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Weak limits of the measures of maximal entropy for Orthogonal polynomials.

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Abstract

In this paper we study the sequence of orthonormal polynomials $\{P_n(\mu; z)\}$ defined by a Borel probability measure μ with non-polar compact support $S(\mu) \subset \mathbb{C}$. For each $n \geq 2$ let ω_n denote the unique measure of maximal entropy for $P_n(\mu; z)$. We prove that the sequence $\{\omega_n\}_n$ is pre-compact for the weak-* topology and that for any weak-* limit ν of a convergent sub-sequence $\{\omega_{n_k}\}$, the support $S(\nu)$ is contained in the filled-in or polynomial-convex hull of the support $S(\mu)$ for μ. And for n-th root regular measures μ the full sequence $\{\omega_n\}_n$ converge weak-* to the equilibrium measure ω on $S(\mu)$.

1 Introduction and general results

In the classical study [5] by Stahl and Totik of general orthogonal polynomials they relate the potential and measure theoretic properties of the asymptotic zero distribution for the sequence of orthonormal polynomials defined by a Borel probability measure μ on $\mathbb{C}$ of infinite, but compact support $S(\mu)$, to the potential and measure theoretic properties of μ and its support. In the paper [4] Christiansen, Henriksen, Pedersen and one of the authors of this paper initiated a study of the relation between the potential theoretic properties of μ and the asymptotic (as n tend to ∞) potential and measure theoretic properties of the Julia sets and filled Julia sets of the orthonormal polynomials P_n. In this paper we extend this with a study of the weak convergence properties of the measures of maximal entropy for the orthonormal polynomials.

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For μ a Borel probability measure on $\mathbb{C}$ with infinite and compact support $S(\mu)$ we denote by $\{P_n(z)\} := \{P_n(\mu; z)\}$ the unique sequence

$$P_n(z) = \gamma_n z^n + \text{ lower order terms.} \tag{1}$$

of orthonormal polynomials wrt. μ. Then P_n is also characterized as the unique normalized polynomial of the form (1) which is orthogonal to all lower degree polynomials or equivalently for which the monic polynomial $p_n(z) = P_n(z)/\gamma_n$ is the unique monic degree n polynomial of minimal norm in $L^2(\mu)$.

For $S \subset \mathbb{C}$ a compact non-polar subset such as $S(\mu)$ above we denote by $\Omega = \Omega(S)$ the unbounded connected component of $\mathbb{C}\setminus S$, by $K = K(S) := \mathbb{C}\setminus\Omega$ the filled-in or just filled S (denoted by the polynomial convex hull in e.g. [5]), by $J = J(S) = \partial K = \partial\Omega \subset S$ the outer boundary of S, by g_Ω the Green's function with pole at infinity for Ω and finally by ω_S the equilibrium measure for S (with $\mathrm{Supp}(\omega_S) = J$). For a measure $\mu \in \mathcal{B}$ we shall write $K(\mu)$ for $K(S(\mu))$, $J(\mu)$ for $J(S(\mu))$ and $\Omega(\mu)$ for $\Omega(S(\mu))$ or simply K, J and Ω, when the measure μ is understood from the context.

Definition 1.1. *We denote by $\mathcal{B}$ the set of Borel probability measure on $\mathbb{C}$ with compact non-polar support.*

We denote by $\mathcal{R}eg$ the set of n-th root regular measures in $\mathcal{B}$, that is

$$\mathcal{R}eg := \left\{\mu \in \mathcal{B} \mid \lim_{n\to\infty} (\gamma_n)^{1/n} = \frac{1}{\mathrm{Cap}(S(\mu))}\right\}.$$

For $\mu \in \mathcal{B}$ and P_n the associated sequence of orthonormal polynomials, we denote by Ω_n the attracted basin of ∞ for P_n, by $K_n = \mathbb{C}\setminus\Omega_n$ and $J_n = \partial K_n = \partial\Omega_n$ respectively the filled Julia set and the Julia set of P_n, see Section 2. for definitions of Julia set and filled Julia set of a polynomial, by g_n the Green's function with pole at ∞ for Ω_n and by ω_n the equilibrium measure for J_n or equivalently the measure of maximal entropy for P_n (see [1]) and see also below for definitions of these terms).

Our main result is

Theorem A. *Let $\mu \in \mathcal{B}$, then the sequence $\{\omega_n\}_{n\geq 2}$ is pre-compact for the weak-* topology and for any limit measure ν of a weakly convergent sub-sequence $\{\omega_{n_k}\}_k$*

$$S(\nu) \subseteq K(\mu).$$

Moreover if $\mu \in \mathcal{R}eg$ then

$$\omega_n \overset{weak^*}{\longrightarrow} \omega_{S(\mu)}.$$

The first part of the theorem is an analogue of the last statement in the following Theorem from [5], recast in the above notation.

Theorem 1.2 ([5, Thm. 2.1.1, the first part of which was first proven by Fejér [2]]). *Suppose $\mu \in \mathcal{B}$. All zeros of the orthonormal polynomials $P_n(z)$, $n \in \mathbb{N}$, are contained in the convex hull $\mathrm{Co}(K)$, and for any compact subset $V \subseteq \Omega$ the* *number of zeros of $P_n(z)$, $n \in \mathbb{N}$, on V is bounded as $n \to \infty$. Consequently any weak* limit point of (the normalized counting meauseres for) the zeros of P_n is supported on $K(\mu)$.*

The second part of the theorem should be compared to [5, Thm. 3.1.4] and [5, Thm. 3.6.1]. Such a comparison reveals that the equilibrium measures ω_n on the Julia sets of $P_n$have much stronger convergence properties than the counting measure on the roots of P_n, at least in the case of regular measures μ.

Also the main Theorem is a measure theoretic version of the following theorem from [4], which says that under a mild extra condtion (see $\mathcal{B}_+$ below) in capacity any limit of a convergent subsequence K_{n_k} (convergent in the Hausdorff topology on the space of compact subsets of $\mathbb{C}$) is contained in $K(\mu)$.

Theorem 1.3 ([4, Thm.1.3]). *For $\mu \in \mathcal{B}_+ := \{\mu \in \mathcal{B} \,|\, \limsup_{n\to\infty} |\gamma|_n^{1/n} < \infty\}$ we have*

$$\limsup_{n\to\infty} K_n \subseteq \mathrm{Co}(K).$$

Moreover, for any $\epsilon > 0$ and $V_\epsilon := \{z \in \mathbb{C} \,|\, g_\Omega(z) \geq \epsilon\}$,

$$\lim_{n\to\infty} \mathrm{Cap}(V_\epsilon \cap K_n) = 0.$$

Our proof follows the approach of Stahl and Totik for Theorem 1.2.

2 Background.

2.1 Potential theory

We use the notation of Randsford [6]. For μ a Borel probability measure on $\mathbb{C}$ with compact support we define its potential as the sub-harmonic function

$$p_\mu(z) := \int_{\mathbb{C}} \log|z-w| d\mu(w) = \log|z| + o(1),$$

which is harmonic on the complement of $S(\mu)$. And we define its energy as

$$I(\mu) := \int_{\mathbb{C}\times\mathbb{C}} \log|z-w| d\mu(w) d\mu(w) = \int_{\mathbb{C}} p_\mu(z) d\mu(z),$$

it satisfies $-\infty \leq I(\mu) \leq \log \mathrm{diam}(S(\mu))$. For a Borel set $B \subset \mathbb{C}$ we denote by $\mathcal{P}(B)$ the set of probability measures μ with compact support $\mathrm{Supp}(\mu) \subset B$, the energy of B as

$$I(B) := \sup\{I(\mu) | \mu \in \mathcal{P}(B)\}$$

and its (logarithmic) capacity as

$$\mathrm{Cap}(B) := \mathrm{e}^{I(B)}.$$

The set B is called polar if $\mathrm{Cap}(B) = 0$ or equivalently $I(B) = -\infty$ and the set B is non-polar otherwise. For $K \subset \mathbb{C}$ a non-polar compact subset there is a unique measure denoted by ω_K, which realises the supremum above. It is called the equilibrium measure for K and its support is contained in the outer boundary of K. And according to Frostman's Theorem, [6, Thm. 3.3.4] its potential p_μ is bounded from below by $I(K) = I(\omega_K)$ and equals $I(K)$ on K except for an F_σ polar subset E of K. The Green's function for $\Omega := \mathbb{C}\backslash K$ is the non-negative sub-harmonic function

$$g_\Omega(z) = p_{\omega_K}(z) - \log(\mathrm{Cap}(K)) = p_{\omega_K}(z) - I(K).$$

The set $K\backslash E$ is precisely the set of Dirichlet regular boundary points of K [6, 4.4.9].

2.2 Polynomial dynamics

For $P(z) = \gamma z^d + \ldots$ a polynomial of degree $d > 1$, an easy computation shows there exists $R = R_P > 0$ such that for any z with $|z| > R : |P(z)| \geq 2|z|$. Thus the orbit of such z under iteration converge to ∞. We denote by Ω_P the basin of attraction for ∞ for P, that is,

$$\Omega_P := \{z \in \mathbb{C} \,|\, P^k(z) \underset{k\to\infty}{\longrightarrow} \infty\} = \bigcup_{k\geq 0} P^{-k}(\mathbb{C}\setminus\overline{D(R)}), \tag{2}$$

where $P^k = \overbrace{P \circ P \circ \ldots \circ P}^{k \text{ times}}$. It follows immediately that Ω_P is open and completely invariant, i.e. $P^{-1}(\Omega_P) = \Omega_P = P(\Omega_P)$. Denote by $K_P := \mathbb{C}\setminus\Omega_P \subseteq \overline{D(R)}$ the filled Julia set for P and by $J_P := \partial\Omega_P = \partial K_P$ the Julia set for P. Then K_P and J_P are compact and also completely invariant. Clearly any periodic point, i.e. solution of an equation $P^k(z) = z$, $k \in \mathbb{N}$ belongs to K_P, so that K_P is non-empty. It follows from (2) that the filled Julia set K_P can also be described as the nested intersection

$$K_P = \bigcap_{k\geq 0} P^{-k}(\overline{\mathbb{D}(R)}). \tag{3}$$

We denote by $g_P\colon \mathbb{C} \longrightarrow [0,\infty)$ the Green's function for Ω_P with pole at infinity. It follows from (3) that the Green's function g_P satisfies

$$g_P(z) = \lim_{k\to\infty} \frac{1}{d^k} \log^+(|P^k(z)|/R) = \lim_{k\to\infty} \frac{1}{d^k} \log^+ |P^k(z)|.$$

Thus g_P vanishes precisely on K_P, i.e. the exceptional set E for g_P is empty and hence every point of J_P is a Dirichlet regular boundary point of Ω_P. Moreover

$$g_P(P(z)) = d \cdot g_P(z) \quad \text{and} \quad \mathrm{Cap}(K_P) = \frac{1}{|\gamma|^{\frac{1}{d-1}}}.$$

According to Brolin, [1], the equilibrium measure ω_P for the (filled) Julia set J_P equals the unique measure of maximal entropy for P.

3 Proof of Theorem A.

Recall that for $\mu \in \mathcal{B}$ and P_n the associated orthonormal polynomials, we denote by Ω_n, J_n, and K_n, respectively, the basin of attraction for ∞, the Julia set, and the filled Julia set of P_n. And moreover by g_n we denote the Green's function for Ω_n and by ω_n the equilibrium measure for K_n with support J_n.

The following Lemma gives an elementary relation between the filled support $K(\mu)$ and the filled Julia sets K_n

Lemma 3.1 ([4, Lemma 2.2]). *Let $\mu \in \mathcal{B}$ and choose $R > 0$ such that $K(\mu) \subset \mathbb{D}(0, R)$. Then there exists N such that for all $n \geq N$:*

$$K_n \subset P_n^{-1}(\overline{\mathbb{D}(0,R)}) \subset \mathbb{D}(0,R).$$

Lemma 3.2. *For a probability measure $\mu \in \mathcal{B}$ the following are equivalent*

1. $$\lim_{n\to\infty} (\gamma_n)^{1/n} = \frac{1}{\mathrm{Cap}(S(\mu))}.$$

2. $$\mathrm{Cap}(K_n) \underset{n\to\infty}{\longrightarrow} \mathrm{Cap}(S(\mu)).$$

3. $$I(\omega_n) \underset{n\to\infty}{\longrightarrow} I(\omega_{S(\mu)})$$

Proof. Since $\mathrm{Cap}(K_n) = \frac{1}{|\gamma_n|^{1/(n-1)}} = \mathrm{e}^{I(\omega_n)}$ the lemma follows from the observation that

$$\lim_{n\to\infty} (\gamma_n)^{1/n} = \frac{1}{\mathrm{Cap}(S(\mu))} \iff \lim_{n\to\infty} (\gamma_n)^{1/(n-1)} = \frac{1}{\mathrm{Cap}(S(\mu))}.$$

□

We shall use the following Lemma which is a refinement of [5, Lemma 1.3.2]

Lemma 3.3. *Let $V, S \subset \mathbb{C}$ be two compact subsets with $V \cap K(S) = \emptyset$ and let $b, 0 < b < 1$ be arbitrary. Then there exists $M = M(b, V, S) \in \mathbb{N}$ such that for M arbitrary points $x_1, x_2, \ldots x_M \in V$ there exists M points $y_1, y_2, \ldots y_M \in \mathbb{C}$ for which the rational function*

$$r(z) = \prod_{j=1}^{M} \frac{z - y_j}{z - x_j}$$

has supremum norm on S bounded by b:

$$||r||_S \leq b.$$

Proof. The above cited Lemma 1.3.2 in [5] shows that the statement holds not for arbitrary b, but for some value of b call it a, $0 < a < 1$ and some corresponding value $m = m(V, S) \in \mathbb{N}$.

To pass from this to the general statement of the lemma let a and m be given as in Stahl and Totik's Lemma, and let $b, 0 < b < 1$ be arbitrary. Choose $n \in \mathbb{N}$ such that $a^n < b$ and set $M = nm$. Given $x_1, \ldots x_M \in V$ apply [5, Lemma 1.3.2] to each of the n sets of points $\{x_{j+mk} | 1 \leq j \leq m\}$, $0 \leq k < n$ and multiply. □

Proposition 3.4. *For any $R > 0$, for any $\mu \in \mathcal{B}$ and any compact set $V \subset \mathbb{C}$ with $V \cap K(S(\mu)) = \emptyset$ there exists $M = M(R, \mu, V) \in \mathbb{N}$ such that for any $w, |w| \leq R$ and for any n the number of pre-images of w in V under the orthonormal polynomial P_n is less than M. That is*

$$\#(P_n^{-1}(w) \cap V) < M.$$

Proof. Fix $R > 0$ and any compact set $V \subset \mathbb{C}$ with $V \cap K(S(\mu)) = \emptyset$. Let $b = 1/(1 + R)$ and let $M = M(b, V, S(\mu))$ be as in Lemma 3.3 above. For each n let $p_n(z) = P_n(z)/\gamma_n$ denote the monic orthogonal polynomial of degree n for μ. Then p_n is the unique degree n monic polynomial of minimal norm $1/\gamma_n$ in $L^2(\mu)$. Suppose towards a contradiction that for some $w, |w| \leq R$ and some n the equation $P_n(z) = w$ has at least M solutions $x_1, \ldots x_M \in V$. Let r be the rational function given by Lemma 3.3 such that $||r||_{S(\mu)} \leq b$ and set $q(z) := r(z) \cdot (P_n(z) - w)/\gamma_n$. Then q is a monic polynomial of degree n. Since P_n is orthogonal to all polynomials of lower degree, and in particular to the constant polynomial w we compute

$$\begin{aligned} ||q||_{L^2(\mu)} &\leq \frac{||r||_{S(\mu)}}{\gamma_n} ||P_n(z) - w||_{L^2(\mu)} \leq \frac{b}{\gamma_n} \sqrt{1 + |w|^2} \\ &\leq \frac{\sqrt{1 + R^2}}{1 + R} \cdot ||p_n||_{L^2(\mu)} < ||p_n||_{L^2(\mu)}. \end{aligned}$$

This contradicts the fact that p_n is the degree n monic polynomial with minimal $L^2(\mu)$ norm. □

Corollary 3.5. *For any $\mu \in \mathcal{B}$ and any compact set $V \subset \mathbb{C}$ with $V \cap K(\mu) = \emptyset$ there exists $M \in \mathbb{N}$ such that any of the orthonormal polynomials P_n, $n \geq 2$ and any $z \in K_n$ the number of pre-images of z in V under P_n is less than M. That is*

$$\#(P_n^{-1}(z) \cap V) < M.$$

Proof. It follows from Lemma 3.1 that there exists $R > 0$ such that

$$K_n \subset P_n^{-1}(\overline{\mathbb{D}(R)}) \subset \mathbb{D}(R),$$

for all $n \geq 2$, where K_n is the filled Julia set of P_n. The Corollary follows immediately from this and Proposition 3.4 above. □

Proof. (of Theorem A) According to Lemma 3.1 the sequence of equillbrium measures ω_n for K_n have uniformly bounded support and so the sequence of such measures is pre-compact for the weak-* topology. Furthermore according to Brolin, [1] the equilibriums measure ω_n is also the unique invariant balanced measure for P_n, i.e. it is the unique probability measure ω on $\mathbb{C}$ such that for any measurable function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$

$$\int_{\mathbb{C}} f(z)d\omega(z) = \frac{1}{n}\int_{\mathbb{C}}\left(\sum_{w, P_n(w)=z} f(w)\right) d\omega(z). \tag{4}$$

Let $V \subset \mathbb{C}$ be a compact subset with $V \cap K(\mu) = \emptyset$ and let $M \in \mathbb{N}$ be as in Corollary 3.5. Then for the measureable function 1_V, the indicator function for V we have

$$\omega_n(V) = \int_{\mathbb{C}} 1_V(z)d\omega_n(z) = \frac{1}{n}\int_{\mathbb{C}}\left(\sum_{w, P_n(w)=z} 1_V(w)\right) d\omega_n(z) \leq \frac{M}{n} \underset{n\to\infty}{\longrightarrow} 0.$$

This proves the first statement, that for any weak limit ν of a convergent subsequence $\{\omega_{n_k}\}_k : S(\nu) \subseteq K(\mu)$. If furthermore $\mu \in \mathcal{R}eg$ then by the definition of n-th-root regularity of μ, Lemma 3.2 and [6, Lemma 3.3.3])

$$I(\nu) \geq \limsup_{n\to\infty} I(\omega_n) = I(\omega_{S(\mu)})$$

Hence $\nu = \omega_{S(\mu)}$, since $\omega_{S(\mu)}$ is the unique measure of maximal energy $I(S(\mu))$. Since there is a unique possible limit we have in fact

$$\omega_n \overset{\text{weak}^*}{\longrightarrow} \omega_{S(\mu)}.$$

□

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