. However, sometimes it is difficult to enumerate all the possible internal functions of participating agents. Therefor, a more practical approach is to provide the conditions for internal functions with which desired properties remain the same in the new system. Before exploring it, we use $out(q, \Gamma_A)[i]$ to denote a set of states, each of which is the ith state of any computation in $out(q, \Gamma_A)$. That is, $out(q, \Gamma_A)[i] = \{q' \in Q \mid \exists \lambda \in out(q, \Gamma_A) : q' = \lambda[i]\}$.

Proposition 3.1. *Given a SOMAS $(\mathcal{S}, M, \Gamma)$ and a new SOMAS $(\mathcal{S}, M', \Gamma)$ under the change of participating agents, Γ is robust w.r.t. a temporal formula ψ if and only if for $q \in Q$ and $A \subseteq \Sigma$ if $\mathcal{S}, M, \Gamma, q \models \langle A \rangle \psi$ then one of the following statements should be satisfied:*

1. $\forall a \in A, \lambda \in out(q, \Gamma_A), q' \in \lambda : m_a(q') = m'_a(q')$;

2. if $\exists i, a \in A, \lambda \in out(q, \Gamma_A), q' \in \lambda : m_i(q') \neq m'_i(q')$ and $i \in \tau_a(q')$ then $\gamma_a(M_a(q')) = \gamma_a(M'_a(q'))$;

3. $\mathcal{S}, M', \Gamma, q \models \langle A \rangle \psi$.

Proof. Because $\mathcal{S}, M, \Gamma, q \models \langle A \rangle \psi$, by its semantics, we have that ψ holds in every computation in $out(q, \Gamma_A)$. Next, we need to inductively prove that $out'(q, \Gamma_A)$ in the new system contains the same

computations as $out(q, \Gamma_A)$ in the original system. Firstly, computations from both $out'(q, \Gamma_A)$ and $out(q, \Gamma_A)$ start from state q. Secondly, suppose $out(q, \Gamma_A)[i] = out'(q, \Gamma_A)[i]$. For the first and the second statement:

1. Because for any $a \in A$ and $q' \in out(q, \Gamma_A)[i]$ it is the case that $m_a(q') = m'_a(q')$, $\tau_a(q')$ returns the same set of agents from which agent a gets input information and thus $\gamma_a(\cdot)$ returns the same action in both systems in state q'.

2. Because even though there exists $i, a \in A, q' \in out(q, \Gamma_A)[i] : m_i(q') \neq m'_i(q')$ and $i \in \tau_a(q')$ it is the case that $\gamma_a(M_a(q')) = \gamma_a(M'_a(q'))$, which means that $\gamma_a(\cdot)$ returns the same action in both systems in state q'.

Hence, $out(q, \Gamma_A)[i+1] = out'(q, \Gamma_A)[i+1]$. So we can conclude that $out'(q, \Gamma_A) = out'(q, \Gamma_A)$. Because ψ holds in every computation in $out(q, \Gamma_A)$, ψ also holds in every computation in $out'(q, \Gamma_A)$. Therefore, if one of the statements is satisfied, we have $\mathcal{S}, M', \Gamma, q \models \langle A \rangle \psi$. $\qquad \square$

In other words, a set of local rules is robust under the change of participating agents if and only if for any coalition if it ensures a temporal formula, then either the internal functions of any agents in that coalition return the same values in both systems, or γ returns the same action for any agent in that coalition along in both systems, or the coalition ensures the temporal formula in the new system. As we can see, the first statement is the most demanding one because the values of internal functions are not allowed to change, and the second statement allows the values of internal functions to change but γ should return the same action, and the third statement is the least demanding because it relaxes the first and the second statements, which might result different computations compared to the original system and thus the formula should be checked again.

Example 3. *In the two-train example, suppose a_1 is more urgent than a_2 in state q_0. According to the local rules, both trains can go through the tunnel along with computation $q_0, q_2, q_3 \ldots$. Now two new trains enter the system. The internal function of each agent returns the same value with all states as before except state q_4. In this case, both trains can go through the tunnel along with computation $q_0, q_2, q_3 \ldots$ because for any states in computation $q_0, q_2, q_3 \ldots$ and any agent the internal function returns the same value as before, which satisfies the first statement in Proposition 3.1. Next, suppose the two trains have new urgencies but it is still the case that a_1 is more urgent than a_2 in state q_0, which satisfies the second statement in Proposition 3.1, both trains can go through the tunnel along with computation $q_0, q_2, q_3 \ldots$. If a_1 is less urgent than a_2, $\gamma_1(M'_1(q_0))$ returns action wait for a_1 and $\gamma_2(M'_2(q_0))$ returns go for a_2, which leads to computation $q_0, q_1, q_3 \ldots$. It is different from the original system and satisfies the third statement in Proposition 3.1. However, if both trains have the same urgency, they will choose to wait according to their local rules, resulting in the undesired state q_0 forever. As we can see, the local rules cannot ensure that the system will reach a desired state no matter what kind of agents participate in the system.*

4 Conclusion

When a self-organizing multi-agent system is deployed in an open environment, agents can enter or exit the system as they want, which might bring the system to an undesired state. Therefore, it is of great importance to verify the behavior of a new system under the change of participating agents. In this paper, we develop a framework of a self-organizing multi-agent system, where agents follow their local rules to communicate with other agents in order to make their next moves. Based on the dependence relation between agents, we provide formal properties that can be used to verify whether a self-organizing multi-agent system is robust under the change of participating agents. In the future, we will generalize our notion of robustness such that it allows us to verify not only agents' contributions but also more general system properties. As we can use a directed graph to represent the dependence relation between agents in terms of communication, we will investigate whether graph theory can be applied to facilitate the verification of robustness.

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A Clock-Optimal Hierarchical Monitoring Automaton for MITL

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1 Introduction

Metric Interval Temporal Logic (MITL) (cf. [2]) is a popular temporal logic for specifying quantitative timing properties of real-time systems. The key to solving the satisfiability and model-checking problem for MITL is the construction of a "monioring automaton". A monitoring automaton for an MITL formula φ is a timed automaton $\mathcal{A}$ along with a constraint g over the configurations of $\mathcal{A}$, that satisfies the following property: for every timed word σ, $\mathcal{A}$ should have a unique accepting run over σ, and the constraint g should "monitor" the truth of φ along this run, in the sense that g is satisfied at a point in the run iff φ is true in σ at that point. From a monitoring automaton for a formula, one can easily obtain a formula automaton for the formula that accepts precisely the (initial) models of the given formula.

In this paper we present an inductive monitoring automaton construction for MITL formulas which is similar in spirit to the one in [7] but with some key differences. Firstly, the construction we propose is more efficient in the usage of clocks and it typically uses one clock less than the constructions in [2, 7] for the timed modalities like Ψ_I. Secondly, our construction handles the full fragment of MITL, including open/closed intervals and past operators (this is in contrast to [7] which handles only closed intervals among other restrictions). Finally, our constructions are carried out in the framework of hierarchical timed Büchi automata (HTBA's) which we introduce in this paper.

An important aspect of our construction is that it is provably optimal in the number of clocks it uses in the component automaton for each operator in the inductive construction. We prove this using a novel lower bound argument that shows that any monitoring automaton for a formula of the form $\Psi_{(l,r)}a$ must use at least $2n + 1$ clocks. The complete details are available in a technical report [4].

2 Preliminaries

We assume the definitons of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{Q}_{\geq 0}$. An *infinite timed word* [1] over an alphabet Σ is an infinite word over $\Sigma \times \mathbb{R}_{\geq 0}$ of the form $(a_0, t_0)(a_1, t_1) \cdots$, satisfying the following conditions: for each $i \in \mathbb{N}$ we have $t_i < t_{i+1}$ (*monotonicity*); and for each $t \in \mathbb{R}_{\geq 0}$ there exists an $i \in \mathbb{N}$ such that $t < t_i$ (*progressiveness*). We denote the set of all infinite timed words over Σ by $T\Sigma^\omega$. Alternatively, an infinite timed word $\sigma = (a_0, t_0)(a_1, t_1) \cdots$ can be viewed as a function $\chi_\sigma : \mathbb{R}_{\geq 0} \to \Sigma \cup \{\delta\}$ given by $\chi_\sigma(t) = a_i$ if there exists an i such that $t = t_i$ and δ otherwise, where δ denotes "time elapse" or "no action."

For a finite alphabet Σ, the syntax of an MITL formula over Σ is given by: $\varphi ::= a \in \Sigma \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi U_I \varphi \mid \varphi S_I \varphi$, where $I \in \mathcal{I}_\mathbb{Q}$ and is not of the form $[l, l]$. Let $\sigma \in T\Sigma^\omega$, $a \in \Sigma$, and $t \in \mathbb{R}_{\geq 0}$. Then the satisfaction relation $\sigma, t \models \varphi$ is given as follows: $\sigma, t \models a$ iff $a = \chi_\sigma(t)$, $\sigma, t \models \psi U_I \eta$ iff $\exists t' >$

$t : \sigma, t' \models \eta, \ t' - t \in I$, and $\forall t'' : t < t'' < t', \ \sigma, t'' \models \psi$, and $\sigma, t \models \psi S_I \eta$ iff $\exists t' < t : \sigma, t' \models \eta, \ t - t' \in I$, and $\forall t'' : t' < t'' < t, \ \sigma, t'' \models \psi$, with boolean operators handled in the expected way.

We say that a timed word σ satisfies an MITL formula φ, written $\sigma \models \varphi$, if and only if $\sigma, 0 \models \varphi$, and set $L(\varphi) = \{\sigma \in T\Sigma^\omega \mid \sigma \models \varphi\}$. We will make use of the standard temporal abbreviations of $U \equiv U_{(0,\infty)}$ and $S \equiv S_{(0,\infty)}$, as well as $\Psi_I \psi \equiv \top U_I \psi$, $\Phi_I \psi \equiv \neg \Psi_I \neg \psi$, $\text{fi}_I \psi \equiv \top S_I \psi$, and $\text{ff}_I \psi \equiv \neg \text{fi}_I \neg \psi$. In the sequel we will use the following expressively equivalent syntax for MITL (see [6, 5]): $\varphi ::= a \in \Sigma \mid \neg \varphi \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \psi U \eta \mid \psi S \eta \mid \Psi_I \varphi \mid \text{fi}_I \varphi$.

3 Hierarchical Timed Büchi Automata

We now introduce *hierarchical timed Büchi automata* which run over timed words. We fix an alphabet Σ for the rest of this section. Let C be a finite set of clocks. A *valuation* for the clocks in C is a map $\mathbf{v} : C \to \mathbb{R}_{\geq 0}$. We denote the set of valuations for C by V_C. A *clock constraint g* over C is a boolean combination of atomic constraints of the form $x \bowtie c$, where x is a clock in C, $\bowtie \in \{<, \leq, =, >, \geq\}$, and c is a rational constant. We denote the set of clock constraints over C by $\Phi(C)$.

A *timed edge Büchi automaton* (TBA) $\mathcal{A}$ over Σ is a structure of the form $(Q, \Sigma, C, E, S, F, tcp)$ where Q is a finite set of states, C a finite set of clocks, $E \subseteq Q \times (\Sigma \cup \{\epsilon\}) \times \Phi(C) \times 2^C \times Q$ a finite set of edges, $S \subseteq E$ a set of initial edges, $F \subseteq E$ a set of final edges, and $tcp : Q \to \Phi(C)$ a "time can progress" condition on states.

A run of $\mathcal{A}$ over an infinite timed word $\sigma = (a_0, t_0)(a_1, t_1) \cdots$ is a tuple of the form (ρ, ν) where $\rho : \mathbb{R}_{\geq 0} \to Q \cup E$ and $\nu : \mathbb{R}_{\geq 0} \to \Phi(C)$, where ρ and ν satisfy the following conditions: let $T = \{t_0, t_1, \ldots\}$. Then (1) there exists a set of non-negative points $U = \{u_0, u_1, \ldots\}$ such that the following holds: (a) $0 \leq u_0 < u_1 \ldots$ and $T \subseteq U$, (b) for all $t \in \mathbb{R}_{\geq 0}, \rho(t) \in E$ iff $t \in U$ and if $\rho(t)$ is of the form (q, a_i, g, X, q') iff $t = t_i$ and is of the form (q, ϵ, g, X, q') iff $t \in U - T$, (c) let $\rho(u_i) = (q_i, a'_i, g_i, X_i, q'_i)$ for each $i \in \mathbb{N}$. Then the valuation of the clocks is defined as follows: for all $x \in C$ and $\forall t \in \mathbb{R}_{\geq 0} \ \nu(t)(x) = 0$ if $t = 0$, t if $t \in (0, u_0]$, $t - u_i$ if $t \in (u_i, u_{i+1}]$ and $x \in X_i$ and $\nu(u_i)(x) + t - u_i$ if $t \in (u_i, u_{i+1}]$ and $x \in C - X_i$. (2): if $\rho(u_0) = (q, a', g, X, q')$ then $\rho(t) = q$ if $t \in [0, u_0)$. And for all $i > 0$ if $\rho(u_i) = (q, a', g, X, q')$ then $\rho(t) = q$ if $t \in (u_{i-1}, u_i)$, and $\rho(t) = q'$ if $t \in (u_i, u_{i+1})$. (3): $\forall t \in \mathbb{R}_{\geq 0}$ if $t \notin U$ then $\nu(t) \models tcp(\rho(t))$ and (4): $\rho(u_0) \in S$. The run (ρ, ν) is called *accepting* if $\rho(t) \cap F \neq \emptyset$ for infinitely many t. We denote by $L(\mathcal{A})$ the set of all timed words on which $\mathcal{A}$ has an accepting run. We say $\mathcal{A}$ is *universal* if $L(\mathcal{A}) = T\Sigma^\omega$, and *unambiguous* if there is at most one *accepting* run over any timed word $\sigma \in T\Sigma^\omega$.

Let $L = [\mathcal{B}_n, \ldots, \mathcal{B}_1]$ be a list of timed edge Büchi automata where each $\mathcal{B}_i$ is of the form $(Q_i, \Sigma, C_i, S_i, E_i, F_i, tcp_i)$. Then a *joint configuration* of $\mathcal{B}_n, \ldots, \mathcal{B}_1$ is a tuple of the form $((r_n, v_n), \ldots, (r_1, v_1))$ where $r_i \in Q_i \cup E_i$ and $v_i \in V_{C_i}$ for each $1 \leq i \leq n$. A *configuration constraint* w.r.t. L is a boolean combination of atomic constraints of the form $\mathcal{B}_i.r$ or $\mathcal{B}_i.x \bowtie c$ where $r \in Q_i \cup E_i, x \in C_i$ for $1 \leq i \leq n$ and $c \in \mathbb{Q}_{\geq 0}$. We denote the set of all *configuration constraints* w.r.t. L by $\mathcal{G}(L)$. A configuration constraint is evaluated over a joint configuration of $\mathcal{B}_n, \ldots, \mathcal{B}_1$. Let $(r_n, v_n), \ldots, (r_1, v_1)$ be a joint configuration of $\mathcal{B}_n, \ldots, \mathcal{B}_1$. Then the atomic constraints are interpreted as follows: (1): $((r_n, v_n), \ldots, (r_1, v_1)) \models \mathcal{B}_i.r$ iff $r_i = r$, and (2): $((r_n, v_n), \ldots, (r_1, v_1)) \models \mathcal{B}_i.x \bowtie c$ iff $v_i(x) \bowtie c$.

A *hierarchical timed Büchi automaton* (or HTBA) $\mathcal{H}$ is a structure of the form $[C_n, \ldots, C_1]$, where each C_i is of the form $(\mathcal{B}_i, G_i)$ with $\mathcal{B}_i = (Q_i, \Sigma, C_i, E_i, S_i, F_i, tcp_i)$ a timed edge Büchi automaton and $G_i : Q_i \cup E_i \to \mathcal{G}([\mathcal{B}_i, \ldots, \mathcal{B}_1])$ a labelling of edges and states of $\mathcal{B}_i$ with "level i" configuration constraints.

Fig. 1 shows an example of an (edge) TBA. In the figures which appear in this paper we use the following conventions. Edges can be thick (initial) or thin (non-initial), and non-dashed (final) or dashed (non-final). An edge from a state p to a state q labelled "$e : h, g, a, X$" represents a transition (p, a, g, X, q) named e, with the external guard h. By convention, a missing component of h or g means "$\top$" in the respective positions, a missing X component means "$\{\}$", and a missing action means $\Sigma \cup \{\epsilon\}$. Further,

on each state p with *tcp* condition h, we assume a self-loop labelled $h, \Sigma, \{\}$. The *tcp* condition is written inside the state (or given by a *tcp* condition table with the figure). An omitted *tcp* condition is to be interpreted as "⊤". Thus, the edge e_1 in the figure is an initial and non-final edge, e_3 is a non-initial, final edge, and e_2 and e_4 are non-initial, non-final edges. The *tcp* condition of the state p is ⊤ while the $tcp(q) = x < 1$. The language accepted by the automaton, $L(\mathcal{B})$, is the set of all timed words which begin with an a occurring at exactly one time unit and which have infinitely many b's. In addition to that if $\sigma \in L(\mathcal{B})$ then for every a that occurs in σ there exists a b one time unit later and vice-versa.

Example 1. *Fig. 2 shows an example* HTBA $\mathcal{H}_{unit}$ *with two component automata* C_1 *and* C_2. *In* C_1 *only the edge* e_b *is initial so* C_2, *even though both its edges are initial, is forced by* C_1 *to take* f_2 *as the initial transition. In* C_1 *only* e_a *is final while in* C_2 *both the edges are final. The language accepted by the automaton is the set of all timed words of the form* $(b, t_0)(a, 1)(b, t_1)(a, 2)(b, t_2)(a, 3) \cdots$.
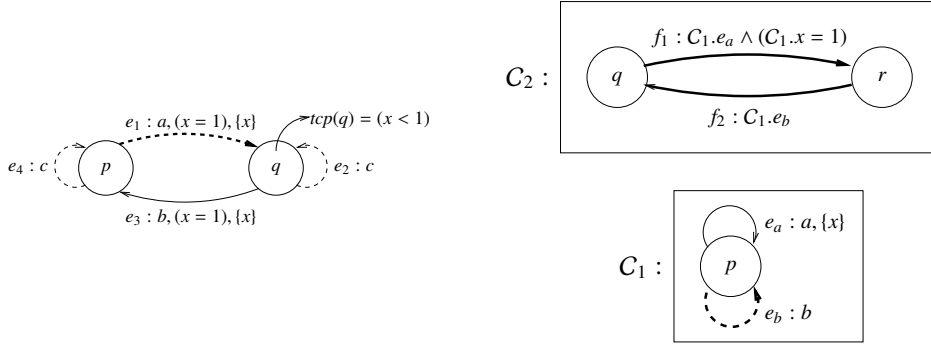

Figure 1: Edge TBA $\mathcal{B}$.

Figure 2: HTBA $\mathcal{H}_{unit}$.

Let $\sigma \in T\Sigma^\omega$. Then a run of $\mathcal{H}$ over σ is a joint run of $\mathcal{B}_n, \ldots, \mathcal{B}_1$ except that on each transition the configuration constraints have to be satisfied. Formally, a run ρ of $\mathcal{H}$ over σ is a tuple $((\rho_n, \nu_n), \ldots, (\rho_1, \nu_1))$ satisfying the following conditions: (1): each (ρ_i, ν_i) is a run of $\mathcal{B}_i$ on σ, and (2): for all $t \in \mathbb{R}_{\geq 0}$ we require that $((\rho_n(t), \nu_n(t)), \ldots, (\rho_1(t), \nu_1(t))) \models \bigwedge_{i=1}^{n} G_i(\rho_i(t))$. Thus, at each time point t the external edge constraints and *tcp* conditions are to be satisfied. We will sometimes write (ρ, ν) to represent a run $((\rho_n, \nu_n), \ldots, (\rho_1, \nu_1))$ over $\mathcal{H}$, and for a point $t \in \mathbb{R}_{\geq 0}$ we use $(\rho(t), \nu(t))$ to denote the tuple $((\rho_n(t), \nu_n(t)), \ldots, (\rho_1(t), \nu_1(t)))$ in the rest of the paper.

A run $((\rho_n, \nu_n), \ldots, (\rho_1, \nu_1))$ of $\mathcal{H}$ is called *accepting* if each (ρ_i, ν_i) is an accepting run of $\mathcal{B}_i$. A timed word σ is accepted by $\mathcal{H}$ if $\mathcal{H}$ has an accepting run over σ. We define $L(\mathcal{H})$, the language accepted by $\mathcal{H}$, to be the set of all timed words accepted by $\mathcal{H}$. We say $\mathcal{H}$ is *universal* if $L(\mathcal{H}) = T\Sigma^\omega$ and unambiguous if $\mathcal{H}$ has at most one accepting run over every timed word.

Let $\mathcal{H} = [C_n, \ldots, C_1]$, where each C_i is of the form $(\mathcal{B}_i, G_i)$, be an HTBA. Then we define a *configuration constraint* over $\mathcal{H}$ to be a configuration constraint over the list of automata $\mathcal{B}_n, \ldots, \mathcal{B}_1$.

Definition 1. *Let* φ *be an* MITL *formula over* Σ. *Let* $\mathcal{H}$ *be an* HTBA *over* Σ *and let* g *be a configuration constraint over* $\mathcal{H}$. *Then* $(\mathcal{H}, g)$ *is called a monitoring* HTBA *for* φ *iff for every timed word* $\sigma \in T\Sigma^\omega$ *the following conditions are satisfied. (1): there exists a unique accepting run of* $\mathcal{H}$ *over* σ. *We denote this run by* $(\rho_\sigma, \nu_\sigma)$. *(2): the guard* g *monitors the truth of the formula along* $(\rho_\sigma, \nu_\sigma)$ *in the sense that* $\forall t \geq 0 : \sigma, t \models \varphi \iff (\rho_\sigma(t), \nu_\sigma(t)) \models g$.

We note that such an HTBA $\mathcal{H}$ is necessarily unambiguous and universal. We define a *monitoring* TBA $(\mathcal{A}, g)$, where $\mathcal{A}$ is a TBA and g is a constraint over the configurations of $\mathcal{A}$, analogously.

4 Monitoring Automaton Construction for MITL

Theorem 1. *For a formula φ we can effectively construct a monitoring* HTBA $(\mathcal{H}_\varphi, g_\varphi)$ *for* φ.

We proceed by induction on the structure of the MITL formula. We consider only the main case, the operator $\Psi_{(l,r)}$, here. We will use the following notations and conventions. If $\mathcal{H} = [C_n, \ldots, C_1]$ and $\mathcal{H}' = [\mathcal{D}_m, \ldots, \mathcal{D}_1]$ are lists of timed edge Büchi automata and C is a timed edge Büchi automaton then by $[C, \mathcal{H}]$ we mean the HTBA $[C, C_n, \ldots, C_1]$ and by $[\mathcal{H}, \mathcal{H}']$ we mean the HTBA $[C_n, \ldots, C_1, \mathcal{D}_m, \ldots, \mathcal{D}_1]$. In the automata we write the edge labels on the transitions only if they are necessary for the discussion.

Consider a formula φ of the form $\Psi_{(l,r)}\psi$ where $0 < r - l \leq l < r < \infty$. Let $n = \lceil l/(r - l)\rceil$. We will use a pair of automata C_1 and C_2 to guess and verify the intervals in which φ is *false*. C_1 resets a pair of clocks (x and y in that order) non-deterministically to guess each interval in which φ is false and C_2 verifies that C_1's guesses are indeed correct by resetting a clock z as soon as y turns l and then verifying that ψ is false through out the interval $(x > l) \wedge (y < l)$ and $(y \geq l) \wedge (z < r - l)$. So we are able to reuse the clocks x and y as soon as they become l as opposed to [2, 7] where clock y is active till it turns r. When the next clock x' turns l the clock z would have turned $r - l$, so we can reuse z when x' becomes l by latest. Thus we need only $2n + 1$ clocks when compared to $2n + 2$ clocks required by the constructions in [2, 7].
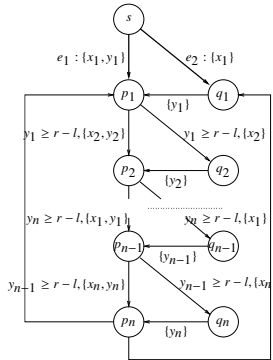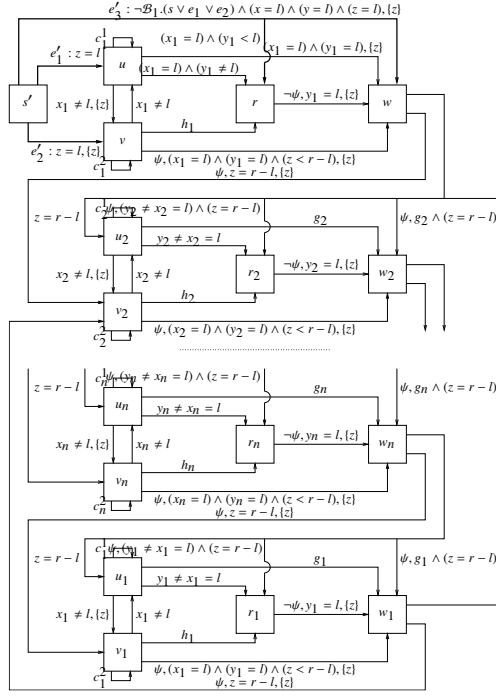

Figure 3: C_1.

Let $(\mathcal{H}_\psi, g_\psi)$ be a monitoring HTBA for ψ. Let C_1 and C_2 be the component automata shown in Fig. 3 and Fig. 4 respectively. Let $\mathcal{H} = [C_2, C_1, \mathcal{H}_\psi]$ and $g = C_1.s \vee \bigvee_{i=1}^{n} C_1.p_i$. Then $(\mathcal{H}, g)$ is a monitoring automaton for φ.

Let us now explain the roles the states play in C_2 in general. The truth value of the formula ψ in the interval $[0, l)$ in σ does not have any bearing on the truth of the formula φ in σ. So, the automaton elapses time in the state s' until z turns l. The states u and v verify that the first interval in which the formula φ is guessed to be true by C_1 does not have sub intervals of size $r - l$ such that ψ is false everywhere in the sub interval with the help of their *tcp* conditions. The state r sees to it that that once x_1 becomes l there is no point where ψ is true until y becomes l while the state w makes sure that ψ remains false for a further $r - l$ time units. Thus, together these two states (along with the transition between them) ensure that the interval in which φ is guessed to be false (by C_1) is indeed correct. Other states play similar roles to validate the guesses of C_1.

Correctness of the construction: We now give a sketch of proof of correctness of our construction for the formula $\varphi = \Psi_{(l,r)}\psi$ when $0 < r - l \leq l < r < \infty$. Let $\sigma \in T\Sigma^\omega$ and let (ρ, ν) be the unique accepting run of $\mathcal{H}_\psi$ over σ. For an $i \in \{1, \ldots, n\}$ let $j = i - 1$ if $i > 1$ and n otherwise, and $k = i + 1$ if $i < n$ and 1 otherwise. Suppose there exist runs (π_1, ν_1) of C_1 and (π_2, ν_2) of C_2 over σ such that $((\pi_2, \nu_2), (\pi_1, \nu_1), (\rho, \nu))$ is a run of $\mathcal{H}$ over σ. Then we observe the following: **Observation 1:** let $t_0^i < t_1^i < t_2^i < \ldots$, etc. be the points where the clock x_i is reset successively in (π_1, ν_1) and let $u_0^i < u_1^i < u_2^i < \ldots$ etc. be the points where y_i is reset successively in (π_1, ν_1). Then $t_0^i \leq u_0^i < t_1^i \leq u_1^i < \cdots$ and $t_0^1 < t_0^2 < \cdots < t_0^n < t_1^1 < t_1^2 < \cdots$. Thus a y_i reset always follows an x_i reset either immediately or sometime after it. Also the x clocks in C_1 are reset cyclically, i.e. if we ignore the resets of the y clocks then always x_j is reset before x_i and if at all a clock is reset after x_i is reset then it is x_k. Similarly the resets of y clocks are also cyclical. **Observation 2:** the distance between successive resets of x_i is at least l because of the following: **(a):** $n = \lceil l/(r - l)\rceil$; **(b):** the x clocks are reset cyclically; and **(c):** in C_1 the time elapsed in p states are at least $r - l$ so the distance between the resets of consecutive y_j and x_i is at least $r - l$ and so, $t_{d+1}^i \geq t_d^i + n \cdot (r - l) \geq t_d^i + l$. Similarly the distance between successive resets of y_i is

Figure 4: C_2.

at least l. **Observation 3:** since C_1 ensures that the distance between the consecutive resets of y_j and x_i is at least $r - l$, the clock z in C_2, which is reset when y_j becomes l, turns $r - l$ on or before x_i turns l for the next time. **Observation 4:** note that in both C_1 and C_2 none of x_i's and y_i's are compared against a value which is greater than l (recall that $r - l \leq l$ by assumption). This enables us to reuse these clocks as soon as they become l, which helps to make our the construction clock optimal. It can now be proved that $(\mathcal{H}, g)$ is indeed a monitoring HTBA for φ. Due of lack of space we omit the proof.

5 Optimality of the Construction for $\Psi_{\langle l,r \rangle} \psi$

We show that any inductive construction of a monitoring automaton for the operator $\Psi_{\langle l,r \rangle}$, where $l, r \in \mathbb{N}$, requires at least $2n + 1$ clocks. This we do by first constructing a timed word σ from a given formula $\varphi = \Psi_{\langle l,r \rangle} a$ and then show that any monitoring TBA for φ will require at least $2n + 1$ clocks to monitor the truth of φ along σ.

Theorem 2. *Any monitoring* TBA *with integer guards for a formula of the form* $\Psi_{\langle l,r \rangle} a$, *where* $l, r \in \mathbb{N}$, *needs at least* $2n + 1$ *clocks.*

Due to lack of space we give the proof of this theorem in Appendix. A.

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A   Optimality of the Construction for $\Psi_{\langle l,r \rangle} \psi$

We show that any inductive construction of a monitoring automaton for the operator $\Psi_{\langle l,r \rangle}$, where $l, r \in \mathbb{N}$, requires at least $2n + 1$ clocks. This we do by first constructing a timed word σ from a given formula $\varphi = \Psi_{\langle l,r \rangle} a$ and then show that any monitoring TBA for φ will require at least $2n + 1$ clocks to monitor the truth of φ along σ.

Consider the formula $\varphi = \Psi_{\langle l,r \rangle} a$ where $l, r \in \mathbb{N}$ and $r - l \leq l$. Let k be the least positive integer such that $r - l$ divides $l + k$ and let ϵ be an irrational number such that $0 < \epsilon < k/2n$. Let $\sigma = (a, t_0)(a, t_1) \cdots$ where: **(a)** $t_0 = r + \epsilon$; **(b)** $\forall i \in \mathbb{N}$, $t_{2i+1} = t_{2i} + \epsilon$; and **(c)** $\forall i \in \mathbb{N}$, $t_{2i+2} = t_{2i+1} + r - l + \epsilon$. Observe that no two action points lie at an integer distance from each other.

Fig. 5 illustrates the timed word σ when $l = 2.5$ and $r = 3.5$. For convenience we have taken ϵ to be 0.5 (which does not meet the prescribed condition).
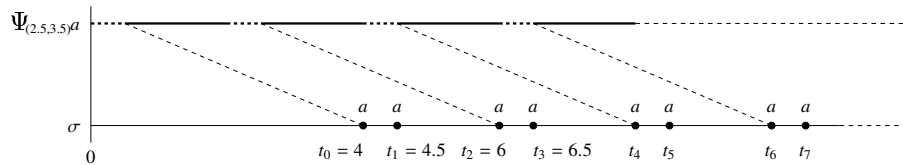
Figure 5: Timed word σ with $l = 2.5$, $r = 3.5$ and $\epsilon = 0.5$.

Let $(\mathcal{A}, g)$ be a monitoring TBA for φ which uses only integer constants. We recall that $\mathcal{A}$ is universal and unambiguous by the definition of a monitoring automaton. Let C be the set of all the clocks in $\mathcal{A}$.

Let us fix the unique accepting run (ρ, ν) of $\mathcal{A}$ over σ for the rest of the section. Let u be one of the action points in σ and let $t \in \mathbb{R}_{\geq 0}$ such that $t < u$. Then we define the set $IDiff_u(t)$ to be the set of all clocks in C whose last reset before t along the run (ρ, ν) lies at an integer distance from u. Formally, we define $IDiff_u(t) = \{x \in C \mid u - (t - \nu(t)(x)) \in \mathbb{N}\}$.

Let $X \subseteq C$ and let h be a conjunctive guard over X. Then we denote by $h(X)$ the guard h restricted to clock constraints over X. For example if $h = (1 < x < 2) \wedge (0 < y < 2)$ and $X = \{x\}$ then

$h(X) = (1 < x < 2)$.

As the first step to show that our construction for $\Psi_{(l,r)}a$ is clock optimal we now prove that $\forall i \in D_0 : \forall t \in (t_i - r, t_i) : IDiff_{t_i}(t) \neq \emptyset$ and $\forall j \in D_1 : \forall t \in (t_j - l, t_j) : IDiff_{t_j}(t) \neq \emptyset$. Suppose there exists an $i \in D_0$ and a point $b \in (t_i - r, t_i)$ such that $IDiff_{t_i}(b) = \emptyset$. Then, in the following series of lemmas we prove that $\mathcal{A}$ is not an unambiguous automaton (the other case when $j \in D_1$ can be handled similarly).

Let T_0 be the set of all time points where the automaton has made a transition in the interval $[0, b)$ in the run (ρ, v) and let T_1 be the set of points where a clock is either reset or turns an integer in the interval $[0, b)$. Let $\lambda = min\{t_2 - t_1 \mid t_1, t_2 \in T_0 \cup T_1 \text{ and } t_1 < t_2\}$ and let δ be a real number such that $0 < \delta < min\{\lambda, \epsilon\}$. Let $D_0 = \{2i \mid 0 \leq i \leq n\}$ and $D_1 = \{2i - 1 \mid 1 \leq i \leq n\}$. Then

Lemma 1. *For all $i \in D_0$ there exists a clock that is reset at $t_i - r$ or turns integer at $t_i - r$ in (ρ, v). Similarly for all $j \in D_1$ there exists a clock that is reset at $t_i - l$ or turns integer at $t_i - l$ in (ρ, v).*

Proof. Suppose there exists an $i \in D_0$ such that no clock is reset or turns integer, at $t_i - r$ in (ρ, v). Note that the truth value of φ changes from false to true at $t_i - r$. Since g monitors φ along (ρ, v) its truth must also change from false to true at $t_i - r$. By assumption no clock is integral at this point, so there must be a transition at $t_i - r$ for the truth value of g to change. Since there is no action at this point it must be an ϵ-transition with an edge guard say h and no resets.

Consider the run (ρ', v') where ρ' is identical to ρ except that the transition $\rho(t_i-r)$ is taken at $t_i-r-\delta$ in ρ'. We claim that (ρ', v') is also an accepting run of $\mathcal{A}$ over σ. Firstly, ρ stays in the same state in the interval $[t_i - r - \delta, t_i - r)$ due to the choice of δ. Secondly, since no clock turns integer at $t_i - r$ we have that $\forall h \in \Phi(C) : (\rho(t_i-r), v(t_i-r)) \models h \iff (\rho'(t_i-r-\delta), v'(t_i-r-\delta)) \models h$. Finally, as no clocks were reset at $t_i - r$ we also have that $\forall t \geq t_i - r : \forall h \in \Phi(C) : (\rho(t), v(t)) \models h \iff (\rho'(t), v'(t)) \models h$. Hence (ρ', v') is also an accepting run of $\mathcal{A}$ over σ. This contradicts the assumption that $\mathcal{A}$ is unambiguous. The proof for the other case is similar and the lemma follows. $\square$

Let (ρ', v') be obtained from (ρ, v) by advancing (or "preponing") by δ all the transitions before b which reset a clock at an integer distance from t_i. Note that by lemma 1 there exists a clock that is reset at an integer distance from t_i in $(0, b)$, so (ρ', v') and (ρ, v) are different. Also, observe that if t is a transition point in (ρ, v) such that t is at an integer distance from t_i then t cannot be an action point due to the properties of σ. So $\rho(t)$ must be an ϵ-transition. Then

Lemma 2. *Let h be a guard with integer constants and let $t \in T_0$. Then, $(\rho(t), v(t)) \models h \iff (\rho'(t - \delta), v'(t - \delta)) \models h$ if $t_i - t \in \mathbb{N}$ and $(\rho(t), v(t)) \models h \iff (\rho'(t), v'(t)) \models h$ otherwise.*

Proof. Let $X = \{x \mid v(t_i)(x) \in \mathbb{N}\}$ and $Y = C - X$. Depending on the relative distance of t from t_i there are two different cases.

- t is at an integer distance from t_i. Then we have two subcases:
    - $x \in X$: then the point of previous reset of x lies at an integer distance from t_i. Since all the clock resets in $[0, b)$ which are at integer distance from t_i have been advanced by δ in ρ' it follows that $(\rho(t), v(t)) \models g(x) \iff (\rho'(t - \delta), v'(t - \delta)) \models g(x)$.
    - $y \in Y$: then $v(t)(y) \notin \mathbb{N}$. And because of the way δ was chosen we have $v(t - \delta)(y) \notin \mathbb{N}$ and $[v'(t-\delta)(y)] = [v(t)(y)]$ as well. Therefore $(\rho(t), v(t)) \models g(y) \iff (\rho'(t-\delta), v'(t-\delta)) \models g(y)$.

- t is not at an integer distance from t_i. Again there are two subcases.
    - $x \in X$: once again due to the choice of δ we can show that $v'(t)(x) \notin \mathbb{N}$ and $[v'(t)(x)] = [v(t)(x)]$. Therefore $(\rho(t), v(t)) \models g(x) \iff (\rho'(t), v'(t)) \models g(x)$.

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– $y \in Y$: since the resets of all clocks in Y coincide in both the runs in the interval $[0, b)$ it follows that $(\rho(t), \nu(t)) \models g(y) \iff (\rho'(t), \nu'(t)) \models g(y)$.

$\square$

Lemma 3. (ρ', ν') *is an accepting run of $\mathcal{A}$ over σ.*

Proof. Let $t \in [0, b)$ and let $\rho(t)$ be a transition in $\mathcal{A}$. Then from lemma 2 we can infer that $(\rho'(t - \delta), \nu'(t - \delta)) \models g$ if t is at a integer distance from t_i, and $(\rho'(t), \nu'(t)) \models g$ otherwise. Since $IDiff_{t_i}(b) = \emptyset$ there exists a point $t \in T_0$ such that t is at a non integer distance from t_i and x has not been reset in the interval (t, b). But this implies that if $\mathcal{A}$ were to describe the run (ρ', ν') in $[0, b)$ on σ then the configuration reached by $\mathcal{A}$ at b on (ρ, ν) and (ρ', ν') coincide. As (ρ', ν') and the run (ρ, ν) of $\mathcal{A}$ over σ coincide from b onwards it follows that (ρ', ν') is a valid run of $\mathcal{A}$ over σ. $\square$

Therefore we conclude that

Lemma 4. $\forall i \in D_0 : \forall t \in (t_i - r, t_i) : IDiff_{t_i}(t) \neq \emptyset$ *and* $\forall j \in D_1 : \forall t \in (t_j - l, t_j) : IDiff_{t_j}(t) \neq \emptyset$.

Lemma 5. *Let* $\gamma(i) = r$ *if* $i \in T_0$ *and* $\gamma(i) = l$ *if* $i \in T_1$. *Let* $i, j \in T_0 \cup T_1$ *with* $i \neq j$. *Then* $IDiff_{t_i}(t) \cap IDiff_{t_j}(t) = \emptyset$ *for all* $t \in (t_i - \gamma(i), t_i) \cap (t_j - \gamma(j), t_j)$.

Proof. Suppose there exists a clock x such that $x \in IDiff_{t_i}(t) \cap IDiff_{t_j}(t)$. Then we have $(t_i - (t - \nu(t)(x)) \in \mathbb{N}$ and $(t_j - (t - \nu(x)(x)) \in \mathbb{N}$. But this implies $|t_j - t_i| \in \mathbb{N}$ which, by the construction of σ, is not true. $\square$

Since $\epsilon < k/2n$ we observe that there exists a $t \in \bigcap_{i \in T_0} (t_i - r, t_i) \cap \bigcap_{i \in T_1} (t_i - l, t_i)$. Then $\forall i \in T_0 \cup T_1 :$ $IDiff_{t_i}(t) \neq \emptyset$ by lemma 4. As $|T_0 \cup T_1| = 2n + 1$ lemma 5 implies that $\mathcal{A}$ must have at least $2n + 1$ clocks. This completes the argument for the case $r - l \leq l$.

Note that as open/closed interval was of no consequence in the proof we can also conclude that any monitoring automaton for the formula of the form $\Psi_{\langle l, r \rangle} \psi$ requires at least $2n + 1$ clocks. Similarly, we can also show that any monitoring automaton for $\Psi_{\langle l, r \rangle} a$ when $0 < l < r - l < \infty$ will require at least 3 clocks, and any monitoring automaton for the cases $l = 0$ or $r = \infty$ (except when $l = 0$ and $r = \infty$) will require at least 1 clock. Therefore

Theorem 3. *Any monitoring* TBA *with integer guards for a formula of the form* $\Psi_{\langle l, r \rangle} a$, *where* $l, r \in \mathbb{N}$, *needs at least* $2n + 1$ *clocks.*

Now to discharge the integer assumption on $\mathcal{A}$ as well as on l and r we observe the following. For a TBA $\mathcal{A}$ and a guard g, and a positive rational constant c, let $c \cdot \mathcal{A}$ and $c \cdot g$ be the automaton and the guard obtained by multiplying the constants in $\mathcal{A}$ and g by c respectively. Then it is easy to see that if $(\mathcal{A}, g)$ is a monitoring TBA for a formula of the form $\Psi_{\langle l, r \rangle} a$ then $(c \cdot \mathcal{A}, c \cdot g)$ is a monitoring TBA for the formula $\Psi_{\langle c \cdot l, c \cdot r \rangle} a$. Now, if there a monitoring TBA $(\mathcal{A}, g)$ for an MITL formula of the form $\Psi_{\langle l, r \rangle} a$, where l and r are rational constants, that uses at most $2n$ clocks then we can get the monitoring TBA $(c \cdot \mathcal{A}, c \cdot g)$, where c is the least common multiple of the denominators of l, r and the constants in $\mathcal{A}$, for the formula $\Psi_{\langle c \cdot l, c \cdot r \rangle} a$ [3] (with brackets matching with the brackets in $\Psi_{\langle l, r \rangle} a$) with at most $2n$ clocks which contradicts Theorem 3.

Theorem 4. *Any monitoring* TBA *for a formula of the form* $\Psi_{\langle l, r \rangle} a$ *needs at least* $2n + 1$ *clocks.*

Finally, we observe that in any inductive construction of a monitoring HTBA for MITL the component for the operator $\Psi_{\langle l, r \rangle}$ needs at least $2n + 1$ clocks. If not, we could flatten the resulting HTBA for $\Psi_{\langle l, r \rangle} a$ to obtain a monitoring TBA for $\Psi_{\langle l, r \rangle} a$ that uses less than $2n + 1$ clocks, contradicting Theorem 4.

AGM Belief Revision About Logic

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1 Background and Motivation

Sometimes philosophers revise their beliefs regarding which principles of logic are correct. For example, one might come to abandon the law of excluded middle out of constructivist inclinations or reject the material conditional on grounds of irrelevance. Yet this kind of belief revision cannot be modeled by existing belief revision systems because each system assumes that revisions occur *within* logical frameworks, not *between* them. Original AGM revision assumes classical logic [1], subsequent non-classical AGM systems presuppose their various logics, and even fully general approaches to revision such as that of [2] (which assumes only the Tarskian postulates) still require that revision occur within a particular logic. To model how belief revision about logic occurs, a new system is needed.

To motivate the need for a new type of system more concretely, consider the following instance of belief revision:

(1) On election night I thought Donald Trump would win, but as the mail-in ballots got counted I came to believe Joe Biden would win.

In traditional AGM, when one wants to revise a belief set K by some proposition p, one first removes $\neg p$ from K (along with anything that entails it), then adds p to K and closes this new set under classical consequence (to ensure that the final belief set contains all the consequences of p). This first part of this process is called "contraction," written $K - p$ (for some K and p) and the second part is called "expansion," written $K + p$." Example (1) can be modeled within this framework by letting d be "Donald Trump will win," b be "Joe Biden will win" and letting K be the initial belief set, where $d \in K$. The revision $K * b$ can be modeled via the Levi identity as $(K - \neg b) + b$. Since Biden losing entails Trump winning, contracting by $\neg b$ will ensure that $d \notin K - \neg b$. Subsequently expanding by b will ensure that $b \in (K - \neg b) + b$.

Now compare example (1) with the following:

(2) I used to think that some instances of *modus ponens* failed, but after giving it some more thought I came to believe that *modus ponens* is unconditionally valid.

This sort of belief revision (i.e. revision about which logical principles are valid) is not uncommon in philosophy. It occurs in the upper echelons of academia when professional philosophers change their beliefs about the correctness of logical principles (e.g. [4, p. 683, 685]), as well as in first-year logic and mathematics courses, where students encounter classical reasoning and gain beliefs about the correctness of inferences they didn't previously have. A first pass comparison of (1) and (2) might suggest that the process of belief revision is the same in each example. In both cases the agent begins with some particular belief(s) (that Trump will win in (1) and that some instances of MP fail in (2)), but when the agent decides they want to believe something else (that Biden will win in (1) and that MP is valid in (2)) they remove these initial beliefs and everything that entails them and then accept the new beliefs.

However, two main differences between (1) and (2) become clear once we attempt (and fail) to model example (2) within the context of traditional AGM. To begin with, in traditional AGM every belief set is closed under classical consequence, and thereby is closed under *modus ponens*. Accordingly, we cannot even get off the ground formalizing this example within classical AGM because we cannot construct a non-trivial belief set for which *modus ponens* fails. A second difference between (1) and (2) is that in (1) the objects that belief sets are revised by are individual sentences (*d* and *b*), whereas in (2) this is not the case. *Modus ponens* is not a single sentence of some propositional language, but rather a schematic rule. The construction of traditional AGM revision operators also prevents us from modeling examples like (2), since traditional revision operators take a belief set and a sentence as input and output a belief set.

Unfortunately, these difficulties cannot be surmounted by existing non-classical AGM systems because, even if such systems allow for the failure of principles of classical logic, there are almost always background logical principles that cannot be violated. For example, AGM systems based on relevant logics (e.g. [3]) allow for failures of irrelevant classical validities but not relevant ones. Even the extremely general AGM system of [2], which does not assume *any* background logic, is still inadequate because it's revision operators do not allow for change regarding which logical principles a belief set is closed under. To model belief revision about logic, we need a new framework with new revision operators.

2 Belief Revision About Logic

2.1 Groundwork

In traditional AGM, revision operators are governed by postulates that appropriately constrain the behavior of the operators. To develop an appropriate operator that revises beliefs about logic, we also need some appropriate postulates. A first one is the following:

Closure: When revising their beliefs about logic, an agent's new collection of beliefs should be closed under the new logic.

This postulate is necessary if we are to take an agent's belief set to be an accurate representation of their beliefs about logic. Without the postulate, an agent could "revise" their beliefs to accept or reject various principles without this revision being reflected in the formalism. We should also expect another postulate which makes sure that any beliefs incompatible with these new principles are omitted from the new belief set. To get such a postulate, we need to be clear about what "incompatible" means. In traditional belief revision, the most common practice is to remove any beliefs that, in combination with the new ones, entail triviality. This is a decent baseline for badness, and when revising logics triviality is something that should be avoided in all but one situation:

Coherency: An agent's new collection of beliefs is trivial iff the agent is revising their beliefs to accept the trivial logic.

This postulate constrains agents in such a way that they will prioritize the avoidance of triviality over the preservation of non-logical information that will trivialize under the new logic. For example, it rules out situations like someone preserving inconsistent beliefs through a revision when moving from a paraconsistent to an explosive logic. This is an idealized assumption about agents (since such situations certainly happen by mistake), but given the abstractness of the project this is a reasonable idealization. These two postulates provide two minimum

requirements we would expect to hold of a revision operator. We should also have a third postulate that provides an upper bound on what can be contained in a new belief set:

Origin: All an agent's new beliefs are either non-logical beliefs preserved from the old collection of beliefs, or consequences of the new logic in combination with these non-logical beliefs.

This postulate ensures that (1) the *old* logic has no bearing on what the new beliefs are and (2) there is no junk information from elsewhere that can wind up in the new belief set. Without this postulate, either of these two things might fail. We might also wish to add a bound on what can be excluded from the new belief set. Such a postulate would ensure as little information as possible is removed in the course of the revision. In particular, we would want to ensure that as much non-logical information as possible is preserved through the revision. One such postulate would be the following:

Maximality: The only non-logical beliefs excluded from the new belief set are those that must be excluded on pain of triviality.

This postulate ensures that we do not lose non-logical information we do not need to lose. A failure of this postulate can result in unnecessary loss of non-logical beliefs (even the loss of all non-logical beliefs). Collectively, these four postulates provide a decent characterization of how we would want a revision operator to behave. Closure ensures an agent's new beliefs about logic are reflected in their new belief set, Coherency prevents inappropriate irrational revision, Origin prevents inappropriate information from being contained in the new belief set, and Maximality ensures as much appropriate information as possible is contained in the new belief sets. We do *not* claim that these postulates constitute the best possible characterization of a good revision operator, but they at least are an interesting and plausible starting point.

With these conceptual preliminaries aside, we can begin the formal work. Firstly, we need to be clear on what "logics" are. In what follows we identify logics with Tarskian consequence operators:

Definition 1. *Tarskian Operators.*
A Tarskian consequence operator (for some language $\mathcal{L}$) is a function $Cn : \mathscr{P}(\mathcal{L}) \longrightarrow \mathscr{P}(\mathcal{L})$ that obeys the following, for arbitrary $X, Y \subseteq \mathcal{L}$:

1. *$X \subseteq Cn(X)$ (Inclusion)*

2. *If $X \subseteq Y$, then $Cn(X) \subseteq Cn(Y)$ (Monotonicity)*

3. *$Cn(X) = Cn(Cn(X))$ (Idempotency)*

This decision is motivated largely by concerns of generality. By working with abstract consequence operators we don't need to worry about the details of proof theory or semantics, allowing for consideration of a greater variety of logics. The Tarskian postulates themselves are motivated by concerns of elegance and interest. The proofs of subsequent results depend heavily on them, and so dropping these postulates results in a much weaker overall theory.

The next point to be clear on is what "non-logical" beliefs are and how they relate to logical ones. Belief bases themselves are inadequate for this task, but combinations and subsets of different belief bases do the trick:[1]

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[1] Belief bases alone don't work because they are incompatible with the Coherency requirement. For example, if $\{p, \neg p\}$ is a belief base for some non-trivial belief set closed under a paraconsistent logic, then if we wished to revise to an explosive logic, we cannot preserve this whole base through the revision without trivializing the resulting belief set. It must be split up, with either only p or only $\neg p$ preserved.

Definition 2. *Non-Logical Information.*
Given some belief set K^L closed under the operator Cn_L, a belief base for K^L is a set B for which

1. *$Cn_L(B) = K^L$*

2. *For all $\phi \in B$, $\phi \notin Cn_L(B \setminus \{\phi\})$*

The set of all non-logical information for a belief set K^L is the set $\mathbb{B}^{K^L} = \bigcup \{B : B$ is a belief base for $K^L \}$

Finally, we need to be clear on what revision operators are:

Definition 3. *Let $\mathbb{L}$ be the set of all Tarskian consequence operators for some language $\mathcal{L}$. A revision operator is a partial function $\circledast : (\mathscr{P}(\mathcal{L}) \times \mathbb{L}) \times \mathbb{L}) \longrightarrow \mathscr{P}(\mathcal{L})$, where $\circledast(X, L, L')$ is defined iff X is closed under L.*

With these tools in hand we can express the above for postulates precisely:

Definition 4. *Postulates for revision*

1. *$K^L \circledast L' = Cn_{L'}(K^L \circledast L')$ (Closure)*

2. *$K^L \circledast L'$ is trivial iff L' is the trivial logic. (Coherency)*

3. *There exists some $A \subseteq \mathbb{B}^{K^L}$ such that $K^L \circledast L' \subseteq Cn_{L'}(A)$. (Origin)*

4. *For all $\phi \in \mathbb{B}^{K^L}$, if L' is not the trivial logic then $\phi \notin K^L \circledast L'$ iff $Cn_{L'}((K^L \circledast L') \cup \{\phi\})$ is trivial. (Maximality)*

2.2   Technical Results

Now we can begin constructing operators themselves. To do so we take inspiration from partial meet contraction in traditional AGM. Central to partial meet contraction is the "remainder set" for some belief set K and sentence p, and similarly here we require the use of a "remainder set" for some belief set K^L, logic L and logic L'. In traditional AGM a remainder set $K \perp p$ is effectively a collection of candidate new belief sets that don't contain p, and we can do something similar with remainder sets here. A remainder set $K^L \perp L'$ is a collection of candidate new belief sets, each of which will be closed under the new logic L' and will preserve non-logical information from the old belief set K^L. More precisely:

Definition 5. *Remainder Set.*
$K^L \perp L' = \{Cn_{L'}(X) : (1) \ X \subseteq \mathbb{B}^{K^L}$, (2) $Cn_{L'}(X) \neq \mathcal{L}$, and (3) for all Y such that $X \subset Y \subseteq \mathbb{B}^{K^L}$, $Cn_{L'}(Y) = \mathcal{L}\}$ if such Xs exist, and is equal to $\{\mathcal{L}\}$ otherwise.

The intention of the third clause is to ensure that as much non-logical information as possible from the old belief set is contained in each new candidate belief set. This is needed if the resulting revision operator based on this remainder set is to satisfy the Maximality postulate. One last tool is needed for the construction of revision operators:

Definition 6. *Preference Orderings.*
$\leq$ is a total, well-founded preorder on $\mathscr{P}(\mathcal{L})$. $\leq$ is said to be semi-strict iff for all $X, Y \in \mathscr{P}(\mathcal{L})$, if $X \nsubseteq Y$ and $Y \nsubseteq X$, then $X < Y$ or $Y < X$.

Intuitively, preference orderings tell us which collections of formulas an agent finds plausible. These preference orderings can be combined with remainder sets to provide us with a basic revision operator:

Definition 7. *Basic Revision.*
$K^L \circledast L' = \bigcap min_{\leq}(K^L \perp L')$

Breaking this down a bit, the revision process begins by identifying all the candidate new belief sets, which collectively constitute the remainder set. We then identify all of the most preferable candidates (via the preference ordering). Finally, the revised belief set $K^L \circledast L'$ is the collection of beliefs that is common to all of the most preferable candidates. This operator behaves fairly nicely. In particular:[2]

Theorem 1. *Basic revision satisfies Closure, Coherency, and Origin.*

Notably, however, basic revision doesn't satisfy Maximality. To get this, we need to impose the further restriction on basic revision that it's preference ordering $\leq$ is semi-strict. Call the resulting operation *strong revision*. Strong revision not only satisfies Maximality, but is characterized by the four postulates:[3]

Theorem 2. *A revision operator $\circledast$ is a strong revision operator iff $\circledast$ satisfies postulates Closure, Origin, Coherency, and Maximality.*

Accordingly, via strong revision we've succeeded in our main goal of providing a formal system that models belief revision about logic. As an example of how revision of this sort might play out in practice, we can return to example (2) from section 1. Let K^L be my initial belief set. Since at this point the agent believes that *modus ponens* fails, there will be some instances of it that are not contained in K^L (namely, either for some ϕ and ψ, either $\phi, \phi \rightarrow \psi \in K^L$ but $\psi \notin K^L$ or $\phi, \phi \rightarrow \psi \in K^L$ and $\neg\psi \in K^L$). Revising my belief set K^L to accept *modus ponens* can then be modeled with strong revision to obtain a new belief set $K^L \circledast L'$, where L' will be a new logic that does validate *modus ponens*. The resulting belief set will preserve as much non-logical information as (non-trivially) possible from the old belief set and will be closed under *modus ponens*.

As a concluding thought, it's worth noting that the framework provided above is only one part of a larger picture. What we have done is provide a framework for how one's collection of beliefs *as a whole* can be appropriately revised when a change of logical principles occurs. This framework assumes that for every revision, the the new logic L' is readily available for use. What we have not considered here is how to start from some logic L and transition from it to the new logic L'. In effect, the system models revision of *beliefs about logic* not revision about *logics themselves*. An interesting follow up project to the present one would be to construct a separate system that accomplishes this goal. If this could be accomplished, then (ideally) it could be integrated with the present system to form a more complete picture: To revise one's beliefs about logic, one could begin with one's initial logic L, revise L directly using this new framework to obtain a new logic L', and then use L' in conjunction with the present system to revise one's total collection of beliefs.

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[2]The proof of theorem 1 requires the use of a first lemma for the proof of Coherency: (1) $\mathcal{L} \in K^L \perp L'$ iff L' is the trivial logic. With this lemma, the proof of theorem 1 simply involves checking the postulates hold.

[3]The proof of theorem 2 requires two additional lemmas, where (2) is used in the proof of (3): (2) For any distinct $X, Y \in K^L \perp L'$, $X \nsubseteq Y$ and $Y \nsubseteq X$. (3) A basic revision operator $\circledast$ satisfies Maximality iff the ordering $\leq$ that $\circledast$ employs is semi-strict. Then the left-to-right direction of theorem 2 follows from theorem 1 plus this third lemma. For the right-to-left direction, we let $\circledast$ be any operator that satisfies the four postulates, let $min_{\leq}(K^L \perp L') = \{X \in K^L \perp L' : K^L \circledast L' \subseteq X\}$ and then check that $K^L \circledast L' = \bigcap min_{\leq}(K^L \perp L')$.

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Relative Expressive Powers of First Order Modal Logic and Term Modal Logic

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Abstract

First Order Modal logic(FOML) is a natural language to reason about modal properties of predicates. In FOML, an example formula would be $\forall x \exists y \ \Box \big(Q(x,y) \big)$. Term modal logic(TML) was introduced to reason about unboundedly many agents and a typical formula in TML looks like $\forall x \exists y \ \Box_x \big(Q(x,y) \big)$. Considering the close similarities between the two logics, in this paper, we explore their relative expressive powers over $\mathcal{K}$-frames.

We prove that every k-variable TML formula can be expressed as a k-variable FOML formula. Conversely, every k-variable FOML formula can be expressed as a $k+1$-variable formula in TML. This proves that both the logics have the same expressive power over unboundedly many variables. On the other hand, for every k we show that the k-variable fragment of FOML is strictly more expressive than k-variable fragment of TML.

1 Introduction

First Order Modal Logic(FOML)[1] combines the power of first order quantifiers and modalities. With this combination, FOML can be used to express the de dicto and de re properties in a natural way as $\Box \exists x.\alpha(x)$ and $\exists x.\Box \alpha(x)$ respectively. In FOML, the set of agents (modal indices) is considered to be finite and fixed in the syntax similar to its propositional counter part.

To model systems with unboundedly many agents, Term Modal Logic (TML) was introduced in [2]. In TML, the modalities are indexed by terms and we have formulas such as $\exists x.\Box_x \phi(x)$. Thus, TML allows us to quantify over modal indices.

Note that syntactically, the only difference between FOML and TML appears in the index of the modalities. The modal index in FOML is fixed (of the form $\Box$) but the modal index in TML are terms (of the form $\Box_x$). As expected, the FOML and TML structures differ only in the description of the accessibility relation. The label of the accessibility relation in FOML structures is fixed but in TML structures, the labels of the accessibility relation change dynamically.

Thus, the two logics are closely related with some subtle differences. Hence a natural question arises about the relative expressive powers of the two logics. In this paper, we explore this aspect in detail for $\mathcal{K}$ frames.

We prove that every k-variable TML formula can be translated as a k-variable FOML formula. Conversely, every k-variable FOML formula can be expressed as a $k+1$-variable formula in TML. These results together imply that both logics have the same expressive power over unboundedly many variables. On the other hand, for every k we show that the k-variable fragment of FOML is strictly more expressive than the k-variable fragment of TML.

The results rely on the fact that the two logics satisfy tree model property over $\mathcal{K}$ frames. To prove these results, we define some natural translations of syntax and mapping of structures of one logic to the other such that the translated formulas are preseved by the corresponding structures.

2 Syntax and Semantics

Let $\mathcal{P}$ be a collection of predicates and let $\mathcal{V}$ be a countable set of variables. We use the same set of predicates and variables for both FOML and TML so that the translations can be viewed naturally.[1]

Definition 2.1. *The syntax of First order modal logic is given by:*

$$\alpha := Q(x_1, \ldots, x_k) \mid \neg\alpha \mid \alpha \wedge \alpha \mid \exists x\ \alpha \mid \Diamond\ \alpha$$

The syntax of Term modal logic is given by:

$$\phi := Q(x_1, \ldots, x_k) \mid \neg\phi \mid \phi \wedge \phi \mid \exists x\ \phi \mid \Diamond_x\ \phi$$

where $Q \in \mathcal{P}$ has arity k and $x, x_1, \ldots, x_k \in \mathcal{V}$.

The $\vee, \rightarrow$ and $\forall$ are defined in the standard way. The dual of $\Diamond$ modality for FOML (similarly $\Diamond_x$ for TML) is given by $\Box\alpha := \neg\Diamond\neg\alpha$ (similarly $\Box_x\phi := \neg\Diamond_x\neg\phi$). Note that the syntax of the two logics differ only in the index of the modal operator.

Let $\mathcal{D}$ be a countable set called potential domain in FOML (called potential agent set in TML).

Definition 2.2 (FOML and TML structures). *An FOML structure is described by the tuple $\mathcal{M}^{\mathcal{F}} = (\mathcal{W}, \delta, \mathcal{R}^{\mathcal{F}}, \rho)$ and a TML structure is given by $\mathcal{M}^{\mathcal{T}} = (\mathcal{W}, \delta, \mathcal{R}^{\mathcal{T}}, \rho)$ where:*

- *$\mathcal{W}$ is a non-empty countable set called* worlds *and $\rho : (\mathcal{W} \times \mathcal{P}) \mapsto \bigcup_k 2^{\mathcal{D}^k}$ is the* valuation function *such that if $Q \in \mathcal{P}$ has arity k then $\rho(w, Q) \subseteq [\delta(w)]^k$*

- accessibility relation *is given by $\mathcal{R}^{\mathcal{F}} \subseteq (\mathcal{W} \times \mathcal{W})$ (in case of FOML) and $\mathcal{R}^{\mathcal{T}} \subseteq (\mathcal{W} \times \mathcal{D} \times \mathcal{W})$ (in case of TML).*

- *$\delta : \mathcal{W} \mapsto 2^{\mathcal{D}}$ assigns to each $w \in \mathcal{W}$ a non-empty set called local domain set in FOML (called local agent set in TML) such that whenever[2] $(w, v) \in \mathcal{R}^{\mathcal{F}}$ (similarly whenever $(w, d, v) \in \mathcal{R}^{\mathcal{T}}$) we have $\delta(w) \subseteq \delta(v)$.*

From the definition, it is clear that the structures of FOML and TML differ only in the description of the accessibility relation. Hence the atomic predicates, negation, conjunction and quantifiers are evaluated in the same way for both FOML and TML. The semantics differ only in the evaluation of the modal formulas.

To evaluate formulas on structures, we need a variable assignment $\sigma : \mathcal{V} \mapsto \mathcal{D}$. We say that σ is *relevant* at $w \in \mathcal{W}$ (for both FOML and TML) if $\sigma(x) \in \delta(w)$ for all $x \in \mathcal{V}$.

Definition 2.3 (FOML and TML semantics). *Given an FOML structure $\mathcal{M}^{\mathcal{F}}$ and $w \in \mathcal{W}$, and σ relevant at w, for all FOML formula α define $\mathcal{M}^{\mathcal{F}}, w, \sigma \models \alpha$ inductively as follows:*

$\mathcal{M}^{\mathcal{F}}, w, \sigma \models Q(x_1, \ldots, x_n)$	$\Leftrightarrow$	$(\sigma(x_1), \ldots, \sigma(x_n)) \in \rho(w, Q)$
$\mathcal{M}^{\mathcal{F}}, w, \sigma \models \neg\alpha$	$\Leftrightarrow$	$\mathcal{M}^{\mathcal{F}}, w, \sigma \not\models \alpha$
$\mathcal{M}^{\mathcal{F}}, w, \sigma \models (\alpha \wedge \beta)$	$\Leftrightarrow$	$\mathcal{M}^{\mathcal{F}}, w, \sigma \models \alpha$ *and* $\mathcal{M}^{\mathcal{F}}, w, \sigma \models \beta$
$\mathcal{M}^{\mathcal{F}}, w, \sigma \models \exists x\ \alpha$	$\Leftrightarrow$	*there is some $d \in \delta(w)$ such that $\mathcal{M}^{\mathcal{F}}, w, \sigma_{[x \mapsto d]} \models \alpha$*
$\mathcal{M}^{\mathcal{F}}, w, \sigma \models \Diamond\ \alpha$	$\Leftrightarrow$	*there is some $u \in \mathcal{W}$ such that $(w, u) \in \mathcal{R}^{\mathcal{F}}$ and $\mathcal{M}^{\mathcal{F}}, u, \sigma \models \alpha$*

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[1]Note that we do not consider constants and function symbols in the vocabulary. Also, we exclude equality from the syntax.

[2]The *monotonicity* condition on the δ function with respect to the accessibility relation in both FOML and TML are imposed to handle the interpretation of free variables. For more details refer [1, 3].

Given a TML *structure* $\mathcal{M}^{\mathcal{T}}$ *and* $w \in \mathcal{W}$, *and for all* TML *formulas* ϕ *define* $\mathcal{M}^{\mathcal{T}}, w, \sigma \models \phi$ *where the atomic case of predicates, negation, conjunction and quantifiers are evaluated in the same way as in* FOML. *For modal formulas,*

$$\boxed{\begin{array}{lll} \mathcal{M}^{\mathcal{T}}, w, \sigma \models \Diamond_x \ \phi & \Leftrightarrow & \textit{there is some } u \in \mathcal{W} \textit{ such that} \\ & & (w, \sigma(x), u) \in \mathcal{R}^{\mathcal{T}} \textit{ and } \mathcal{M}^{\mathcal{T}}, u, \sigma \models \phi \end{array}}$$

Let $\mathcal{K}^{\mathcal{F}}$, $\mathcal{K}^{\mathcal{T}}$ be the collections of all FOML and TML structures respectively which are rooted trees.[3]

3 Interpreting TML in FOML

Given a TML formula ϕ, we can obtain an FOML formula by simply ignoring the modal indices. The index information can be encoded as a new unary predicate.

Definition 3.1 (Embedding TML into FOML). *Given a* TML *formula* ϕ, *let* $E \in \mathcal{P}$ *be a new unary predicate not occurring in* ϕ. *The translation of* ϕ *into an* FOML *formula is inductively defined as follows:*

$\mathsf{Tr}_1(Q(x_1, \ldots, x_k))$	$=$	$Q(x_1, \ldots, x_k)$	$\mathsf{Tr}_1(\neg \phi)$	$=$	$\neg \ \mathsf{Tr}_1(\phi)$
$\mathsf{Tr}_1(\phi \wedge \psi)$	$=$	$\mathsf{Tr}_1(\phi) \wedge \mathsf{Tr}_1(\psi)$	$\mathsf{Tr}_1(\exists x \ \phi)$	$=$	$\exists x \ \mathsf{Tr}_1(\phi)$
$\mathsf{Tr}_1(\Diamond_x \phi)$	$=$	$\Diamond \left(E(x) \wedge \mathsf{Tr}_1(\phi) \right)$			

For instance $\exists x \ \Box_x \left(\forall y \ \Diamond_y Q(x,y) \right)$ is translated to $\exists x \Box \left(E(x) \rightarrow \forall y \ \Diamond(E(y) \wedge Q(x,y)) \right)$. Note that the translation preserves the number of variables.

For the structures, define the mapping $\pi : \mathcal{K}^{\mathcal{T}} \rightarrow \mathcal{K}^{\mathcal{F}}$ where every $\mathcal{M}^{\mathcal{T}} \in \mathcal{K}^{\mathcal{T}}$ is mapped to $\mathcal{M}^{\mathcal{F}}$ which is obtained by ignoring the edge labels on the accessibility relation in $\mathcal{M}^{\mathcal{T}}$ and for all $w, u \in W$ and $d \in \mathcal{D}$ we have $(w, d, u) \in \mathcal{M}^{\mathcal{T}}$ iff $\rho(u, E) = \{d\}$. Since we are considering only tree models, this mapping is sufficient to preserve the translated formulas. Note that both $\mathcal{M}^{\mathcal{T}}$ and $\pi(\mathcal{M}^{\mathcal{T}})$ have the same world set $\mathcal{W}$.

Theorem 3.2. *Let* ϕ *be any* TML *formula, for all* TML *structures* $\mathcal{M}^{\mathcal{T}} \in \mathcal{K}^{\mathcal{T}}$ *and for all* $w \in \mathcal{W}$ *and for all* σ *we have* $\mathcal{M}^{\mathcal{T}}, w, \sigma \models \phi$ *iff* $\pi(\mathcal{M}^{\mathcal{T}}), w, \sigma \models \mathsf{Tr}_1(\phi)$.

Corollary 3.3. *Every* k-*variable* TML *formula can be translated to a* k-*variable* FOML *formula such that the translated formulas are preserved by the mapping* $\pi : \mathcal{K}^{\mathcal{T}} \rightarrow \mathcal{K}^{\mathcal{F}}$.

4 Interpreting FOML in TML

Given a FOML formula α, to obtain a TML formula, we need to index the modality with some variable. This can be done by introducing a new variable that does not occur in α and existentially quantify the new variable at the outer most level.

Definition 4.1 (Embedding FOML into TML). *Given an* FOML *formula* α, *let* $z \in \mathcal{V}$ *be a fresh variable not occurring in* α. *The translation of* α *into an* TML *formula is inductively defined as follows:*

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[3]Since we are considering $\mathcal{K}$-frames, every pointed TML structure is bisimilar to its tree unravelling and the bisimulation preserves the formulas[3]. Similar result holds for FOML also. Hence, we restrict to the class of structures to be rooted tree models for both the logics.

$\mathsf{Tr}_2(Q(x_1,\dots,x_k))$	$=$	$Q(x_1,\dots,x_k)$	$\mathsf{Tr}_2(\neg\alpha)$	$=$	$\neg\ \mathsf{Tr}_2(\alpha)$
$\mathsf{Tr}_2(\alpha\wedge\beta)$	$=$	$\mathsf{Tr}_2(\alpha)\wedge\mathsf{Tr}_2(\beta)$	$\mathsf{Tr}_2(\exists x\ \alpha)$	$=$	$\exists x\ \mathsf{Tr}_2(\alpha)$
$\mathsf{Tr}_2(\Diamond\alpha)$	$=$	$\Diamond_z\big(\mathsf{Tr}_2(\alpha)\big)$			

A sentence α is translated to $\alpha' = \exists z\ \mathsf{Tr}_2(\alpha)$.

For example, $\exists x\ \Box\big(\forall y\ \Diamond Q(x,y)\big)$ is translated to $\exists z\ \exists x\Box_z\big(\forall y\ \Diamond_z Q(x,y)\big)$. Note that α' has 1 extra variable compared to α.

To translate the structures, let $a_0 \in \mathcal{D}$ be a special domain element and without loss of generality assume that for all FOML structures $\mathcal{M}^{\mathcal{F}} \in \mathcal{K}^{\mathcal{F}}$ (which is a tree) rooted at r, we have $a_0 \in \delta(r)$.[4] Define the mapping $\mu : \mathcal{K}^{\mathcal{F}} \to \mathcal{K}^{\mathcal{T}}$ where every $\mathcal{M}^{\mathcal{F}} \in \mathcal{K}^{\mathcal{F}}$ is mapped to $\mathcal{M}^{\mathcal{T}}$ which has the same underlying tree structure as $\mathcal{M}^{\mathcal{F}}$ and for all $w, u \in W$ we have $(w, u) \in \mathcal{R}^{\mathcal{F}}$ iff $(w, a_0, u) \in \mathcal{R}^{\mathcal{T}}$. In other words, $\mu(\mathcal{M}^{\mathcal{F}})$ is the same as $\mathcal{M}^{\mathcal{F}}$ where all edges are labelled by a_0. Again note that $\mathcal{M}^{\mathcal{T}}$ and $\mu(\mathcal{M}^{\mathcal{T}})$ have the same set of $\mathcal{W}$.

Theorem 4.2. *Let α be any FOML formula, for all FOML structure $\mathcal{M}^{\mathcal{F}} \in \mathcal{K}^{\mathcal{F}}$ and for all $w \in \mathcal{W}$ we have $\mathcal{M}^{\mathcal{F}}, w, \sigma \models \alpha$ iff $\mu(\mathcal{M}^{\mathcal{F}}), w, \sigma_{[z\to a_0]} \models \mathsf{Tr}_2(\alpha)$.*

Corollary 4.3. *Every k-variable FOML formula can be translated to a $k+1$-variable TML formula such that the translated formulas are preserved by the mapping $\mu : \mathcal{K}^{\mathcal{F}} \to \mathcal{K}^{\mathcal{T}}$.*

Corollary 3.3 and 4.3 together concludes that over $\mathcal{K}$-frames, both FOML and TML have the same expressive power over unbounded number of variables. Now we prove that with bounded number of variables, FOML is strictly more expressive than TML.

Fix some k. We define two FOML models $\mathcal{M}_1^{\mathcal{F}}$ and $\mathcal{M}_2^{\mathcal{F}}$ and a formula α with k variables such that $\mathcal{M}_1^{\mathcal{F}}, r_1 \models \alpha$ and $\mathcal{M}_2^{\mathcal{F}}, r_2 \not\models \alpha$. But for all k-variable TML formulas ϕ both $\mu(\mathcal{M}_1^{\mathcal{F}})$ and $\mu(\mathcal{M}_2^{\mathcal{F}})$ agree on ϕ at r_1 and r_2 respectively.

Let $Q \in \mathcal{P}$ be a unary predicate. Define $\mathcal{M}_1^{\mathcal{F}}$ and $\mathcal{M}_2^{\mathcal{F}}$ as shown in Figure 1(with unlabeled edges). Note that $\delta_1(r_1) = \delta_2(r_2) = \{a_0, a_1, \dots a_k\}$ and $\rho_1(r_1) = \rho_2(r_2)$. Also, every u_j in $\mathcal{M}_1^{\mathcal{F}}$ is identical to v_j in $\mathcal{M}_2^{\mathcal{F}}$. Clearly, $\mathcal{M}_1^{\mathcal{F}}, r_1 \models \exists x_1 \dots \exists x_k\ \Box\big(\bigvee_{i=1}^k Q(x_i)\big)$ (by picking $a_1 \dots a_k$ as witness for $x_1 \dots x_k$ respectively). But note that $b \notin \delta_2(r_2)$. Hence, we have $\mathcal{M}_2^{\mathcal{F}}, r_2 \not\models \exists x_1 \dots \exists x_k\ \Box\big(\bigvee_{i=1}^k Q(x_i)\big)$.

Now we show that $\mu(\mathcal{M}_1^{\mathcal{F}})$ and $\mu(\mathcal{M}_2^{\mathcal{F}})$ agree on all k variable TML formulas. Note that all the edges in $\mu(\mathcal{M}_1^{\mathcal{F}})$ and $\mu(\mathcal{M}_2^{\mathcal{F}})$ are labeled with a_0. Let $\mu(\mathcal{M}_1^{\mathcal{F}}) = \mathcal{M}_1^{\mathcal{T}}$ and $\mu(\mathcal{M}_2^{\mathcal{F}}) = \mathcal{M}_2^{\mathcal{T}}$.
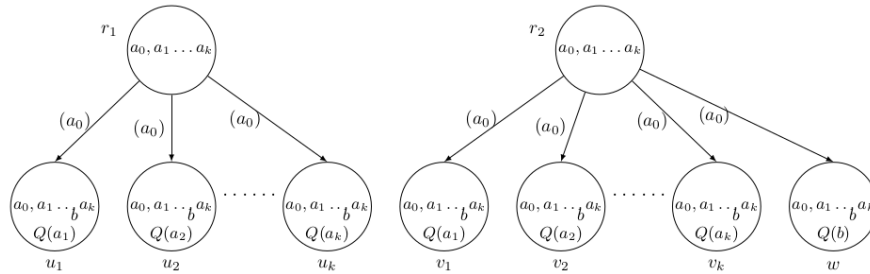
Figure 1: The FOML models $\mathcal{M}_1^{\mathcal{F}}$ and $\mathcal{M}_2^{\mathcal{F}}$ have no edge labeling and the translated models $\mu(\mathcal{M}_1^{\mathcal{F}})$ and $\mu(\mathcal{M}_2^{\mathcal{F}})$ have edges labeled with a_0. Note that $\mathcal{M}_1^{\mathcal{F}}, r_1 \models \exists x_1 \dots \exists x_k\ \Box\big(\bigvee_{i=1}^k Q(x_i)\big)$ but we have $\mathcal{M}_2^{\mathcal{F}}, r_2 \not\models \exists x_1 \dots \exists x_k\ \Box\big(\bigvee_{i=1}^k Q(x_i)\big)$

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[4]Suppose $a_0 \notin \delta(r)$, since $\delta(r) \neq \emptyset$, we can rename some $d \in \delta(r)$ as a_0.

Lemma 4.4. *Let $\mathcal{M}_1^{\mathcal{T}}$ and $\mathcal{M}_2^{\mathcal{T}}$ be as defined above. Then for all $\sigma : [x_1 \ldots x_k] \to [a_0 \ldots a_k]$ and for all k-variable* TML *formulas ϕ, the following holds:*

1. *For all $1 \le l \le k$, $\mathcal{M}_1^{\mathcal{T}}, u_l, \sigma \models \phi$ iff $\mathcal{M}_2^{\mathcal{T}}, v_l, \sigma \models \phi$.*

2. *For all $1 \le j \le k$ If $a_j \notin range(\sigma)$ then $\mathcal{M}_1^{\mathcal{T}}, u_j, \sigma \models \phi$ iff $\mathcal{M}_2^{\mathcal{T}}, w, \sigma \models \phi$.*

Claim (1) follows since u_j and v_j are identical. For (2), note that a_j, b are accessible to formulas only via quantification. Moreover, u_j is identical to w with a_j and b interchanged. Hence the claim holds.

Lemma 4.5. *For all k-variable* TML *formula ϕ and for all $\sigma : [x_1 \ldots x_k] \to [a_0, a_1, \ldots a_k]$, $\mathcal{M}_1^{\mathcal{T}}, r_1, \sigma \models \phi$ iff $\mathcal{M}_2^{\mathcal{T}}, r_2, \sigma \models \phi$.*

Proof. We induct on the structure of ϕ. The atomic predicates, negation, conjunction and quantifier cases are standard since $\delta_1(r_1) = \delta_2(r_2)$ and $\rho_1(r_1) = \rho_2(r_2)$.

For the case $\Diamond_{x_i} \phi$, first note that for all $\sigma : [x_1 \ldots x_k] \to [a_0 \ldots a_k]$ and there is some $a_j \notin range(\sigma)$. Also, if $\sigma(x_i) \ne a_0$ then the claim holds trivially (since both models will make the formula false). So assume that $\sigma(x_i) = a_0$. This implies $a_j \ne a_0$.

Now if $\mathcal{M}_1^{\mathcal{T}}, r_1, \sigma \models \Diamond_{x_i} \phi$ then we have some u_l such that $\mathcal{M}_1^{\mathcal{T}}, u_l, \sigma \models \phi$ and by Lemma 4.4(1) we have $\mathcal{M}_2^{\mathcal{T}}, v_l, \sigma \models \phi$. Hence $\mathcal{M}_2^{\mathcal{T}}, r_2, \sigma \models \Diamond_{x_i} \phi$.

If $\mathcal{M}_2^{\mathcal{T}}, r_2, \sigma \models \Diamond_{x_i} \phi$ then we have two cases. If there is some v_l such that $\mathcal{M}_2^{\mathcal{T}}, v_l, \sigma \models \phi$ then by Lemma 4.4(1), $\mathcal{M}_1^{\mathcal{T}}, u_l, \sigma \models \phi$. Otherwise we have $\mathcal{M}_2^{\mathcal{T}}, w, \sigma \models \phi$ and by Lemma 4.4(2) (since $a_j \notin range(\sigma)$) we have $\mathcal{M}_1^{\mathcal{T}}, u_j, \sigma \models \phi$. Hence in both cases, $\mathcal{M}_1^{\mathcal{T}}, r_1, \sigma \models \Diamond_{x_i} \phi$. $\square$

Theorem 4.6. *Over $\mathcal{K}$ frames, both* TML *and* FOML *have same expressive powers with unboundedly many variables. But, for every k, the k-variable fragment of* FOML *is strictly more expressive than the k-variable fragment of* TML.

5   Conclusion

We have discussed the results for $\mathcal{K}$-frames. It needs to be worked out how these translations behave under various frame restrictions and also for constant domain structures.

The result that the k-variable fragment of FOML is strictly more expressive than the k-variable fragment of TML is dependent on the mapping $\mu : \mathcal{K}^{\mathcal{F}} \to \mathcal{K}^{\mathcal{T}}$. Instead of uniformly labeling all the edges with label a_0, there could be other schemes for labeling the edges. We conjecture that under any mapping $\mu' : \mathcal{K}^{\mathcal{F}} \to \mathcal{K}^{\mathcal{T}}$ that preserves formulas (under corresponding formula translation), the k-variable fragment of FOML will be strictly more expressive than the k-variable fragment of TML. To substantiate the conjecture, note that the 2-variable fragment of FOML is undecidable but the 2-variable fragment of TML is decidable[3].

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Covid-19 and Knowledge Based Computation*

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1 Introduction

The problem of dealing with Covid-19, until a vaccine is universally administered, is to decrease the rate of transmission while getting some social and economic activity going. Infection passes from one person A to another person B when A is infected and B is susceptible. That is to say that B is not infected and not yet immune. Social activity also takes place when one person interacts with another. Perhaps A is a taxpayer and B is a tax consultant. Then filing the tax return may take the form of the two of them meeting. Much can be done electronically without such a meeting, but if A is a person who needs a haircut and B is a barber then it seems that it is necessary for the two to meet in order that A gets his haircut.

Authorities can block infectious encounters through some kind of lockdown. For instance church services can be banned. Beaches can be closed. At the same time essential services have to be allowed. You must be allowed to go to a grocery store because after all, you do have to eat. But non-essential encounters may be blocked so that for instance all barber shops may be required to be closed. But this has the consequence that someone who needs to get a haircut is not even allowed to visit a barber who has been infected and is now immune. The ham handed rule is *block one, block all!*. It is better if contacts depend on *knowledge* of risk.

Random graphs: communication in society takes place along a graph. The study of random graphs begins with a paper by Erdös and Rényi, 1959 in which they looked into the question of *what does it mean to say that a graph is random* and what properties do random graphs (in their sense) have. Others have followed up their work and in addition to the Erdös Rényi graphs there are also other kinds of random graphs like those due to Watts-Strogatz and Barabási-Albert. Since human beings are connected in a graph structure (both for infection and for transmission of knowledge) , it is important to ask what kind of graph we are talking about.

The issue here is the spread of infection in a graph and what we can do to stop or decrease this spread. Knowing the graph of human encounters is clearly relevant.

Knowledge Based Computation: A knowledge based program could have instructions like *do x if you know that A* where it is not necessarily specified *how* it is known whether A is true. Such programs were investigated in [4,6]. But such instructions can arise in distributed computing. The question for us is this. If we do not want blanket instructions like *do not go to the beach* or *do not go to church*, but replace them by *go to church only if you* know *that it is safe*, then we are replacing a blanket instruction by a knowledge based one. And clearly the person following this instruction needs the knowledge *whether* it is safe to go to church.

Testing is one way to gather this knowledge but attention needs also to be paid to spreading that knowledge to those individuals who have to carry out such instructions. Consider for example the instruction, *pay x amount of tax if your bank interest was y*. You need to know

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what y is but you do not and so your bank is required to furnish you with that knowledge. So filling out your tax return is actually a knowledge based algorithm and others have the obligation to provide you with the requisite knowledge. Similarly with Covid-19, the burden is on the state to provide you with information which allows you to decide whether it is safe to go to your barber. See [3,6] for details of knowledge based computation. See also [5].

The program: If we allowed custom made encounters which are safe, then much more economic and social activity could proceed. And here knowledge based programs can come to the rescue. Suppose everyone follows and is required to follow the two following instructions:

Version A

1. If I know that I am infected then I should not visit someone who, I know, is susceptible. $K_i(I(i) \wedge S(j)) \rightarrow \neg V(i, j)$ (and symmetrially)

2. If I know that I am susceptible then I should not visit someone who, I know, is infected. $K_i(S(i) \wedge I(j) \rightarrow \neg V(i, j)$ (and symmetrically)

These two conditions are permissive in case of ignorance. Safer versions of these two conditions are as follows:

Version B

1. If I do not know that I am not infected then I should not visit someone who, for all I know, is susceptible. $\neg K_i(\neg I(i) \wedge \neg S(j)) \rightarrow \neg V(i, j)$

2. If I do not know that I am not susceptible then I should not visit someone who, as far as I know, is infected. $\neg K_i(\neg S(i) \wedge \neg I(j)) \rightarrow \neg V(i, j)$

Note that in case of ignorance of my status, B1 would forbid me from visiting whereas A1 would allow it. So A1 is more permissive and less safe. But A is compatible with more economic activity.

If there is a great deal of testing and the results of the tests are generally available, then the two versions A and B will converge. As more information becomes available, what is allowed under B would grow and what is allowed under A would shrink. With perfect knowledge they will coincide.

Also the results need not be made *generally* available. I need to know that *my* barber is safe but I do not need to know about the status of a barber in California. But it is not enough if *only* governor Cuomo knows if my barber is safe. *I* must have that information myself, but only about my own barber. So there could be a way to balance privacy concerns with knowledge distribution.

Knowledge based algorithms give us a clear path to balancing safety and liveness. What is needed is to get the required information about who is infectious, who is susceptible and who is (currently) immune and *make sure* that this information is available to those individuals who need to make decisions about their own activities.

Knowledge based algorithms are implicit in many current activities like testing, or contact tracing. But since the formal, knowledge based aspect is not made explicit, some insights can be missed.

2 Model

Think of the model as consisting of three sets S, I, R (for susceptible, infected and recovered) which are disjoint and whose union is W (the set of all nodes). At each moment of time t these sets have certain values S(t), I(t), R(t). From time t to t+1 some changes can take place in these three sets. Thus a susceptible person may become infected, an infected person may recover, but a recovered person stays recovered. (As we assume for now). A susceptible person only becomes infected if there is an edge at time t between it and an infected node. Then an edge is *unsafe* if one of its nodes is susceptible and the other one is infected. Now suppose we have a knowledge base and some of the nodes know whether they or their neighbors are infected or susceptible or recovered. Then they can carry out algorithm A or B as the case may be.

Under protocol A you can utilize The edge (i, j) provided you *do not know* that the edge is unsafe. Under protocol B you only utilize that edge if you *know* that the edge is safe. Condition B is more strict. Now utilization of an edge can result in some nodes becoming infected under protocol A but not under B. At the same time an edge being utilized can result in a certain positive value from the social activity. So protocol A also has an advantage.

Should we use protocol A or protocol B? If we do not know very much about who is infected and who is susceptible and so on but we do know that the number of infected nodes is very very small, then with high probability an edge is safe and therefore protocol A is a good protocol to use. It will add a lot of value and will not result in much infection. If a large proportion of nodes are infected then this method A is not safe and then we should go to protocol B. Also, even if the number of infected nodes is currently small, it could increase and we need to continue to test to make sure that it remains small. Continuing to make the knowledge base larger and using protocol A with discretion might be the most benefial method to use.

But protocol B is useful only if you have a lot of knowledge because knowledge is required for the edge to be used and if there is not enough knowledge then edges will not be used very much and there will not be enough social activity. Use of protocol B therefore requires a lot of testing and a *dissemination* of the results obtained,

There are three kinds of edges between individuals, *hard edges* like that between husband and wife living in the same house. These edges are largely involuntary. And there are *soft edges* where it is up to the two parties concerned whether they will have an encounter. An *activity* edge is either a hard edge which is used for an encounter or a soft edge where two nodes x and y interact to carry out some activity.

The goal then is to allow activity edges *only* when it is safe and necessary but to allow as many of them as possible. Moreover, these permissions can be custom made. If I am recovered or my barber is not infected then I can get a haircut. If you are susceptible and your barber might be infected then you cannot get a haircut from him. It is then not necessary to block *all* visits to barbers. Such considerations are implicit in seeking testing and in making use of it. But a formal framework helps to make it explicit.

Algorithm: These considerations bring us to this question. Suppose I am a policy maker, I know something about the hard edges and the soft edges, I know the size of W more or less and have a limited capacity to do testing. Say I can test 10% of W. Then how should I best use my resources? What information shall I gather and to whom shall I convey it? It could well happen that we come up with an NP-complete problem but it may also be that there are algorithms which are computationally cheap and which would give me a ballpark idea of how to proceed. The problem seems sufficiently simple (logically speaking) so that approximate solutions or scalable solutions would be wihin my grasp.

Evaluation: Let us make the model richer by representing it as $\mathcal{M} = (W, R, I, S, E, H)$ where H represents the hard edges and E represents the soft edges. The policy maker P knows all the members of H and if an element e in H has the form $(i, j) : i, j \in W$ then both i, j know that there is such an edge e. The edges are symmetric so there is no difference between (i, j) and (j, i). The soft edges in E are situational, they may exist at one moment and not at the next. Both i, j have to agree to the edge $e = (i, j)$ for the edge to be *alive*. If you want to see your barber, you may need to call him and make an appointment. He will not agree if he knows that you are infected, or that you are susceptible and he is infected.

It is not necessary for the policy maker to know all the soft edges. Suppose certain formulas of the form $S(i)$ or $I(i)$ or $R(i)$ are known to the policy maker. If I am i and I want to make contact with j then perhaps I can consult a database and find out the status of j. So the total number of formulas that the policy maker P needs to know are only at most a constant times the size of W rather than that of $W \times W$.

Let us suppose that at time t, the disjoiint sets of infected, susceptible and recovered are $I(t), S(t), R(t)$ respectively.
We asume that $I(t) \cup S(t) \cup R(t) = W$. $R(t) \subseteq R(t+1)$,
Thus those who are recovered at time t remain recovered.
(This may not be realistic but we assume it for now). $S(t+1) \subseteq S(t) \cup I(t+1)$.
Everyone who is susceptible at t remains so, or is infected at $t+1$.
$R(t+1) \subseteq R(t) \cup I(t)$ If you are recovered at time $t+1$ then either you were already recovered or you just recovered.
This is an incomplete set. More conditions need to be added.

Note that if j is in I at time $t+1$ but not in it at time t then there must have been either a hard edge (i, j) at time t with i infected, or such a live soft edge. You can be infected from your spouse whether you chose or not, but you can only be infected from your barber if the two of you *chose* to meet.

Suppose that P at time t has a knowledge base KB of *some* formulas of the form $R(i, t), S(i, t), I(i, t)$ which tells her who is infected, who is susceptible and who is recovered. She shares these formulas with some members of W or makes them available in case of need. Then they carry out algorithm B. No new people will be infected but also some people who were infected recover (or die).

Proposition 2.1. *Assume that protocol B is used.*

- *If H is empty and all members of W observe protocol B, then $I(t+1) \subseteq I(t)$. No one is infected who was not already.*

- *If H is not empty let C_i be the hard edge component of node i. That is, C_i is the smallest set such that $i \in C_i$ and If $(j, k) \in H$ and $j \in C_i$ then $k \in C_i$. Then if i is infected, then assuming no recovery, all of C_i will be eventually infected.*

Proof:

- *Note that protocol B allows no unsafe soft edges. And since H is empty, there are indeed no unsafe edges. So no new person gets infected.*

- *If there are indeed hard edges then infection can spread along a hard edge. But starting from i, it can only spread inside the component of i. $\square$*

Note that if some infected people recover sponanteously then it will not be the case that everyone in C_i will be infected in the limit. Even if there is a chain from i to j of hard edges, some people in that chain might recover and j might never get infected.

3   Some questions

- Some elements of temporal logic need to be brought in. For instance if my barber is infected, how did he get infected? There must have been an encounter between him and some infected person when the barber was susceptible.

- if we know that someone is infected, we should contact everyone who had contact with that person.

- as the proportion of people who are not susceptible rises, there will be greater liikelihood of saturation and the growth of the pandemic will slow down. Such issues need to be brought into a logical language. See the paper [7] for history based semantics.

- we should pay more attention to those nodes who have high degree, i.e. many contacts. Examples are busdrivers, prostitutes and waiters.

- We should pay more attention to people who are vulnerable like the elderly and people with pre-existing conditions.

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On the Characterizations of Tarski-type and Lindenbaum-type Logical Structures

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Universal logic, as elaborated by Béziau in [1, 2], is the study of structures of the form $(\mathscr{L}, \vdash)$ where $\mathscr{L}$ is a set and $\vdash \, \subseteq \mathcal{P}(\mathscr{L}) \times \mathscr{L}$. Here $\mathcal{P}(\mathscr{L})$ denotes the power set of $\mathscr{L}$. These structures, henceforth referred to as logical structures, will be our main object of interest in this talk.

Before going further, we would like to mention that the term "universal logic" shouldn't be confused with "universal logic project" (cf. [4]). The former we have defined already, whereas the later roughly is all about developing "a general theory of logics considered as mathematical structures" (cf. [3]). In the later case the mathematical structures under consideration may not be logical structures at all.

The content of this talk is a contribution to universal logic understood in the former sense. In mainstream approaches to a general theory of logics, above definition of logical structures is not considered *as it is* (cf. e.g., [7, 8, 9, 10, 12, 14, 15, 17]). Although in all these cases one can show that one is either dealing with a particular type of logical structure, or the structure under consideration induces a family of logical structures. This ubiquity of logical structures was a major motivation for considering that particular notion of logical structures. In this talk, however, we will be concerned with particular types of logical structures. In what follows, except Definition 1, Definition 4 and Definition 5 all other definitions and results presented here are new to the best of our knowledge.

We begin with the definition of Tarski-type logical structures and deductively closed sets,

Definition 1 (TARSKI-TYPE LOGICAL STRUCTURES). Let $(\mathscr{L}, \vdash)$ be a logical structure. Then $(\mathscr{L}, \vdash)$ is called a *Tarski-type logical structure* (or, simply *Tarski-type*) if $\vdash$ satisfies the following properties.

(a) For all $\Gamma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, if $\alpha \in \Gamma$ then $\Gamma \vdash \alpha$. (Reflexivity)

(b) For all $\Gamma, \Sigma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, if $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Sigma$ then $\Sigma \vdash \alpha$. (Monotonicity)

(c) For all $\Gamma, \Sigma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, if $\Gamma \vdash \beta$ for all $\beta \in \Sigma$ then, $\Sigma \vdash \alpha$ implies that $\Gamma \vdash \alpha$. (Transitivity)

Definition 2 (DEDUCTIVELY CLOSED SETS). Let $(\mathscr{L}, \vdash)$ be a logical structure and $\Sigma \subseteq \mathscr{L}$. Then Σ is called a *deductively closed set* in $\mathscr{L}$ if it satisfies the following properties.

(1) For all $\Gamma \subseteq \Sigma$ and all $\beta \in \mathscr{L}$, if $\Gamma \vdash \beta$ then $\beta \in \Sigma$.

(2) For all $\beta \in \Sigma$, $\Sigma \vdash \beta$.

It turns out that the existence of deductively closed sets are crucial in characterizing Tarski-type logical structures. Our result in this regard is the following theorem.

Theorem 3 (CHARACTERIZATION OF TARSKI-TYPE LOGICAL STRUCTURES). Let $(\mathscr{L}, \vdash)$ be a logical structure. Then the following statements are equivalent.

(1) $(\mathscr{L}, \vdash)$ is Tarski-type.

(2) For all $\Gamma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$ such that $\Gamma \not\vdash \alpha$, there exists a deductively closed $\Sigma \subseteq \mathscr{L}$ such that $\Gamma \subseteq \Sigma$ and $\Sigma \not\vdash \alpha$.

Logics in which one can perform Lindenbaum-like constructions of maximal consistent sets are of special importance because constructions of this type often help us to obtain an easy proof of the completeness theorem for the corresponding logic. In the previous theorem, what we stated can be phrased alternatively as follows: a logical structure $(\mathscr{L}, \vdash)$ is Tarski-type iff for all $\Gamma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, whenever $\Gamma \not\vdash \alpha$, there exists a deductively closed extension Σ such that $\Sigma \not\vdash \alpha$. This statement has a striking similarity with the Lindenbaum Lemma, which is concerned with the extension of a consistent set to a maximal consistent one. Is there any relation between these two types of extensions in general? More specifically, is every Tarski-type logical structure "Lindenbaum-type" in some reasonable sense of the term and vice versa? To answer these questions, we will need to have a good enough notion of "Lindenbaum-type" logical structures. Here we talk about one such notion.

Definition 4 (Strongly-Lindenbaum-Type Logical Structures). Suppose that $(\mathscr{L}, \vdash)$ is a logical structure. Then $(\mathscr{L}, \vdash)$ is called a *strongly-Lindenbaum-type logical structure* (or, simply *strongly-Lindenbaum-type*) if for all $\Gamma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$ the set,

$$\mathsf{T}_\alpha^\Gamma := \{\Sigma : \Gamma \subseteq \Sigma \text{ and } \Sigma \not\vdash \alpha\}$$

has a maximal element whenever it is non-empty.

Definition 5 (α-saturated sets). Let $(\mathscr{L}, \vdash)$ be a logical structure. Let $\Gamma \subseteq \mathscr{L}$ and $\alpha \in \mathscr{L}$. Then Γ is called α-saturated in $\mathscr{L}$ if $\Gamma \not\vdash \alpha$ and for all $\beta \in \mathscr{L} \setminus \Gamma$, $\Gamma \cup \{\beta\} \vdash \alpha$.

The definition of α-saturated sets is taken from [13]. With this definition in hand, we have the theorem:

Theorem 6 (Characterization of Strongly Lindenbaum-type Logical Structures). Let $(\mathscr{L}, \vdash)$ be a logical structure. Then the following statements are equivalent.

(1) For all $\Gamma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, if $\Gamma \not\vdash \alpha$ then there exists a maximal α-saturated set Σ such that $\Gamma \subseteq \Sigma$.

(2) $(\mathscr{L}, \vdash)$ is strongly-Lindenbaum-type.

The logical structures defined in Definition 4 were called "Lindenbaum-type" due to the following considerations:

- For logical structures of the form $(\mathscr{L}, \vdash)$ induced by any Hilbert-style system of classical propositional logic (where $\mathscr{L}$ is the set of wffs and $\vdash$ is the usual proof-theoretic consequence relation) α-saturated sets correspond to maximal consistent sets of wffs.

- Consequently, for those logical structures, (1) of Theorem 6 becomes the usual Lindenbaum Lemma.

Till now, we have gradually introduced Tarski-type and strongly-Lindenbaum-type logical structures. At this point, it is natural to wonder about logical structures which are both Tarski-type as well as strongly-Lindenbaum-type (abbreviated henceforth as TsL logical structures). Here is the corresponding characterization theorem:

Theorem 7 (CHARACTERIZATION OF TsL LOGICAL STRUCTURES). Suppose that $(\mathscr{L}, \vdash)$ is a logical structure. Then the following statements are equivalent.

(1) For all $\Gamma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, there exists a deductively closed α-saturated set $\Sigma \subseteq \mathscr{L}$ such that $\Gamma \subseteq \Sigma$.

(2) $(\mathscr{L}, \vdash)$is both Tarski-type and strongly-Lindenbaum-type.

However, we point out that there are Tarski-type structures which are not strongly-Lindenbaum-type and vice versa. Below we give two such examples.

Example 8 (A TARSKI-TYPE LOGICAL STRUCTURE THAT IS NOT STRONGLY-LINDENBAUM–TYPE). Let X be any infinite set. Define $C : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows:

$$C(\Gamma) := \begin{cases} \Gamma & \text{if } \Gamma \text{ is finite} \\ X & \text{otherwise} \end{cases}$$

for all $\Gamma \subseteq X$. Now let $\vdash_C$ be defined as follows:

$$\text{For all } \Gamma \subseteq X \text{ and } \alpha \in X, \ \Gamma \vdash_C \alpha \text{ iff } \alpha \in C(\Gamma)$$

Then $(X, \vdash_C)$ is Tarski-type but not strongly-Lindenbaum-type.

We point out that Kuratowski closure operators (cf. [11]) corresponding to the cofinite topology on an infinite set always satisfies the above property.

Example 9 (A STRONGLY-LINDENBAUM-TYPE LOGICAL STRUCTURE THAT IS NOT TARSKI–TYPE). Consider any system of classical propositional calculus. Let $(\mathbf{CPC}, \vdash_{\mathbf{CPC}})$ denote the induced logical structure where $\mathbf{CPC}$ denotes the set of wffs and $\vdash_{\mathbf{CPC}}$ the usual proof theoretic consequence relation. Define a new logical structure, say, $(\mathbf{CPC}, \models)$ as follows:

$$\text{For all } \Gamma \subseteq \mathbf{CPC} \text{ and all } \alpha \in \mathbf{CPC}, \ \Gamma \models \alpha \text{ iff } \Gamma \setminus \{\alpha\} \vdash_{\mathbf{CPC}} \alpha$$

Then $(\mathbf{CPC}, \models)$ is strongly-Lindenbaum-type but $\models$ is not reflexive. So $(\mathbf{CPC}, \models)$ is not Tarski-type.

Universal logic has deep connections with Suszko's Thesis (cf. [16]) and many-valued logics (cf. [5]). Investigations in this line, along with the development of a theory of logical structures, remains a job for the future. It would be interesting to consider more general sets instead of $\{0, 1\}$. The case of complete lattice, e.g., is examined in some detail in [6]. It would also be nice to see the graded counterparts of the notions talked about here and their interactions. All these are left for the future.

Acknowledgements

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Feferman-Vaught decompositions for prefix classes of first order logic

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Abstract

The Feferman-Vaught theorem provides a way of evaluating a first order sentence φ on a disjoint union of structures by producing a decomposition of φ into sentences which can be evaluated on the individual structures and the results of these evaluations combined using a propositional formula. This decomposition can in general be non-elementarily larger than φ. We show that for FO sentences in prenex normal form with a fixed number of quantifier alternations, such a decomposition can be obtained in time elementary in the size of φ.

Introduction

The Feferman-Vaught theorem from [4] is a classic result from model theory that gives a method to evaluate a first order (FO) sentence over a generalized product of structures by reducing it to the evaluation of other first order sentences over the individual structures and the evaluation of a monadic second order (MSO) sentence over an index structure. One of the simplest generalized products is disjoint union and here in case of finitely many structures, one can replace the evaluation of the mentioned MSO sentence, with the evaluation of a propositional formula. One can also stratify the result by the rank of the FO sentence φ being evaluated on the disjoint union, that is, one can have the sentences in the alluded "decomposition" of φ to have the same bound on their rank as that for φ. These results and their generalizations to MSO have a variety of applications in computer science, such as in showing the decidability of theories, satisfiability checking and algorithmic meta-theorems (see [14] for a survey).

Computing the Feferman-Vaught decomposition for an FO sentence φ over the binary disjoint union of structures (finite or infinite) takes time that is bounded by an m-fold exponential in the size of φ, where m is the rank of φ [14]. This runtime is thus non-elementary in the size of φ, and cannot be improved in general, owing to a non-elementary lower bound for the size of the decomposition over all finite structures (and hence also arbitrary structures) [1]. The time complexity can however be improved by considering special classes of finite structures, such as those of bounded degree, where it takes at most 3-fold exponential time to compute the decomposition if the degree is at least 3, and 2-fold exponential time if the degree is at most 2 [11].

In this paper, we take a different approach towards getting faster decompositions, by observing the syntax of the formulae considered. A well-studied normal form for FO sentences is the prenex normal form (PNF). A prenex sentence is an FO sentence which begins with a string of quantifiers that is followed by a quantifier-free formula. Every FO sentence is equivalent to a prenex sentence and can be brought into such a PNF form in time polynomial in the size of the FO sentence [10]. Let Σ_n and Π_n denote the classes of all PNF sentences that contain $n-1$ alternations of quantifiers (equivalently, n blocks of quantifiers) in the quantifier prefix, and whose leading quantifier is existential and universal respectively. It turns out that various properties of interest in computer science can be expressed using Σ_n or Π_n sentences for very low values of n, indeed with n as just 2. Examples include parameterized problems such as k-Vertex cover, k-Clique and k-Dominating Set which are all Σ_2 expressible (more examples can be found in Appendix A of [17]). In program verification, the Σ_2 fragment is called *Effectively Propositional Logic* (EPR) for which there exist practical implementations of DPLL-based decision procedures for checking

satisfiability [15, 2, 8]. In databases, Π_2 sentences are the syntactic form of source-to-target dependencies in the data exchange setting, and also of views in data integration [3, 12]. Again, over special classes of structures such as those of bounded degree as aforementioned, every FO sentence is equivalent to a Boolean combination of Σ_2 sentences. Thus considering a fixed number of quantifier alternations is a well-motivated restriction.

Towards the central result of this paper, we consider a "tree" generalization of Σ_n and Π_n formulae, that we denote $\mathrm{T}\Sigma_n$ and $\mathrm{T}\Pi_n$. For any FO formula, any root to leaf path in parse tree of the formula can be seen as a word over the quantifier symbols $\exists$ and $\forall$, the logical connectives $\wedge, \vee$ and $\neg$, the predicate symbols of τ along with "$=$", and a set of variables. We define $\mathrm{T}\Sigma_n$ as the class of all FO formulae ψ in negation normal form (NNF, where negations appear only at the atomic level), such that the word corresponding to any root to leaf path in the parse tree of ψ has the form $\exists \cdot (\exists^* \wedge \forall^* \vee)^* w$ where the number of quantifier alternations in the word is at most $n-1$, and w contains no quantifiers. Likewise for $\mathrm{T}\Pi_n$, this word has the form $\forall \cdot (\forall^* \vee \exists^* \wedge)^* w$ with at most $n-1$ quantifier alternations and w as before. Clearly $\mathrm{T}\Sigma_n$ and $\mathrm{T}\Pi_n$ generalize the Σ_n and Π_n classes of formulae considered in NNF. We can now state the main result of this paper (cf. Theorem 3.1). Below $\mathsf{tower}(n, \cdot)$ denotes the n-fold exponential function, and $\mathrm{T}\Sigma_n[m]$ and $\mathrm{T}\Pi_n[m]$ respectively denote the classes of $\mathrm{T}\Sigma_n$ and $\mathrm{T}\Pi_n$ sentences of quantifier rank at most m.

Theorem 1.1. *For every $\mathrm{T}\Sigma_n[m]$ or $\mathrm{T}\Pi_n[m]$ sentence φ, one can compute a Feferman-Vaught decomposition for φ in time $\mathsf{tower}(n, O((n+1) \cdot |\varphi|^2))$. Further, the FO sentences appearing in the decomposition of φ belong to $\mathrm{T}\Sigma_n[m]$ if φ is a $\mathrm{T}\Sigma_n[m]$ sentence, and to $\mathrm{T}\Pi_n[m]$ if φ is a $\mathrm{T}\Pi_n[m]$ sentence.*

In other words, computing the Feferman-Vaught decomposition has an elementary dependence on the size of φ when the number of quantifier alternations in the mentioned "tree PNF" form of φ is bounded. Further, this decomposition is stratified (in the sense mentioned earlier) by both the rank of φ as well as the number of quantifier alternations in the tree PNF form. As a consequence, we obtain that the $\mathrm{T}\Sigma_n[m]$ theory of the disjoint union of two structures is determined by the $\mathrm{T}\Sigma_n[m]$ theories of the individual structures. Likewise for the $\mathrm{T}\Pi_n[m]$ theory (cf. Corollary 3.2). Using a similar reasoning and as a related result, we show that the number of $\mathrm{T}\Sigma_n[m]$ or $\mathrm{T}\Pi_n[m]$ formulae with a given number of free variables, considered modulo equivalence, is an elementary function of m when n is bounded (cf. Proposition 3.3). This is in contrast to the non-elementary lower bound for this number for general FO sentences of rank bounded by m [13, Chapter 3].

Related work: It is known that bounding the number of quantifier alternations allows obtaining finite automata for MSO sentences over words, in elementary time [20], in contrast with general non-elementary lower bounds in this context [19]. The same restriction on Presburger arithmetic again yields faster decision procedures [16, 9]. Finally, the two variable fragment of FO also admits an elementary (doubly exponential) Feferman-Vaught decomposition for disjoint union [6].

Notation and terminology

We assume the reader is familiar with the standard syntax and semantics of FO [13]. Let τ be a fixed vocabulary that contains only relation symbols. We define the classes $\mathrm{T}\Sigma_n$ and $\mathrm{T}\Pi_n$ of FO formulae over τ that respectively generalize the Σ_n and Π_n prefix classes over this vocabulary when the quantifier-free parts of the formulae in the latter classes are in negation normal form (NNF). We define $\mathrm{T}\Sigma_n$ and $\mathrm{T}\Pi_n$ using simultaneous induction. For the base case of $n = 0$, the classes $\mathrm{T}\Sigma_n[0]$ and $\mathrm{T}\Pi_n[0]$ are both equal to the class of all quantifier-free formulae over τ in NNF. So this class is built up from atomic formulae of the form $R(x_1, \ldots, x_k)$ and $x_1 = x_2$ and the negations of these, where R is a k-ary predicate symbol in τ and $x_1, \ldots, x_k$ are variables, using the Boolean connectives $\wedge$ and $\vee$. Inductively assume $\mathrm{T}\Sigma_n$ and $\mathrm{T}\Pi_n$ have been defined for $n \geq 0$. Then:

- A $\mathrm{T}\Sigma_{n+1}$ formula is either a finite conjunction of $\mathrm{T}\Pi_n$ formulae, or a formula of the form $\exists y \varphi_1$ where φ_1 is a $\mathrm{T}\Sigma_{n+1}$ formula.

- A $T\Pi_{n+1}$ formula is either a finite disjunction of $T\Sigma_n$ formulae, or a formula of the form $\forall y \varphi_1$ where φ_1 is a $T\Pi_{n+1}$ formula.

We denote respectively by $T\Sigma_n[m]$ and $T\Pi_n[m]$, the classes of all $T\Sigma_n$ and $T\Pi_n$ formulae that have quantifier rank (i.e. quantifier nesting depth) at most m. An example of a $T\Sigma_3[4]$ formula is

$$\varphi(x) \coloneqq \exists z\big(\forall u_1 \forall v_1 \exists w_1 (E(z,x) \wedge E(v_1,w_1) \wedge E(u_1,v_1)) \bigwedge \forall u_2 (z = u_2 \vee \exists w_2 E(w_2,u_2))\big)$$

We now recall the notions of reduction sequences and models for these from the literature. Note that reduction sequences as we present them below are an adaptation of the special case of 2-reduction sequences from [11], and the adaptation follows the ideas in [7].

Let $\mathcal{L} \subseteq$ FO be a logic. Given numbers $r \geq 0$, $i \in [r] = \{1, \ldots, r\}$ and $j \in [2]$, let $\psi_{i,j}$ be an $\mathcal{L}$-formula over τ whose free variables are contained in a sequence $\bar{x}_j$ of variables. We assume $\bar{x}_1$ and $\bar{x}_2$ to be disjoint. Let $\Delta_j(\bar{x}_j) = (\psi_{1,j}, \ldots, \psi_{r,j})$. Let $X_{i,j}$ be a propositional variable, $\mathcal{X} = \{X_{i,j} \mid i \in [r], j \in [2]\}$, and β be a propositional formula over the variables of $\mathcal{X}$. We call the triple $D(\bar{x}_1, \bar{x}_2) = (\Delta_1(\bar{x}_1), \Delta_2(\bar{x}_2), \beta)$ an $\mathcal{L}$-reduction sequence.

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be τ-structures that are disjoint, and for $j \in [2]$, let $\bar{a}_j$ be a tuple of elements from $\mathfrak{A}_j$. We say that $(\mathfrak{A}_1, \mathfrak{A}_2, \bar{a}_1, \bar{a}_2)$ is a model of the $\mathcal{L}$-reduction sequence $D(\bar{x}_1, \bar{x}_2) = (\Delta_1(\bar{x}_1), \Delta_2(\bar{x}_2), \beta)$, denoted $(\mathfrak{A}_1, \mathfrak{A}_2, \bar{a}_1, \bar{a}_2) \vDash D(\bar{x}_1, \bar{x}_2)$, if $|\bar{a}_j| = |\bar{x}_j|$ for $j \in [2]$, and there exists an assignment $\mu : \mathcal{X} \to \{0, 1\}$ such that $\mu \vDash \beta$ and for $i \in [r]$ and $j \in [2]$,

$$\mu(X_{i,j}) = 1 \quad \leftrightarrow \quad (\mathfrak{A}_j, \bar{a}_j) \vDash \psi_{i,j}(\bar{x}_j).$$

The disjoint union of two τ-structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$, denoted $\mathfrak{A}_1 \uplus \mathfrak{A}_2$, is the τ-structure whose universe is the union of the universes of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and in which every predicate of τ is interpreted as the union of its interpretations in $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Given an $\mathcal{L}$ formula $\varphi(\bar{x}_1, \bar{x}_2)$ over τ, we say that an $\mathcal{L}$-reduction sequence $D(\bar{x}_1, \bar{x}_2)$ is a *Feferman-Vaught decomposition of $\varphi(\bar{x}_1, \bar{x}_2)$ over disjoint union*, or simply a Feferman-Vaught decomposition of $\varphi(\bar{x}_1, \bar{x}_2)$ if disjoint union is clear from context, if it holds that for any two disjoint τ-structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$, if $\bar{a}_j$ is a $|\bar{x}_j|$-tuple from $\mathfrak{A}_j$ for $j \in [2]$, then

$$(\mathfrak{A}_1 \uplus \mathfrak{A}_2, \bar{a}_1, \bar{a}_2) \vDash \varphi(\bar{x}_1, \bar{x}_2) \quad \leftrightarrow \quad (\mathfrak{A}_1, \mathfrak{A}_2, \bar{a}_1, \bar{a}_2) \vDash D(\bar{x}_1, \bar{x}_2)$$

A function that will be relevant for us in this paper is $\mathsf{tower} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which is the function defined as follows: $\mathsf{tower}(0, k) = k$, and $\mathsf{tower}(n, k) = 2^{\mathsf{tower}(n-1,k)}$. So $\mathsf{tower}(n, k)$ is an n-fold exponential function of k. Finally, we abbreviate in the standard way the expression 'if and only if' by 'iff', and 'respectively' by 'resp.'.

Main theorem

Theorem 3.1. *For every $m, n \geq 0$ and every $T\Sigma_n[m]$ (resp. $T\Pi_n[m]$) formula $\varphi(\bar{x}_1, \bar{x}_2)$ over τ, there is a $T\Sigma_n[m]$-reduction sequence (resp. $T\Pi_n[m]$-reduction sequence) $D(\bar{x}_1, \bar{x}_2)$ such that:*

1. *$D(\bar{x}_1, \bar{x}_2)$ is a Feferman-Vaught decomposition of $\varphi(\bar{x}_1, \bar{x}_2)$ over disjoint union.*

2. *$D(\bar{x}_1, \bar{x}_2)$ can be computed from $\varphi(\bar{x}_1, \bar{x}_2)$ in time $\mathsf{tower}(n, O((n+1) \cdot |\varphi|^2))$ and the size of $D(\bar{x}_1, \bar{x}_2)$ is $\mathsf{tower}(n, O((n+1) \cdot |\varphi|))$.*

Proof. We show the result by induction on n for $\varphi(\bar{x}, \bar{y}) \in T\Sigma_n$; the proof when $\varphi(\bar{x}, \bar{y}) \in T\Pi_n$ is similar. The proof of part (1) of the theorem follows the exposition in [7]. The base case is easy; the details can be found in [18]. Assume as induction hypothesis that the statement of the theorem holds for all formulae χ that belong to $T\Pi_n$ (and are of any rank) or belong to $T\Sigma_{n+1}[m']$ for $m' < m$. Assume further that if $D_\chi = (\Delta_1^\chi, \Delta_2^\chi, \beta_\chi)$ is the reduction sequence for χ given by the theorem, then β_χ contains no negations, and if χ contains a leading existential quantifier (i.e.

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such a quantifier at the root of the parse tree of χ), then β_χ is a disjunction of conjuncts with each conjunct being a conjunction of exactly two positive literals, one a variable corresponding to a formula in Δ_1^χ and the other a variable corresponding to a formula in Δ_2^χ. (This holds in the base case.) Consider now a formula $\varphi(\bar{x}_1, \bar{x}_2)$ of $\mathrm{T}\Sigma_{n+1}[m]$, of rank m. We have two cases as below.

(A) The first case is when $\varphi(\bar{x}_1, \bar{x}_2) := \bigwedge_{k \in [t]} \varphi_k(\bar{x}_{1,k}, \bar{x}_{2,k})$ where $t \geq 1$, φ_k is a formula of $\mathrm{T}\Pi_n[m]$ and $\bar{x}_{j,k}$ is a subtuple of $\bar{x}_j$ for $j \in [2]$ and $k \in [t]$. By induction hypothesis, there exist reduction sequences $D_k(\bar{x}_{1,k}, \bar{x}_{2,k}) = (\Delta_1^k(\bar{x}_{1,k}), \Delta_2^k(\bar{x}_{2,k}), \beta_k)$ with the properties mentioned in the hypothesis. Then the desired reduction sequence for $\varphi(\bar{x}_1, \bar{x}_2)$ is $D(\bar{x}_1, \bar{x}_2) = (\Delta_1(\bar{x}_1), \Delta_2(\bar{x}_2), \beta)$ where $\Delta_j(\bar{x}_j) = \Delta_j^1(\bar{x}_{j,1}) \cdot \Delta_j^2(\bar{x}_{j,2}) \cdot \ldots \cdot \Delta_j^t(\bar{x}_{j,t})$ for $j \in [2]$, and $\beta = \bigwedge_{k \in [t]} \beta_k$. Here $\cdot$ denotes concatenation of tuples. It is easy to verify that $D(\bar{x}_1, \bar{x}_2)$ satisfies the conditions mentioned in induction hypothesis. (The time taken and the size calculations for $D(\bar{x}_1, \bar{x}_2)$ can be done analogously as in the base case.)

(B) The second case is when $\varphi(\bar{x}_1, \bar{x}_2) := \exists z \varphi_1(\bar{x}_1, \bar{x}_2, z)$ where φ_1 is a formula of $\mathrm{T}\Sigma_{n+1}[m-1]$. We observe that the free variables of φ_1 can be seen as being amongst the tuple $(\bar{y}_1, \bar{y}_2)$ where either $\bar{y}_1 = \bar{x}_1 \cdot z$ and $\bar{y}_2 = \bar{x}_2$, or $\bar{y}_1 = \bar{x}_1$ and $\bar{y}_2 = \bar{x}_2 \cdot z$. Corresponding to each of these views, we have by induction hypothesis that there exists reduction sequences $D_1(\bar{x}_1 \cdot z, \bar{x}_2) = (\Delta_1^1(\bar{x}_1 \cdot z), \Delta_2^1(\bar{x}_2), \beta_1)$ and $D_2(\bar{x}_1, \bar{x}_2 \cdot z) = (\Delta_1^2(\bar{x}_1), \Delta_2^2(\bar{x}_2 \cdot z), \beta_2)$ satisfying the properties mentioned in the induction hypothesis above. Consider β_k; writing it as an OR of ANDs, we have that

$$\beta_k \leftrightarrow \beta_k' := \bigvee_{l \in [N_k]} C_l^k \quad \text{where} \quad C_l^k := \bigwedge_{i \in S_{l,1}^k} X_{i,l,1}^k \wedge \bigwedge_{i \in S_{l,2}^k} X_{i,l,2}^k \tag{1}$$

Above N_k is the number of conjuncts in β_k'; the sets $S_{l,1}^k$ and $S_{l,2}^k$ are mutually exclusive and their union is the set of all literals in C_l^k; and $\Delta_j^k(\bar{x}_j) = (\psi_{i,l,j}^k)_{i \in S_{l,j}^k, \, l \in [N_k]}$. We now define the formulae $\xi_{l,1}^k(\bar{x}_1)$ and $\xi_{l,2}^k(\bar{x}_2)$ for $k \in [2]$ and $l \in [N_k]$ as below.

$$\begin{aligned}
\xi_{l,1}^1(\bar{x}_1) &:= \exists z \bigwedge_{i \in S_{l,1}^1} \psi_{i,l,1}^1(\bar{x}_1, z) & \xi_{l,2}^1(\bar{x}_2) &:= \bigwedge_{i \in S_{l,2}^1} \psi_{i,l,2}^1(\bar{x}_2) \\
\xi_{l,1}^2(\bar{x}_1) &:= \bigwedge_{i \in S_{l,1}^2} \psi_{i,l,1}^2(\bar{x}_1) & \xi_{l,2}^2(\bar{x}_2) &:= \exists z \bigwedge_{i \in S_{l,2}^2} \psi_{i,l,2}^2(\bar{x}_2, z)
\end{aligned} \tag{2}$$

In the event that $S_{l,j}^k = \varnothing$, we put $\xi_{l,j}^k(\bar{x}_j) := \mathsf{True}$. Let $Y_{l,j}^k$ be a new propositional variable for $l \in [N_k]$ and $j, k \in [2]$. Consider the reduction sequence $D(\bar{x}_1, \bar{x}_2) = (\Delta_1(\bar{x}_1), \Delta_2(\bar{x}_2), \beta)$ where

$$\Delta_j(\bar{x}_j) = (\xi_{1,j}^1, \ldots, \xi_{N_1,j}^1, \xi_{1,j}^2, \ldots, \xi_{N_2,j}^2) \quad ; \quad \beta := \bigvee_{k \in [2]} \bigvee_{l \in [N_k]} (Y_{l,1}^k \wedge Y_{l,2}^k) \tag{3}$$

We claim that $D(\bar{x}_1, \bar{x}_2)$ is indeed a reduction sequence for $\varphi(\bar{x}_1, \bar{x}_2)$ as required by the theorem statement. This together with the fact that β as constructed above contains no negations and further is a disjunction of conjuncts of the form mentioned in the induction hypothesis, would serve to complete the induction. We have two cases: (i) φ_1 is a finite conjunction of $\mathrm{T}\Pi_n[m-1]$ formulae, and (ii) φ_1 is a $\mathrm{T}\Sigma_{n+1}[m-1]$ formula of the form $\exists z' \varphi_1'$ where φ' is a $\mathrm{T}\Sigma_{n+1}[m-2]$ formula. We give below the proof that $D(\bar{x}_1, \bar{x}_2)$ is a $\mathrm{T}\Sigma_{n+1}[m]$-reduction sequence for case (ii), and a sketch of the calculations for the size of $D(\bar{x}_1, \bar{x}_2)$ and the time taken to compute it for case (i). The full analysis of both cases can be found in [18].

If φ_1 is a $\mathrm{T}\Sigma_{n+1}[m-1]$ formula of the form $\exists z' \varphi_1'$ as described above, then by the induction hypothesis, $\psi_{i,l,j}^k$ is also a $\mathrm{T}\Sigma_{n+1}[m-1]$ formula, and since φ_1 contains a leading existential quantifier, the formula β_k can be assumed to be of the form $\bigvee_{i \in [r]} Z_{i,1}^k \wedge Z_{i,2}^k$ for some $r \geq 1$ where $Z_{i,j}^k$ corresponds to a formula of Δ_j^k for $j \in [2]$. Then β_k is already in the OR of ANDs form so that $|S_{l,j}^k| = 1$ for all the values of j, k, l considered (cf. (1)). Indeed then $\xi_{l,j}^k$ is a $\mathrm{T}\Sigma_{n+1}[m]$ formula. So that $D(\bar{x}_1, \bar{x}_2)$ is a $\mathrm{T}\Sigma_{n+1}[m]$-reduction sequence.

If φ_1 is a finite conjunction of $\mathrm{T}\Pi_n[m-1]$ formulae, then we observe that for $k \in [2]$, if $\mathcal{X}_k$ is the set of variables of β_k, then every pair $(\xi_{l,1}^k, \xi_{l,2}^k)$ corresponds to a unique subset of $\mathcal{X}_k$. The

size of this pair is at most twice the sum of the sizes of the formulae of D_k that correspond to the variables in the mentioned subset of $\mathcal{X}_k$; so the size of the pair is at most twice the size of the reduction sequence D_k. Also the value of N_k is at most $2^{|\mathcal{X}_k|}$ and $|\mathcal{X}_k|$ is at most the size of D_k. Using these observations and the induction hypothesis, we can show the following. Below $|\cdot|$ denotes the size of the parameter, and $\mathsf{Time}(\cdot)$ denotes the time taken to compute the parameter.

$$|\Delta_1| + |\Delta_2| \leq \sum_{k \in [2]} \text{Number of pairs } (\xi^k_{l,1}, \xi^k_{l,2}) \cdot \text{size of } (\xi^k_{l,1}, \xi^k_{l,2})$$

$$\leq 4 \cdot \mathsf{tower}(n, c \cdot (n+1) \cdot |\varphi_1|) \cdot \mathsf{tower}(n+1, c \cdot (n+1) \cdot |\varphi_1|)$$

$$|\beta| \leq \sum_{k \in [2]} \text{Number of pairs } (\xi^k_{l,1}, \xi^k_{l,2}) \cdot \text{size of } (Y^k_{l,1} \wedge Y^k_{l,2})$$

$$\leq 6 \cdot \mathsf{tower}(n, c \cdot (n+1) \cdot |\varphi_1|) \cdot \mathsf{tower}(n+1, c \cdot (n+1) \cdot |\varphi_1|)$$

$$\Rightarrow \quad |D(\bar{x}_1, \bar{x}_2)| \leq \mathsf{tower}(n+1, c \cdot (n+2) \cdot |\varphi|)$$

$$\mathsf{Time}(D(\bar{x}_1, \bar{x}_2)) \leq \sum_{k \in [2]} \mathsf{Time}(D_i) + \text{Time taken to write out } D(\bar{x}_1, \bar{x}_2)$$

$$\leq \mathsf{tower}(n+1, c \cdot (n+2) \cdot |\varphi|^2)$$

This completes the induction and the proof. $\qquad\qquad\square$

Corollary 3.2. *Let $\mathcal{L}$ be one of the logics $\mathrm{T}\Sigma_n$ or $\mathrm{T}\Pi_n$. Given τ-structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and $m \in \mathbb{N}$, the $\mathcal{L}[m]$ theory of $\mathfrak{A}_1 \uplus \mathfrak{A}_2$ is determined by the $\mathcal{L}[m]$ theories of $\mathfrak{A}_1$ and $\mathfrak{A}_2$.*

The proof of this corollary can be found in [18]. We conclude with a calculation of a bound on the number of formulae in $\mathrm{T}\Sigma_n$ and $\mathrm{T}\Pi_n$ when the rank and the number of free variables of the formulae in these classes are bounded. We again refer the reader to [18] for the proof.

Proposition 3.3. *Let $\mathcal{L}$ be one of the logics $\mathrm{T}\Sigma_n$ or $\mathrm{T}\Pi_n$. Then upto logical equivalence, for $m, t \geq 0$, the number of formulae in $\mathcal{L}[m](\tau)$ whose free variables are among a given t-tuple $\bar{x}$ of variables, is $\mathsf{tower}(n+2, |\tau| \cdot (n+1) \cdot (m+t)^p)$ where p is the maximum arity of the predicates of τ.*

Future work

As directions ahead, we would like to generalize Theorem 3.1 to extensions of FO such as MSO, and obtain a similar theorem (and extension) for Cartesian products as well. We would also like to apply Theorem 3.1 to obtain Feferman-Vaught decompositions of general FO sentences over bounded degree structures given that over these structures, any FO sentence is equivalent to a Boolean combination of Σ_2 sentences, and compare the times taken to produce these decompositions with those in [11] which are optimal with respect to the tower height. Finally, we are interested in investigating the model checking problem for $\mathrm{T}\Sigma_n$ and $\mathrm{T}\Pi_n$ over graphs of bounded clique-width. It is known from [5] that under believed complexity theoretic assumptions, there is in general no algorithm that can solve the model checking problem for FO sentences φ over graphs of bounded clique-width in time $f(|\varphi|) \cdot n^r$ where n is the number of vertices in the graph, $r \geq 0$ and f is an elementary function of $|\varphi|$ (this holds over even all finite trees which have clique-width at most 3). Intuitively, it seems that the unrestricted number of quantifier alternations in the input FO sentence has a role to play in the mentioned result, given the fact that the number of FO sentences modulo equivalence, of a given rank and arbitrary quantifier alternations, is non-elementary in the rank. In this light, Proposition 3.3 motivates the following question which we would like to answer.

Problem 4.1. *For any fixed $k, n \geq 0$, does there exist an algorithm that, given a graph G of clique-width at most k and a $\mathrm{T}\Sigma_n$ or $\mathrm{T}\Pi_n$ sentence φ, decides whether G satisfies φ in time $f(|\varphi|) \cdot |G|^r$ where $r \geq 0$ and f is an elementary function of $|\varphi|$?*

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Plenitude

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Abstract

Oliver and Smiley's mid-plural logic is extended with indefinite descriptions.

1 Introduction

Plural logic has become popular, from the extensive literature we cite [14, 1, 3, 27, 8], for a recent criticism see [9]. Oliver and Smiley wrote a book on the subject [16] which we take for granted as our base and will frequently refer to. Here is a brief summary of what we require.

The basic idea in their extension of first-order logic to a two-sorted free logic is that variables are divided into singular (apple) and plural (**apples**) sorts. Combinations **a,b**, **a.b** and **-a** exhibit Boolean structure of plurals. The idiosyncratic notation reflects Oliver and Smiley's desire to avoid ontological commitment to sets. Plural variables do not come under the scope of quantification in *mid-plural* logic [16, Chapter 12], it is shown that there is an expressiveness jump both for plural definite descriptions (**the apples**) and plural quantifiers ($\forall$**apples**) rendering axiomatization impossible. By restricting quantification, Oliver and Smiley obtain completeness for mid-plural logic following a carefully worked out Henkin argument.

Our approach is as follows. When talking of existence, one can say there is only a single x such that $A(x)$, or there exist exactly two $x \neq y$ such that $A(x)$ and $A(y)$, or there exist exactly three distinct x, y, z such that $A(x)$, $A(y)$ and $A(z)$, and so on upto some finite number. Then we start running out of variables to talk. So we suggest $Many$ **a**, where **a** is of plural sort, to describe a large but finite number of singular variables ranging over the elements of **a**.

$Many$ and Few are indefinite descriptions, they do not stand for a fixed quantity. In a domestic application one may say there are many apples in the refrigerator, when the number may be something like a dozen. In a societal application one may say there were many people in the market, when the number may be something like a thousand. Even in a single discourse, for the statements that there are many apples in the refrigerator and there were many people in the market, the magnitude of the two plurals may not be comparable.

There exist $Many(\mathbf{a},\mathbf{b})$, that is, there are many individuals which are among **a** or **b**, does not imply that $Many$ **a** or that $Many$ **b**. This is discussed below. $Many$ **a** $\wedge Many$ **-a** could be consistent, for example in a close election. $\neg Many$ **a** $\wedge \neg Few$ **a** could be consistent, we do not insist on having to decide among them. The range inbetween is called a *penumbra* [6], following an idea from [18] a predicate $Bet_{Few,Many}$ **a** could be introduced.

Few **a** $\wedge Few$ **-a** could be consistent because the domain may have only few elements. But this reduces to zeroth-order logic where quantifiers are not required. So for simplicity we assume the *plenitude* of the domain, that there are many values in the domain.

We follow Philip Peterson's work [19] which had $Few\ x$, $Many\ x$ and $Most\ x$ quantifiers and bound variables ranging over individuals. In his logic $Many\ x\ A \supset Few\ x\ A$ is valid and $Few\ x\ A \equiv Most\ x\ \neg A$. We accept that Few is a *degree* of plenitude lower than $Many$ [23].

Harvey Friedman interpreted there exists a set with many elements as a *generalized quantifier*: there exists a set of positive measure, and proved a completeness theorem [10, 11, 22].

The treatment here is not quantified but *propositional*, linguistically justified by an interest in *finite* plurals. This article addresses Open Problem 2.7 of Moss and Raty [15].

2 Syntax

The signature of our logic contains all the symbols in mid-plural logic (which extends first-order logic) plus the indefinite plural descriptions. Thus there are singular variables for which we use $x, y, \ldots$, plural variables for which we use $\mathbf{a}, \mathbf{b}, \ldots$. There are also constants, function and predicate symbols forming terms of mixed sort for which we use $a, b, \ldots$, exhaustive and indefinite descriptions, combinations of plurals, connectives $\wedge, \vee, \neg, \supset, \equiv$ and quantifiers $\forall, \exists$.

Terms are as in Oliver and Smiley. They include variables and constants, function terms $f(a_1, a_2, ..., a_n)$, exhaustive descriptions $x{:}A$—which are not written $\{x{:}A\}$ to refrain from ontological commitment to sets—and plural combinations $\mathbf{-a}$, $\mathbf{a.b}$, $\mathbf{a\text{-}b}$ and $\mathbf{a,b}$ for complement, intersection, difference and union.

Atomic formulas include predicates $P(a_1, a_2, ..., a_n)$, inclusion $a \preccurlyeq b$, read as *a among b*, and indefinite descriptions $Many\ \mathbf{a}$, $Few\ \mathbf{a}$, $Most\ \mathbf{a}$ (the last definable as $Few\ \mathbf{-a}$) and, if required, $Bet_{Few,Many}\ \mathbf{a}$.[1] A finite number of plenitude degrees (say $few, many, plenty$), possibly with penumbral predicates, can be allowed with small changes to the axioms and proofs below.

Atomic formulas equality $a = b$ and constants $true$ and $false$ are definable as two-way inclusion, $x = x$, etc. Boolean formulas are as usual. Exhaustive and indefinite descriptions can define quantifiers, $\exists x A$ as $x{:}A = x{:}A$, $\forall x A$ as $x{:}A = x{:}true$, $Many\ x\ A$ as $Many(x{:}A)$.

$A^\star$ distributes predicate A over a plural, that is, $A^\star(\mathbf{a})$ when every $x \preccurlyeq \mathbf{a}$ satisfies $A(x)$. Using exhaustive description, "There are many apples in the refrigerator" can be written as $Many(x{:}\ apple(x) \wedge inFridge(x))$. By defining plurals such as $\mathbf{apples} = x{:}apple(x)$, this can be written as $Many(\mathbf{apples.inFridge})$. "Many apples are red" is $Many(\mathbf{apples}.red^\star)$.

Following Oliver and Smiley, a model has multi-valued interpretation of plural terms. Our interpretation is richer because it has to be said whether those multi-values are many, are few, so the interpretation now has degrees "many" or "few" or inbetween. A plural term $\mathbf{a.b}$ has degree many *only if* both constituents also do. A plural term $\mathbf{a,b}$ has degree many *if* either constituent does. The lack of the necessity/sufficiency of the conditions makes plurals indefinite.

3 Examples

One might say that there are few continents because one can count them on one's fingers. We want to avoid explicit information on counts. Extending this information to plural terms is not inferential. 8 planets is not many, whereas it appears that 80 planets would be many, a ten-fold division of many instances may result in each quotient having not many instances. For instance one may accept the truth of $Many\ \mathbf{apples}$, $Many\ \mathbf{inOrchard}$ and $Many\ \mathbf{inFridge}$ and of $Many\ \mathbf{apples.inOrchard}$, but not of $Many\ \mathbf{apples.inFridge}$. We accept that $Many\ \mathbf{a.b}$ implies $Many\ \mathbf{a}$ and $Many\ \mathbf{b}$.

The *pigeonhole principle* $Many\ \mathbf{a,b} \supset (Many\ \mathbf{a} \vee Many\ \mathbf{b})$ is not deemed valid. (Friedman had a corresponding axiom, see [22].) This rules out a proof of a König lemma of the form: If in a tree with many nodes, all nodes have few chldren, then there must be a path in the tree with many nodes. This is because, beginning with a root with many descendants, an induction step would come to the few children of a node having many descendants. This is a collective and *non-distributive* property of these children [26], it may be that each of 8 (few) children has

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[1]In supervaluations $Few\ \mathbf{a} \vee Many\ \mathbf{a}$ [6], or in subvaluations $Few\ \mathbf{a} \wedge Many\ \mathbf{a}$ [13], could serve.

8 (few) descendants but together they have 64 (many). Multiplication of few by few could yield a collective many, see Gabbay and Schlechta in the context of non-monotonic logic [12].

To switch to something non-mathematical, there is a lively controversy among astronomers and the general public on whether there are many planets, hinging on what defines a planet. Thus, whether there is a planet which goes around the Sun once in 10000 years depends on the belief that there are many planets, as follows.

The belief that there are many planets is based on the principle that planethood is primarily determined by size. Other astronomical details such as rigidity and not being a satellite enter the picture, they are ignored here. Bodies with diameter more than 900 km and not "clearing" their orbits are called dwarf planets. Ceres and Pluto are dwarf planets. The orbit of Pluto intersects that of the traditional planet Neptune, but by taking Pluto to be a dwarf planet, Neptune's orbit is clearing. It is believed that there are more dwarf planets to be found, since they are smaller and difficult to observe.

Let the older planets in our solar system (except Pluto) come under the plural

$$\textbf{clearing} = x\colon diam(x) > 900 \wedge clear(x),$$

stating that their size is above 900 km, and they "clear" their orbit of other planets. The new category of dwarf planets is the plural

$$\textbf{dwarf} = x\colon diam(x) > 900 \wedge \neg clear(x).$$

The state of our solar system is the formula (most planets orbit the Sun in < 300 Earth years):

$$A = \quad Many \ \textbf{dwarf} \wedge Few \ \textbf{clearing} \wedge \forall x(x \preccurlyeq \textbf{clearing} \supset orbit(x) < 300) \wedge$$
$$pluto \preccurlyeq \textbf{dwarf} \wedge ceres \preccurlyeq \textbf{dwarf} \wedge orbit(pluto) < 300 \wedge orbit(ceres) < 300.$$

There are two definitions of planets, $\textbf{planet}_1 = \textbf{clearing}$ then implies $Few \ \textbf{planet}_1$, and $\textbf{planet}_2 = \textbf{dwarf,clearing}$ implies $Many \ \textbf{planet}_2$. Astronomers are divided on whether to include the dwarf planets among the planets (see [2] for a popular account):

$$B = (\textbf{planet} = \textbf{planet}_1) \vee (\textbf{planet} = \textbf{planet}_2).$$

Sedna, discovered on 14 November 2003, is about the size of Ceres, although it is not yet classified as a dwarf planet because we do not know about its rigidity. Here is what we know.

$$C = (diam(ceres) > 900 \equiv diam(sedna) > 900) \wedge orbit(sedna) > 10000.$$

Therefore $A, C \vdash sedna \preccurlyeq \textbf{planet}_2$. Since $Many \ \textbf{planet}$ and $Few \ \textbf{planet}$ cannot both be true (this is formalized as an axiom below), we get $A, B, C, Many \ \textbf{planet} \vdash sedna \preccurlyeq \textbf{planet}$.

By first-order logic, $A, B, C, Many \ \textbf{planet} \vdash \exists x(x \preccurlyeq \textbf{planet} \wedge orbit(x) > 10000)$.

4 Axioms

In addition to the axioms and inference rules in Oliver and Smiley, we add the following. Our semantics will rely on an intuitive notion of degrees of plenitude.

1. $Many \ (x\colon true)$ asserts there are many elements in the domain. This is an assumption.

2. $\forall x(x \preccurlyeq \textbf{a} \supset x = \textbf{a}) \supset Few \ \textbf{a}$ says that singletons are few, so few will end up getting a lower degree than any other.

3. $\mathbf{a} \preccurlyeq \mathbf{b} \supset (Many\ \mathbf{a} \supset Many\ \mathbf{b}) \wedge (Few\ \mathbf{b} \supset Few\ \mathbf{a})$ expresses monotonicity of plenitude. It follows that $Many\ \mathbf{a.b} \supset Many\ \mathbf{a}$ and $Many\ \mathbf{a} \supset Many\ \mathbf{a,b}$, and under the same antecedent $\mathbf{a} \preccurlyeq \mathbf{b}$, that $Many\ \mathbf{-b} \supset Many\ \mathbf{-a}$.

4. $Few\ \mathbf{a} \supset \neg Many\ \mathbf{a}$ *partitions* the plural sort into degrees: few, $many$ and the penumbral $Bet_{Few,Many}$ (which could be ruled out by another axiom). Plural terms are created by exhaustive description of a formula A, so implicitly with every formula there is a degree.

5. $\neg Many\ \mathbf{a} \supset Many\ \mathbf{-a}$ relates *negation* to plenitude. If there are not many individuals among a, there will be many in the complement given that there are many elements in the domain. This partitions $Many\ \mathbf{a}$ into three degrees based on the degrees of $\mathbf{-a}$: $Many\ \mathbf{a} \wedge Many\ \mathbf{-a}$ and the remaining two obtained from the previous axiom.

6. $\forall x((Few\ \mathbf{a} \supset Few\ \mathbf{a,}x \vee Bet_{Few,Many}\ \mathbf{a,}x) \wedge (Many\ \mathbf{a} \supset Many\ \mathbf{a}\text{-}x \vee Bet_{Few,Many}\ \mathbf{a}\text{-}x) \wedge (Bet_{Few,Many}\ \mathbf{a} \supset (Few\ \mathbf{a}\text{-}x \vee Bet_{Few,Many}\ \mathbf{a}\text{-}x) \wedge (Many\ \mathbf{a,}x \vee Bet_{Few,Many}\ \mathbf{a,}x)))$ sets up the degree ordering in the presence of a penumbral predicate.

5 Semantics

Recall the semantics given for $(x\mathbf{:}A)$ in Oliver and Smiley. Let val be an assignment. It is lifted to exhaustive descriptions using the clause:

$val(x\mathbf{:}A)$ are the individuals $val'(x)$ for every x-variant val' of val such that $val' \models A$.

We add a Tarskian definition. The semantics is intuitive and the axioms try to capture that intuition. The third clause illustrates how penumbral predicates could be handled.

Definition 5.1. $val \models Many\ \mathbf{a}$ if and only if there are many among $val(\mathbf{a})$.

$val \models Few\ \mathbf{a}$ if and only if there are few among $val(\mathbf{a})$.

$val \models Bet_{Few,Many}\ \mathbf{a}$ iff there are more than few and less than many among $val(\mathbf{a})$.

Theorem 5.2 (Completeness). *The axiomatization above extending Oliver and Smiley's is weakly complete.*

Proof. We work with finite theories. The Henkin-style completeness proof for mid-plural logic in Oliver and Smiley is followed, only the differences introduced by our syntax are specified.

In the Lindenbaum construction of the model from a *finite* consistent set Δ, when a distinct singular $h \preccurlyeq \mathbf{a}$ for Henkin constant h is encountered in the enumeration as a possible addition to a finite consistent set Δ, its consistency with respect to Δ has to be checked.

Suppose $Many\ \mathbf{a}$ in Δ. Find maximal $\mathbf{b}$ under the $\preccurlyeq$ order such that $\Delta \vdash \mathbf{b} \preccurlyeq \mathbf{a} \wedge \neg Many\ \mathbf{b}$. Count distinct singular $h \preccurlyeq \mathbf{a}$, this must be above the count of distinct singular $h \preccurlyeq \mathbf{b}$. If required create many fresh singular witnesses h and add $h \preccurlyeq \mathbf{a}, h \preccurlyeq \mathbf{-b}, \exists x(x = h)$ to Δ (sufficient to cross intermediate degrees and penumbras), and also add the *plenitude conditions* $h \neq h'$ for every two such h, h'. By monotonicity and negation $Many(\mathbf{a,}h)$ is consistent when $Many\ \mathbf{a}$ is.

Suppose $Few\ \mathbf{a}$ in Δ. Find minimal $\mathbf{b}$ under the $\preccurlyeq$ order such that $\Delta \vdash \mathbf{a} \preccurlyeq \mathbf{b} \wedge \neg Few\ \mathbf{b}$. Count distinct singular $h \preccurlyeq \mathbf{a}$, this must be below the count of distinct singular $h \preccurlyeq \mathbf{b}$ and adding one to the former should preserve the separation of counts.

Suppose a penumbral $Bet_{Few,Many}\ \mathbf{a}$ in Δ. Find maximal $\mathbf{b}$ and minimal $\mathbf{c}$ under the $\preccurlyeq$ order such that $\Delta \vdash \mathbf{b} \preccurlyeq \mathbf{a} \wedge Few\ \mathbf{b}$ and $\Delta \vdash \mathbf{a} \preccurlyeq \mathbf{c} \wedge Many\ \mathbf{c}$ and preserve separation of counts.

Indefinite semantics of $Many$ and Few are used here. What is required for completeness is that there are choices which lead to a maximal consistent set.

In the Truth Lemma for the model constructed in Oliver and Smiley [16, Chapter 12, Lemma 9], we have three additional cases to prove. Here is one of the requirements:

The assignment val $\models$ Many $\mathbf{a}$ if and only if the formula Many $\mathbf{a}$ is one of the truth set Δ.

Starting from the right, if $Many$ $\mathbf{a}$ is one of the truth set Δ, by our construction, many h such that $h \preccurlyeq \mathbf{a}$ are in Δ. By our plenitude conditions they are distinct. So many x-variants val' of val with $x \preccurlyeq \mathbf{a}$ exist. Since $\mathbf{a} = x{:}x \preccurlyeq \mathbf{a}$, there are many among $val(\mathbf{a})$ and $val \models Many$ $\mathbf{a}$.

For the other direction, contrapositively, if $\neg Many$ $\mathbf{a}$ is one of the truth set Δ, our construction and the partition axiom ensure that at all stages there are not many h such that $h \preccurlyeq \mathbf{a}$ were added to Δ. Thus there are not many x-variants val' of val with $x \preccurlyeq \mathbf{a}$. Since $\mathbf{a} = x{:}x \preccurlyeq \mathbf{a}$, there are not many among $val(\mathbf{a})$ and $val \not\models Many$ $\mathbf{a}$. $\qquad\square$

Remark 5.3. Should one use a Skolem function for quantified formulas like $\forall x A(x, \mathbf{a})$ when the scope of the free plural $\mathbf{a}$ is within the quantifier? Thus in $\forall x(Many$ $\mathbf{a} \wedge A(x, \mathbf{a}))$ where $\mathbf{a}$ appears only inside the quantifier, the plenitude of $\mathbf{a}$ will depend on the value of x.

6 Discussion

We thank Kit Fine, Anantha Padmanabha, Rohit Parikh, Abhisekh Sankaran, Byeong-Uk Yi and two anonymous referees for reading our earlier version and commenting on it.

Possible meanings for indefinite descriptions are through comparison classes [21], intervals [20] or (semi)lattices [14, 4]. The structure of these scales, for example, the two scales of plenitude in $Many(\mathbf{apples.inFridge}) \wedge Many(\mathbf{apples.inOrchard})$, is not clear.

Indefinite descriptions can be applied to other areas, we take as our basis Kit Fine's Rutgers lectures on vagueness and sorites arguments [7]. For vagueness there have been suggestions to use nonclassical logics [5, 7], supervaluations [6], games [24, 17], subvaluations [13] and bounded arithmetic [17]. Our approach extends that of degrees and intervals, anticipating the problem Fine mentions of "penumbral connection". Thus, in talking about two colours between which there is a penumbra, one could use a singular logic with predicates such as $Orange(x)$, $Red(x)$ and $Bet_{Orange,Red}(x)$ over an underlying linear order which may be discrete or dense.[2] Supervaluations which make more "precise" valuations correspond to shrinking the range of the penumbral predicate. In the most "precisified" view the penumbra shrinks to nothing. The approach of epistemic uncertainty is encompassed by leaving the boundaries between these various predicates indefinite.

Fine argues for global indeterminacy rather than local, that is, when doing a global "march" from orange to red, one is forced to choose a boundary. We rely on an irreflexive symmetric adjacency relationship between predicates [18], allowing an indefinite $Bet_{Orange,Red}$ between adjacent colours, but not $Bet_{Yellow,Red}$ between distant colours (there could be applications for such predicates). The transitive order $\preccurlyeq$ in this article provides a more global context.

Parikh explains his development of Yessenin-Volpin's ultrafinitism [25] to show the utility of vague reasoning irrespective of semantics [17]. This article also seeks to increase expressiveness of the logical language disturbing the classical framework as little as possible. Whether there is cognitive access to plenitude [28] is an interesting question.

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[2] For example, the additive 3-byte RGB colour model in computer graphics has values $Orange = (255, 128, 0)$ and $Red = (255, 0, 0)$.

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Multiple Task Specification inspired from Mīmāṁsā for Reinforcement Learning Models
(Work in progress)

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Abstract

In this paper we propose a model comprising agent and environment incorporating action specification and rewards inspired from the Indian philosophy of Mīmāṁsā. The specification includes atomic instructions as well as composite ones such as instructions with conjunction, exclusive instructions and instructions in sequence. These instructions follow a separate logical structure and can be combined with environment in the form of propositions. The performance of action is evaluated through three values S, V and N, which are mapped to rewards from the perspective of agent in the Reinforcement Learning models. This helps the agent to learn faster and reduce the computational complexity.

This is a work in progress we are working on the various rewarding schemes and value functions from the perspective of reinforcement learning models.

1 Introduction

In Reinforcement Learning (RL) strategies, an intelligent goal directed agent is employed which perceives the environment and performs suitable action. The agent in addition to performing tasks, learns continuously and makes appropriate decision to choose actions. On performing suitable actions, the agent is programmed to receive rewards, which help to direct the agent to reach the goal. A number of reward and value functions are available in the literature [5]. However, these rewards are calculated from the environment region which are connected through a state-space. The agent perceives the state and performs a single action that effect the state. There has been some work focusing on specifying multiple tasks through temporal logic [1], where the tasks are specified through propositions and states.

An alternate 3-valued logical formalism namely Mīmāṁsā Inspired Representation of Actions (MIRA) has been proposed, which includes the specification of both atomic and composite actions through instructions [3], [4]. The formalism also provides the evaluation values to determine whether the action has been performed with the presence of intention of goal.

This paper proposes a model incorporating MIRA formalism to the goal directed agent model in reinforcement learning. The contribution is two-fold : (i) Instead of single action, composite actions could be specified and (ii) the reward mechanism is directed from the agent rather than the environment.

The paper is organized as follows. First, the temporal action based multiple task specification and evaluation mechanism is described in Section 2. Then a short description of types of action and the characteristics of result of performing those from the perspective of Mīmāṁsā are provided in Section 3. An overview of the formalism of MIRA is presented in Section 4. The model incorporating MIRA formalism is proposed in Section 5 and the paper concludes in Section 6.

2   Related Work

Considerable work has been carried out in Reinforcement Learning models [5]. In the work of teaching multiple tasks with LTL (Linear Temporal Logic), an approach for specification of multiple actions and rewards has been proposed [1]. The multiple tasks specification includes tasks occurring in sequence and tasks that are to be performed in any order. With the semantic constructs of *True* and *False*, the progression of learning is evaluated with LTL specification of $prog(\sigma_i, \varphi)$, where σ_i is assignment of truth values of LTL proposition φ. With this construct an off policy learning is proposed, which helps the agent to extract sub tasks and learn simultaneously. A multitask co-safe LTL specification is defined as a tuple $\mathcal{T} = <S, A, T, \mathcal{P}, L, \Phi, \gamma>$. Here, $<S, A, T, \gamma>$ are defined as in Markovian Decision Process (S - finite set of states, A - finite set of actions, γ - discount factor and T -transistion probability). Φ is the set of tasks, $\mathcal{P}$ is the set of proposition symbols and L is the labeling function $L : S \to 2^p$ which maps each state to the truth values based on proposition p. The reward function is then introduced such that if the sequence of states $s_0, s_1, s_2, ...s_n$ attain a *true* value ($\sigma_{0:n} \models \varphi$) according to L, then reward is 1. Else, the value of reward is maintained at 0 as shown in Equation 1.

$$R_\varphi(<s_0, s_1, ..., s_n>) = \begin{cases} 1 & \sigma_{0:n-1} \not\models \varphi \text{ and } \sigma_{0:n} \models \varphi \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The major disadvantage with this method is the labeling function $L : S \to 2^p$, which leads to a state space evaluation of 2^p. Moreover, only from the truth values of states the information of task being performed or not is made available.

To overcome these disadvantages, we plan to adopt the formalism of MIRA which can be used for multiple task specification. In addition, since the formalism also supports action evaluation, we intend to use these towards rewarding schemes. The advantage is that it may provide a stateless approach or reduction in the number of states. The concept of reward for this paper is inspired from Mīmāṁsā. A brief description of the types of action and rewarding methods according to Mīmāṁsā are provided in the next section.

3   Karma in Mīmāṁsā

Mīmāṁsā, one of the Indian philosophies provides an extensive method of interpretation of Vedic injunctions to perform the sacrificial rites [2]. One of the methods of interpretation is explained through three types of action performance (*karma*). These are regular duty (*nitya karma*), occasional duty (*naimittika karma*) and desired duty (*kāmya karma*). While regular duties are to be performed at all times, Occasional duties are to be performed at certain times. For example, performing *Sandhyavandhana* is *nitya karma* and performing ablutions to ancestors belong to *naimittika karma*. *kāmya karma* is to be performed for any desired objective such as to bring forth rain [6].

The performance of *nitya karma* and *naimittika karma* are specified in two ways.

1. Performance of *karma* yields good results (*karanay abhyudhayam*)

2. Non performance of *karma* yields bad results (*akaranay prathyavāya janakam*)

The performance of *kāmya karma* is optional, depending on the intention in achieving some objective. This structure of yielding results is specified in the formalism of MIRA as the

evaluation parameters, namely *Satisfaction (S)*, if the action is performed with the intention of goal; *Violation (V)*, if action is not performed; and no intention of achieving the goal as *N*. An overview of this formalism is provided in the next section.

4   Overview of MIRA

A formalism of Mīmāṁsā Inspired Representation of Actions has been proposed [3], [4], where the specification of instructions and the action performance are indicated through syntax and semantics, respectively. This is a 3-valued logical formalism given by the values S, V, N. The language is given by:

$$\mathcal{L}_i = (I, R, P, B) \tag{2}$$

where $I = (I^v \cup I^n)$ is a set of imperatives consisting of positive $I^v = \{i_1^+, i_2^+, \cdots, i_n^+\}$ and negative imperatives $I^n = \{i_1^+, i_2^+, \cdots, i_n^+\}$. R and P are the borrowed from the language of proposition logic, which are the sets of reasons and purposes defined by $\{r_1, r_2, \cdots, r_n\}$ and $\{p_1, p_2, \cdots, p_n\}$, respectively. B indicates binary connectives $\{\wedge, \vee, \rightarrow_r, \rightarrow_i, \rightarrow_p\}$. With the given vocabulary of language, the formula of imperatives $\mathcal{F}_i$ is given by:

$$\mathcal{F}_i = i | (i \rightarrow_p p) | (i \rightarrow_p p_1) \wedge (j \rightarrow_p p_2) | (i \rightarrow_p \theta) \oplus (j \rightarrow_p \theta) | (\varphi \rightarrow_i \psi) | (\tau \rightarrow_r \varphi) \tag{3}$$

Here, i represents a single instruction, $(i \rightarrow_p p)$ indicates instruction enjoining goal, $(i \rightarrow_p p_1) \wedge (j \rightarrow_p p_2)$ and $(i \rightarrow_p \theta) \oplus (j \rightarrow_p \theta)$ denote instructions that are mandatory, where sequence of i and j does not matter and exclusive instructions, which indicate *either-or* type, respectively. $(\varphi \rightarrow_i \psi)$ represent two instructions in a sequence and $(\tau \rightarrow_r \varphi)$ signifies a condition in the form of proposition formula τ for action(s) to be performed through instructions specified by φ.

The semantics of MIRA is described in terms of action performance, where $\varphi \in \mathcal{F}_i$ take the value of S, V or N, $\mathcal{E}(\varphi) = \{S, V, N\}$; $\tau \in R$ and $\theta \in P$ take the value of $\top$ (True) and $\bot$ (False), $\mathcal{E}(\tau) = \mathcal{E}(\theta) = \{\top, \bot\}$. Use of binary connectives B on $\{S, V, N, \top, \bot\}$ lead to the output values $\{S, V, N\}$.

Thus, instructions that denote actions can be represented using the language $\mathcal{L}_i$ and can be evaluated to S, V or N. If the action is performed, the evaluation attains the value of S indicating *Satisfaction*, if the action is not performed, the formula attains the value of V denoting *Violation* and if there is no intention to reach the goal, the formula evaluates to N.

As an example for instruction specification, the image representing the plan of a building consisting of five rooms (numbered from 0 to 4) connected through a two-way door can be considered (Figure 1). The robot can navigate through the doors and move from one room to another.

Assuming the agent in room 2, to reach the part5, the specification according to MIRA formalism can be given as:

$$(room2 \rightarrow_r (goto3 \rightarrow_i ((goto1 \rightarrow_p room5) \oplus (goto4 \rightarrow_p part5)))) \tag{4}$$

The valuation can be done in such a way that the ultimate goal of the agent is to reach *part*5 from *room*2. If the agent proceeds through room4, then the evaluation results in Equation 5.

$$(\top \rightarrow_r (S \rightarrow_i ((V \oplus S)))) = S \tag{5}$$
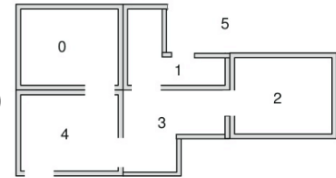

Figure 1: Image of a building

The specification of this formalism and evaluation of action performance in S,V and N can be used towards the rewarding mechanisms in a reinforcement learning agent. A prototype of this model, which is proposed with MIRA formalism is described in the next section.

5   Task Evaluation in terms of rewards

The formalism of MIRA can help to specify actions and compute rewards when performed correctly in reinforcement learning strategies.

The model of agent and environment can be adapted across MIRA formalism, where the Environment maps to the sets R and P and the action from agent correspond to I. The agent and environment interaction according to the Reinforcement Learning model is slightly tweaked to address the formalism of MIRA as shown in Figure 2.
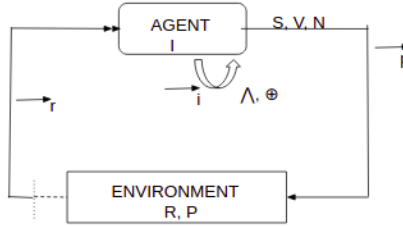

Figure 2: Agent and Environment with MIRA formalism

At each time-step $t = 0, 1, 2, 3$, the agent is allowed to perform actions from the inputs of environment if desirable. The environment at time step t can be an instant of $\tau \in R$, which is sensed by the agent. This sensing is represented by the binary connective $(\rightarrow_r)$. Based on the policy determined by reinforcement strategies, the particular action(s) is decided by the agent. The action selection includes composite actions which can be specified with the help of $\rightarrow_i$, $\oplus$ and $\wedge$. The effect of actions leads to purposes that reflect in the environment and is represented by the connective $\rightarrow_p$.

The rewards as per reinforcement learning is mapped from the environment [5]. As shown in Equation 1, the reward using LTL is assigned to 1 or 0 through the perception of the sequence of states.

But through our proposed model, the evaluation of actions in terms of S, V and N can be mapped towards rewards such that if the tasks performed evaluate to S, then it is marked against reward. This evaluation occurs at two levels. In the first level, $\mathcal{F}_i$ is evaluated to $\{S, V, N\}$. With these values, the reward is evaluated to $\{1, 0\}$ in the second level.

Definition 1. *A reward is mapped to the value of 1, if the evaluation of tasks as per MIRA formalism attains the value of S and 0 otherwise as given in Equation 6*

$$R_\psi = \begin{cases} 1, & if\ \mathcal{E}(\psi) = S \\ 0, & otherwise \end{cases} \tag{6}$$

where ψ denotes the tasks represented through MIRA formalism.

For example, Equations 4 and 5 provide the specification and evaluation of tasks, respectively leading to the value of S. The reward to the agent in this case is 1 as per Equation 6.

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Comparing Equations 1 and 6, it can be seen that the sequence of states are not perceived according to our proposed model, thereby leading to the reduction in state-space.

Moreover, the value of N can help the agent to orient towards goal, which may lead to faster learning which may be evident from the following scenario. For example, if the agent picks up the path of $room2 \to room3 \to room4 \to room0$, the evaluation leads to N indicating the agent cannot reach the goal of $room5$ (Equation 7 and 8).

$$room2 \to_r (goto3 \to_i (goto4 \to_i (goto0 \to_p room0))) \tag{7}$$

$$evaluates\ to\ \top \to_r (S \to_i (S \to_i (S \to_p N))) = N \tag{8}$$

This is because, the present RL model based on LTL uses exploratory or exploitable approach rather than a goal directed one.

Thus the rewards can be triggered from the agent rather than the environment in the proposed approach, leading to multiple advantages such as lesser state-space and goal directed learning as opposed to exploratory learning.

6   Conclusion and Future Work

In this work, the formalism of MIRA is introduced for specifying multiple tasks in Reinforcement Learning model. Multiple task specification includes tasks that can be performed without any order, tasks to be performed in a sequence and tasks that can be carried out in one way or the other. In addition, the goal orientation of the agent can be captured through the specification of $i \to_p \theta$ and the corresponding semantic evaluation value, N. The performance and non-performance of actions indicated through semantic values S and V, respectively can be mapped towards the evaluation of rewards.

This work is a preliminary attempt to construct a Reinforcement Learning model through the principles of Mīmāṁsā. We are currently working to strengthen this model through update evaluation at regular intervals and by providing different rewarding mechanisms. Our focus is also to experimentally analyse the model against other benchmark algorithms.

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Analytic Multi-Succedent Sequent Calculus for Combining Intuitionistic and Classical Propositional Logic

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Abstract

This paper proposes a sequent calculus, denoted by $\mathsf{G}(\mathbf{C} + \mathbf{J})$, for a combination of the intuitionistic propositional logic and the classical propositional logic. Such a combination was studied in Humberstone (1979) and Del Cerro and Herzig (1996). Our sequent calculus has the following two features. First, the system, which is based on multi-succedent sequents, is constructed by the idea of adding the classical implication to Maehara's multi-succedent sequent calculus of intuitionistic logic. The context of the right rule for the intuitionistic implication, however, should be restricted to propositional variables and intuitionistic implications. Second, $\mathsf{G}(\mathbf{C} + \mathbf{J})$ enjoys the cut-elimination theorem, which allows us to establish the subformula property. Hence, it is shown proof-theoretically that $\mathsf{G}(\mathbf{C} + \mathbf{J})$ is a conservative extension of each of the classical propositional logic and the intuitionistic propositional logic.

1 Introduction

This paper studies a combined system of intuitionistic propositional logic and classical propositional logic from proof-theoretic viewpoints. [1] In particular, this paper proposes a cut-free sequent calculus, denoted by $\mathsf{G}(\mathbf{C} + \mathbf{J})$, for a combination of the intuitionistic propositional logic and the classical propositional logic. As far as the authors know, such a combination was studied so far by [5, 4, 8]. While [1, 2] also combined the intuitionistic propositional logic and the classical propositional logic, the semantics given in [1] and the semantics in [2] are different from Kripke semantics given in [5, 4, 8]. This paper follows the semantic treatment of [5, 4, 8]. The classical negation is also added to other non-classical logics, i.e., logic of strict implication [6] and subintuitionistic logic [3].

Humberstone [5]'s syntax consisted of classical and intuitionistic negations ($\neg_c$, $\neg_i$, respectively), conjunction and disjunction, where the intuitionistic implication is defined as $\neg_i(A \wedge \neg_c B)$. Humberstone interpreted the classical implication $\neg_c A$ over intuitionistic Kripke model as: $w \models \neg_c A$ iff $w \not\models A$. He also provided sound and complete natural deduction calculus and proved the completeness theorem via a canonical model construction.

Del Cerro and Herzig [4] proposed a Hilbert axiomatization of a combined system of intuitionistic propositional logic and classical propositional logic. They pointed out that a Hilbert-style axiomatization could *not* be obtained by simply combining axioms of the two logics. This is because one of the intuitionistic axioms $A \rightarrow_i (B \rightarrow_i A)$ (where $\rightarrow_i$ is the intuitionistic implication) becomes invalid when we add the classical implication to the syntax of the intuitionistic propositional logic. As a result, they proposed their appropriate axiomatization $\mathbf{C} + \mathbf{J}$ by employing the idea of conditional logic. Del Cerro and Herzig proved that $\mathbf{C} + \mathbf{J}$ is sound and complete for the following Kripke semantics for

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C + J: it is based on the same Kripke semantics as Humberstone [5], where the satisfaction of the classical implication $\to_c$ is defined naturally as: $w \models A \to_c B$ iff $w \models A$ implies $w \models B$ (the addition of the classical implication breaks the *persistency* or *monotonicity* of a formula in Kripke semantics, which leads to the invalidity of the axiom $A \to_i (B \to_i A)$). Based on this, they showed *semantically* that **C + J** is conservative over intuitionistic propositional logic and classical propositional logic. Moreover, Del Cerro and Herzig established the decidability of **C + J** by translating it into modal logic **S4**.

Lucio [8] provided a sequent calculus, denoted by **FO**$^\supset$, for the first-order expansion of **C + J**. It is noted that the logical connectives of **FO**$^\supset$ consists of intuitionistic implication, classical implication and classical negation and universal quantifier. One of the key ideas of [8] consists in employing the notion of *structured* single succedent sequent of the form: $\Gamma_1; \cdots; \Gamma_n \Rightarrow A$, where Γ_i is a finite set of formulas. For example, $B_1, B_2; C_1, C_2, C_3; D \Rightarrow A$ is a structured sequent and its semantic interpretation (see [8, Definition 16]) is: for any w_1, w_2, w_3 such that $w_1 \leqslant w_2 \leqslant w_3$, if $w_1 \models B_i$ for all i, $w_2 \models C_j$ for all j and $w_3 \models D$, then $w_3 \models A$. This special notion of sequent enables us to define sequent rules for both intuitionistic and classical implication without any context restriction. It is noted that some rule for **FO**$^\supset$ does not satisfy the subformula property. Lucio [8] proved that **FO**$^\supset$ is sound and complete for the intended Kripke semantics.

This paper proposes a cut-free sequent calculus $\mathsf{G}(\mathbf{C + J})$ for the combination, studied by Humberstone [5] and Del Cerro and Herzig [4], of the intuitionistic propositional logic and the classical propositional logic. In particular, the calculus is shown to be equipollent with Del Cerro and Herzig's Hilbert axiomatization (see Fact 3, Theorems 1 and Theorem 2). The cut-freeness of $\mathsf{G}(\mathbf{C + J})$ (Theorem 3) leads us to the subformula property, which gives us a *purely proof-theoretic* argument for that $\mathsf{G}(\mathbf{C + J})$ is conservative over the classical propositional logic and the intuitionistic propositional logic. Unlike [8], this paper employs the ordinary notion of multi-succeedent sequent, i.e., $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets. It is remarked that all the inference rules of $\mathsf{G}(\mathbf{C + J})$ are *analytic*, i.e., enjoys the subformula property, while some rules of **FO**$^\supset$ are *not* analytic.

To define $\mathsf{G}(\mathbf{C + J})$, we first observe, within the intuitionistic propositional logic, that the right rule for the intuitionistic implication

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to_i B}$$

can be restricted to the following form without changing the provable sequents:

$$\frac{A, C_1 \to_i D_1, \ldots, C_m \to_i D_m, p_1, \ldots, p_n \Rightarrow B}{C_1 \to_i D_1, \ldots, C_m \to_i D_m, p_1, \ldots, p_n \Rightarrow A \to_i B}.$$

On the top of this reformulated multi-succedent sequent calculus of the intuitionistic logic, we add the ordinary left and right rules of the classical implication. Without this restriction, the invalid formula $A \to_i (B \to_i A)$ in Kripke semantics could be derived. Therefore, this restriction is compatible with Del Cerro and Herzig [4]'s observation of the impossibility of a simple combination of Hilbert systems of the classical and intuitionistic logics. To obtain our sequent calculus $\mathsf{G}(\mathbf{C + J})$, we do not need to employ the idea of conditional logic as did in [4], but we need to specify the essential "core" of the right rule for the intuitionistic implication.

2 Syntax, Kripke Semantics and Hilbert System

This section reviews a Kripke semantics proposed in [5, 4] in term of the following syntax. Our syntax $\mathcal{L}$ consists of a countably infinite set Prop of propositional variables and the following logical connectives: falsum $\bot$, disjunction $\vee$, conjunction $\wedge$, intuitionistic implication $\to_i$, and classical implication $\to_c$. We

denote by $\mathcal{L}_\mathbf{C}$ and $\mathcal{L}_\mathbf{J}$ the resulting syntax dropping $\to_i$ and $\to_c$ from $\mathcal{L}$, respectively. The set Form of all formulas in our syntax is defined inductively as follows:

$$A ::= p \,|\, \bot \,|\, A \vee A \,|\, A \wedge A \,|\, A \to_i A \,|\, A \to_c A,$$

where $p \in \mathsf{Prop}$. We denote by $\mathsf{Form}_\mathbf{C}$ and $\mathsf{Form}_\mathbf{J}$ the set of all classical formulas and the set of all intuitionistic formulas, respectively. We define $\top := \bot \to_i \bot$, $\neg_c A := A \to_c \bot$ and $\neg_i A := A \to_i \bot$.

Let us move to the semantics for our syntax $\mathcal{L}$. A *model* is a tuple $M = (W, R, V)$ where W is a non-empty set of possible worlds, R is a preorder on W, i.e., R satisfies reflexivity and transitivity, $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation function satisfying the following *persistency* condition: $w \in V(p)$ and wRv jointly imply $v \in V(p)$ for all worlds $w, v \in W$. Given a model $M = (W, R, V)$, a world $w \in W$ and a formula A, the *satisfaction relation* $w \models_M A$ is inductively defined as follows:

$$
\begin{aligned}
w &\models_M p & &\text{iff} & &w \in V(p), \\
w &\not\models_M \bot & & & & \\
w &\models_M A \wedge B & &\text{iff} & &w \models_M A \text{ and } w \models_M B, \\
w &\models_M A \vee B & &\text{iff} & &w \models_M A \text{ or } w \models_M B, \\
w &\models_M A \to_i B & &\text{iff} & &\text{for all } v \in W, (wRv \text{ and } v \models_M A \text{ jointly imply } v \models_M B). \\
w &\models_M A \to_c B & &\text{iff} & &w \models_M A \text{ implies } w \models_M B.
\end{aligned}
$$

Let us say that A is *valid* if $w \models_M A$ for all models $M = (W, R, V)$ and all worlds $w \in W$. We say that a formula A is *persistent* if, for every model $M = (W, R, V)$ and every $w, v \in W$, $w \models_M A$ and wRv jointly imply $v \models_M A$.

Consider a model $M = (W, R, V)$ such that $W = \{w_1, w_2\}$, $R = \{(w_1, w_1), (w_1, w_2), (w_2, w_2)\}$, $V(p) = \{w_2\}$. In this model, $w_1 R w_2$ and $w_1 \models_M \neg_c p$ holds, but $w_2 \not\models_M \neg_c p$ holds. Moreover, $w_1 \not\models_M \top \to_i \neg_c p$. These arguments give us the following propositions.

Proposition 1. A formula $\neg_c p$ is not persistent.

Proposition 2. Neither $\neg_c p \to_i (\top \to_i \neg_c p)$ nor $\neg_c p \to_c (\top \to_i \neg_c p)$ is valid.

Proposition 2 means that an intuitonistic tautology $A \to_i (B \to_i A)$ is no longer valid when we add the classical negation $\neg_c$ to the intuitionistic syntax $\mathcal{L}_\mathbf{J}$. This is why Del Cerro and Herzig [4] needs to restrict this axiom scheme (see the axiom Per in Table 1) in their Hilbert system $\mathbf{C} + \mathbf{J}$ given in Table 1. They established that $\mathbf{C} + \mathbf{J}$ is sound and complete for Kripke semantics as above.

Table 1: Hilbert System $\mathbf{C} + \mathbf{J}$ in [4]

(CL)	All instances of classical tautologies
(CK)	$(A \to_i (B \to_c C)) \to_c ((A \to_i B) \to_c (A \to_i C))$
(ID)	$A \to_i A$
(CMP)	$(A \to_i B) \to_c (A \to_c B)$
(Per)	$A \to_c (B \to_i A)^\dagger$
	$\dagger$: every occurrence of $\to_c$ in A is in the scope of $\to_i$.
(MP)	From A and $A \to_i B$ we may infer B
(RCN)	From A we may infer $B \to_i A$

Fact 3 (Del Cerro and Herzig [4])**.** For all formulas A, A is valid iff A is a theorem of $\mathbf{C} + \mathbf{J}$.

It is remarked that Humberstone [5] showed that his natural deduction system for combinint classical and intuitionistic logic is sound and complete for the same Kripke semantics.

3 Multi-Succedent Sequent Calculus

Table 2: Sequent Calculus $\mathsf{G}(\mathbf{C}+\mathbf{J})$

Axioms

$$\frac{}{A \Rightarrow A} \ (Id) \quad \frac{}{\bot \Rightarrow} \ (\bot)$$

Structural Rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \ (\Rightarrow w) \quad \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \ (w \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} \ (\Rightarrow c) \quad \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \ (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \ (Cut)$$

Propositional Logical Rules

$$\frac{A, C_1 \to_i D_1, \ldots, C_m \to_i D_m, p_1, \ldots, p_n \Rightarrow B}{C_1 \to_i D_1, \ldots, C_m \to_i D_m, p_1, \ldots, p_n \Rightarrow A \to_i B} \ (\Rightarrow \to_i) \quad \frac{\Gamma_1 \Rightarrow \Delta_1, A \quad B, \Gamma_2 \Rightarrow \Delta_2}{A \to_i B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \ (\to_i \Rightarrow)$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to_c B} \ (\Rightarrow \to_c) \quad \frac{\Gamma_1 \Rightarrow \Delta_1, A \quad B, \Gamma_2 \Rightarrow \Delta_2}{A \to_c B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \ (\to_c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \ (\Rightarrow \wedge) \quad \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \ (\wedge \Rightarrow_1) \quad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \ (\wedge \Rightarrow_2)$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \ (\Rightarrow \vee_1) \quad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B} \ (\Rightarrow \vee_2) \quad \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \ (\vee \Rightarrow)$$

In what follows, we use the ordinary notion of multi-succedent sequent. A *sequent* is a pair of finite multisets denoted by $\Gamma \Rightarrow \Delta$, which is read as "if all formulas in Γ are true then some formulas in Δ are true."

Table 2 provides our multi-succedent sequent calculus $\mathsf{G}(\mathbf{C}+\mathbf{J})$. Our basic strategy of constructing $\mathsf{G}(\mathbf{C}+\mathbf{J})$ is to add classical implication to the multi-succedent sequent calculus $\mathbf{mLJ}$ of intuitionistic propositional logic, proposed by Maehara [9]. However, if the ordinary left and right rules of classical implication were added to $\mathbf{mLJ}$, the soundness of the resulting calculus would fail, i.e., $\neg_c p \to_c (\top \to_i \neg_c p)$ would be derivable, which is not valid by Proposition 2. This is the reason why the original right rule

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to_i B}$$

of intuitionistic implication of $\mathbf{mLJ}$ is restricted to the right rule given in Table 2. This restriction is still sufficient to derive the axiom (Per) in Hilbert System $\mathbf{C}+\mathbf{J}$.

Remark 4. If the main connective of each formula in Γ of the above ordinary right rule for intuitionistic implication is not the classical implication, we can derive the above ordinary right rule for $\to_i$. For

example, let Γ be $p \wedge q, C \rightarrow_i D$. Then we recover the original rule as follows:

$$
\cfrac{\cfrac{\cfrac{p \Rightarrow p}{p,q \Rightarrow p} \ (w \Rightarrow) \quad \cfrac{q \Rightarrow q}{p,q \Rightarrow q} \ (w \Rightarrow)}{p,q \Rightarrow p \wedge q} \ (\Rightarrow \wedge) \quad A, p \wedge q, C \rightarrow_i D \Rightarrow B}{\cfrac{\cfrac{A, p, q, C \rightarrow_i D \Rightarrow B}{p,q,C \rightarrow_i D \Rightarrow A \rightarrow_i B} \ (\Rightarrow \rightarrow_i)}{p \wedge q, C \rightarrow_i D \Rightarrow A \rightarrow_i B} \ (\wedge \Rightarrow, c \Rightarrow)} \ (Cut)
$$

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Theorem 1. If A is a theorem of $\mathbf{C} + \mathbf{J}$ then $\Rightarrow A$ is derivable in $\mathsf{G}(\mathbf{C} + \mathbf{J})$.

We say that a sequent $\Gamma \Rightarrow \Delta$ is *valid* if, for every model $M = (W, R, V)$ and every $w \in W$, whenever $w \models_M A$ for all $A \in \Gamma$, it holds that $w \models_M C$ for some $C \in \Delta$.

Theorem 2. If $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G}(\mathbf{C} + \mathbf{J})$ then $\Gamma \Rightarrow \Delta$ is valid.

By Fact 3 and Theorems 1 and 2, $\mathsf{G}(\mathbf{C} + \mathbf{J})$ is semantically complete, i.e., if A is valid then $\Rightarrow A$ is derivable in $\mathsf{G}(\mathbf{C} + \mathbf{J})$. Therefore, $\mathbf{C} + \mathbf{J}$ and $\mathsf{G}(\mathbf{C} + \mathbf{J})$ are *equipollent*.

Let us denote by $\mathsf{G}^-(\mathbf{C} + \mathbf{J})$ a sequent calculus obtained from $\mathsf{G}(\mathbf{C} + \mathbf{J})$ by removing the rule (Cut). With the help of a variant of "Mix rule" by Gentzen ("*extended cut rule*" used in [7, 11]) to take care of contraction rules, the following *cut elimination theorem* is obtained.

Theorem 3. If $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G}(\mathbf{C} + \mathbf{J})$, then $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G}^-(\mathbf{C} + \mathbf{J})$.

By Theorem 3, the subformula property is also obtained. This ensures that $\mathsf{G}(\mathbf{C} + \mathbf{J})$ is a conservative extension of both classical propositional logic and intuitionistic propositional logic.

4 Further Direction

To emphasize a merit of our proof-theoretic approach, we are currently working on decidability and Craig interpolation theorem of $\mathsf{G}(\mathbf{C} + \mathbf{J})$. The decidability of $\mathbf{C} + \mathbf{J}$ is already shown by Del Cerro and Herzig [4] with the help of the translation from $\mathbf{C} + \mathbf{J}$ into $\mathbf{S4}$. However, decidability argument (originally due to Gentzen (cf. [10])) of a cut-free sequent calculus could give a more direct argument.

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A Logic for Instrumental Desire

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1 Introduction

Inferences involving desire modalities play a significant role in philosophy, logic, linguistics and artificial intelligence etc. In this paper, we interpret desire from a perspective of instrumentalism[1]. A desire can directly causally affect the preferences that make the agent pleasure. And we argue that causality is important in evaluating desire modalities. In this way, a desire-causality model is given and a modal logic based on it is proposed.

The desire modality is usually treated as a certain propositional attitude in logic. In [13], Hintikka formulated desire by a *buletic* accessibility relation. This approach is inadequate to characterize desire, because in those accessible worlds, there may be a proposition that the agent does not want but still exists. In the BDI model, desires which are simply expressed as a set of goals represent the motivational state of the agent, like[2, 7], while its nature has not received much attention. The necessity for an independently semantics motivation to analyze desire modalities was proposed in [12]. Then some targeted models and semantic analysis were devised, e.g., [8, 12, 24]. From the perspective of decision theory, a desire behavior is usually regarded as an outcome of the entanglement of preferences and beliefs. In [14], the authors proposed a logic of conditional desires based on the ordering relations of preference and normality respectively. Likewise, [16, 17, 23] give various logical approaches to combine preference with belief or other informational attitude, and study both their static and dynamic structure. Different from the qualitative investigation in preference logic, [4, 5] discuss the desire revision based on the possibilistic framework. In our paper, we try to amplify the difference between desire and preference, namely we try to illustrate the desire that an agent has is not necessarily her preference. We also employ a total preorder formalizing the preference relation in the model, but it alone does not characterize desire. To explain our motivation, let's take an example:

> *Robin was in charge of an important project in the company recently, and the tremendous pressure made him sleepless every night. Robin is a healthy - conscious person and poor sleep will make him sluggish the next day which bothers Robin most. So having a good sleep is what he needs most at the moment. Therefore, on this sleepless night, Robin wants to take some sleep-helping pills to help him sleep well, even if these pills have some side effects*

In this example, Robin's wanting to take sleeping pills is obviously not for its own sake, but results from instrumental desire: the desire of taking pills depends on the result that the pills brings about. The question is how to formalize such a property.

A possible answer to account for Robin's desire is that the desire of taking pills can be seen as a result of comparing the actual world (in which the pills are not taken) and a hypothetical

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[1]There is a widely discussion about instrumental desire in philosophy, see [1, 6, 19]

situation in which the pills are taken: Robin desires taking sleeping pills only if the hypothetical situation with the pills taken is "more prefered" than the actual world[2]. However, how to decide which hypothetical situation should the actual world be compared to? A possible solution is that we can make use of certain similarity ordering as in [15], such that the actual world is only compared to the most similar world(s) in which the pills are taken: Robin desires taking the pills because (i) the most similar world with the pills taken is a possible world in which Robin has a good sleep after he takes the pills (ii) the possible world in which Robin has a good sleep after taking the pills is more preferred than the actual world (it is reasonable to consider a possible world in which Robin takes the pills but still has a bad sleep as a remote possible world.)

It seems that this approach successfully accounts for Robin's desire of taking sleeping pills. However it has some problems. Consider the following example:

Robin's family is not rich and he cannot afford a sport car. He is thinking about buying a birthday gift for his daughter.

Consider two possible worlds: in world W_1, he is very rich and buys a sport car as a birthday gift for her daughter; in world W_2, he is not rich but still buys a sport car as birthday gift for her daughter (maybe by borrowing money). If we consider W_1 is closer to the actual world than W_2, then according to the previous formalization of desire, Robin desires buying a sport car as birthday gift for her daughter, because obviously W_1 is more preferred than W_2 (we suppose that everyone likes to be rich, and Robin is no exception). But it is unreasonable to say that "Robin desires buying a sport car as birthday gift for her daughter" given the situation in the actual world. Therefore we have to refute that W_1 is more similar than W_2 compared with the actual world. However we do not have much reason to justify the claim on the difference in their similarity.

Causality is needed to account for the examples: In the first example, the pills are desired because the taking sleeping pills causes a good sleep; In the second example, though a possible world in which Robin buys a sport car for his daughter as gift is more likely to be possible world in which Robin is rich, Robin's being rich is not caused by buying a sport car.

Explicitly embedding causality into account of desire has some benefits. What an object brings about is crucial for evaluating whether the object is desired by an agent: a key factor of Robin's desire of sleeping pills is that taking sleeping pills brings about a good sleep. This is the core of understanding desire from an instrumental perspective. On the other hand, when we evaluate an agent's desire, it is important to assume some factors should keep constant, see[21, 9]. Causality is important to decide which factors should be constant. In the second example, an important factor in Robin's not desiring to buy a sport car is that buying a sport car cannot make Robin rich (so that the actual world is only compared with a world in which Robin is not rich but buys a sport car). This factor can only be revealed from a causal perspective.

In a word, in this paper we argue that wanting ϕ for an agent a means ϕ can bring about a certain preference of a. Moreover the desire we are talking about is pleasure-based[3], i.e., what we prefer are things that make us pleasure. So our desire is to realize our happiness. In the next section, to formalize the instrumental desire we proposed, the syntax and semantics will be given. And there are some interesting valid inferences under this framework.

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[2]In [12], Heim proposed a way of interpreting desire along this line
[3]A pleasure-based desire has a widely discussion in philosophy, see [3, 20], etc.

2 Desire-Causality Model and the Logic ID

We will introduce a 'desire-causality model' in this section. In the model, we use a total order over all the possible worlds to represent the preference. And in order to reflect the underlying philosophical idea of 'the preference that makes pleasure', we set that this order is stable and this preference is for its own sake. Then to capture the causal relationship, we will adopt a causal model. Since our approach take s causal affect into account, we make use of the interventionist approach to causality from [10, 18]. The causal affects can be characterized by a set of structural functions.

The basic facts will be represented by a set of variables. A signature $\mathcal{S} = \langle \mathcal{U}, \mathcal{V}, \mathcal{R} \rangle$ characterizes the variables that are taken into account. Following the formalization from [10, 18], $\mathcal{S}$ consists of three parts:

- $\mathcal{U} = \{U_1, \ldots, U_m\}$ is a finite set of *exogenous* variables,

- $\mathcal{V} = \{V_1, \ldots, V_n\}$ is a finite set of *endogenous* variables

- $\mathcal{R}(X)$ is the non-empty range of the variable $X \in \mathcal{U} \cup \mathcal{V}$.

After the signature gets fixed, we can define a desire-causality model to formalize instrumental desire.

Definition 1 (Desire-Causality Model). *Let $\mathcal{S} = \langle \mathcal{U}, \mathcal{V}, \mathcal{R} \rangle$ be a signature for atomic propositions. A desire-causality model is a tuple $\langle \mathcal{F}, \mathcal{A}, < \rangle$.*

- *$\mathcal{F} = \{f_{V_j} \mid V_j \in \mathcal{V}\}$ assigns, to each endogenous variable V_j, is a mapping from all assignments[4] to $\mathcal{U} \cup \mathcal{V} \setminus \{V_j\}$ to $\mathcal{R}(V_j)$. $\mathcal{F}$ is assumed to be recursive[5]*

- *$\mathcal{A}$ is the **valuation** function, assigning to every $X \in \mathcal{U} \cup \mathcal{V}$ a value $\mathcal{A}(X) \in \mathcal{R}(X)$. For each $V_j \in \mathcal{V}$, V_j complies with $\mathcal{F}_{V_j}$, that is: $\mathcal{A}(V_j) = \mathcal{F}_{V_j}(\mathcal{A}_{V_j})$ where $\mathcal{A}_{V_j}$ is the partial assignment in $\mathcal{A}$ to $\mathcal{U} \cup \mathcal{V} \setminus \{V_j\}$ (for any $X \in \mathcal{U} \cup \mathcal{V} \setminus \{V_j\}$, $\mathcal{A}(X) = \mathcal{A}_{V_j}(X)$)*

- *$<$ is a total order over all possible assignments to $\mathcal{U} \cup \mathcal{V}$.*

The set of exogenous variables $\mathcal{U}$ in $\mathcal{S}$ intends to represent $\mathcal{S}$ those whose value is causally independent from the value of every other variable. The set of endogenous variables $\mathcal{U}$ in $\mathcal{S}$ intends to represent those whose value is completely determined by the value of other variables in the system. The set of structural functions $\mathcal{F}$ represents the (deterministic) causal rules (so it is only for endogenous variables): for each endogenous variable, the structural function for this variable tells its value, once given the status of (assignment to) all other variables. $\mathcal{A}$ represents the current state with the assignment to all variables. $<$ represents agents' preference for its own sake. Following the traditional approach (e.g. [11, 22, 25]) of representing preference with an ordering, we read $\mathcal{A}_2 < \mathcal{A}_1$ as the assignment $\mathcal{A}_1$ is more preferred than $\mathcal{A}_2$ for its own sake.

For simplicity, we will write $X_1, ..., X_n$ as $\vec{X}$ and write a sequence $X_1 = x_1, ..., X_n = x_n$ as $\vec{X} = \vec{x}$. We will also write $X_1 = x_1 \wedge ... \wedge X_n = x_n$ as $\vec{X} = \vec{x}$ when the context allows,

Definition 2 (Language). *Formulas ϕ of the language $\mathcal{L}$ based on $\mathcal{S}$ are given by*

$$\phi ::= X = x \mid \neg \phi \mid \phi \wedge \phi \mid FP(\vec{X} = \vec{x}) \mid D(\vec{X} = \vec{x}) \mid (\vec{X} = \vec{x}) \,\Box\!\!\rightarrow \phi$$

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[4]An assignment a to a set of variables S is a mapping from S to $\bigcup_{X \in S} \mathcal{R}(X)$ such that for each $Z \in S$, $a(Z) \in \mathcal{R}(Z)$
[5]For the definition of recursiveness, see [10]

The formula $FP(\vec{X}=\vec{x})$ should be read as the variables in $\vec{X}$ are preferred for its own sake when they were set to the values $\vec{x}$ And $D(\vec{X}=\vec{x})$ represents the desire that wanting $\vec{X}$ to be $X_1,...,X_n$ to be x_n. Note that we do not consider the iteration of desire in this paper. $(\vec{X}=\vec{x}) \square\!\!\rightarrow \phi$ denotes the counterfactual conditional which means if $X_1=x_1,...,X_n=x_n$ were true, then ϕ would be the case.

Definition 3 (Intervention). *Let $M = \langle \mathcal{F}, \mathcal{A}, < \rangle$ be a DC-model based on $\mathcal{S}$; let $\vec{X}=\vec{x}$ be an assignment on $\mathcal{S}$. The DC-model $M_{\vec{X}=\vec{x}} = \langle \mathcal{F}_{\vec{X}=\vec{x}}, \mathcal{A}^{\mathcal{F}}_{\vec{X}=\vec{x}}, < \rangle$, resulting from an intervention setting the values of variables in $\vec{X}$ to $\vec{x}$, is such that*

- *$\mathcal{F}_{\vec{X}=\vec{x}}$ is as $\mathcal{F}$ except that, for each endogenous variable X_i in $\vec{X}$, the function f_{X_i} is replaced by a constant function f'_{X_i} that returns the value x_i regardless of the values of all other variables.*

- *$\mathcal{A}^{\mathcal{F}}_{\vec{X}=\vec{x}}$ is the unique valuation where (i) the value of each exogenous variable not in $\vec{X}$ is exactly as in $\mathcal{A}$, (ii) the value of each exogenous variable X_i in $\vec{X}$ is the provided x_i, and (iii) the value of each endogenous variable complies with its new structural function (that in $\mathcal{F}_{\vec{X}=\vec{x}}$).[6]*

In fact, an intervention that sets a variable in X to the value x is an operation that maps a given model M to a new model $M_{X=x}$, in this way, X is cut off from all its causal dependencies and fixed to the value x.

Definition 4 (Truth Conditions). *Let $\langle \mathcal{F}, \mathcal{A}, < \rangle$ be a DC-model based on $\mathcal{S}$, the truth condition of the formulas in $\mathcal{L}$ are given by (boolean cases are defined as usual):*

- *$\langle \mathcal{F}, \mathcal{A}, < \rangle \models X=x$ iff $\mathcal{A}(X) = x$*

- *$\langle \mathcal{F}, \mathcal{A}, < \rangle \models FP(\vec{X} = \vec{x})$ iff for any two assignments $\mathcal{A}_1$ and $\mathcal{A}_2$ to $\mathcal{U} \cup \mathcal{V}$ such that $\mathcal{A}_1(\vec{X}) = \vec{x}$ and $\mathcal{A}_2(\vec{X})$ is not $\vec{x}$, $\mathcal{A}_2 < \mathcal{A}_1$*

- *$\langle \mathcal{F}, \mathcal{A}, < \rangle \models (\vec{X}=\vec{x}) \square\!\!\rightarrow \phi$ iff $\langle \mathcal{F}_{\vec{X}=\vec{x}}, \mathcal{A}^{\mathcal{F}}_{\vec{X}=\vec{x}} \rangle \models \phi$*

- *$\langle \mathcal{F}, \mathcal{A}, < \rangle \models D(\vec{X}=\vec{x})$ iff $\mathcal{A}^{\mathcal{F}} < \mathcal{A}^{\mathcal{F}}_{\vec{X}=\vec{x}}$*

The second item says that the valuation that setting the variables in $\vec{X}$ to $\vec{x}$ are preferred to the one does not do that setting, which reflects the idea of 'for its own sake'. And the final item defined the desire modality intuitively: wanting $\vec{X}$ to be $X_1,...,X_n$ to be x_n means the possible assignment that sets variables in $\vec{X}$ to be $\vec{x}$ is preferred for its own sake to the real one.

Now let us revisit the example *Robin's desire of taking sleeping pills*. Let $X=1$ stands for 'taking sleeping-help pills' and $X=0$ stands for not; and $Y=1$ stands for 'having a good sleep' and $Y=0$ stands for not. 'Robin wants to take sleeping-help pills' can be formulated as $D(X=1)$. This model can be described by Figure 1.

In Figure 1, we can see that for any assignments $\mathcal{A}_i$ and $\mathcal{A}_j$ $(i, j \in \{1, 2, 3\})$ such that $\mathcal{A}_i(Y) = 1$ and $\mathcal{A}_j(Y) = 0$, we have $\mathcal{A}_j < \mathcal{A}_i$. And hence we have $PF(Y=1)$. Since $(X=1) \square\!\!\rightarrow (Y=1)$, $\langle \mathcal{F}_{X=1}, \mathcal{A}^{\mathcal{F}}_{X=1}, < \rangle \models Y = 1$, there is a transition from $\mathcal{A}$ to $\mathcal{A}_2$ and $\mathcal{A}_2$ is actually $\mathcal{A}^{\mathcal{F}}_{X=1}$. Since $\mathcal{A} < \mathcal{A}_2$, we have $\langle \mathcal{F}, \mathcal{A}, < \rangle \models D(X = 1)$.

Now we apply the same way of modelling to Robin's desire of buying a sport car. Let $S=1$ stands for buying a sport car and $S=0$ for not; And $R=1$ stands for Robin's being rich and $R=0$ for not. The corresponding desire-causality model is illustrated in Figure 2.

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[6]Note that, since $\mathcal{F}$ is recursive, the valuation $\mathcal{A}^{\mathcal{F}}_{\vec{X}=\vec{x}}$ is uniquely determined.
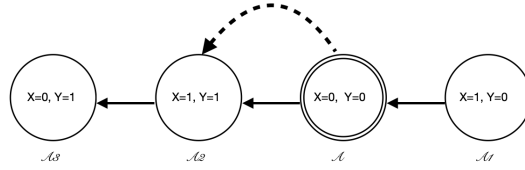
Figure 1: Robin's desire of taking sleeping pills

The double circle represents the real assignment; the solid arrow is the total order < and the dotted arrow is the counterfactual transition by setting the value of X to 1.
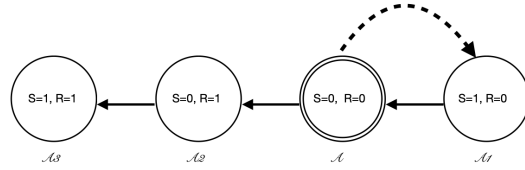

Figure 2: Robin's desire of buying a sport car

We can check that $D(S{=}1)$ does not hold at $\mathcal{A}$, which is the actual world. The result fits our intuition that Robin does not desire buying a sport car.

3 Expected Outcomes

We will axiomatize the logic system of ID and prove its completeness. Under this framework, we have some interesting inferences to capture the instrumental desire.

(1) $((X{=}x \;\square\!\!\rightarrow Y = y) \wedge PF(Y{=}y)) \rightarrow D(X{=}x)$

(2) $(\vec{X}{=}\vec{x} \wedge D(\vec{Y}{=}\vec{y})) \rightarrow \neg(\vec{Y}{=}\vec{y} \;\square\!\!\rightarrow D(\vec{Z}{=}\vec{z}))$
     if $\vec{X}{=}\vec{x}$, $\vec{Y}{=}\vec{y}$, $\vec{Z}{=}\vec{z}$ are all full assignment of $\mathcal{U} \cup \mathcal{V}$

(3) $\vec{X}{=}\vec{x} \;\square\!\!\rightarrow D(\vec{Y}{=}\vec{y}) \rightarrow D(\vec{X}{=}\vec{x} \wedge \vec{Y}{=}\vec{y}))$, if $\vec{X}$ and $\vec{Y}$ are disjoint

(4) $(\vec{X}{=}\vec{x} \;\square\!\!\rightarrow \vec{Y}{=}\vec{y}) \wedge D(\vec{X}{=}\vec{x}) \rightarrow D(\vec{X}{=}\vec{x} \wedge \vec{Y}{=}\vec{y})$, if $\vec{X}$ and $\vec{Y}$ are disjoint

Also the connection between our logic ID and the CP logic (*ceteris paribus* logic) can be observed, and it will be taken into account in the full version. One possible way to extend our framework is to change the preference structure, e.g., the way of preference lifting, which will be explored in the following work.

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