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Filled Julia sets of Chebyshev polynomials

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Abstract

We study the possible Hausdorff limits of the Julia sets and filled Julia sets of subsequences of the sequence of dual Chebyshev polynomials of a non-polar compact set $K \subset \mathbb{C}$ and compare such limits to K. Moreover, we prove that the measures of maximal entropy for the sequence of dual Chebyshev polynomials of K converges weak* to the equilibrium measure on K.

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1 Introduction

Just like orthogonal polynomials, Chebyshev polynomials are central objects in numerical analysis, and yet very little seems to be known about their dynamical properties. In this paper, we study the limit behavior of the filled Julia sets of dual Chebyshev polynomials and obtain results similar to those known for orthogonal polynomials (see [3] and [14]).

Let $K \subset \mathbb{C}$ be an infinite compact set. The Chebyshev polynomial for K of degree n is the unique monic polynomial T_n of minimal supremum-norm $||T_n||_{K,\infty}$. We denote by Ω the unbounded connected component of $\mathbb{C} \setminus K$ and define J as the outer boundary of K, that is, $J = \partial\Omega \subset \partial K \subset K$. Moreover, we denote by $\mathrm{Po}(K) := \mathbb{C} \setminus \Omega$ the filled-in K, also known as the polynomial convex hull of K, and by $\mathrm{Co}(K)$ the Euclidean convex hull of K.

We shall work with the *dual Chebyshev polynomials*, that is, the polynomials $\mathcal{T}_n$ given by

$$\mathcal{T}_n(z) = T_n(z)/||T_n||_{K,\infty}$$

and write

$$\mathcal{T}_n(z) = \gamma_n z^n + \text{lower order terms},$$

where $\gamma_n = 1/||T_n||_{K,\infty}$. Note that $\mathcal{T}_n$ is the unique degree n polynomial which is bounded by 1 on K and for which the leading coefficient is positive and maximal amongst all such polynomials. In other words, these polynomials solve the maximization problem dual to the minimization problem of T_n (see [5]).

Denote by $\mathcal{K}$ the set of non-empty compact subsets of $\mathbb{C}$. For $\{K_n\}_n \subset \mathcal{K}$ a uniformly bounded sequence of compact sets, we define compact limit sets $I := \liminf_{n\to\infty} K_n$ and $S := \limsup_{n\to\infty} K_n$, with $I \subset S$, by

$$\liminf_{n\to\infty} K_n := \{z \in \mathbb{C} \,|\, \exists \{z_n\},\, K_n \ni z_n \underset{n\to\infty}{\longrightarrow} z\}, \tag{1}$$

$$\limsup_{n\to\infty} K_n := \{z \in \mathbb{C} \,|\, \exists \{n_k\},\, n_k \nearrow \infty \text{ and } \exists \{z_{n_k}\},\, K_{n_k} \ni z_{n_k} \underset{k\to\infty}{\longrightarrow} z\}. \tag{2}$$

The set I may be empty, whereas S is always non-empty. We will say that the sequence $\{K_n\}_n$ converges to K and write $\lim_{n\to\infty} K_n = K$ if and only if

$$I = S = K.$$

This defines a metrizable topology called the Hausdorff topology on $\mathcal{K}$. For a brief introduction to the underlying complete metric on $\mathcal{K}$, see Section 2.4.

In [6], Douady proved that if $\{P_n\}_n$ is a sequence of polynomials of fixed degree $k \geq 2$ and if $P_n \to P$, then

$$J \subset \liminf_{n\to\infty} J_n \quad \text{and} \quad \limsup_{n\to\infty} K_n \subset K,$$

where J_n, J are the Julia sets and K_n, K the filled Julia sets for P_n and P. In [3], we proved potential theoretical analogs of these relations for the sequence of orthonormal polynomials defined by a Borel probability measure with non-polar compact support in $\mathbb{C}$. These results were subsequently complemented with convergence statements for the sequence of measures of maximal entropy by Petersen and Uhre [14].

In this paper, we prove versions of these theorems in the setting of Chebyshev polynomials.

Theorem 1.1 *Let $K \subset \mathbb{C}$ be a non-polar compact set and let $\{\mathcal{T}_n\}_n$ be the associated sequence of dual Chebyshev polynomials. Then the corresponding sequence of filled Julia sets $\{K_n\}_{n\geq 2}$ is pre-compact in $\mathcal{K}$ and for any limit point K_∞ of a convergent subsequence $\{K_{n_k}\}_k$, we have that*

$$K \subset \mathrm{Po}(K_\infty) \subset \mathrm{Po}(\limsup_{n\to\infty} K_n) \subset \mathrm{Co}(K).$$

We conjecture – but are not able to prove – that $K_\infty \setminus K$ is a polar subset. From this it would follow that $J = \partial\,\mathrm{Po}(K) \subset \liminf_{n\to\infty} J_n$, which is similar to the orthogonal polynomial case.

Theorem 1.1 gives upper and lower bounds on the possible limits of the filled Julia sets. The following theorem shows that in measure theoretical sense we do have convergence.

Theorem 1.2 *Let $K \subset \mathbb{C}$ be a non-polar compact set, let $\mathcal{T}_n$ be the corresponding dual Chebyshev polynomials, and let ω_n (for $n \geq 2$) denote the unique measure of maximal entropy for $\mathcal{T}_n$. Then the sequence $\{\omega_n\}_n$ converges weak* to the equilibrium measure on K:*

$$\omega_n \xrightarrow{weak^*} \omega_K.$$

As for notation, we denote by Ω_n the attracted basin of ∞ for $\mathcal{T}_n$, by $K_n = \mathbb{C} \setminus \Omega_n$ the filled Julia set, and by $J_n = \partial K_n = \partial \Omega_n$ the Julia set. Recall that for $n \geq 2$, the common equilibrium measure ω_n for K_n and J_n is also the unique measure of maximal entropy for $\mathcal{T}_n$ (see, e.g., [2]).

In figure 1, we have pictured some filled Julia sets for the Chebyshev polynomials of $K = \{e^{i\theta} : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$ to illustrate the content of our main results. See [7, 12] for further details on the structure of these polynomials.

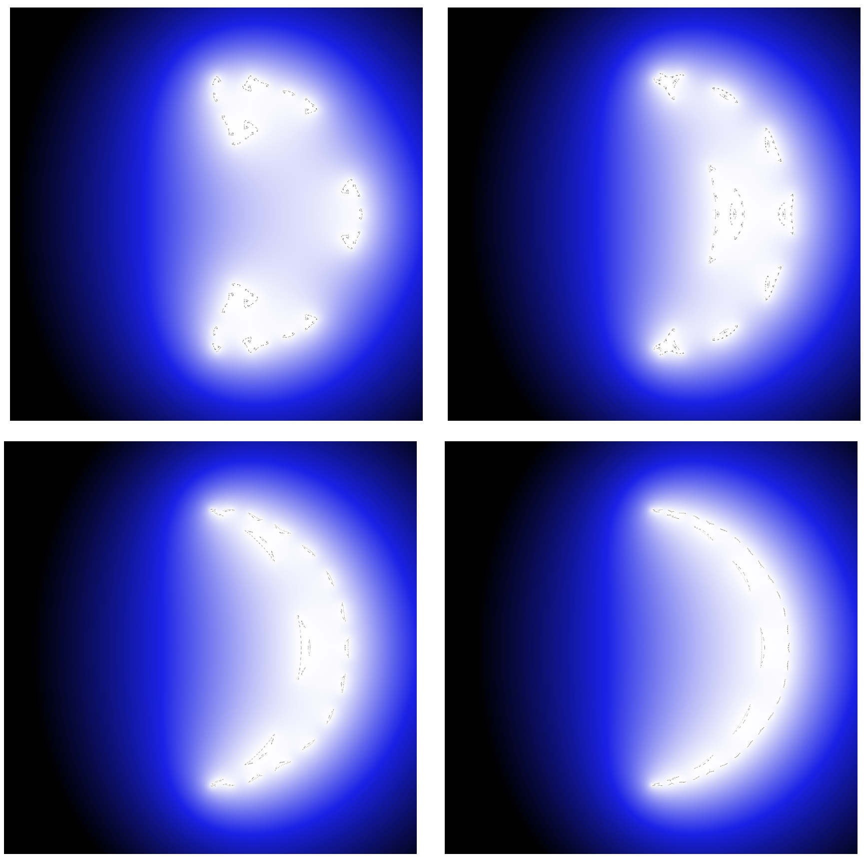

Figure 1. The graphics illustrate the filled Julia sets K_5, K_{10}, K_{20}, and K_{40} of numerically approximated Chebyshev polynomials associated to the arc $K = \{e^{i\theta} : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$. The windows are chosen so that the real and imaginary parts of z ranges between -1.5 and 1.5. The filled Julia set is drawn in black. The complement is drawn in blue, with a brightness decided by the Green's function associated to K_n, such that for values of g close to zero, the color is light, whereas the color get increasingly darker as g increases. The main theorem states that any accumulation point of a sequence $z_n \in K_n$ must be contained in the right half-disk, whereas the polynomial convex hull of any Hausdorff limit point K_∞ of the sequence $\{K_n\}_n$ contains the arc K.

2 Background

We start by introducing the main players, namely potential theory, polynomial dynamics, Chebyshev polynomials, and Hausdorff distance.

2.1 Potential theory in brief

We shall only be concerned with the case where K is non-polar. The outer boundary J of a non-polar compact set K supports a unique harmonic measure ω_K for Ω with respect to ∞ (on the Riemann sphere $\widehat{\mathbb{C}}$). This measure equals the so-called equilibrium measure of both J and K, that is, the unique Borel probability measure supported on K of maximal energy (following the sign convention of [15]). Indeed, defining the energy of a compactly supported Borel probability measure ν on $\mathbb{C}$ as the extended real number

$$I(\nu) = \iint_{\mathbb{C}\times\mathbb{C}} \log|z-w| \, \mathrm{d}\nu(z) \, \mathrm{d}\nu(w) \in [-\infty, \infty),$$

then the maximal energy, among all Borel probability measures supported on K, exists and is realized by ω_K. The maximal energy $I(K) := I(\omega_K)$ is finite by the non-polar hypothesis on K.

We denote by g_Ω the Green's function for Ω with pole at ∞, that is, the function

$$g_\Omega(z) = \int_{\mathbb{C}} \log|z-w| \, \mathrm{d}\,\omega_K(w) - I(\omega_K). \tag{3}$$

The equillibrium measure ω_K is also the distributional Laplacian Δg_Ω of the Green's function. The capacity of K is defined by $\mathrm{Cap}(K) = \exp(I(K))$ and g_Ω is the unique non-negative subharmonic function which is harmonic and positive on Ω, satisfies

$$g_\Omega(z) = \log|z| - \log \mathrm{Cap}(K) + o(1) \quad \text{at infinity}, \tag{4}$$

and which is zero precisely on $K \setminus E$, where E is the exceptional set defined by

$$E = \{z \in J \mid z \text{ is } \textit{not} \text{ a Dirichlet regular boundary point}\} \subset J.$$

The set E is an F_σ polar set, see [15, Theorems 4.2.5 and 4.4.9].

2.2 Polynomial dynamics in brief

For any polynomial P of degree $d > 1$, there exists $R > 0$ such that $|P(z)| > |z|$ for $|z| \geq R$. It follows that for $|z| > R$ the orbit of z under iteration by P converges to ∞. The basin of attraction of ∞ for P, denoted Ω_P, can therefore be written as

$$\Omega_P = \{z \in \mathbb{C} \mid P^k(z) \underset{k\to\infty}{\longrightarrow} \infty\} = \bigcup_{k\geq 0} P^{-k}(\mathbb{C} \setminus \overline{\mathbb{D}(R)}). \tag{5}$$

Here $P^k = \overbrace{P \circ P \circ \ldots \circ P}^{k \text{ times}}$, whereas P^{-k} denotes the inverse image and $\mathbb{D}(R)$ is the open ball of radius R centered at 0. It follows immediately that Ω_P is open and completely invariant, that is,

$$P^{-1}(\Omega_P) = \Omega_P = P(\Omega_P).$$

Denote by $K_P = \mathbb{C} \setminus \Omega_P \subseteq \overline{\mathbb{D}(R)}$ the filled Julia set for P and by $J_P = \partial\Omega_P = \partial K_P$ the Julia set for P. Then K_P and J_P are compact and also completely invariant. Clearly, any periodic point (i.e., a solution of the equation $P^k(z) = z$ for some $k \in \mathbb{N}$) belongs to K_P, so that K_P is non-empty. It follows from (5) that the filled Julia set K_P can also be described as the nested intersection

$$K_P = \bigcap_{k \geq 0} P^{-k}(\overline{\mathbb{D}(R)}). \tag{6}$$

To ease notation, we denote the Green's function for Ω_P with pole at infinity by g_P (in lieu of the more cumbersome g_{Ω_P}). It follows from (6) that g_P satisfies

$$g_P(z) = \lim_{k\to\infty} \frac{1}{d^k} \log^+(|P^k(z)|/R) = \lim_{k\to\infty} \frac{1}{d^k} \log^+ |P^k(z)|. \tag{7}$$

Here $\log^+$ is the positive part of log (i.e., $\log^+ x = \max\{\log x, 0\}$ for $x \geq 0$). Thus g_P vanishes precisely on K_P and hence ([15, Theorem 4.4.9]) every point in J_P is a Dirichlet regular boundary point of Ω_P. Moreover, denoting the leading coefficient of P by γ, we see that

$$g_P(P(z)) = d \cdot g_P(z) \quad \text{and} \quad \mathrm{Cap}(K_P) = \frac{1}{|\gamma|^{\frac{1}{d-1}}}. \tag{8}$$

2.3 Chebyshev polynomials in brief

Recall that K_n is the filled Julia set of the dual Chebyshev polynomial $\mathcal{T}_n$. We have that

$$\lim_{n\to\infty} \mathrm{Cap}(K_n) = \lim_{n\to\infty} \frac{1}{\gamma_n^{\frac{1}{n-1}}} = \lim_{n\to\infty} \frac{1}{\gamma_n^{\frac{1}{n}}} = \mathrm{Cap}(K),$$

where the last equality sign is a classical result of Szegő [9], see also (11) below. Equivalently,

$$\lim_{n\to\infty} I(K_n) = I(K). \tag{9}$$

The Bernstein–Walsh Lemma (see, e.g., [15, Theorem 5.5.7]) states that

$$\forall\, n \in \mathbb{N}, \forall\, z \in \mathbb{C} : \frac{1}{n} \log |\mathcal{T}_n(z)| \leq g_\Omega(z).$$

Note that since $g_\Omega \geq 0$, it follows trivially that also

$$\forall\, n \in \mathbb{N}, \forall\, z \in \mathbb{C} : \frac{1}{n} \log^+ |\mathcal{T}_n(z)| \leq g_\Omega(z). \tag{10}$$

The sequence $\mathcal{T}_n$ is n-th root regular, that is

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathcal{T}_n(z)| = g_\Omega(z) \tag{11}$$

locally uniformly on $\mathbb{C} \setminus \mathrm{Co}(K)$, see [11, Theorem 3.9 (Chapter III)] or [4, Theorem 3.2].

Finally, recall that for every n the zeros of $\mathcal{T}_n$ are located in $\mathrm{Co}(K)$ (see [8] or, e.g., [16, proof of Lemma 4]).

2.4 Hausdorff distance in brief

Denote by $\mathcal{K}$ the set of non-empty compact subsets of $\mathbb{C}$. The Hausdorff distance on $\mathcal{K}$ is the natural choice in dynamical systems (see, e.g., [6]) and we shall also use it here. Let us briefly recall the main definitions. For $L, M \in \mathcal{K}$, the Hausdorff semi-distance from L to M is given by

$$\mathrm{d_H}(L, M) := \sup\{\mathrm{d}(z, M) \,|\, z \in L\} = \sup_{z\in L} \inf_{w\in M} |z - w|$$

and the Hausdorff distance between the two sets is given by

$$\mathrm{D_H}(L, M) := \max\{\mathrm{d_H}(L, M), \mathrm{d_H}(M, L)\}.$$

The pair $(\mathcal{K}, \mathrm{d_H})$ is a complete metric space. A bounded sequence $\{K_n\}_n \subset \mathcal{K}$ of compact sets is convergent if and only if

$$\liminf_{n\to\infty} K_n = \limsup_{n\to\infty} K_n,$$

where lim inf and lim sup are defined as in (1) and (2). Moreover, any bounded sequence $\{K_n\}_n \subset \mathcal{K}$ is sequentially pre-compact, that is, any subsequence has a convergent sub-subsequence. For compact subsets of the Riemann sphere, $\widehat{\mathbb{C}}$, we use instead the spherical metric. For compact subsets of $\mathbb{C}$ the spherical and the Euclidean metrics induce the same topology on $\mathcal{K}$.

3 Filled Julia sets and Green's functions for guided sequences of polynomials

We start by introducing the notion of a *guided* sequence of polynomials.

Definition 3.1 *Let K be a non-polar compact set. We define a polynomial sequence of the form $\{P_n = \delta_n z^n + \text{lower order terms}\}_n$ to be K-guided if*

1. *the set Z consisting of all zeros of all P_n is bounded, and*
2. *there exist constants $a > 0$ and $b \in \mathbb{R}$ such that*

$$\liminf_{n\to\infty} \frac{1}{n} \log^+ |P_n(z)| \geq a \cdot g_\Omega(z) - b \tag{12}$$

locally uniformly on $\mathbb{C} \setminus \mathrm{Co}(K)$, *where* g_Ω *is the Green's function for the unbounded component of the complement of* K.

Example 3.2 *For a non-polar compact set* K, *the sequence of dual Chebyshev polynomials* $\{\mathcal{T}_n\}_n$ *is a* K*-guided sequence with* $a = 1$ *and* $b = 0$. *This follows from n-th root regularity* (11) *and the fact that* $Z \subset \mathrm{Co}(K)$.

We now proceed with a general result which implies that there is a uniform upper bound on the size of the filled Julia sets K_n.

Proposition 3.3 *Let* K *be a non-polar compact set and let* $\{P_n\}_n$ *be a* K*-guided sequence of polynomials. Then there exists* $R > 0$ *and* $N \in \mathbb{N}$ *such that*

$$\forall\, n \geq N: \quad K_n \subset P_n^{-1}(\overline{\mathbb{D}(R)}) \subset \mathbb{D}(R). \tag{13}$$

This statement is a vast generalisation of [3, Lemma 2.1]. Although the proof is similar, we provide it here for completeness.

Proof : Fix $R > 0$ such that $Z \subset \mathbb{D}(R)$ and $2\epsilon = \min\{a g_\Omega(z) - b \mid |z| = R\} > 0$. Then $C(0, R)$ is a compact subset of $\mathbb{C} \setminus K$ and, in fact, $Z \cup K \subset \mathbb{D}(R)$. Hence, by the hypothesis (12), we can choose N_1 such that

$$\forall\, n \geq N_1,\, \forall\, z \in C(0, R) : \frac{1}{n} \log |P_n(z)| \geq \epsilon. \tag{14}$$

Since all zeros of each P_n is contained in $Z \subset \mathbb{D}(R)$, it follows from the minimum principle for harmonic functions that (14) holds for $|z| \geq R$. Choose any $N \geq N_1$ such that $R < \mathrm{e}^{N\epsilon}$. Then

$$\forall\, n \geq N,\, \forall\, z,\, |z| \geq R : |P_n(z)| \geq \mathrm{e}^{N\epsilon} > R.$$

Hence $P_n^{-1}(\overline{\mathbb{D}(R)}) \subset \mathbb{D}(R)$ for all $n \geq N$. The inclusion $K_n \subset P_n^{-1}(\overline{\mathbb{D}(R)})$ follows immediately from (6). The proof is complete. □

Corollary 3.4 *Let* $K \subset \mathbb{C}$ *be a non-polar compact set and* $\{\mathcal{T}_n\}_n$ *the sequence of dual Chebyshev polynomials for* K. *Then the corresponding sequence of filled Julia sets* $\{K_n\}_{n\geq 2}$ *is bounded and hence sequentially pre-compact in* $\mathcal{K}$. *Moreover,*

$$\limsup_{n\to\infty} K_n \subset \mathrm{Co}(K).$$

Proof : According to Example 3.2 and Proposition 3.3, there exists $R > 0$ and N such that (13) is satisfied. Thus the sequence $\{K_n\}_n$ is bounded in $\mathbb{C}$ and hence pre-compact in $\mathcal{K}$.

Let Δ be any bounded topological disk with $\overline{\Delta} \subset \widehat{\mathbb{C}} \setminus \mathrm{Co}(K)$. Then the boundary $\partial\Delta$ is a compact subset of $\mathbb{C} \setminus \mathrm{Co}(K)$. Hence $2\epsilon := \min\{g_\Omega(z) \mid z \in \partial\Delta\} > 0$. Arguing as in the proof of Proposition 3.3, we now find that

$$\forall\, n \geq N, \forall\, z \in \partial\Delta : \frac{1}{n} \log|\mathcal{T}_n(z)| \geq \epsilon.$$

Consequently,

$$\forall\, n \geq N, \forall\, z \in \partial\Delta : |\mathcal{T}_n(z)| \geq \mathrm{e}^{N\epsilon} > R. \tag{15}$$

Since $Z \subset \mathrm{Co}(K)$, we have $Z \cap \overline{\Delta} = \emptyset$ and therefore (15) holds on $\overline{\Delta}$ by the minimum principle for non-vanishing holomorphic functions. Moreover, $\overline{\Delta} \cap K_n = \emptyset$ for $n \geq N$ by (13). Since Δ was arbitrary, we have $\limsup_{n\to\infty} K_n \subset \mathrm{Co}(K)$. □

Proposition 3.5 *Let $0 < C \leq 1 \leq R$. Then for any polynomial P of degree $n \geq 2$ with filled Julia set K satisfying* $\mathrm{Cap}(K) \geq C$ *and*

$$K \subset P^{-1}(\overline{\mathbb{D}(R)}) \subset \mathbb{D}(R),$$

the Green's function $g = g_P$ satisfies

$$\left\| g(z) - \frac{1}{n} \log^+ |P(z)| \right\|_\infty \leq \frac{M}{n},$$

where $M = \log \frac{4R}{C}$.

This generalizes [3, Prop. 2.3] in that there is no reference to any particular sequence of polynomials nor extremality property.

Proof : Evidently, $\mathrm{Cap}(K) < R$ by the general properties of capacity so that

$$|\log \mathrm{Cap}(K)| \leq \log \frac{R}{C}.$$

By (3) and (4), the Green's functions g can be written as

$$g(z) = \log|z| - \log \mathrm{Cap}(K) + \int_K \log|1 - w/z|\, d\omega(w),$$

where ω is the equilibrium measure on K.

For $|z| \geq 2R$ and $w \in K$, we have $|w/z| < 1/2$ so that $\forall\, z,\ |z| \geq 2R$:

$$|g(z) - \log|z|| \leq |\log \mathrm{Cap}(K)| + \log 2 \leq \log \frac{R}{C} + \log 2 < M. \tag{16}$$

We divide the complex plane into the set $A = \{z \,|\, |P(z)| \leq 2R\}$ and its complement and estimate $|g(z) - \frac{1}{n} \log^+ |P(z)||$ separately on each set.

For all $z \in \mathbb{C} \setminus A$, we have $|P(z)| > 2R \geq 2$ so that $\log^+ |P(z)| = \log|P(z)|$ and

$$\left| g(z) - \frac{1}{n} \log^+ |P(z)| \right| = \left| \frac{1}{n} g(P(z)) - \frac{1}{n} \log|P(z)| \right| < \frac{M}{n}.$$

Here we have used (8) and (16) applied to $P(z)$. For any $z \in \partial A$, we have $|P(z)| = 2R$ so that

$$\begin{aligned}
0 \le g(z) &= \frac{1}{n} g(P(z)) \\
&= \frac{1}{n}\left(\log(2R) - \log \operatorname{Cap}(K) + \int_K \log|1 - w/P(z)|\, d\omega(w)\right) \\
&< \frac{1}{n}\left(\log \frac{2R}{C} + \log 2\right) = \frac{M}{n}.
\end{aligned}$$

Hence, by the maximum principle for subharmonic functions, $g(z) < M/n$ on A. Similarly,

$$0 \le \frac{1}{n}\log^+ |P(z)| \le \frac{\log 2R}{n} < \frac{M}{n}$$

on A by construction. Finally,

$$\left| g(z) - \frac{1}{n}\log^+ |P(z)| \right| \le \max\{g(z), \frac{1}{n}\log^+|P(z)|\} \le \frac{M}{n}$$

on A. This completes the proof. □

The above proposition gives us uniform bounds on the Green's functions g_n for the dual Chebyshev polynomials $\mathcal{T}_n$ (cf. (7)).

Corollary 3.6 *For a non-polar compact set K and the corresponding dual Chebyshev polynomials $\mathcal{T}_n$ with Green's functions g_n, there exists $N \in \mathbb{N}$ and $M > 0$ such that*

$$\forall\, n \ge N: \quad \left\| g_n(z) - \frac{1}{n}\log^+ |\mathcal{T}_n(z)| \right\|_\infty \le \frac{M}{n}. \tag{17}$$

Proof : By Example 3.2, there exists $R > 1$ and $N \in \mathbb{N}$ such that the filled Julia sets K_n for $\mathcal{T}_n$ satisfy (13). Moreover, $\operatorname{Cap}(K_n) \to \operatorname{Cap}(K) > 0$ as $n \to \infty$ so there exists $C \in (0,1)$ such that $C < \operatorname{Cap}(K_n)$ for all n. Thus (17) follows by applying Proposition 3.5 to each $\mathcal{T}_n$. □

Corollary 3.7 *For a non-polar compact set K and the corresponding dual Chebyshev polynomials $\mathcal{T}_n$ with Green's functions g_n, there exists $N \in \mathbb{N}$ and $M > 0$ such that*

$$\forall\, n \ge N,\ \forall\, z \in \mathbb{C}: \quad g_n(z) \le g_\Omega(z) + M/n. \tag{18}$$

In particular,

$$\limsup_{n\to\infty} g_n(z) \le g_\Omega(z)$$

uniformly on $\mathbb{C}$. Moreover,

$$\lim_{n\to\infty} g_n(z) = g_\Omega(z)$$

locally uniformly on $\mathbb{C} \setminus \mathrm{Co}(K)$ and for every $n \geq N$, we have

$$K \subseteq \{z \mid g_n(z) \leq M/n\}. \tag{19}$$

Proof : As for (18), combine the Bernstein–Walsh Lemma in the form (10) with the above Corollary. Then the lim sup statement immediately follows. Similarly, the lim inf statement follows from the n-th root regularity of the Chebyshev polynomials and the above Corollary. As for (19), combine the Corollary with $||\mathcal{T}_n||_{K,\infty} = 1$ so that $\frac{1}{n}\log^+ |\mathcal{T}_n(z)| \equiv 0$ on K. □

4 Proofs of the main theorems

We are now ready to present the proofs of our main results.

4.1 Proof of Theorem 1.1.

The notion of Caratheodory convergence is the last ingredient we need to prove Theorem 1.1. Recall that a pointed domain is a pair (U, z), where $z \in U \subset \widehat{\mathbb{C}}$ and U is a domain (i.e., connected and open).

Definition 4.1 *Let $(U_n, z_n)_n$ be a sequence of pointed domains. We say that the sequence converges to the pointed domain (U_∞, z_∞) if*

1. *$z_n \to z_\infty$,*
2. *any compact subset $K \subset U_\infty$ is a subset of U_n for all but finitely many n, and*
3. *for any open connected set V with $z_\infty \in V$, if $V \subset U_n$ for infinitely many n, then $V \subset U_\infty$.*

Remark 4.2 *There is an alternative characterization of Caratheodory convergence. The pointed domains $(U_n, z_n)_n$ converge to (U_∞, z_∞) if and only if the following two properties hold:*

- *i) $z_n \to z_\infty$, and*
- *ii) for any compact subset $K \subset \widehat{\mathbb{C}} \setminus \{z_\infty\}$ and for any subsequence $(n_k)_k$, if $\widehat{\mathbb{C}} \setminus U_{n_k} \longrightarrow K \subset \widehat{\mathbb{C}} \setminus \{z_\infty\}$ in the Hausdorff distance on compact subsets of $\widehat{\mathbb{C}}$, then the domain U_∞ is equal to the component of $\widehat{\mathbb{C}} \setminus K$ containing z_∞.*

Using the formulation of Caratheodory convergence as in the remark, one can show that even if $z_n \to z_\infty$, a pointed sequence $(U_n, z_n)_n$ may fail to converge either because (for some subsequence) $\mathrm{d}(z_n, \partial U_n) \to 0$ or because there are at least two subsequences U_{n_k} and U_{n_m} such that $\widehat{\mathbb{C}} \setminus U_{n_k}$ and $\widehat{\mathbb{C}} \setminus U_{n_m}$ converge to K_1 and K_2, respectively, but the connected components of $\widehat{\mathbb{C}} \setminus K_1$ and $\widehat{\mathbb{C}} \setminus K_2$ containing z_∞ are different. However, if $\mathrm{d}(z_n, \partial U_n)$ is bounded uniformly from below by some $r > 0$, then the sequence $(U_n, z_n)_n$ is sequentially pre-compact. In particular, any sequence of pointed domains (Ω_n, ∞) in the Riemanns sphere with $K_n \subset \mathbb{D}(R)$ for some fixed $R > 0$ is sequentially compact.

Given a sequence $(g_n)_n$ of Green's functions with pole at ∞ for domains $\Omega_n \subset \widehat{\mathbb{C}}$ with compact complements $K_n = \mathbb{C} \setminus \Omega_n$, there is no general relation between limits of subsequences g_{n_k} and the question of Caratheodory convergence of the corresponding subsequence of pointed domains (Ω_{n_k}, ∞). However, the following Proposition gives an upper bound on such Caratheodory limits.

Proposition 4.3 *Let $(g_n)_n$ be a sequence of Green's functions with pole at ∞ for domains $\Omega_n \subset \widehat{\mathbb{C}}$, and let $K' = \{z \mid g_n(z) \to 0\}$. Then*

$$K' \cap \Omega_\infty = \emptyset$$

for any domain Ω_∞ such that a subsequence $(\Omega_{n_k}, \infty)_k$ converges to (Ω_∞, ∞) in the sense of Caratheodory.

Proof : It suffices to consider the case where $(\Omega_n, \infty) \longrightarrow (\Omega_\infty, \infty)$ in the sense of Caratheodory. Let $z_0 \in \Omega_\infty$ be arbitrary. Choose a domain U, $\{z_0, \infty\} \subset U \subset \overline{U} \subset \Omega_\infty$, having a Green's function g_U with pole at ∞. We may take U to be a Jordan disk. In view of definition 4.1 part 2, there exists n_0 such that $\overline{U} \subset \Omega_n$ for all $n \geq n_0$. By [15, Cor. 4.4.5], $0 < g_U(z) \leq g_{\Omega_n}(z)$ for any $z \in U$ when $n \geq n_0$. In particular, $0 < g_U(z_0) \leq g_n(z_0)$ for all $n \geq n_0$. Hence $z \notin K'$. □

The upper bound in Theorem 1.1 follows from Corollary 3.4. So it only remains to prove that

$$\Omega \supset \Omega_\infty, \quad \text{or equivalently,} \quad K \cap \Omega_\infty = \emptyset$$

for any Caratheodory limit point (Ω_∞, ∞) of a convergent subsequence $(\Omega_{n_k}, \infty)_k$. However, according to Corollary 3.7 the Green's functions g_n for Ω_n satisfies that $g_n(z) \to 0$ on K. So the statement follows from Proposition 4.3.

4.2 Proof of Theorem 1.2.

We need a few more auxillary results in order to prove this result. First of all, we shall use the following Lemma which is a refinement of [13, Lemma 1.3.2]. For a proof, the reader is referred to [14, Lemma 3.3].

Lemma 4.4 *Let $V, K \subset \mathbb{C}$ be compact sets with V contained in the unbounded component of $\mathbb{C} \setminus K$, and let $b \in (0, 1)$ be arbitrary. Then there exists $M = M(b, V, K) \in \mathbb{N}$ such that for M arbitrary points $x_1, x_2, \ldots, x_M \in V$, there exists M points $y_1, y_2, \ldots y_M \in \mathbb{C}$ for which the rational function*

$$r(z) = \prod_{j=1}^{M} \frac{z - y_j}{z - x_j}$$

has supremum norm on K bounded by b, that is, $||r||_K \leq b$.

We shall also make use of the following result.

Proposition 4.5 *Let $K \subset \mathbb{C}$ be a non-polar compact set with corresponding dual Chebyshev polynomials $\mathcal{T}_n$. For any $R > 0$ and any compact set $V \subset \mathbb{C}$ with $V \cap \mathrm{Po}(K) = \emptyset$, there exists $M = M(K, R, V) \in \mathbb{N}$ such that for any w, $|w| \leq R$, and for any n, the number of pre-images of w in V under the polynomial $\mathcal{T}_n$ is less than M. In symbols,*

$$\#[\mathcal{T}_n^{-1}(w) \cap V] < M.$$

Proof : Fix $R > 0$ and any compact set $V \subset \mathbb{C}$ with $V \cap \mathrm{Po}(K) = \emptyset$. Let $b = 1/(1+R)$ and let $M = M(b, V, \mathrm{Po}(K))$ be as in Lemma 4.4. For each n, let $T_n(z)$ be the usual monic Chebyshev polynomial of degree n for K. Then T_n is the unique degree n monic polynomial of minimal sup-norm $1/\gamma_n$ on K.

Suppose towards a contradiction that for some w, $|w| \leq R$, and some n, the equation $\mathcal{T}_n(z) = w$ has at least M solutions $x_1, \ldots x_M \in V$. Let r be the rational function given by Lemma 4.4 such that $||r||_K \leq b$ and set $q(z) := r(z) \cdot (\mathcal{T}_n(z) - w)/\gamma_n$. Then q is a monic polynomial of degree n and

$$\begin{aligned} ||q||_K &\leq \frac{||r||_K}{\gamma_n} ||\mathcal{T}_n(z) - w||_K \leq \frac{b}{\gamma_n}\sqrt{1+|w|} \\ &\leq \frac{\sqrt{1+R}}{1+R} \cdot ||T_n||_K < ||T_n||_K. \end{aligned}$$

This contradicts the fact that T_n has minimal sup-norm on K among all monic degree n polynomials. $\square$

Corollary 4.6 *Let $K \subset \mathbb{C}$ be a non-polar compact set and suppose $V \subset \mathbb{C}$ with $V \cap \mathrm{Po}(K) = \emptyset$. Then there exists $M = M(K; V) \in \mathbb{N}$ such that for any of the dual Chebyshev polynomials $\mathcal{T}_n$ with $n \geq 2$ and any $z \in K_n$, the number of pre-images of z in V under $\mathcal{T}_n$ is less than M.*

Proof : It follows from Proposition 3.3 that there exists $R > 0$ such that

$$K_n \subset \mathcal{T}_n^{-1}(\overline{\mathbb{D}(R)}) \subset \mathbb{D}(R)$$

for all $n \geq 2$, where K_n is the filled Julia set of $\mathcal{T}_n$. The Corollary follows immediately from this and Proposition 4.5. $\square$

Proof : (of Theorem 1.2) According to Corollary 3.4, the sequence of equillibrium measures ω_n for K_n have uniformly bounded support and so the sequence of such measures is pre-compact for the weak-* topology. Furthermore, according to Brolin [2] (see also Lyubich [10]), the equilibriums measure ω_n is also the unique invariant balanced measure for $\mathcal{T}_n$, that is, it is the unique probability measure ω on $\mathbb{C}$ such that for any measurable function $f : \mathbb{C} \longrightarrow \mathbb{C}$,

$$\int_{\mathbb{C}} f(z) d\omega(z) = \frac{1}{n} \int_{\mathbb{C}} \left(\sum_{w, \mathcal{T}_n(w) = z} f(w) \right) d\omega(z).$$

Let $V \subset \mathbb{C}$ be a compact subset with $V \cap \mathrm{Po}(K) = \emptyset$ and let $M \in \mathbb{N}$ be as in Corollary 4.6. Then for the measureable function 1_V (i.e., the indicator function for V), we have

$$\omega_n(V) = \int_{\mathbb{C}} 1_V(z) d\omega_n(z) = \frac{1}{n} \int_{\mathbb{C}} \left(\sum_{w, \mathcal{T}_n(w)=z} 1_V(w) \right) d\omega_n(z) \leq \frac{M}{n} \underset{n\to\infty}{\longrightarrow} 0.$$

This proves that for any weak limit ν of a convergent subsequence $\{\omega_{n_k}\}_k$, the support $S(\nu)$ is contained in $\mathrm{Po}(K)$. Furthermore, by (9) and [15, Lemma 3.3.3]), we have

$$I(\nu) \geq \limsup_{k\to\infty} I(\omega_{n_k}) = \lim_{n\to\infty} I(\omega_n) = I(\omega_K) = I(\mathrm{Po}(K)).$$

Hence $\nu = \omega_K$ since ω_K is the unique measure of maximal energy $I(K)$. As the limit is unique, we in fact have

$$\omega_n \xrightarrow{\text{weak}^*} \omega_K$$

and this proves Theorem 1.2. □

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