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A critique of circular reasoning

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Research article

Hippocrates of Chios – His Elements and His Lunes

A critique of circular reasoning

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In memory of
Johan Rosing (1907–1978)
and J. A. Bundgaard (1898–1976)
teachers who shaped me but
left me free

Abstract: Hippocrates of Chios, active during the later fifth century BCE, is stated in Proclus's "catalogue of geometers" to have been the first writer of elements, and is also known to have worked on the squaring of "lunes", plane figures contained by a convex and a concave circular arc. Customarily this is taken to mean that he wrote a book in the axiomatic style of Euclid's *Elements*, and a Euclidean reading of the text on lunules has been used to produce a list of such Euclidean propositions as must already have been in Hippocrates's *Elements*.

The present article, analyzing Hippocrates's procedures closely, makes the observation that little or nothing of what Hippocrates makes use of had not been known and used in practical geometry, in part for more than a millennium, in part for at least a century. It suggests that Hippocrates's argumentation, instead of being rooted in an axiomatic system, was based on the "locally obvious", such knowledge as his audience would be familiar with and could be supposed to accept as evident. His "elements", far from being an axiomatic system, would be a list of such locally obvious techniques and insights.

We know Hippocrates's work on lunes from Simplicios's sixth-century commentary to Aristotle's *Physics*. Simplicios proposes two versions, first a report of what Alexander of Aphrodisias had told in his commentary to the same work and passage, next (as he claims) the description of the work that Eudemos had offered, adding only Euclidean proofs where he supposes Eudemos has omitted them. Comparison of the two show that they are both genuine, none of them derived from the other. It is suggested that Alexander draws on Hippocrates's teaching, being based either on lecture notes of his or on students' notes; the Eudemos-version may instead go back to what Hippocrates published more officially.

Keywords: Hippocrates of Chios; lunes; pre-axiomatic Greek geometry; “elements” in Greek geometry; Simplicios; Alexander of Aphrodisias

Mathematics Subject Classification: 01A20

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I. How – and what – do we know?

The earliest Greek mathematician from whose hand we possess at least an edited version of an original work is Hippocrates of Chios. A widely used general history of mathematics characterizes him as “the most famous mathematician of his century” (namely the fifth century BCE) [Kline 1972: 40]. Thomas Heath, in his *History of Greek Mathematics*, generally well-based on sources, after reporting Iamblichos’s story about an anonymous Pythagorean who had lost his property and therefore was allowed to teach geometry for money, and combining it with two partially discordant stories about how Hippocrates lost his wealth, feels it to be a fair conclusion [Heath 1921: 22] that

Hippocrates of Chios, the first writer of *Elements*, who also made himself famous by his quadrature of lunes, his reduction of the duplication of the cube to the problem of finding two mean proportionals, and his proof that the areas of circles are in the ratio of the squares on their diameters, also taught for money, and for a like reason.

So, Hippocrates wrote the first collection of *Elements*, taught (geometry, it seems from the context) – and did so for money. In any case according to Heath.

Before we ask about the sources that concern Hippocrates’s mathematics and his mathematical activity, we may regard those for his loss of fortune. One version comes from Philoponos’s commentary to Aristotle’s *Physics* 185^a16 [ed. trans. Osborne 2006: 51]:

Hippocrates was a merchant of Chios, who was a victim of piracy and lost everything. He travelled to Athens to bring a case against the pirates and while he was staying in Athens, for a long time on account of the court case, he attended philosophy classes, and reached such a high standard in geometry that he attempted to discover the quadrature of the circle.

This was written almost a millennium after the supposed event. Aristotle, at a century’s distance only, has this offhand remark (*Ethica Eudemia* 1247^a17–20, trans. [Simpson 2013: 179]):

Hippocrates was clearly a geometrician but in other matters he seemed stupid and foolish and while on a sea voyage lost much gold coin, because of his simplicity so they say, to the collectors of the 2 percent customs duty at Byzantium.

“... as we are told” – that is, we have to do with an anecdote, whose veracity it is impossible to judge from the versions quoted after a century’s oral circulation (not to speak of Philoponos’s version with its impossible idea of fighting pirates in court, probably picked from some florilegium). It may as well have been invented *ex nihilo* by somebody wanting to denigrate Hippocrates as an unreliable

fool as being based on facts. Absent from Aristotle's account, we observe, is the idea that Hippocrates should have learned his geometry in Athens. Since he shares his homeland Chios with the slightly older Oinopides (on whom repeatedly below), the existence of an adequate environment there for learning geometry is quite plausible – Paul Tannery [1887: 109] (followed in many more words but with no more solid arguments by Maria Timpanaro Cardini [2010: 231]) doubts that there were at the times schools of the type in Athens, and instead believes Hippocrates to have been a student of Oinopides who then, when in Athens, started teaching what he had learned.

As to Hippocrates's teaching, what do we *really* know? The only hint in the sources that he taught is a passage in Aristotle's *Meteorology*¹ stating that “those around Hippocrates and his disciple Aischylos” shared the opinion of the Pythagoreans concerning the nature of comets – namely that they are to be counted among the planets. However, since Aristotle is able to give a detailed account of Hippocrates's doctrine, we are on much safer ground when accepting Hippocrates as a teacher – but as a teacher of *astronomy*.

Hippocrates's writing of the first collection of *Elements* obviously supports the assumption that he taught geometry too – but how, and what, do we know about these *Elements*?

Our source is Proclus (*Commentary to Elements I*, 66.4–8,² trans. [Morrow 1970: 54]):

Following [Oinopides of Chios and Anaxagoras came] Hippocrates of Chios, who invented the method of squaring lunules, and Theodorus of Cyrene became eminent in geometry. For Hippocrates wrote a book on elements,³ the first of whom we have any record who did so.

Proclus is almost as late as Philoponos, but his source (for this as well as for the whole so-called “catalogue of geometers” covering the period before Euclid) was once supposed to be Eudemos's history of geometry, written in the late fourth century BCE.⁴ Strong arguments, however, speak against *direct* use of Eudemos's work.

¹ My translation (as all translations with no identified translator in the following) from [Bekker 1831: 342^b36–343^a1]. “Those around” was the standard way to refer to the circle of those who studied with a philosopher or similar teacher. For some reason, [Webster 1931] (followed by the revised version in [Barnes 1984]) as well as [Lee 1952] omit “those around”, thus removing the only [kind of] evidence for Hippocrates having been a regular teacher (Lee has it in his Greek text, showing that he did not suspect the words to be interpolated).

In 344^b15, “those around Hippocrates” are mentioned again, still in connection with their opinion about comets; this time Lee translates “the school of Hippocrates”.

² Here and everywhere in the following, my references to the *Commentary* point to page and line in the Greek text in [Friedlein 1873].

³ “Wrote a book on elements” translates στοιχεῖα συνέγραψεν. Συγγράφω may mean “compose a writing or a work in writing” (thus [GEL]) and plausibly does so in the present context. That he wrote a book is thus a likely interpretation; but noteworthy is the indefinite στοιχεῖα, precisely rendered by Morrow. Proclus does *not* say that Hippocrates wrote “the first *Elements*”, as inherent in Heath's italicization in “first writer of *Elements*”.

⁴ Thus [Cantor 1880: 113], repeated verbatim in the second and third edition – “a fragment of Eudemos, or at least an extract from his historical-geometrical writings”. The list of names goes on until Philip of Mende, a student of Plato, and then jumps to Euclid, where Proclus has to construct information as best he can (cf. quotation on p. 161). The only writer on the history of geometry or mathematics whom we know about and who would know about Philip of Mende and not about Euclid is Eudemos. Cf. [Sachs 1917: 24].

Firstly, there is the question whether Eudemos's work was at all at hand in Proclus's time; Paul Tannery [1912/1883: 345] claimed that *all* quotations from Eudemos postdating the fourth century CE are indirect – inasfar as concerned with his history of geometry mostly derived from a compilation produced by Sporos of Nicaea toward the end of the third century. In particular Tannery asserted that Proclus only knew Eudemos through Geminus or Porphyry.

Heiberg [1884: 345], followed by Ivor Bulmer-Thomas [1971: 463], objected that Simplicios and Eutocios speak (in other connections, to one of which we shall return) of using Eudemos in such words that they must have possessed the original and not an epitome or abridgment. Heiberg and Bulmer-Thomas are therefore convinced that Eudemos's history was still accessible at least to these two in the early sixth century. Personally I find that argument less than convincing; not a few colleagues in recent times refer to work they only know indirectly or through later editions as if they were using the original publication.⁵ Moreover, if writing in an environment where it was known that only an epitome had survived, it would not even be venial sin to omit that information – Simplicios and Eutocios were writing for their own public, not for historians of the 19th or 21st century.⁶

Be that as it may, since a century there seems to be full agreement that Proclus's “Eudemian summary” or “catalogue of geometers” – from which comes the information that Hippocrates produced the first book about elements – cannot come from Eudemos's hand. As formulated by Thomas Heath [1921: I, 118],

a perusal of the summary itself is sufficient to show that it cannot have been written by Eudemos; the most that can be said is that, down to a certain sentence, it was probably based, more or less directly, upon data appearing in Eudemos's *History*. At the sentence in question there is a break in the narrative, as follows:

Those who have compiled histories bring the development of this science up to this point. Not much younger than these is Euclid, who put together the *Elements*, collecting many of the theorems of Eudoxus, perfecting many others by Theaetetus, and bringing to irrefragable demonstration the propositions which had only been somewhat loosely proved by his predecessors.

Since Euclid was later than Eudemos, it is impossible that Eudemos can have written this; while the description “those who have compiled histories”, and who by implication were a little older than Euclid, suits Eudemos excellently. Yet the style of the summary after the break does not show any such change from that of the earlier portion as to suggest different authorship.

⁵ I shall abstain from naming the estimated colleague who has used my own work in a similar way.

⁶ Who objects to unreserved references to Apollonios's *Conics* or Diophantos's *Arithmetic*, even though nobody has seen the complete texts for a thousand years? We all know that what exist are incomplete survivals, so why repeat?

As a parallel outside mathematics and its history, think of this fragment of a footnote in [Gadamer 1990: 70]:
In der wohl zuletzt geschriebenen Einleitung zu seiner Schleiermacherbiographie gebraucht Dilthey oft
»Erlebnis« Vgl. Ges. Schriften Bd. 13, 1, S. XXXV–XLV.

Any student of mine would have been told (kindly but emphatically) to make a decent reference. But Hans-Georg Gadamer wrote for colleagues who all had Dilthey's *Gesammelte Schriften* in their institute library if not in their office.

This argument, though less fully, is repeated by Ivor Bulmer-Thomas [1971: 463]. Already Gustav Junge [1907], though his main concern had been the Proclean passage that ascribes the discovery of the theory of the irrational to Pythagoras, had given strong arguments that Proclus's catalogue of geometers, though ultimately and in the main going back to Eudemos, is strongly redacted and contaminated, if not by Proclus himself then by somebody living centuries after Eudemos as well as Euclid.

That raises another question – did Hippocrates himself speak of “elements”, and if so, what did that mean to him and his contemporaries? The term fits a derivation from the use of στοιχείον for a letter as member of the alphabet (thus not referring to the letters of a running text, for which γράμμα is used) – cf. [Burkert 1959: 170]. In his commentary to *Euclid's* work, Proclus explains (72.23–73.12, ed. trans. [Morrow 1970: 59f]), paraphrasing and at least in the end expanding Plato's younger contemporary Menaechmos:

The term “element,” however, can be used in two senses, as Menaechmus tells us. For what proves is called an element of what is proved by it; thus in Euclid the first theorem is an element of the second, and the fourth of the fifth. [...] An element so regarded is a kind of lemma. But in another sense “element” means a simpler part into which a compound can be analyzed. In this sense not everything can be called an element of anything [that follows from it], but only the more primary members of an argument leading to a conclusion, as postulates are elements of theorems. This is the sense of “element” that determines the arrangement of the elements in Euclid's work, some of them being elements of plane geometry, and some elements of stereometry.

Burkert [1959: 192] gets out of this that in this second sense *elements* (στοιχεῖα) are the postulates, and that this is the meaning of Euclid's title. He couples this to a derivation of the word not from the use about letters but from an underlying verb στοιχέω, “to stand orderly” (etc.). As he continues, “the essence of mathematics, as it was discovered by the Greeks, consists in its propositions forming an uninterrupted logical system” (*einen lückenlosen logischen Zusammenhang*). He still knows of no proof that Hippocrates already spoke of his own work as Στοιχεῖα, but “since it was comparable to Euclid's Στοιχεῖα, if not in realization then at least in conception, what other designation could he have given to it?”

Apart from presenting a complete misreading of Proclus's text, Burkert here offers an only slightly veiled case of circular reasoning – “since Hippocrates's work was Greek and therefore at least in aspiration axiomatic just as that of Euclid, it carried the same name, and therefore constituted an uninterrupted logical system”. Worse, perhaps, Burkert overlooks that the notion of “postulates” as a coherent group (as they are in Euclid's *Elements*, and as asked for by Burkert's understanding) is post-Aristotelian.

Aristotle does not know the concept in general, but he is aware of Euclid's Postulate II – that can be seen in *Physica* 207^b29–31 (trans. Hardie & Gaye in [Aristotle, *Works*, II]):

[mathematicians] do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish.

Moreover, Aristotle is explicitly aware of the need for something like Postulate V. In *Analytica priora* 64^b38–65^a7 (ed. trans. Tredennick in [Cook & Tredennick 1938: 486f]), where a problem about parallels serves to exemplify circular reasoning, the way out is stated to take as an axiom [ἄξιόω] the proposition one wants to but cannot prove. The fifth postulate thus had not yet been formulated although Aristotle recognizes the dilemma; moreover, the verb used is not αἰτέω, “to request” (from which αἰτήμα, conventionally translated “postulate”).

All in all, we may reasonably conclude that Hippocrates produced a collection of basic propositions that might serve to prove others or make more advanced constructions, and probably put them together in some kind of book. At least in later times this collection was recognized as belonging to the same genre as Euclid's *Elements*. We have no means to discover, however, whether Hippocrates himself spoke of στοιχεῖα, “elements” nor – much more interesting – whether this collection was already built at least ideally as an axiomatic structure.⁷

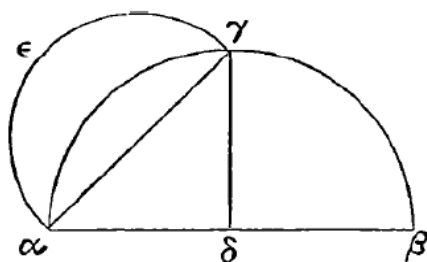
Very often, historians have followed Aristotle's advice (not meant by Aristotle for such cases) and presupposed what they want to prove as an axiom. As an historian with an Aristotelian bent, I shall try instead to see what thought and second thoughts about available evidence allows us to infer. That is the topic of part II.

II. Hippocrates's lunes, and what they tell us

Our source for Hippocrates's lunes in Simplicios's sixth-century (CE) commentary to Aristotle's *Physics*. Simplicios, on the other hand, reports from two sources – Alexander of Aphrodisias (writing ca 200 CE), and Eudemos (with the already-mentioned proviso that he may know the latter from a secondary compilation or abridgement only – a proviso I shall not repeat but ask the reader to keep in mind).

Since Alexander is only reported by Simplicios, not quoted, most of those who discuss the text do not translate Simplicios's words but restrict themselves to secondary reporting. Simplicios's full report from Alexander is evidently found in [Diels 1882], but my Greek is insufficient to make use of this critical edition. I therefore reproduce the English translation given by George Allman [1889: 67f] (on its part based on [Bretschneider 1870: 103–105] as well as the Diels edition); I also borrow Allman's diagrams, based on those of Diels, while the section labelling is mine:

- A Let a semicircle $\alpha\beta\gamma$ be described on the straight line $\alpha\beta$; bisect $\alpha\beta$ in δ ; from the point δ draw a perpendicular $\delta\gamma$ to $\alpha\beta$, and join $\alpha\gamma$; this will be the side of the square inscribed in the circle of which $\alpha\beta\gamma$ is the semicircle. On $\alpha\gamma$ describe the semicircle $\alpha\epsilon\gamma$. Now since the



square on $\alpha\beta$ is equal to double the square on $\alpha\gamma$ (and since the squares on the diameters are to each other as the respective circles or semicircles), the semicircle $\alpha\gamma\beta$ is double the

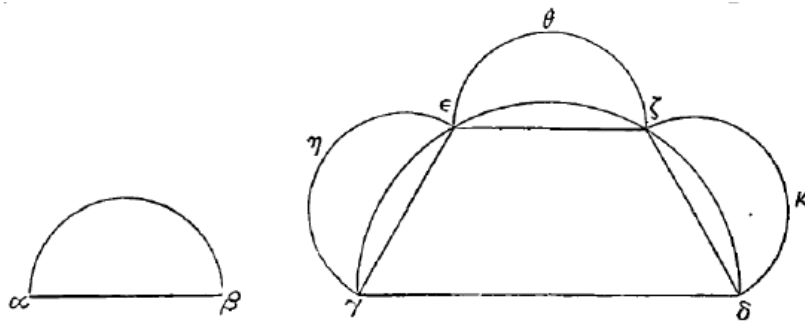
⁷ Reviel Netz [2004: 276], deliberately “minimalist”, is even more radical in his doubts: Proclus's reference to Hippocrates's *Elements*

may be interpreted to mean that Hippocrates wrote an early version of Euclid's *Elements*. However this could also mean that Hippocrates was the first to write down proofs for propositions which Eudemos considered as part of the subject-matter of “*Elements*”. Perhaps even: no more than the first to enunciate some such propositions

– assuming, as Netz then points out in a footnote, “that the word “*Elements*” entered the Proclean summary from Eudemos, and was not Proclus' own interpretation of Eudemos”.

semicircle $\alpha\epsilon\gamma$. The quadrant $\alpha\gamma\delta$ is, therefore, equal to the semicircle $\alpha\epsilon\gamma$. Take away the common segment lying between the circumference $\alpha\gamma$ and the side of the square; then the remaining lune $\alpha\epsilon\gamma$ will be equal to the triangle $\alpha\gamma\delta$; but this triangle is equal to a square. Having thus shown that the lune can be squared, Hippocrates next tries, by means of the preceding demonstration, to square the circle thus:—

- B Let there be a straight line $\alpha\beta$, and let a semicircle be described on it; take $\gamma\delta$ double of $\alpha\beta$, and on it also describe a semicircle; and let the sides of a hexagon, $\gamma\epsilon$, $\epsilon\zeta$, and $\zeta\delta$ be inscribed in it. On these sides describe the semicircles $\gamma\eta\epsilon$, $\epsilon\theta\zeta$, $\zeta\kappa\delta$. Then each of these semicircles described on the sides of the hexagon is equal to the semicircle on $\alpha\beta$, for $\alpha\beta$ is equal to each side of the hexagon. The four semicircles are equal to each other, and together are then four times the semicircle on $\alpha\beta$. But the semicircle on $\gamma\delta$ is also four times that on $\alpha\beta$. The semicircle on $\gamma\delta$ is, therefore, equal to the four semicircles – that on $\alpha\beta$, together with the three semicircles on the sides of the hexagon. Take away from the semicircles on the sides of



the hexagon, and from that on $\gamma\delta$, the common segments contained by the sides of the hexagon and the periphery of the semicircle $\gamma\delta$; the remaining lunes $\gamma\eta\epsilon$, $\epsilon\theta\zeta$, and $\zeta\kappa\delta$, together with the semicircle on $\alpha\beta$, will be equal to the trapezium $\gamma\epsilon$, $\epsilon\zeta$, $\zeta\delta$. If we now take away from the trapezium the excess, that is a surface equal to the lunes (for it has been shown that there exists a rectilinear figure equal to a lune), we shall obtain a remainder equal to the semicircle $\alpha\beta$; we double this rectilinear figure which remains, and construct a square equal to it. That square will be equal to the circle of which $\alpha\beta$ is the diameter, and thus the circle has been squared.

The treatment of the problem is indeed ingenious; but the wrong conclusion arises from assuming that as demonstrated generally which is not so; for not every lune has been shown to be squared, but only that which stands over the side of the square inscribed in the circle; but the lunes in question stand over the sides of the inscribed hexagon. The above proof, therefore, which pretends to have squared the circle by means of lunes, is defective, and not conclusive, on account of the false-drawn figure ($\psi\epsilon\upsilon\delta\omicron\gamma\rho\acute{\alpha}\phi\eta\mu\alpha$) which occurs in it.

Firstly, Alexander is obviously right that the squaring of the circle is fallacious, exactly for the reason given.⁸ There is widespread agreement that Hippocrates cannot be responsible for the mistake – in the words of Heath [1921: I, 187], “It is evident that this account does not represent Hippocrates's own argument, for he would not have been capable of committing so obvious an error”.

⁸ Lloyd [1987: 126 and *passim*] strongly doubts that Alexander should have attributed any fallacy of Hippocrates, and supposed this to be a disingenuous interpretation on the part of Simplicios. However, a passage in Themistios's fourth-century paraphrase of the *Physics* [ed., trans. Todd 2012: 126], clearly based on Alexander's commentary, albeit somewhat damaged in the manuscript, confirms Simplicios's reading.

But whether this is the fallacy hinted at in the Aristotelian passage which Simplicios discusses is not at all certain, cf. [Lloyd 1987]. Since this question does not touch at what we are doing here, there is not reason to elaborate.

Secondly, let us probe what Hippocrates bases his argument on (his “elements”, in Proclus's/Menaichmos's first sense). In section (A), the only appeal to preceding knowledge is found in the passage

Now since the square on $\alpha\beta$ is equal to double the square on $\alpha\gamma$ (and since the squares on the diameters are to each other as the respective circles or semicircles), the semicircle $\alpha\gamma\beta$ is double the semicircle $\alpha\epsilon\gamma$.

On one hand, this makes use of the “Pythagorean rule” as applied to the diagonal of a square; on the other of the proportionality of circular and semicircular areas to the square on the diameter. Both principles had been known by Near Eastern practical geometers at least since the beginning of the second millennium BCE – both of them in the general cases of, respectively, rectangles with *any ratio* between the sides and areas of *any kind* if only similar.⁹ Beyond that, the proof makes use of what we may call the arithmetic of areas (additivity and subtractivity), something not even Euclid considered worth arguing for specifically but just included in his Common Notions 2 and 3 (“If equals be added to equals, the wholes are equal”, etc.),¹⁰ and without which it is difficult to give any meaning to area measurement. Finally, it is taken for granted that a line can be bisected and that a perpendicular can be raised (spoken of separately, but together the two amount to the producing a perpendicular bisector); how to do this with ruler and compass was a fairly recent discovery of Oinopides.¹¹ It is far from certain, however, that Hippocrates thought of Oinopides's construction; if he did so he would know that in order to bisect $\alpha\beta$ he already needed to construct the perpendicular bisector. He may well have had older (practitioners') methods in mind, where (e.g.) halving is produced by means of a string that is folded, and the perpendicular drawn by means of a set square.

Section (B) begins by performing a construction in which it is presupposed that the side of a regular hexagon equals the semidiameter of the circumscribed circle. This was also old knowledge – see (for example) a Susa tablet of Old Babylonian date in [Bruins & Rutten 1961: 24 and pl. 2], and a Mycenaean gold roundel from 1500–1550 BCE in [Høyrup 2000: 30], conspicuously revealing the traces of being compass-produced; the decoration of a small box from the same place, perhaps one generation earlier, shows how this elementary geometrical fact will be obvious to anybody playing

⁹ The “Pythagorean rule” in general terms is quoted in the text Db₂-146 from ca 1775 BCE, see [Høyrup 2002: 261]; the proportionality of areas (also circles and their parts) to the square of a characteristic linear dimension is the basis for the geometric section of tables of technical constants going together with the sexagesimal place-value-system since the outgoing third millennium. For the coefficients, see for example [Robson 1999: 34–56].

¹⁰ Here as in the following, quotations from the *Elements* in translation are taken from [Heath 1926], but checked in [Heiberg 1883].

¹¹ Our source for this is Proclus's *Commentary* 283.7–10, trans. [Morrow 1970: 220f]. Since Proclus is able to relate that Oinopides spoke of the perpendicular as a “line drawn gnomonwise” (while his “catalogue of geometers” offers no such precise information), we can probably trust the ascription.

Proclus speaks of Oinopides only in relation to the Euclidean construction of a perpendicular on a line from a point outside it, but the trick for this involves creating a segment of which the requested perpendicular is the bisector; similarly for the construction of a perpendicular from a given point on a line.

with a compass drawing multiple adjacent circles (*ibid.* p. 32). As we see, the principle is taken for granted, its use is not even made explicit.

The proportionality of semicircular areas to the square on the diameter (as well as the arithmetic of areas) is also made use of in part (B), but not the “Pythagorean rule”.

Simplicios goes on [ed. trans. Allman 1889: 69]:

Eudemus, however, tells us in his *History of Geometry*, that Hippocrates demonstrated the quadrature of the lune, not merely the lune on the side of the square, but generally, if one might say so: if, namely, the exterior arc of the lune be equal to a semicircle, or greater or less than it. I shall now put down literally (κατὰ λέξιν) what Eudemus relates, adding only a short explanation by referring to Euclid's *Elements*, on account of the summary manner of Eudemus, who, according to archaic custom, gives concise proofs.¹²

A number of editor-translators of the ensuing expanded Eudemos text have tried to clean it of the additions, not always agreeing in detail on what to omit.¹³ Here I shall reproduce the translation found in [Thomas 1939: 237–253]:¹⁴

[Eudemus] writes thus in the second book of the *History of Geometry*.

C The quadratures of lunes, which seemed to belong to an uncommon class of propositions by reason of the close relationship to the circle, were first investigated by Hippocrates, and seemed to be set out in correct form; therefore we shall deal with them at length and go through them. He made his starting-point, and set out as the first of the theorems¹⁵ useful to his purpose, that similar segments of circles have the same ratios as the squares on their bases. And this he proved by showing that the squares on the diameters have the same ratios as the circles.

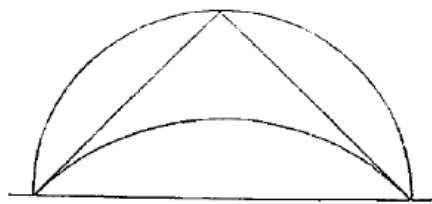
¹² Here, Allman observes in a footnote that “Simplicios did not adhere to his intention, or else some transcriber has added to the text” – cf. the specifications of Geoffrey Lloyd [1987: 116], who takes it for granted that Simplicios himself is responsible.

¹³ [Tannery 1912/1883]; [Diels 1882] (edition only); [Allman 1889]; [Rudio 1907] (full Simplicios text, with indication of what Ferdinand Rudio supposes not to come from Eudemos); [Thomas 1939]; [Becker 1936] (edition only) translated in [Netz 2004]. zs

When referring in what follows to passages due to Simplicios I have inspected the complete text in [Bretschneider 1870] and [Rudio 1907].

¹⁴ For the present purpose it is mostly unimportant that Thomas used modernizing terminology – for instance, replacing “the [line] from the centre” by “radius”, or that equality δυνάμει becomes equality “in square”. It can actually be debated whether the translation “potential”, referring to the Aristotelian concept, is not more misleading, cf. [Høyrup 1990], for which reason I shall also avoid it in my own discussion. In order to keep present to mind that Greek had no term corresponding to “radius” I shall use instead “semidiameter” in my own text.)

¹⁵ “Theorems” is Thomas's invention; a literal translation would be “first of what was useful ...”; Netz [2004: 248] has “first among the things useful ...”. On “prove” in the next sentence, see note 24 and preceding text.



D Having first shown this he described in what way it was possible to square a lune whose outer circumference was a semicircle. He did this by circumscribing about a right-angled isosceles triangle a semicircle and about the base a segment of a circle similar to those cut off by the sides.¹⁶ Since the segment about the base is equal to the sum of those about the sides, it follows that when the part of the triangle above the segment about the base is added to both the lune will be equal to the triangle. Therefore the lune, having been proved equal to the triangle, can be squared. In this way, taking a semicircle as the outer circumference of the lune, Hippocrates readily squared the lune.

¹⁶ Simplicios states that this presupposes *Elements* III.33 and involves Euclid's definition of similarity of segments (III, def. 11) as “those in which the angles are equal to one another” – the angles in question being the inscribed angle encompassing the corresponding chord, that is, those of def. 8,

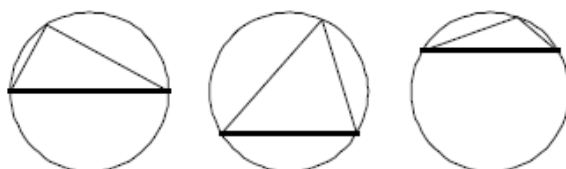
An angle in a segment is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the base of the segment, is contained by the straight lines so joined.

Section (A) and the appurtenant diagram shows how things can be made without reference to Euclid. All that is needed is to complete the diagram of (C) so as to make it correspond to the left half of the diagram of (A): that can be done by drawing the mirror images of the shorter chords of (C) in the long chord. Their intersection is easily seen to be the centre of the required circle. However, as argued in detail below in connection with the more intricate case of (E), Hippocrates does not need to have the arc constructed.

Rudio (p. 49; and so evidently Bretschneider, p. 110) includes a passage which he considers Eudemian even though it is eliminated as dubious by Thomas. It explains (translating the same word *τμήμα* first as “sector”, next as “segment”) that

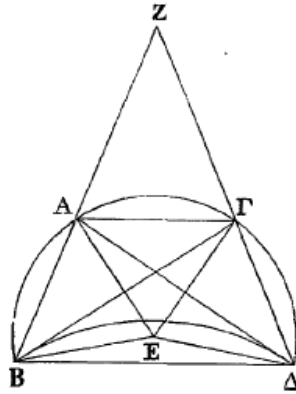
similar sectors are those which make up the same part of the circle, for example half-circle to half-circle and third-circle to third-circle. Therefore similar segments also take up the same angles. And all half-circles are indeed right, and those of the larger are smaller than right, and indeed smaller according to how much larger the segments are, and those of the smaller are larger, and indeed larger according to how much smaller the segments are.

This diagram illustrates what is meant:



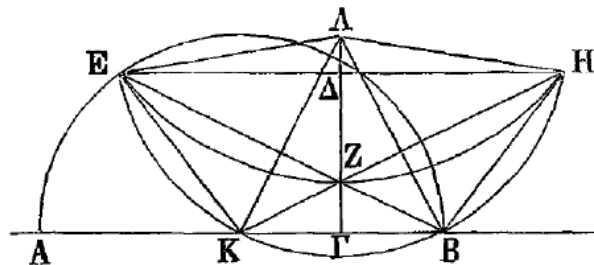
The inconsistent use of *τμήμα* and the combination of a reference to parts of circles with the Euclidean definition of similarity suggests that the passage is not only Eudemian but actually goes back to Hippocrates (mathematical terminology tends to become more consistent in enduring use); but that it is contaminated – perhaps already by Eudemos, perhaps also or only by Simplicios.

- E Next in order he assumes [an outer circumference] greater than a semicircle [obtained by] constructing a trapezium having three sides equal to one another while one, the greater of the parallel sides, is such that the square on it is three times the square on each of those sides, and then comprehending the trapezium in a circle¹⁷ and circumscribing about its greatest side a segment similar to those cut off from the circle by the three equal sides. That the said segment is greater than a semicircle is clear if a diagonal is drawn in the trapezium. For this diagonal, subtending two sides of the trapezium, must be less than the sum of the



squares on the diagonal and that one of the other sides which is subtended by the said [greatest] side together with the diagonal. For the squares on BΓ, ΓΔ are greater than three times, and the square on BΔ is equal to three times, the square on ΓΔ. Therefore the angle standing on the greatest side of the trapezium is acute. Therefore the segment in which it is is greater than a semicircle. And this segment is the outer circumference of the lune.¹⁸

- F If [the outer circumference] were less than a semicircle, Hippocrates solved this also, using the following preliminary construction. Let there be a circle with diameter AB and centre K. Let ΓΔ bisect BK at right angles; and let the straight line EZ be placed between this and the circumference verging towards B so that the square on it is one-and-a-half times the square on one of the radii. Let EH be drawn parallel to AB, and from K let [straight lines] be drawn joining E and Z. Let the straight line [KZ] joined to Z and produced meet EH at H, and again let [straight lines] be drawn from B joining Z and H. It is then manifest that EZ produced will pass through B – for by hypothesis EZ verges towards B – and BH will be equal to EK.



This being so, I say that the trapezium EKBH can be comprehended in a circle.

¹⁷ Here, Simplicios inserts a proof that this can be done.

¹⁸ Simplicios here observes that Eudemos omits the actual proof of the squaring of the lune. It follows from $\square(B\Delta) = 3\square(BA)$, via the corresponding equality involving the appurtenant segments, and some area arithmetic.

G Next let a segment of a circle be circumscribed about the triangle EZH; then clearly each of the segments on EZ, ZH will be similar to the segments on EK, KB, BH.

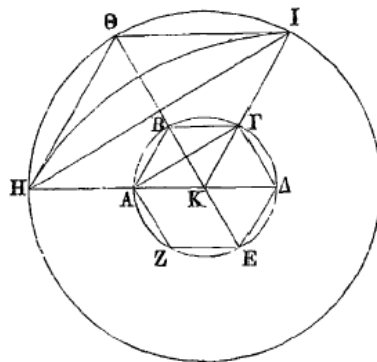
This being so, the lune so formed, whose outer circumference is EKBH, will be equal to the rectilinear figure composed of the three triangles BZH, BZK, EKZ. For the segments cut off from the rectilinear figure, inside the lune, by the straight lines EZ, ZH are (together) equal to the segments outside the rectilinear figure cut off by EK, KB, BH. For each of the inner segments is one-and-a-half times each of the outer, because, by hypothesis, the square on EZ is one-and-a-half times the square on the radius, that is, the square on EK or KB or BH. Inasmuch then as the lune is made up of the three segments and the rectilinear figure *less* the two segments – the rectilinear figure including the two segments but not the three – while the sum of the two segments is equal to the sum of the three, it follows that the lune is equal to the rectilinear figure.

H That this lune has its outer circumference less than a semicircle, he proves by means of the angle EKH in the outer segment being obtuse. And that the angle EKH is obtuse, he proves thus.

| | |
|---------------------|--------------------------------|
| Since | $EZ^2 = \frac{3}{2}EK^2$ |
| and | $KB^2 > 2BZ^2$, ¹⁹ |
| it is manifest that | $EK^2 > 2KZ^2$. |
| Therefore | $EZ^2 > EK^2 + KZ^2$. |

The angle at K is therefore obtuse, so that the segment in which it is is less than a semicircle.

I Thus Hippocrates squared every lune, seeing that [he squared] not only the lune which has for its outer circumference a semicircle, but also the lune in which the outer circumference is greater, and that in which it is less, than a semicircle.²⁰



¹⁹ As Thomas points out, no proof is given for this. However, it is easily seen that if $KB^2 = 2BZ^2$, then angle KZB would be right, and ZE would be equal to BZ, already smaller than KB. If $KB^2 < 2BZ^2$, ZE would be even smaller. To see this we only need the relation between side and diagonal in a square, and if Hippocrates, Eudemos and Simplicios did not consider it obvious from the diagram they might still see no need for a specified argument.

Heath [1921: I, 195] offers a much more sophisticated proof, which we may safely discard as reconstruction of what was on Hippocrates's mind.

²⁰ Netz [2004: 252] suspects section (I) to be due to Simplicios. In any case Simplicios goes on after (I) and before (J) (translated from [Rudio 2007: 69]):

But it was not exclusively over the side of a square, as Alexander has related, and [Hippocrates] did not at all undertake to square the lune over the side of the hexagon, as Alexander also asserts.

J But he also squared a lune and a circle together in the following manner. Let there be two circles with K as centre, such that the square on the diameter of the outer is six times the square on the diameter of the inner. Let a [regular] hexagon ABΓΔEZ be inscribed in the inner circle, and let KA, KB, KΓ be joined from the centre and produced as far as the circumference of the outer circle, and let KA, KB, KΓ be joined. Then it is clear that HΘ, ΘI are sides of a [regular] hexagon inscribed in the outer circle. About HI let a segment be circumscribed similar to the segment cut off by HΘ. Since then $HI^2 = 3\Theta H^2$ (for the square on the line subtended by two sides of the hexagon, together with the square on one other side, is equal, since they form a right angle in the semicircle, to the square on the diameter, and the square on the diameter is four times the side of the hexagon, the diameter being twice the side in length and so four times as great in square), and $\Theta H^2 = 6AB^2$, it is manifest that the segment circumscribed about HI is equal to the segments cut off from the outer circle by HΘ, ΘI, together with the segments cut off from the inner circle by all the sides of the hexagon. For $HI^2 = 3H\Theta^2$, and $\Theta I^2 = H\Theta^2$, while ΘI^2 and $H\Theta^2$ are each equal to the sum of the squares on the six sides of the inner hexagonal, since, by hypothesis, the diameter of the outer circle is six times that of the inner. Therefore the lune HΘI is smaller than the triangle HΘI by the segments taken away from the inner circle by the sides of the hexagon. For the segment on HI is equal to the sum of the segments on HΘ, ΘI and those taken away by the hexagon. Therefore the segments [on] HΘ, ΘI are less than the segment about HI by the segments taken away by the hexagon. If to both sides there is added the part of the triangle which is above the segment about HI, out of this and the segment about HI will be formed the triangle, while out of the latter and the segments [on] HΘ, ΘI will be formed the lune. Therefore the lune will be less than the triangle by the segments taken away by the hexagon. For the lune and the segments taken away by the hexagon are equal to the triangle. When the hexagon is added to both sides, this triangle and the hexagon will be equal to the aforesaid lune and to the inner circle. If then the aforementioned rectilinear figures can be squared, so also can the circle with the lune.²¹

Section (C) seems to claim that Hippocrates *proved* “that the squares on the diameters have the same ratios as the circles” – a claim that was accepted by Heath, as we saw above in the initial quotation. Unfortunately we do not know which kind of proof is referred to, and it is thus for good reasons that Netz [2004: 249] translates the initial words of section (D) “This being shown to his satisfaction” (while admitting that the Greek αὐτῷ may also just mean “by him”).

Actually, the verb translated first “proved” and then “showing” (δείκνυμι) in the end of (C) may just as well mean “point out” or “explain”.²² When it reappears in the beginning of (D), the verb

²¹ Netz [2004: 283 n. 47] sees the structure “if then ..., so ...” as belonging in “a context in which it is not self-evident that all rectilinear figures can be squared”, contrary to the second part of my argument in note 26 (below) and to what is presupposed by Hippocrates as reported by Alexander in (B). But Netz’s reading is far from compulsory. The standard prayer to the gods in the *Iliad* runs, not “Do Z in my favour” but “If I have done X for you [implying: which you cannot deny], then do Z in my favour” – precisely the same rhetorical enthymeme. By taking their starting point with already mature rhetoric and already mature mathematics, Apostolos Doxiadis and Michalis Sialaros [2013] overlook this early influence of the eristic-rhetorical ambiance on the formation of the mathematical language.

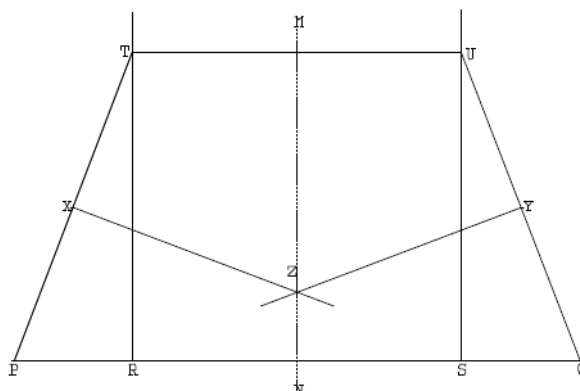
²² This fits to the full Netz’s further conclusion [2004: 275], that “Hippocrates left this as a starting point, without providing a mathematical proof”

refers to the assertion about similar segments (the immediate presupposition for what follows) rather than to the proportionality of circular areas to the squares on the diameters. The choice of the translation “proving” is another case of circular reasoning taking it for granted that Hippocrates, as a Greek mathematician, followed the ideals of later Greek mathematics. So is evidently the insertion of a “theorem”, cf. note 15.

The proof in (D) is strangely mutilated: “Since the segment about the base is equal to the sum of those about the sides ...” – but this is not intuitively obvious from the diagram accompanying this section unless we refer it to the familiar geometry of a square with diagonal; the diagram of section (A) accompanied by the definition of similarity by “parts” of the circle (not by angles) is definitely more suited, since here it is obvious that we deal with half-circles. It looks as if the whole section (D) is a re-elaboration of what is reported from Alexander in (A) – a re-elaboration that has deleted some of the traces of the underlying thinking.

Be that as it may, it is clear that the basis for the argument in D is the same as in (A) – “Pythagorean rule” and proportionality of similar areas to the square on a characteristic linear dimension.²³

(E) is not quite as simple. At first, Hippocrates constructs a trapezium with sides s , s , s and $\sqrt{(3s^2)}$. He does not explain how to do it, but the necessary tools were, firstly, the Pythagorean rule; secondly, the *construction* of a perpendicular – as already mentioned a recent discovery due to Oinopides of Chios.



We do not know how Hippocrates performed the construction, but the diagram shows a possible way using exactly these tools. Firstly, the long side is constructed from the short side, possibly by means of repeated application of the Pythagorean rule (but a mean proportional between s and $3s$ is not to be excluded²⁴). This long side is drawn as PQ . Symmetrically around its mid-point the short

²³ Simplicios has an added reference to *Elements* I.47, the “Pythagorean theorem”. It is possible that this is no mere addition (one of those promised in the quotation before note 12) but indeed a rewriting of a reference to the rule in the Eudemian text – cf. the same note.

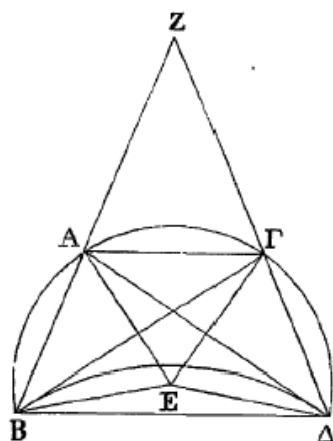
²⁴ Hippocrates is also supposed to be the one who reduced the doubling of the cube to the finding of two mean proportional; if we accept that we may safely assume that he knew how to find a single mean proportional. H. G. Zeuthen [1896: 84] has his doubts, but afterwards nobody, it seems. The source for the ascription is Proclus's *Commentary* 213.2–9 [trans. Morrow 1970:167], and Proclus's formulation is indeed not quite limpid. But internal evidence in the text also provides evidence that Hippocrates knew to construct a mean proportional. When he has shown that a lune equals an isosceles trapezium he stops. Transforming an isosceles trapezium into a rectangle by a simple cut-and-paste operation is unproblematic – but transforming the rectangle into a square asks in principle for the finding of the mean proportional between the sides, which Hippocrates must hence have considered a

side is marked off as RS . In R and S , perpendiculars to PQ are raised, and with semidiameter equal to RS and centres P and Q circles are drawn, which meet the perpendiculars in T and U . Because RT and SU are parallel, TU is equal to RS , whence also to PT and QU .

We do not need to worry about the problem of parallels (meaning “along each other”). At the time nobody did – as we have seen, Aristotle mentions unacknowledged circular reasoning about them a century later, which shows that they were only discovered to present a difficulty well after Hippocrates's times. We have thus constructed the trapezium asked for.

Next we need to circumscribe a circle about the trapezium. Even this seems to be just stated by Hippocrates – at least, there is no trace of an argument in the Eudemos text used by Simplicios, who constructs a proof of his own – text in [Rudio 1907: 52–55]. A simpler feasibility proof (so simple that Hippocrates may have seen it intuitively) can be made from the above diagram: We draw the perpendicular bisectors to PT and QU (Oinopides again), which meet in Z (if we need evidence that they are not parallel we observe that angles PTU and QUT are obtuse, since $TM < PN$, $UM < QN$). Since Z is equidistant from P and T , and from Q and U , and for symmetry reasons located on the perpendicular bisector NM of PQ , all four points P , T , U and Q are equidistant from Z , and thus located on the same circle.²⁵

The next step is to construct a moon. Now we may look at Simplicios's diagram (more complicated than needed for our purpose, since Simplicios used it in his above-mentioned Euclid-based proof that the trapezium is cyclic – text in [Rudio 1907: 52–55]). Above $B\Delta$ a segment similar to those over the shorter sides BA , $A\Gamma$ and $\Gamma\Delta$ is drawn. Once more, no explanation is given how this is to be done; but if we forget about the Euclidean definition of similarity of segments, it is easy – what we need is to draw through B and Δ a circle whose semidiameter is $\sqrt{3}$ times $BE = AE = \Gamma E = \Delta E$, the radii of the smaller segments. As a matter of fact, however, the argument does not even



matter of course. We should remember, however, that *Elements* VI.13 is not the only way to find the mean proportional between the sides of a rectangle. A simple cut-and-paste procedure may transform the rectangle into a gnomon (a trick familiar since the early second millennium BCE), and then application of the Pythagorean rule yields immediately the square equal to the difference between the two squares determining the gnomon – that is, equal to the gnomon, and thus to the rectangle from which it was produced.

²⁵ Giora Hon and Bernard Goldstein [2008] argue at length that the ancient Greeks had no concept of symmetry because *συμμετρία* does not mean “symmetry” (while pretending that they argue at the level of concepts and not of words). Whoever has looked at an ancient Greek temple facade (be it on location or on a photo) will be aware that this view can be safely disregarded.

require that we know how to perform the construction. Hippocrates simply shows in what follows that the lune between the outer circle and this arc, *in whatever way it may have been made*, can be squared. No worse than applying unawares the axiom of choice as done in much modern mathematics, and done for that matter by Euclid (“let a point D be taken at random on the other side of the straight line AB ...”/“... τυχόν σημείον ...”, *Elements* I.12).

The squarability of the lune is argued as follows: the area of the larger segment (still, *in whatever way it may have been made*) is three times each of the smaller segments (once more a consequence of the proportionality of areas to the squares on a characteristic linear extension); the usual kind of area arithmetic then shows the lune contained between the circle and the perimeter of the larger segment to equal the trapezium. As Simplicios observes [ed. trans. Rudin 1907: 56f], this is not even made explicit in the Eudemian text, and Simplicios reasonably believes Hippocrates to have found it too simple to be worth an explanation; after all, the principle was already explained in (D).

This was given as an example of a lune contained by a circumference larger than the semicircle. That it really *is* larger is argued thus by Hippocrates: he takes it for granted that the angle $BA\Gamma$ is obtuse (we already saw why this is obvious from the hypothetical construction of the trapezium). Therefore $B\Gamma$ must be larger in square than the sum of the squares on BA and $A\Gamma$.

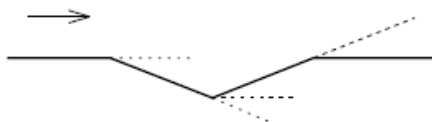
Details imminently. But first we may think of a passage in Plato's *Republic* (510C, trans. [Shorey 1930: II. 111]):

For I think you are aware that students of geometry and reckoning and such subjects first postulate the odd and the even and the various figures and three kinds of angles and other things akin to these in each branch of science, regard them as known, and, treating them as absolute assumptions, do not deign to render any further account of them to themselves or others, taking it for granted that they are obvious to everybody.

The speaker is Socrates, but since the passage comes from Book VI it has nothing to do with what the historical Socrates had said (in the case of Book I there is at least the possibility that it reflects Socrates's own opinions and perhaps his words). It is still *possible* that Plato tries to adapt the argument to what could have been said in Socrates's time. In any case, between two and five decades after Hippocrates wrote his treatise, it seems that obtuse, right and acute angles were still primitive concepts with an axiom-like status.

This goes beyond how Old Babylonian practical geometers would think. They certainly knew to distinguish “good” from “bad” corners – the former were essential in area calculations, the latter were unimportant and could be distorted *ad libitum* in field plans (cf. for example [Høyrup 2002: 103–105]). But angles as measurable quantities were known and used by Greek practical geometers (in *casu*, architects) in the sixth century BCE, as evident from Eupalinos's tunnel on Samos: having to avoid soft rock, Eupalinos at a certain point has to turn some 20° to the right; after excavating for a while in this new direction, he turns twice as much to the left, and goes on just as long in this new direction, and then turns 20° to the left (this at least is the unmistakable ideal planning; further complications of the rock has led to some deviations) – see [Kienast 1995: 142–145], and this drawing, based on Kienast's diagram but with added dotted lines and arrow indicating the digging direction:²⁶

²⁶ Weighing all available evidence (pp. 177–182), Kienast concludes that the tunnel must be dated between ca 550 and ca 530 BCE. This is too early to have received inspiration from any doctrines developed by Pythagoras after he left Samos for Croton around 530 BCE (even if we decide to disregard the strong arguments against Pythagoras

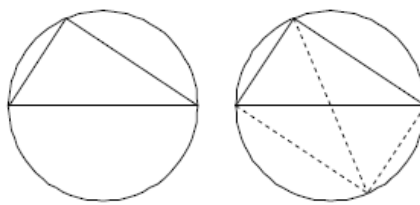


Back to Hippocrates, and perhaps to something close to his own words. If angle $BA\Gamma$ had been right, then $B\Gamma$ would have equalled BA and $A\Gamma$, all three in square. This is once again the Pythagorean rule. Now, since angle $BA\Gamma$ is obtuse, it must be larger (since it is not said by how much, this is *not* the “extended Pythagorean theorem” of *Elements* II.12, which is not needed). Therefore, the square on $B\Gamma$ exceeds the double of the squares on BA , $A\Gamma$ and $\Gamma\Delta$ taken singly. But the sum of these equals the square on $B\Delta$. Therefore, the sum of the squares on $B\Gamma$, $\Gamma\Delta$ and $B\Delta$ falls short of the sum of the squares on $B\Gamma$ and $\Gamma\Delta$, and in consequence angle $B\Gamma\Delta$ is acute. This implies, it is stated, that the lower arc of the circumscribed circle (not drawn) is smaller than a semicircle, whence the upper arc $BA\Gamma\Delta$ (a lune on which has just been squared) must be larger than a semicircle, as claimed.

What is the foundation for the latter step? We might appeal to *Elements* III.20,

In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base,

whose proof asks for nothing but a bit of arithmetic (this time concerning angles) once we know the sum of the angles of a triangle.



But less can do. All we need for Hippocrates's conclusion is that the (“circumferential” or

being a “mathematician” offered by [Junge 1907] and [Burkert 1962]). Moreover, the angle geometry involved has nothing to do with the kind of mathematical discoveries ascribed to Pythagoras by late sources (early sources ascribe none); if anything, it shows some affinity with what Proclus ascribes in the “catalogue of geometers” at a thousand years' distance to Thales [trans. Morrow 1970: 124, 195, 233, 275]. Most informative is the attribution of *Elements* I.26 to Thales (Proclus, *Commentary* 275.14–16, trans. [Morrow 1970: 275]),

If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle,

– *not* because the writer (whether Eudemos or a later redactor) knows about any written source or oral tradition telling this but because the method by which he is reported to have determined the distance of ships at sea shows that Thales *must* have used it.

Great names are fly paper, they attract attribution of great feats. In the same vein, Thales was supposed to have predicted a solar eclipse, Anaxagoras the fall of a meteorite, and Anaximander an earthquake – see [Longrigg 1976: 296, 298 n. 11]. All we can conclude is thus:

- Firstly, that *some* Greeks used angles well before Hippocrates for practical distance measurement (measured *how* we do not know – “degrees” can be safely ruled out);
- and secondly, that Eudemos or his redactor had a tendency to claim a background in proved theory for practical methods.

“inscribed”) angle spanned by a diameter seen from a point on the circumference is right. That was old knowledge, and intuitively obvious if we complete the triangle corresponding to the angle as a rectangle and draw the second diagonal, as in the preceding diagram. For symmetry reasons it is obvious that the two diagonals meet at a point which is equidistant from the four corners, and thus at the centre of the circumscribed circle, of which both are diameters. This is clearly used (without being made explicit) in the Old Babylonian text BM 85194, in two problems that determine: first the chord from a circular circumference (which, via a factor, is equivalent to the diameter) and the arrow; next the arrow from the circumference and the chord – see [Høyrup 2002: 272–274]. There is no need for transmission of this insight from the Bronze Age – rediscovery is not difficult. Once we know this, it is in any case clear that an acute angle spans less than the semicircle, as asserted.

(F) is more difficult to follow, and makes it evident that Euclid was not the first Greek mathematicians to delete the traces of his analysis, presenting only a synthesis which makes us admire the genius of the geometer but prevents us from knowing how he got his ideas (cf. also what was said about the relation between sections (A) and (D) before note 25).²⁷ Simplicios adds elaborate explanations of relatively trivial matters; the failing elucidation of an intricate point (below, paragraph after note 31) left unexplained by Eudemos makes us doubt the depth of his mathematical understanding.

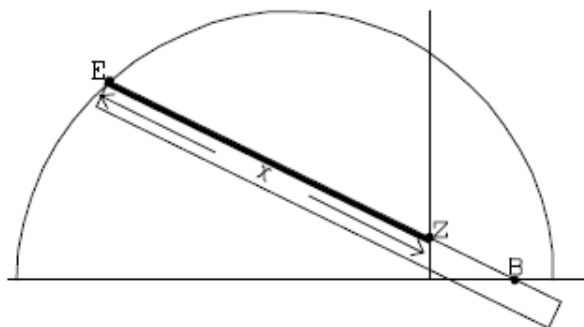
The essential circle in the argument is the one containing the arc EKBH, with the three chords EK, KB and BH being equal (as are also the appurtenant segments). But in order to determine the adequate size of these Hippocrates has to start with another (semi-)circle KAEB, with centre K and semidiameter AK = KB. In Γ , mid-point between K and B, the perpendicular $\Gamma\Delta$ is raised. Now a length x is constructed such that the square on x is $1\frac{1}{2}$ times the square on the semidiameter AK – that is, x is the mean proportional between $A\Gamma$ and AK.²⁸ Next a line is drawn through B and cutting $\Gamma\Delta$ in Z and the circular circumference in E, in such a way that ZE equals x . As pointed out by Thomas (p. 244 n. a), this is a problem of the second degree, which can be solved by ruler and compass – but apparently not by methods at Hippocrates's disposal. Instead he used a verging construction, a “mechanical” procedure where x is marked from one end of a ruler, and this end is moved along the circumference while the ruler is kept in contact with B, until the marked distance falls on the point Z (diagram next page). After the creation of the machinery of *Elements* II.1–10, the so-called “geometric algebra”, verging constructions were no longer used for problems that could be solved by ruler and compass, and they were reserved for “higher” problems; but it appears that during some earlier epoch they had been acceptable – cf. [Zeuthen 1896: 80–82]. Hippocrates in any case makes use of one.

²⁷ Speaking of Hippocrates's “analysis” does not imply that he should already have had an explicit notion of analysis versus synthesis; even Aristotle's remark in the *Nicomachean Ethics* 1112^b20–21 (trans. W. D. Ross in [Aristotle, *Works* 9],

the person who deliberates seems to investigate and analyse [about means to achieve an end] in the way described as though he were analysing a geometrical construction

does not imply more than a *practice* of making an analysis (a preliminary diagram) as the first step when trying to elucidate a geometric problem, and is in need of no metamathematical conceptualization.

²⁸ Once again, it can be constructed either directly as a mean proportional or via repeated use of the Pythagorean theorem. The formulation “ $1\frac{1}{2}$ times the square” points toward the latter possibility but hardly proves it.



Once Z is found, simpler steps follow. An isosceles trapezium EKBHE is constructed, whose diagonals intersect in Z. That it is isosceles is neither argued nor stated explicitly, but it follows from the fact that Z is located on the perpendicular bisector of KB. “This being so”, the trapezium can be inscribed in a circle. Simplicios gives a proof (a full page in Rudin’s translation), but we have already seen how it follows from simple considerations, and there is no reason to suppose that Hippocrates did, and then hid, something more complex. Since this is the second time he performs this inscription without argument, the possibility to inscribe an isosceles trapezium in a circle appears to have been one of Hippocrates’s “elements”, at least in the first sense of the Proclus-Euclid quotation after note 2.²⁹

Next a circular arc is circumscribed about EZH, producing two segments EZ and ZH. These, it is stated, are “clearly” similar to those on EK, KB and BH, and accordingly no explanation is given by Hippocrates/Eudemos. Nor does Simplicios tell his readers why, and we may suppose he did not know. If we presuppose that similar segments correspond to similar inscribed angles, it is easily seen that the segment on EK is similar to that on EZ: the point E is located on both circle circumferences, and the angles EHZ are EHK are identical; but these are the inscribed angles for the segments on EZ and EK, which therefore are similar. If Hippocrates did not know about the relation between segments and inscribed angles, he must have found it evident in some other way. As observed on p. 174, however, simple angle arithmetic combined with knowledge of the sum of angles in a triangle leads easily to the necessary insight; this *may* be another one of Hippocrates's “elements”. Since EK and KB are radii in the same circle, they are equal, and for symmetry reasons also equal to HB. Also for symmetry reasons, EZ is equal to ZH. Moreover, $EZ^2 = \frac{3}{2} EK^2$. The usual area arithmetic now shows that the lune EKBHZE has the same area as the polygon EKBHZE.

That this lune has its outer circumference smaller than a semicircle is argued in (H) in a way similar to what was done in (E). No argument is given for the crucial inequality $KB^2 > 2BZ^2$, but as shown in note 21 this is fairly evident and may have been considered thus by Hippocrates (which shows something about the way he argues).

The claim in (I), whether indeed made by Eudemos or straightened by Simplicios, is obviously not justified, at least not if we translate $\pi\acute{\alpha}\nu\tau\alpha$ $\mu\eta\gamma\acute{\iota}\sigma\kappa\omicron\nu$ as “every lune”. We may expand the translation as “every [kind of] lune”, and then things are not quite as bad. In any case, the passage

²⁹ Actually, in the manuscript the passage translated “this being so, I say that the trapezium EKBH can be comprehended in a circle” follows after the first paragraph of (G). Since Rudio this has been supposed to be a copyist’s blunder. Netz [2004: 250, 282 n.36] argues that the manuscript is impeccable, and that the passage represents second thoughts added by Eudemos; if this is so, Hippocrates took the existence of the circumscribed circle to be so evident that he did not even mention it (but it may have been shown in his diagram, as it is in that of Simplicios).

says nothing about what Hippocrates himself thought he had done, nor about what he ultimately aimed at – and neither Eudemos nor Simplicios probably knew better than we do. If Hippocrates thought he was on the way of squaring the circle, we may think of the idea as “Hippocrates's conjecture”, and thus put it at the same level as Goldbach's conjecture and the Riemann hypothesis. Yet even this would approach circularity – “since Hippocrates was a good mathematician, he has to be judged according to the criteria by which we judge good mathematicians”.

So, let us return to (J), which speaks more directly about what Hippocrates is supposed to have done. Just as (D) has a clear affinity to (A), (J) is somehow related to (B): on one hand, it builds the argument around (here two) regular hexagons; on the other, it does not square a lune but a circle together with a lune. The proof, once again, makes use of these principles:

- the Pythagorean rule;
- the basic geometry of the regular hexagon;
- the proportionality of similar areas to the squares on a characteristic linear extension;
- and simple arithmetic of areas.

It thus teaches us nothing new.

III. Hippocrates's “elements” and arguments

Simplicios did not just add Euclidean references and Euclidean proofs to Eudemos's text, cf. note 12. It is therefore not automatically clear which arguments come from Simplicios and which from Eudemos³⁰ – and, behind these, what from Eudemos and what perhaps from Hippocrates's original text. We can probably trust Simplicios when he only speaks of adding and not of removing (except perhaps when this is implied in some reshaping of an argument); it is less certain that Eudemos's text removes nothing.

With this proviso, we may try to sum up what we have come to recognize as Hippocratean “elements” in either the first or the second sense (the text does not allow us to distinguish, in particular because many are implicit only in Eudemos's report); in Ken Saito's terminology (e.g., [Saito 1997]), we shall try to identify Hippocrates's “toolbox”. Next, we may say something about the style of his arguments.

First of all, we notice that there is no reason to distinguish Alexander's Hippocrates and Eudemos's Hippocrates on this account. Evidently, Eudemos's version is more extensive, and accordingly more informative, but that is the only difference.

Ever-recurrent are:

- the Pythagorean rule;
- the proportionality of similar areas to the squares on a characteristic linear extension;
- and simple arithmetic of areas.

To this comes, in (B) and (J), some basic properties of the regular hexagon, and in (A) and (D) basic properties of the square and the isosceles right triangle. These, as well as the three preceding

³⁰ Not automatically, but mostly with little doubt when a simple linguistic observation is taken into account [Becker 1936: 415–417; Netz 2004: 255]: passages that indubitably come from Eudemos speak (e.g.) of “the line on AB”, as Aristotle would do; text which indubitably comes from Simplicios's hand speaks of “the line AB”. That allows us to ascribe most dubious passages to one or the other with fair certainty (note 22 deals with an exception, a passage in which no lettered lines are spoken about).

points, were known since well above a millennium in Near Eastern practical and scribal geometry.

Also known, but perhaps reinvented in a new context where quantified angles were made use of (a context which we have seen reflected in the Eupalinos tunnel), is knowledge that the inscribed angle corresponding to a circular diameter is right.

Even more recent in origin is the construction of bisecting perpendiculars which seems to inhere in the construction of various symmetric trapezia. To draw a perpendicular from a point on or outside a given line could be done by means of a set square, as had been done before. But for the *bisecting* perpendicular Oinopides's construction is needed, unless a foldable string be used.³¹

Time and again, the Pythagorean rule and the right angle enter in arguments of type “more than” or “less than”. There is not trace of the “extended Pythagorean theorem” (*Elements* II.12–13), but repeatedly it is concluded that if the sum of the squares on two sides of a triangle exceeds the square on the third side, then the angle they enclose is obtuse; similarly, if an inscribed angle is obtuse, then the enclosing segment is smaller than a semicircle. There are some indications in the text that the constancy of inscribed angles containing a given segment is also used to determine the similarity of segments in circles with different diameters, but since the principle is never appealed to explicitly it is not excluded that Hippocrates had a different argument in mind.

To this we may add the possibility to construct isosceles trapezia from their sides and the insight that these can be inscribed in circles; but both follow from the preceding combined with considerations of symmetry (and the possibility to use a compass).

And finally, the verging technique, unexplained and therefore almost certainly meant to be performed mechanically with a ruler as shown in the figure; and possibly but not necessarily the construction of a mean proportional, cf. note 26.

All in all, a very restricted set of tools.

We may confront it with the hypothetical description of the contents of Hippocrates's *Elements* (supposed to be a coherent treatise) presented in [Bulmer-Thomas 1971: 414–416] – basically, all of Euclid's *Elements*, excepted books V and X. Bulmer-Thomas bases himself on two premises. Firstly, everything ascribed to (often undated) Pythagoreans in late sources is supposed to be really Pythagorean, created moreover by the first generation of μαθηματικοί and therefore known by Hippocrates.³² Secondly, if Hippocrates does something which can be justified by means of Euclid's proposition Z, and Euclid proves this by means of propositions X and Y, then these three are supposed all to have been known by Hippocrates and included in *his* version of the *Elements*.

The alternative, which allows us to see Hippocrates's toolbox as here identified as the likely core of his collection of “elements” (whether a systematic treatise or not) is to think of him as arguing from the “locally obvious” – a notion I have suggested in [Høyrup 2019] and explained as

³¹ The fact that the method is never explained might reflect that Hippocrates saw the construction of the perpendicular bisector in itself as a primary operation, not to be argued from other knowledge, which could be of one or the other kind. *Might*.

³² For example, the ascription of the application of areas with excess and deficiency to the “Pythagorean muse” (Proclus, *Commentary* 419.16) leads to the claim that the whole of *Elements* II (as well as *Elements* I) were substantially included in Hippocrates's treatise. In particular, since Hippocrates's verging construction can be replaced by a trick based on *Elements* II.6, this must be how Hippocrates did it (viciously hiding his knowledge by referring instead to the verging).

“presuppositions which the interlocutor – or, in case of writing, the imagined or ‘model’ reader – will accept as obvious”. As an illustrative example (fitting because *we* do not see it as an obvious starting point for a mathematical argument) I chose there the use of the rule of three in Italian late medieval *abbacus* mathematics. *We* will find that the rule

If some computation was said to us in which three things are proposed, then we shall multiply the thing that we want to know with the one which is not of the same (kind), and divide in the other³³

is in itself in need of explanation, and we will readily find one using first-degree algebra or, if we want to stay traditional, basic proportion theory. Many *abbacus* authors, instead, would solve other problems involving proportionality (composite interest, expansion or reduction of fractions, etc.) via an appeal to the rule of three.³⁴ The reason that the opaque rule (opaque because the intermediate step has no concrete meaning) could remain in use for centuries (not only two centuries but at least 17 if we take the Indian origin in account, see [Høyrup 2012]) was that it was used so often first in the *abbacus* school and then in practical computational life (and before that, in Indian and Arabic commercial practice) that it was accepted as obvious by everybody in the environment. “Proceed, and faith will come to you”, as d’Alembert is supposed to have said – we might indeed have taken pre-Cauchy analysis as an example of the “locally obvious” instead of the rule of three.

We should remember that Hippocrates worked only half a generation or so after Oinopides, and thus when something as simple as the erection of a perpendicular by means of ruler and compass was a recent discovery. Hippocrates may well, as suggested by Netz [2004: 246f] have been the first to *write* about geometry; at least he will have been one of the very first. Before him, there will have been no time to develop anything like an axiomatic framework (which cannot exist without a persistent support, that is, writing sufficiently developed to carry a logical argument). As we may observe, there is not the slightest reference to a definition in the Eudemos text,³⁵ all arguments (when they are there at all and not tacitly omitted) are single-level, directly based on the tools from the box listed above.

Hippocrates need not have used *all* the tools he was familiar with in his work on the lunes – here as mostly, the absence of evidence should not be mistaken for evidence of absence; if he was the one who showed the equivalence of the doubling of a cube and the finding of two mean proportionals,³⁶ something more will have been needed. Beyond that, however, we have no means to substantiate any guess about what else he knew and would use on other occasions. Even his verging

³³ My literal translation from [Arrighi 1989: 9]. With no or at most minimal variations, this formulation from ca 1300 remained in use for two centuries.

³⁴ [Høyrup 2019: 27–29] goes through a specific example in detail.

³⁵ Unless, of course, we count the explanation of what is meant by “similar segments” and follow Rudio’s acceptance of this dubious passage as Eudemian – cf. note 11. Even then the formulation looks more like that of a commentator like Eudemos than as something said by Hippocrates himself.

³⁶ The work on lunes offers so far unnoticed support for this ascription. In either version the problem of squaring a circle is reduced to that of squaring a lune. Whereas we have no traces of axiomatics in Hippocrates’s text it seems obvious that he tried to approach difficult problems through transformation into something different and perhaps more approachable.

construction does not prove definitively that he was ignorant of the application of areas with deficiency and excess – he *may* in principle have used the verging construction not because he was ignorant of alternatives but because he considered the verging legitimate, and perhaps because it fell naturally in continuation of his analysis.

After so many pages, the reader may sigh, *parturiunt montes, nascetur ridiculus mus*, “the mountains are in labour, a ridiculous mouse is born”³⁷ – but after all, this mouse is the earliest evidence we have of Greek theoretical geometry, much more informative than Neoplatonic and Neopythagorean fables about Pythagoras. It also makes much more sense than these and the narratives about Greek mathematics having sprung like Athena full-clad in armour from Zeus's head;³⁸ it suggests how *locally* argued mathematics, once it came into writing and was practised with critical questioning, over a century gave rise to the intention to produce an axiomatic system as delineated in Aristotle's *Posterior analytic* and as realized by Euclid – becoming in the end an ideology inducing Simplicios and so many later historians to believe that even Hippocrates had made that kind of mathematics, rendered unfortunately by Eudemos “according to archaic custom” with “concise proofs”; cf. [Høyrup 2019].

IV. Alexander and Eudemos

Simplicios first relates what Alexander of Aphrodisias had reported about Hippocrates's work on lunes, and then what Eudemos has to say – making it shine through (see the quotation on p. 166) that this is the better source, and that Simplicios himself is thus better informed than Alexander (they are, after all, competing commentators on the same Aristotelian passage). Just after (J), what shines through in the beginning is made explicit (translated from [Rudio 1907: 75f]):

But for knowing that which concerns Hippocrates of Chios, higher standing must be given to Eudemos, since he was closer to him in time and had listened to Aristotle.³⁹

³⁷ Horace, *The Art of Poetry* 139, [ed. Nisard 1869: 174].

³⁸ A passage in [Allman 1889: 63f] comes to mind:

Hankel expresses surprise at the fact that this oldest fragment of Greek geometry – 150 years older than Euclid's *Elements* – already bears that character, typically fixed by the latter, which is so peculiar to the geometry of the Greeks.

Fancy a naturalist finding a fragment of the skeleton of some animal which had become extinct, but of which there were living representatives in a higher state of development; and fancy him improving the portion of the skeleton in his hands by making additions to it, so that it might be more like the skeleton of the living animal; then fancy other naturalists examining the improved fragment with so little attention as to exclaim : “Dear me! how strange it is that the two should be so perfectly alike!”.

Hermann Hankel wrote his history of ancient and medieval mathematics in [1874], relying on Bretschneider and therefore not distinguishing between the Simplicios- and the Eudemos-layers of the text. But Allman's mockery has retained much of its validity in later times.

³⁹ That Eudemos had listened to Aristotle is obviously only relevant for understanding that passage in Aristotle which both Alexander and Simplicios comment upon (the question discussed in [Lloyd 1987]), not for deciding which version of Hippocrates's work is genuine.

As it is obvious from the above, however, what Alexander offers is not a simpler or a distorted version of what can be read in Eudemos. The two texts are different and independent of each other, though clearly dealing with the same topic and using the same conceptual tools. The difference between them should be informative.

The print world was (and still is) familiar with “first” and “second, revised and augmented” editions; in the digital world we have the experience of “preliminary versions” which end up in odd corners on the web from where the author has no possibility to remove them.

Neither of these was the situation of Hippocrates; his was the world of incipient manuscript culture, where any preliminary copy of a treatise lent to somebody might start to circulate on its own as long as somebody found it interesting enough to copy it or have it copied. Later in Greco-Roman Antiquity, where booksellers made it possible to buy more or less standardized versions, private copying remained important – not least for scientific and philosophical books, where the public was too restricted to make it worthwhile for a bookseller to keep a master copy [Blanck 2008: 161–165]. However, the relation between the two versions of Hippocrates's work on the lunes does not suggest them to be simply diverging descendants of the same text.

We may recall that Hippocrates is supposed to have taught a school, a group of young people “around” him. In this connection, it may not be totally frivolous to think: firstly, of the relation between Plato's esoteric and exoteric teachings, of which the former may only have reached the written stage through indirect reflections in the exoteric dialogues and through the hands of followers, for example in the “Seventh Letter” (cf. [Gaiser 1968: 3 and *passim*]) – not to speak of the occasional polemics of Aristotle, the run-away follower; and secondly, of the relation between the Aristotelian *πραγματεῖα*, the systematic treatises reflecting the school lectures, and the writings directed at a larger public (lost, except for fragments).

From a later time, but closer in genre, we should notice that Galen (trans. P. E. Easterling, in [Easterling & Know 1985: 20])

describes how the working notes he gave his pupils for their private use were subjected to wholesale distortion and alteration and circulated commercially as genuine works of the master.

There is no reason to suspect Hippocrates of having taught in his school a doctrine different in character from what went into a published book; but any teacher who has written a book on the basis of lectures made earlier on will know that the two become different. It could well be that what Alexander reports is an early version of the investigation of the lunes (perhaps based on what was written down by listeners, perhaps on notes prepared by Hippocrates himself), while Eudemos has access to a revised and much expanded version published afterwards.⁴⁰ Also possible but wholly unlikely (for one thing in view of the relation between (A) and (D) as discussed on p. 171) is the opposite order.

In any case, the existence of two parallel versions of the same investigation may be another trace of Hippocrates having had a school “around” him – an indirect trace, certainly, but this time a trace of a school teaching geometry and not only astronomy.

Conflict of interest

The author declares no conflicts of interest in this paper.

⁴⁰ In this case, Alexander may have erred in attributing the fallacy to Hippocrates, because a student may have been responsible for the over-enthusiastic interpretation of what had been achieved; cf. note 8.

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