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Round and round it goes

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Round and round it goes

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Abstract

We discuss the coupling of a rotating flywheel with a nonrotating flywheel. We find an expression for the tangential force between the wheels and an expression for the change in angular momentum of the system. Then we calculate the fraction of rotational kinetic energy which is transferred from the rotating flywheel to the nonrotating flywheel. The theoretical results are compared with the experimental results in the article Mário S M N F Gomes et al 2018 The 'Spinning disk touches stationary disk' problem revisited: an experimental approach *Eur. J. Phys.* **39** 045709. We then introduce a series of initially nonrotating flywheels between the two original flywheels which are coupled and decoupled beginnning with coupling between the original rotating flywheel and the first wheel in the series and finally coupling between the last flywheel in the series with the original nonrotating flywheel. The condition for maximum energy transfer from the original rotating flywheel to the original nonrotating flywheel for a given number of flywheels will be shown to be achieved when the masses of the flywheels constitute a geometric series if all the flywheels are of the same type. If the flywheels are not of the same type the maximum is achieved when the product of the K–factors of the inertial moments and the masses constitute a geometric series.

1 Introduction

When two flywheels are mechanically coupled to one another there will be som slippage between the two wheels before the points at the periphery achieve the same speed (circumferential speed). We will assume that one of the wheels initially is rotating about its axis and the other has zero angular velocity about its axis. The parameters of the wheels determine how much rotational kinetic energy can be transferred from the rotating wheel to the nonrotating wheel. If we want to transfer energy via more than one wheel, we could ask: What is the condition for transferring maximally energy to the nonrotating wheel?

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In section 2 we just couple two wheels [1], [2] , [3] , [4] and find the ratio of the energy transferred to the nonrotating wheel and the initial rotational kinetic energy of the other wheel. Then we compare some of the theoretical results obtained here with the experimental results in reference 1.
In section 3 we couple and decouple a series of flywheels between the two original wheels and find the condition for the parameters of these flywheels, so that the transfer of energy to the originally nonrotating wheel is as big as possible.
In section 4 we generalize the results obtained in section 2 and 3.
In the article we proceed step by step from the simple case to the more general case.

2 Coupling of two flywheels

We consider two flywheels A and B. A is rotating without friction about the axis a-a with angular velocity Ω_o directed in the positive z-direction. The z-axis is parallel to a-a. The mass of A is M, its moment of inertia about the axis a-a is I and its radius is R. The flywheel B has to begin with zero angular velocity. Its mass is $\mathcal{M}$, its moment of inertia about the axis b-b is $\mathcal{I}$ and its radius is $\mathcal{R}$. The axis b-b is parallel to the axis a-a. B can rotate freely about the axis b-b. We assume the z-axis is vertical. See Fig. 1.

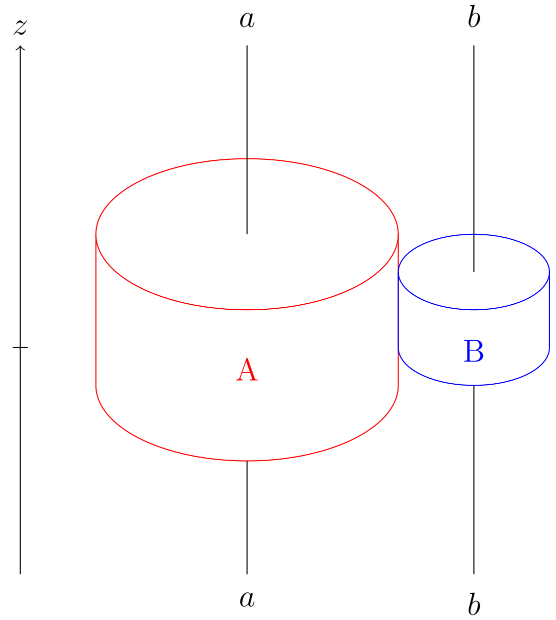

Figure 1: Rotating flywheel A coupled to nonrotating flywheel B.

The moments of inertia of the axles are negligible compared to the moments of inertia of the flywheels.
The flywheels are brought in contact with each other. See Fig. 2.

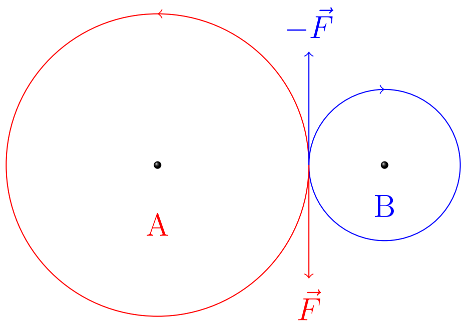

Figure 2: The two flywheels seen in direction of the negative z-axis. The red force attacks the red wheel. The blue force attacks the blue wheel.

We have the following forces on A: A friction force $\vec{F}$ (red in Fig. 2) from B in the horizontal direction orthogonal to the direction from its point of attack to the axis a-a, a horizontal normal force from B directed from the point of contact between the wheels to a-a, a reaction force from the axle and of course the force of gravity. The force of gravity is cancelled by a vertical normal force. The length of $\vec{F}$ is F. The flywheel A has no translational motion, so the total force on A is zero. For the flywheel B we have similar forces. The friction force on B from A is $-\vec{F}$ (blue in Fig. 2) according to Newton's third law.
We will consider the situation where there is a force of friction F between the two wheels as long as there is a difference in velocity of the point of contact of wheel A and the velocity of the point of contact of wheel B. Referring to the situation concidered in reference 1, we assume that the normal force between the wheels and hence the frictional force between the wheels have a magnitude that does not stop the wheels from rotating: Wheel A is still rotating with an angular velocity in the original direction and wheel B is rotating with an angular velocity in the opposite direction. When the points of contact of the two wheels have the same velocity there is no slippage between the wheels and hence no frictional force between the wheels.
We will now use the angular momentum theorem for each of the wheels. For A we take the moments about the axis a-a and for B about the axis b-b. The

angular velocity of A is Ω and the angular velocity of B is ω. We get

$$I\,\frac{d\Omega}{dt} = -R\,F \tag{1}$$

$$\mathcal{I}\,\frac{d\omega}{dt} = -\mathcal{R}\,F \tag{2}$$

The force F is unknown and might be a function of time. By integrating eqs. (1) and (2) and using the initial conditions $\Omega = \Omega_o$ and $\omega = 0$ we have

$$\Omega(t) = -\frac{R}{I}\int_0^t F(t)dt + \Omega_o \tag{3}$$

$$\omega(t) = -\frac{\mathcal{R}}{\mathcal{I}}\int_0^t F(t)dt \tag{4}$$

When the two wheels have the same circumferential speed at the time T we have $R\,\Omega(T) = -\mathcal{R}\,\omega(T)$. This condition determines the integral

$$\int_0^T F(t)dt = \frac{R\,\Omega_o}{\frac{R^2}{I} + \frac{\mathcal{R}^2}{\mathcal{I}}} \tag{5}$$

We can then use this result for the integral in equations (3) and (4) and find the final angular velocities of the wheels, when they have the same circumferential speed

$$\Omega = \frac{1}{1 + \frac{\mathcal{I}\,R^2}{I\,\mathcal{R}^2}}\,\Omega_o \tag{6}$$

$$\omega = -\frac{\frac{R}{\mathcal{R}}}{1 + \frac{\mathcal{I}\,R^2}{I\,\mathcal{R}^2}}\,\Omega_o \tag{7}$$

The right hand side of equation (5) is a fixed number given by the initial angular velocity of wheel A and the characteristic parameters of the two wheels. That means that the integral for the force F over time from $t = 0$ to $t = T$, where the two wheels have the same circumferential speed, is totally independent of how the tangential force F is obtained and of its variation in time.

The time average $\langle F\rangle$ of F is

$$\langle F\rangle = \frac{R\,\Omega_o}{T\left(\frac{R^2}{I} + \frac{\mathcal{R}^2}{\mathcal{I}}\right)} \tag{8}$$

If we assume that F is independent of time, that is $F = F_{\mathrm{const}}$, then eq. (5) implies the following relation between T and F_{const}

$$F_{\mathrm{const}}\, T = \frac{R\,\Omega_o}{\frac{R^2}{I} + \frac{\mathcal{R}^2}{\mathcal{I}}} \tag{9}$$

If F_{const} is known from the normal force between A and B and the coefficient of friction for the surfaces of the flywheels we can calculate T. On the other hand, if T is known, we can calculate F_{const} from this relation.

Some numbers. In the following we will compare our theoretical results with the experimental results in reference 1 under the assumption that the wheels are rotating freely and are not subject to air resistance or resistance from any other media. According to reference 1 the friction is indeed small which also can be seen from Figure 3b in reference 1 which shows that the angular velocities of the two wheels are approximately constant before the wheels are brought in contact with one another and also after they have achieved the same circumferential speed.
Assuming that F is constant and by using the numbers in reference 1: $2\,R = 22.8\,\mathrm{cm}$, $2\,\mathcal{R} = 7.00\,\mathrm{cm}$, $I = 0.0091\,\mathrm{kg\,m^2}$ and $\mathcal{I} = 0.000\,63\,\mathrm{kg\,m^2}$ and the following two numbers estimated from Figure 3b in the article $R\,\Omega_o = 1.8\,\mathrm{rad\,s^{-1}\,m}$ and $T = 5.4\,\mathrm{s} - 2.6\,\mathrm{s} = 2.8\,\mathrm{s}$ we can now calculate F_{const} and the result is

$$F_{\mathrm{const}} = 0.19\,\mathrm{N} \tag{10}$$

in good agreement with the value $F_{\mathrm{const}} = 0.20\,\mathrm{N}$ found in the article by estimating the angular accelerations of the wheels.

Nonconservation of angular momentum. The total angular momentum of the system to begin with is simply

$$L_{\mathrm{i}} = I\,\Omega_o \tag{11}$$

and the final total angular momentum is

$$L_{\mathrm{f}} = I\,\Omega + \mathcal{I}\,\omega = \frac{1 - \frac{\mathcal{I}\,R}{I\,\mathcal{R}}}{1 + \frac{\mathcal{I}\,R^2}{I\,\mathcal{R}^2}}\, L_{\mathrm{i}} \tag{12}$$

both in the z-direction. From that we clearly see that the angular momentum is not conserved by the coupling of the two wheels. Furthermore, if

$$\frac{\mathcal{I}\,R}{I\,\mathcal{R}} > 1 \tag{13}$$

the angular momentum changes sign.
From equation (12) the change ΔL in the angular momentum is

$$\Delta L = -\frac{(R+\mathcal{R})\, R\, \Omega_o}{\frac{R^2}{I} + \frac{\mathcal{R}^2}{\mathcal{I}}} \tag{14}$$

This can also be seen from the angular momentum theorem for the whole system. The following calculation will confirm this. Moments are taken about the axis a-a. The only force contributing to the torque τ is the reaction force from the axle b-b. This force cancels the blue force in Fig. 2. Thus we have

$$\frac{dL}{dt} = -(R+\mathcal{R})\, F \tag{15}$$

which by integration gives

$$\Delta L = -(R+\mathcal{R}) \int_0^T F(t)dt \tag{16}$$

The integral is known from equation (5) and using this result we again find the change in angular momentum as given by equation (14).
For the relative change in angular momentum for the system we have

$$\frac{\Delta L}{L_{\mathrm{i}}} = -\frac{1+\frac{\mathcal{R}}{R}}{1+\frac{I\,\mathcal{R}^2}{\mathcal{I}\,R^2}} \tag{17}$$

If both flywheels are of the same type with moments of inertia about their axis of rotation of the form $\mathscr{I} = \mathscr{K}\mathscr{M}\mathscr{R}^2$, the relative change in angular momentum can be written as

$$\frac{\Delta L}{L_{\mathrm{i}}} = -\frac{1+\frac{\mathcal{R}}{R}}{1+\frac{M}{\mathcal{M}}} \tag{18}$$

The absolute value of this fraction increases for $\frac{M}{\mathcal{M}}$ decreasing and it increases for $\frac{\mathcal{R}}{R}$ increasing. Both results are intuitively reasonable.

Some numbers. From Figure 4b in reference 1 we have $L_{\mathrm{i}} = 0.13\,\mathrm{kg\,m^2\,s^{-1}}$ and $L_{\mathrm{f}} = 0.06\,\mathrm{kg\,m^2\,s^{-1}}$. Thus we have

$$\frac{\Delta L}{L_{\mathrm{i}}} = -0.54 \tag{19}$$

For the theoretical value in equation (17) we find, with the numbers given in reference 1

$$\frac{\Delta L}{L_{\mathrm{i}}} = -0.55 \tag{20}$$

This number is also in good agreement with the experimental value in equation (19).

Transfer of energy. Using equation (7), the final rotational kinetic energy E_{Bf} of B is

$$E_{\text{Bf}} = \tfrac{1}{2}\,\mathcal{I}\,\omega^2 = \frac{\frac{\mathcal{I}\,R^2}{I\,\mathcal{R}^2}}{\left(1+\frac{\mathcal{I}\,R^2}{I\,\mathcal{R}^2}\right)^2}\,E_{\text{Ai}} \tag{21}$$

where $E_{\text{Ai}} = \frac{1}{2}\,I\,\Omega_o^2$ is the initial rotational kinetic energy of A. The fraction of energy transferred to B is

$$\frac{E_{\text{Bf}}}{E_{\text{Ai}}} = \frac{\frac{\mathcal{I}\,R^2}{I\,\mathcal{R}^2}}{\left(1+\frac{\mathcal{I}\,R^2}{I\,\mathcal{R}^2}\right)^2} \tag{22}$$

which is a function of $\frac{\mathcal{I}\,R^2}{I\,\mathcal{R}^2}$. Its maximum is achieved for $\frac{\mathcal{I}\,R^2}{I\,\mathcal{R}^2} = 1$ and has the maximum value

$$\left(\frac{E_{\text{Bf}}}{E_{\text{Ai}}}\right)_{\text{max}} = \frac{1}{4} \tag{23}$$

If the moments of inertia have the form $I = K\,M\,R^2$ and $\mathcal{I} = \mathcal{K}\,\mathcal{M}\,\mathcal{R}^2$, the radii cancel in equation (22), and if furthermore $K = \mathcal{K}$, that is if A and B are of the same type (homogeneous cylinders, homogeneous spheres, spherical shells), the fraction in equation (22) only depends on the ratio of the masses

$$\frac{E_{\text{Bf}}}{E_{\text{Ai}}} = \frac{\frac{\mathcal{M}}{M}}{\left(1+\frac{\mathcal{M}}{M}\right)^2} = \frac{\frac{M}{\mathcal{M}}}{\left(1+\frac{M}{\mathcal{M}}\right)^2} \tag{24}$$

and in this case the maximum is achieved for $M = \mathcal{M}$.

Some numbers. From Figure 4a in reference 1 we can see that $E_{\text{Ai}} = 1.05\,\text{J}$ and $E_{\text{Bf}} = 0.25\,\text{J}$. This gives

$$\frac{E_{\text{Bf}}}{E_{\text{Ai}}} = 0.24 \tag{25}$$

For the theoretical value given in equation (22) with the numbers from reference 1 inserted we find

$$\frac{E_{\text{Bf}}}{E_{\text{Ai}}} = 0.24 \tag{26}$$

in pretty good agreement with the experimental value in equation (26) [5]. The values obtained from the figures in reference 1 are of course to be taken with a grain of salt.

Symmetry. In the following we will encounter examples of the function $s_{\rm m}$ defined by

$$s_{\rm m}(x) = \frac{x}{\left(1 + x^{\frac{1}{m}}\right)^{2\,m}} \quad \text{for} \quad x > 0 \quad \text{and} \quad m = 1,\, 2,\, 3\, \ldots \tag{27}$$

This function has the property that

$$s_{\rm m}(\tfrac{1}{x}) = s_{\rm m}(x) \tag{28}$$

which is seen by the calculation

$$\begin{aligned} s_{\rm m}(x) &= \frac{\frac{1}{x^2}\,x}{\frac{1}{x^2}\left(1 + x^{\frac{1}{m}}\right)^{2\,m}} \\[2ex] &= \frac{\frac{1}{x}}{\left(\left(\frac{1}{x}\right)^{\frac{1}{m}}\right)^{2\,m}\left(1 + x^{\frac{1}{m}}\right)^{2\,m}} \\[2ex] &= \frac{\frac{1}{x}}{\left(1 + \left(\frac{1}{x}\right)^{\frac{1}{m}}\right)^{2\,m}} = s_{\rm m}(\tfrac{1}{x}) \end{aligned} \tag{29}$$

The right–hand side of equation (24) is an example of a function of this type with $x = \frac{M}{\mathcal{M}}$ and $\mathrm{m} = 1$. The physical meaning of this is that the ratio $\frac{E_{\rm Bf}}{E_{\rm Ai}}$ would be the same, if the wheel A and the wheel B were interchanged, in which case the masses M and $\mathcal{M}$ are interchanged.

3 Coupling, decoupling, coupling, decoupling, coupling, ...

We will now study what will happen, if more and more wheels are coupled. The axes about which the wheels can rotate freely are all parallel to each other. To begin with, wheel A and wheel 1 are coupled. When they have the same circumferential speed they are decoupled. Then wheel 1 and wheel 2 are coupled, and when they have the same circumferential speed they are decoupled. We continue this way, until wheel n and wheel B have the same circumferential speed. The initial angular velocity of A is Ω_o and all other wheels have zero initial angular velocity. We will only consider flywheels of the same type and with moments of inertia about their axis of the form

$\mathscr{I} = \mathscr{K}\mathscr{M}\mathscr{R}^2$. For fixed masses of A and B, M and $\mathcal{M}$, we want to find the masses M_1, M_2, ... , M_n of the wheels between A and B which give maximum transfer of rotational kinetic energy from A to B.
In the following, E_Ai is the initial rotational kinetic energy of A, and E_Bf is the final rotational kinetic energy of B.

3.1 One flywheel between A and B

Now we put one flywheel, flywheel 1, between A and B. When A and 1 have the same circumferential speed, they are decoupled. The rotational kinetic energy of 1 is $E_1^\star$ (a $\star$ on an energy means an intermediate result), and from equation (24) we get

$$\frac{E_1^\star}{E_{A\mathrm{i}}} = \frac{\frac{M}{M_1}}{(1+\frac{M}{M_1})^2} \tag{30}$$

Then we couple 1 and B, and when they have the same circumferential speed, we have again, by using equation (24)

$$\frac{E_{B\mathrm{f}}}{E_1^\star} = \frac{\frac{M_1}{\mathcal{M}}}{(1+\frac{M_1}{\mathcal{M}})^2} \tag{31}$$

From the last two equations we conclude

$$\frac{E_{B\mathrm{f}}}{E_{A\mathrm{i}}} = \frac{E_{B\mathrm{f}}}{E_1^\star}\,\frac{E_1^\star}{E_{A\mathrm{i}}} = \frac{\frac{M}{\mathcal{M}}}{(1+\frac{M}{M_1})^2\,(1+\frac{M_1}{\mathcal{M}})^2} \tag{32}$$

This is the fraction we want to maximize for given values of M and $\mathcal{M}$. The numerator is a constant, so we must minimize the denominator. That is, we want to find the minimum for the function g_1 with respect to M_1, where the function g_1 is defined by

$$g_1(M_1) = (1+\tfrac{M}{M_1})\,(1+\tfrac{M_1}{\mathcal{M}}) \tag{33}$$

By differentiating g_1 with respect to M_1 we find

$$\frac{dg_1}{dM_1} = \frac{1}{\mathcal{M}} - \frac{M}{M_1^2} \tag{34}$$

Thus we find that the minimum for g_1 is achieved for $M_1 = \sqrt{M\,\mathcal{M}}$. This is the value of M_1 we must choose to get maximal transfer of rotational kinetic energy from A to B via the wheel 1.
Notice that $M_1 = \sqrt{M\,\mathcal{M}}$ is equivalent to

$$\frac{M_1}{M} = \frac{\mathcal{M}}{M_1} \tag{35}$$

By using equation (32) we can calculate the maximum for the fraction of energy transferred from A to B

$$\left(\frac{E_{B\mathrm{f}}}{E_{A\mathrm{i}}}\right)_{\max} = \frac{\frac{M}{\mathcal{M}}}{\left(1+\sqrt{\frac{M}{\mathcal{M}}}\right)^4} \tag{36}$$

The right–hand side of equation (36) is again a function of the type defined in equation (25) with $x = \frac{M}{\mathcal{M}}$ and m $= 2$. This allows us once more to interchange the two wheels, and the ratio between the energy of the last wheel and the energy of the initial rotating wheel is unchanged.

3.2 Two flywheels between A and B

We now put two flywheels, 1 and 2, between A and B. When A and 1 are decoupled, flywheel 1 has the energy $E_1^\star$, and when 1 and 2 are decoupled, flywheel 2 has the energy $E_2^\star$. As before we have

$$\frac{E_1^\star}{E_{A\mathrm{i}}} = \frac{\frac{M}{M_1}}{(1+\frac{M}{M_1})^2} \tag{37}$$

Then 1 and 2 are coupled. When they have the same circumferential speed, we have

$$\frac{E_2^\star}{E_1^\star} = \frac{\frac{M_1}{M_2}}{(1+\frac{M_1}{M_2})^2} \tag{38}$$

Finally 2 and B are coupled, and when they have the same circumferential speed we get

$$\frac{E_{\mathrm{Bf}}}{E_2^\star} = \frac{\frac{M_2}{\mathcal{M}}}{(1+\frac{M_2}{\mathcal{M}})^2} \tag{39}$$

From the equations (37), (38) and (39) we conclude

$$\frac{E_{B\mathrm{f}}}{E_{A\mathrm{i}}} = \frac{\frac{M}{\mathcal{M}}}{(1+\frac{M}{M_1})^2\,(1+\frac{M_1}{M_2})^2\,(1+\frac{M_2}{\mathcal{M}})^2} \tag{40}$$

Again we want to maximize this fraction, and we can do that by minimizing the denominator. We define the function g_2 by

$$g_2(M_1,\, M_2) = (1+\tfrac{M}{M_1})\,(1+\tfrac{M_1}{M_2})\,(1+\tfrac{M_2}{\mathcal{M}}) \tag{41}$$

In order to determine the stationary point(s) of this function, we calculate the partial derivatives of g_2

$$\frac{\partial g_2}{\partial M_1} = \frac{M_1^2 - M\,M_2}{M_1^2\,M_2}\left(1 + \frac{M_2}{\mathcal{M}}\right) \tag{42}$$

$$\frac{\partial g_2}{\partial M_2} = \left(1 + \frac{M}{M_1}\right)\frac{M_2^2 - M_1\,\mathcal{M}}{M_2^2\,\mathcal{M}} \tag{43}$$

Putting the partial derivatives equal to zero and solving the equations we find

$$M_1^2 = M\,M_2 \quad \wedge \quad M_2^2 = M_1\,\mathcal{M} \tag{44}$$

This implies

$$\frac{M_1}{M} = \frac{M_2}{M_1} = \frac{\mathcal{M}}{M_2} = q \tag{45}$$

with

$$q = \left(\frac{\mathcal{M}}{M}\right)^{\frac{1}{3}} \tag{46}$$

Thus the stationary point is

$$M_1 = \left(M^2\,\mathcal{M}\right)^{\frac{1}{3}} \quad \wedge \quad M_2 = \left(M\,\mathcal{M}^2\right)^{\frac{1}{3}} \tag{47}$$

Equation (45) shows that the masses constitute a geometric series with the common ratio q. Having determined the stationary point for the function g_2, we conclude that the stationary point is a minimum point for g_2 because $g_2(M_1,\, M_2) \to \infty$ for $M_1\,,\, M_2 \to 0$ or ∞. With this result equation (40) determines the maximum for the fraction of energy transferred from A to B via the two flywheels 1 and 2

$$\left(\frac{E_{B\mathrm{f}}}{E_{A\mathrm{i}}}\right)_{\mathrm{max}} = \frac{\frac{M}{\mathcal{M}}}{\left[1 + \left(\frac{M}{\mathcal{M}}\right)^{\frac{1}{3}}\right]^6} \tag{48}$$

Again we see from the right–hand side of equation (48) that we can interchange the wheels A and B (or the masses M and $\mathcal{M}$) because the right–hand side this time is given by the function s_3.

3.3 n flywheels between A and B

The result for two flywheels between A and B in equation (40) can easily be generalized to n flywheels

$$\frac{E_{\mathrm{Bf}}}{E_{\mathrm{Ai}}} = \frac{\frac{M}{\mathcal{M}}}{\left[\left(1+\frac{M}{M_1}\right) \times \Pi_{\mathrm{i}=1}^{n-1}\left(1+\frac{M_i}{M_{i+1}}\right) \times \left(1+\frac{M_n}{\mathcal{M}}\right)\right]^2} \tag{49}$$

In order to maximize the transfer of rotational kinetic energy from A to B we must minimize the function g_{n} defined by

$$g_{\mathrm{n}}(M_1,\, M_2,\, M_3,\, \dots,\, M_{n-1},\, M_n) = \left(1+\frac{M}{M_1}\right) \times \Pi_{\mathrm{i}=1}^{n-1}\left(1+\frac{M_i}{M_{i+1}}\right) \times \left(1+\frac{M_n}{\mathcal{M}}\right) \tag{50}$$

To find the stationary point(s) we calculate the partial derivatives of g_{n}.
For $i = 1$ we have (here and in the following POS is an expression which is always positive)

$$\frac{\partial g_{\mathrm{n}}}{\partial M_1} = \left(-\frac{M}{M_1^2}+\frac{1}{M_2}\right) \times \mathrm{POS} \tag{51}$$

For $i \neq 1$ and $i \neq n$ we have

$$\frac{\partial g_{\mathrm{n}}}{\partial M_i} = \mathrm{POS} \times \left(-\frac{M_{i-1}}{M_i^2}+\frac{1}{M_{i+1}}\right) \times \mathrm{POS} \tag{52}$$

and finally for $i = n$

$$\frac{\partial g_{\mathrm{n}}}{\partial M_n} = \mathrm{POS} \times \left(-\frac{M_{n-1}}{M_n^2}+\frac{1}{\mathcal{M}}\right) \tag{53}$$

The partial derivatives are put equal to zero. From this we get

$$\frac{M_1}{M} = \frac{M_2}{M_1} = \frac{M_3}{M_2} = \cdots = \frac{M_{n-1}}{M_{n-2}} = \frac{M_n}{M_{n-1}} = \frac{\mathcal{M}}{M_n} = q \tag{54}$$

That the stationary point is a point of minimum for the function g_{n} is a consequence of $g_{\mathrm{n}}(M_1,\, M_2,\, M_3,\, \dots,\, M_i,\, \cdots,\, M_{n-1},\, M_n) \to \infty$ for $M_i,\, M_j,\dots \to 0$ or ∞ for any $i,\, j,\, \dots$.
We conclude that we have determined the masses of the flywheels which give maximum transfer of energy from A to B. The masses of the flywheels in the general case thus also constitute a geometric series with the common ratio given by

$$q = \left(\frac{\mathcal{M}}{M}\right)^{\frac{1}{n+1}} \tag{55}$$

and the masses of the flywheels are given by

$$M_i = \left(M^{n+1-i}\,\mathcal{M}^i\right)^{\frac{1}{n+1}} \tag{56}$$

The fraction of energy that maximaly can be transferred to B from A for given masses of A and B can then be determined from equation (49) and equation (56)

$$\left(\frac{E_{B\mathrm{f}}}{E_{A\mathrm{i}}}\right)_{\max} = \frac{\frac{M}{\mathcal{M}}}{\left(1+\left(\frac{M}{\mathcal{M}}\right)^{\frac{1}{n+1}}\right)^{2\,(n+1)}} \tag{57}$$

Again we notice the symmetry of the above equation. Here the right–hand side is given by the function s_{n+1}, and we can also in this case interchange the wheels A and B or the masses of the wheels.

We found earlier that the maximum transfer of energy from one wheel to another is obtained for the speciel case that the two masses are equal. That implies that we have an upper bound for $\left(\frac{E_{B\mathrm{f}}}{E_{A\mathrm{i}}}\right)_{\max}$

$$\left(\frac{E_{B\mathrm{f}}}{E_{A\mathrm{i}}}\right)_{\max} \leq \frac{1}{2^{2\,(n+1)}} \tag{58}$$

which follows from equation (49) with $M = \mathcal{M}$ which requires $M_i = M = \mathcal{M}$ for all i. This can be seen in Fig. 3, where the maximum occurs for the mass ratio $\frac{\mathcal{M}}{M} = 1$. We see that for greater and greater number of intermediate wheels the less energy is transferred to the last wheel. For $\mathcal{M} \ll M$ or $\mathcal{M} \gg M$ the transfer of energy to B is very small which is physically intuitively clear. For mathematical reasons it is also clear because the right hand side of equation (57) is a function of the aforementioned type s_m. For functions of this type we have (see equation (28)) $s_\mathrm{m}(x) \to 0$ for $x \to 0^+$ and then by using the symmetry $s_\mathrm{m}(\frac{1}{x}) = s_\mathrm{m}(x)$ we also have $s_\mathrm{m}(x) \to 0$ for $x \to \infty$.

A comparison with linear collisions. The result obtained so far is similar to the result for energy transfer in one-dimensional collisions of many objects [6], [7], where we have an object A colliding with another object which then collides with another object and so on, until the last object in the series collides with the final object B. All the objects except A have zero initial velocity. To transfer as much kinetic energy as possible from A to B the masses can also in this case be shown to constitute a geometric series.

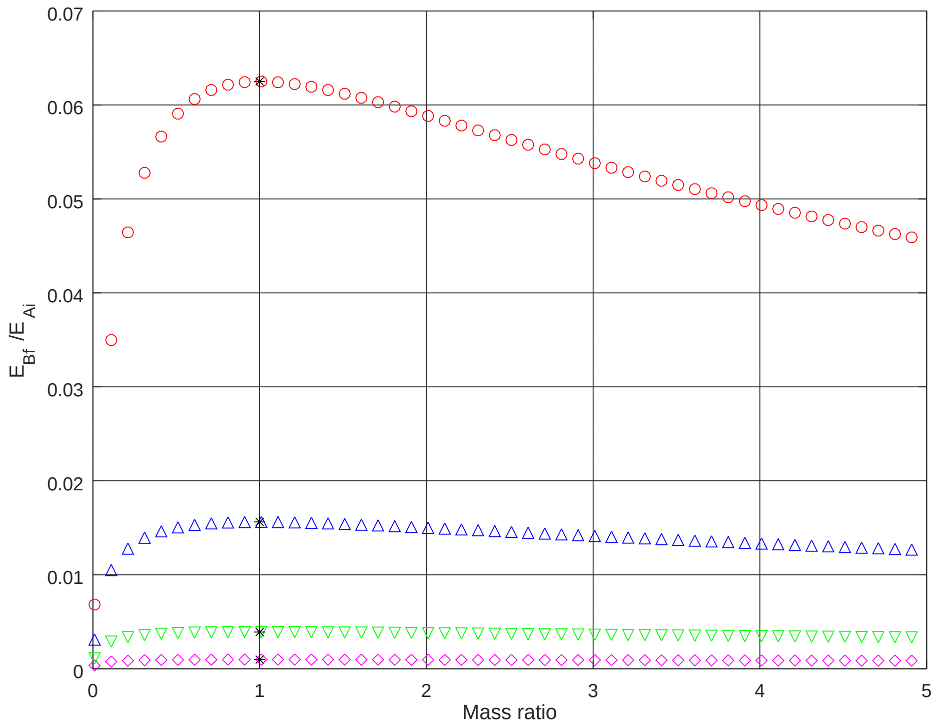

Figure 3: The final energy of B divided with the initial energy of A as a function of the mass ratio $\frac{\mathcal{M}}{M}$ for different numbers of wheels between A and B. Red $\circ$: $n = 1$. Blue $\triangle$: $n = 2$. Green $\bigtriangledown$: $n = 3$. Magenta $\diamond$: $n = 4$. The maximum value is indicated with an $*$.

3.4 The total rotational energy

We will now calculate the total rotational kinetic energy of all the wheels in the final state. The masses of the flywheels constitute a geometric series with the quotient given by equation (55). By using the equations (6) and (7), we are able to find the angular velocities of all the wheels. In the following, a $\star$ on an angular velocity means an intermediate result. We will go through the calculations for three flywheels between A and B. From that we easily generalize the result to n flywheels.
After decoupling A and 1, the final angular velocity Ω of A and the intermediate angular velocity $\omega_1^\star$ of 1 are

$$\Omega = \frac{1}{1+q}\,\Omega_o \quad \text{and} \quad \omega_1^\star = -\frac{\frac{R}{R_1}}{1+q}\,\Omega_o \tag{59}$$

After decoupling 1 and 2, the final angular velocity ω_1 of 1 and the intermediate angular velocity $\omega_2^\star$ of 2 are

$$\omega_1 = -\frac{\frac{R}{R_1}}{(1+q)^2}\,\Omega_o \quad \text{and} \quad \omega_2^\star = \frac{\frac{R}{R_2}}{(1+q)^2}\,\Omega_o \tag{60}$$

After decoupling 2 and 3, the final angular velocity ω_2 of 2 and the intermediate angular velocity $\omega_3^\star$ of 3 are

$$\omega_2 = \frac{\frac{R}{R_3}}{(1+q)^3}\,\Omega_o \quad \text{and} \quad \omega_3^\star = -\frac{\frac{R}{R_3}}{(1+q)^3}\,\Omega_o \tag{61}$$

After decoupling 3 and B, the final angular velocity ω_3 of 3 and the final angular velocity ω of B are

$$\omega_3 = -\frac{\frac{R}{R_3}}{(1+q)^4}\,\Omega_o \quad \text{and} \quad \omega = \frac{\frac{R}{\mathcal{R}}}{(1+q)^4}\,\Omega_o \tag{62}$$

Then the total rotational kinetic energy E_f in the final state is

$$E_\mathrm{f} = \tfrac{1}{2}\,I\,\Omega^2 + \tfrac{1}{2}\,I_1\,\omega_1^2 + \tfrac{1}{2}\,I_2\,\omega_2^2 + \tfrac{1}{2}\,I_3\,\omega_3^2 + \tfrac{1}{2}\,\mathcal{I}\,\omega^2 \tag{63}$$

and the ratio of E_f and the initial rotational kinetic energy of A $E_\mathrm{Ai} = \frac{1}{2}\,I\,\Omega_o^2$ is (use $\frac{I_2\left(\frac{R}{R_2}\right)^2}{I} = q^2$ and similar relations)

$$\begin{aligned}
\frac{E_\mathrm{f}}{E_\mathrm{Ai}} &= \frac{1}{(1+q)^2} + \frac{q}{(1+q)^4} + \frac{q^2}{(1+q)^6} + \frac{q^3}{(1+q)^8} + \frac{q^4}{(1+q)^8} \\
&= \frac{1}{(1+q)^2}\left[1 + \frac{q}{(1+q)^2} + \frac{q^2}{(1+q)^4} + \frac{q^3}{(1+q)^6}\right] + \frac{q^4}{(1+q)^8} \\
&= \frac{1}{(1+q)^2}\,\frac{1-\left[\frac{q}{(1+q)^2}\right]^4}{1-\frac{q}{(1+q)^2}} + \frac{q^4}{(1+q)^8}
\end{aligned} \tag{64}$$

where we have used the fact that the $[\cdots]$-parenthesis is the sum of a geometric series with the common ratio $\frac{q}{(1+q)^2}$.
Equation (64) is easily generalized to n flywheels between A and B

$$\frac{E_\mathrm{f}}{E_\mathrm{Ai}} = \frac{1}{(1+q)^2}\,\frac{1-\left[\frac{q}{(1+q)^2}\right]^{n+1}}{1-\frac{q}{(1+q)^2}} + \frac{q^{n+1}}{(1+q)^{2\,n+2}} \tag{65}$$

For arbitrary values of M and $\mathcal{M}$, we have by equation (55) $q \to 1$ for $n \to \infty$. From equation (65) we then conclude

$$\frac{E_{\mathrm{f}}}{E_{\mathrm{Ai}}} \to \frac{1}{3} \quad \text{for} \quad n \to \infty \tag{66}$$

For the final rotational kinetic energy of B we have from the last term in equation (65) the result we have derived before

$$\left(\frac{E_{\mathrm{Bf}}}{E_{\mathrm{Ai}}}\right)_{\max} = \frac{q^{n+1}}{(1+q)^{2\,n+2}} = \frac{\frac{M}{\mathcal{M}}}{\left(1+\left(\frac{M}{\mathcal{M}}\right)^{\frac{1}{n+1}}\right)^{2\,(n+1)}} \tag{67}$$

4 The general case

We now drop the condition that all the wheels are of the same type. The different K's in the expressions for the moments of inertia might then have a role to play.

By repeating the previous calculations, equation (49) is replaced by

$$\frac{E_{B\mathrm{f}}}{E_{A\mathrm{i}}} = \frac{\frac{K\,M}{\mathcal{K}\,\mathcal{M}}}{\left[\left(1+\frac{K\,M}{K_1\,M_1}\right) \times \Pi_{\mathrm{i}=1}^{n-1}\left(1+\frac{K_i\,M_i}{K_{i+1}\,M_{i+1}}\right) \times \left(1+\frac{K_n\,M_n}{\mathcal{K}\,\mathcal{M}}\right)\right]^2} \tag{68}$$

If we introduce the new variables

$$\overline{M} = K\,M \quad \text{and} \quad \overline{M}_i = K_i\,M_i \quad \text{and} \quad \overline{\mathcal{M}} = \mathcal{K}\,\mathcal{M} \tag{69}$$

equation (68) can be written

$$\frac{E_{B\mathrm{f}}}{E_{A\mathrm{i}}} = \frac{\frac{\overline{M}}{\overline{\mathcal{M}}}}{\left[\left(1+\frac{\overline{M}}{\overline{M}_1}\right) \times \Pi_{\mathrm{i}=1}^{n-1}\left(1+\frac{\overline{M}_i}{\overline{M}_{i+1}}\right) \times \left(1+\frac{\overline{M}_n}{\overline{\mathcal{M}}}\right)\right]^2} \tag{70}$$

We can then as before deduce that the maximum of the ratio in equation (70) is achieved, when the new variables constitute a geometric series with the common ratio

$$q = \left(\frac{\mathcal{K}\,\mathcal{M}}{K\,M}\right)^{\frac{1}{n+1}} \tag{71}$$

and the maximum in the general case is

$$\left(\frac{E_{B\mathrm{f}}}{E_{A\mathrm{i}}}\right)_{\max} = \frac{\frac{K\,M}{\mathcal{K}\,\mathcal{M}}}{\left(1+\left(\frac{K\,M}{\mathcal{K}\,\mathcal{M}}\right)^{\frac{1}{n+1}}\right)^{2\,(n+1)}} \tag{72}$$

This is illustrated in Fig. 4. We see from the figure that for a given value of $k = \frac{\mathcal{K}}{K}$ we have the maximum transfer of energy to wheel B for a value of $\frac{\mathcal{M}}{M}$ which fulfills the condition $\frac{\mathcal{K}\mathcal{M}}{K\,M} = 1$. So if we choose the inverse ratio of $\mathcal{K}$ and K the maximum will appear for the inverse ratio of the ratio of $\mathcal{M}$ and M as we found before. In Fig. 4 this is shown for the couple $k = 0.5$ and $k = 2$ and for the couple $k = 0.25$ and $k = 4$. We also see that the maximum value is the same for the cases considered, as it should be, because it is only dependent on the number of intermediate wheels n.

The right–hand side of equation (72) is an example of the symmetric function in equation (27) with m $= n + 1$ and the variable $x = \frac{K\,M}{\mathcal{K}\mathcal{M}}$. In the general case with different types of flywheels it is thus also possible to interchange the first flywheel (A) and the last flywheel (B) without changing the ratio $\left(\frac{E_{B\mathrm{f}}}{E_{A\mathrm{i}}}\right)_{\mathrm{max}}$. But we can *not* as before just interchange the masses M and $\mathcal{M}$ of the wheels.

We notice that the only K's that matter are K for A and $\mathcal{K}$ for B. The K–factors for the intermidiate wheels are not of any importance for the final result. That is because the wheel A is only a donator of energy and the wheel B is only a receiver of energy, whereas all the other wheels between A and B are both receivers and donators of energy.

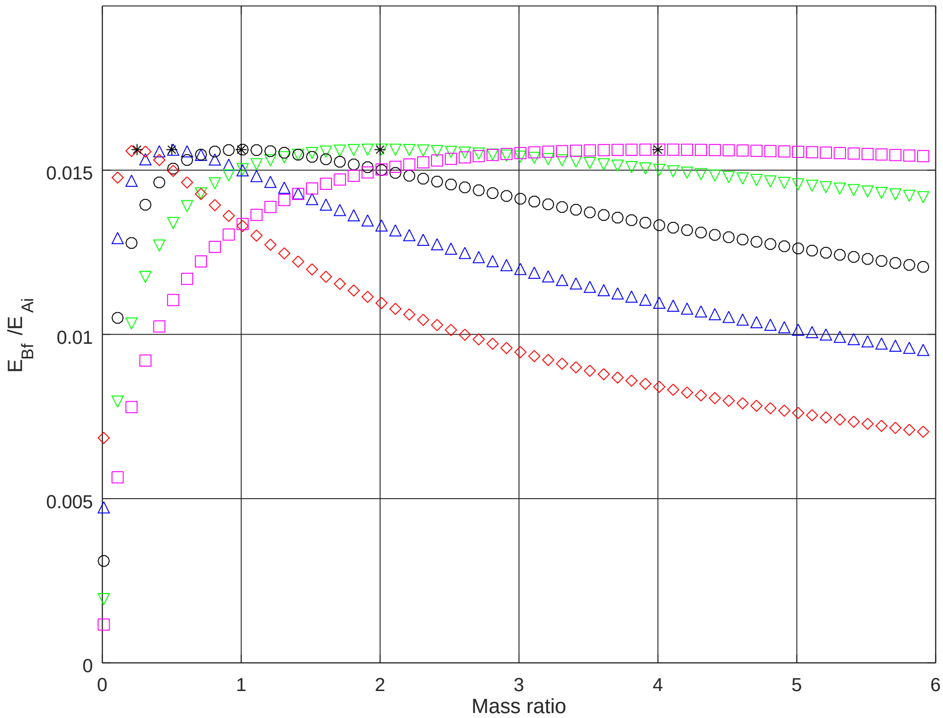

Figure 4: The final energy of B divided with the initial energy of A as a function of the mass ratio $\frac{\mathcal{M}}{M}$ for different values of $k = \frac{\mathcal{K}}{K}$ with two wheels between A and B. Black $\circ$: $k = 1$. Blue $\triangle$: $k = 2$. Green ∇: $k = 0.5$. Red $\diamond$: $k = 4$. Magenta $\square$: $k = 0.25$. The maximum value (here 2^{-6}) is marked with an $*$.

5 Conclusion

We have rederived and expanded the results in reference 1 for the coupling of two flywheels. We have shown that the experimental results in reference 1 are in excellent agreement with the theoretical results derived in section 2. The coupling and decoupling of a series of originally nonrotating flywheels, when we couple two flywheels in the series one after another, has been shown to require that for transferring maximal energy from the original rotating wheel to the last nonrotating wheel, the masses of the flywheels have to constitute a geometric series, if the flywheels are of the same type. Furthermore, we have found that the maximal value of the transfer of energy in the general case with different types of wheels is achieved by a similar condition, namely that the product of the K–factors in the inertial moments and the masses of the wheels constitute a geometric series. We have shown that the maximum value is only dependent on the type of the first and the last wheel and their masses and the number of wheels between these two wheels. The type of the flywheels between the first and the last wheel does not matter at all.

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[3] Lewin W 2017 Solution 1/2 (`https://youtu.be/NOfvX5raV1w`)

[4] Lewin W 2017 Final Solution (`https://youtu.be/OAOJxnW_1lU`)

[5] If we accept equation (24) then $\frac{E_{\mathrm{Bf}}}{E_{\mathrm{Ai}}} \geq 0.24$ implies $\frac{2}{3} \leq \frac{M}{\mathcal{M}} \leq \frac{3}{2}$. Thus the 'almost' maximum transfer of energy from A to B is achieved for a rather wide range of the ratio of the two masses. The masses of the two flywheels in the experiment in reference 1 must have been close to the optimum choice for transfer of energy from A to B.

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[7] J. D. Kerwin 1972 Velocity, Momentum, and Energy Transmissions in Chain Collisions *Am. J. Phys.* **40** 1152