

The Maslov index in symplectic Banach spaces

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The Maslov index in symplectic Banach spaces

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Abstract

We consider a curve of Fredholm pairs of Lagrangian subspaces in a fixed Banach space with continuously varying *weak* symplectic structures. Assuming vanishing index, we obtain intrinsically a continuously varying splitting of the total Banach space into pairs of symplectic subspaces. Using such decompositions we define the Maslov index of the curve by symplectic reduction to the classical finite-dimensional case. We prove the transitivity of repeated symplectic reductions and obtain the invariance of the Maslov index under symplectic reduction, while recovering all the standard properties of the Maslov index.

As an application, we consider curves of elliptic operators which have varying principal symbol, varying maximal domain and are not necessarily of Dirac type. For this class of operator curves, we derive a desuspension spectral flow formula for varying well-posed boundary conditions on manifolds with boundary and obtain the splitting of the spectral flow on partitioned manifolds.

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Preface

The purpose of this Memoir is to establish a universal relationship between incidence geometries in finite and infinite dimensions. In finite dimensions, counting incidences is nicely represented by the *Maslov index*. It counts the dimensions of the intersections of a pair of curves of Lagrangian subspaces in a symplectic finite-dimensional vector space. The concept of the Maslov index is non-trivial: in finite dimensions, the Maslov index of a loop of pairs of Lagrangians does not necessarily vanish. In infinite dimensions, counting incidences is nicely represented by the *spectral flow*. It counts the number of intersections of the spectral lines of a curve of self-adjoint Fredholm operators with the zero line. In finite dimensions, the spectral flow is trivial: it vanishes for all loops of Hermitian matrices.

Over the last two decades there have been various, and in their way successful attempts to generalize the concept of the Maslov index to curves of Fredholm pairs of Lagrangian subspaces in strongly symplectic Hilbert space, to establish the correspondence between Lagrangian subspaces and self-adjoint extensions of closed symmetric operators, and to prove spectral flow formulae in special cases, namely for curves of Dirac type operators and other curves of closed symmetric operators with bounded symmetric perturbation and subjected to curves of self-adjoint Fredholm extensions (i.e., well-posed boundary conditions). While these approaches vary quite substantially, they all neglect the essentially finite-dimensional character of the Maslov index, and, consequently, break down when one deals with operator families of *varying* maximal domain. Quite simply, there is no directly calculable Maslov index when the symplectic structures are weak (i.e., the symplectic forms are not necessarily generated by anti-involutions J) and vary in an uncontrolled way.

In this Memoir we show a way out of this dilemma. We develop the classical method of symplectic reduction to yield an *intrinsic* reduction to finite dimension, induced by a given curve of Fredholm pairs of Lagrangians in a fixed Banach space with varying symplectic forms. From that reduction, we obtain an *intrinsic* definition of the Maslov index in symplectic Banach bundles over a closed interval. This Maslov index is calculable and yields a general spectral flow formula. In our application for elliptic systems, say of order one on a manifold M with boundary Σ , our fixed Banach space (actually a Hilbert space) is the Sobolev space $H^{1/2}(\Sigma; E|_{\Sigma})$ of the traces at the boundary of the $H^1(M; E)$ sections of a Hermitian vector bundle E over the whole manifold. For $H^{1/2}(\Sigma; E|_{\Sigma})$, we have a family of continuously varying weak symplectic structures induced by the principal symbol of the underlying curve of elliptic operators, taken over the boundary in normal direction. That yields a symplectic Banach bundle which is the main subject of our investigation.

Whence, the message of this Memoir is: The Maslov index belongs to finite dimensions. Its most elaborate and most general definitions can be reduced to the

finite-dimensional case in a natural way. The key for that - and for its identification with the spectral flow - is the concept of Banach bundles with weak symplectic structures and intrinsic symplectic reduction. From a technical point of view, that is the main achievement of our work.

Bernhelm Booß-Bavnbek
Chaofeng Zhu

Introduction

Upcoming and continuing interest in the Maslov index. Since the legendary work of V.P. Maslov [65] in the mid 1960s and the supplementary explanations by V. Arnol'd [3], there has been a continuing interest in the Maslov index for curves of Lagrangians in symplectic space. As explained by Maslov and Arnol'd, the interest arises from the study of dynamical systems in classical mechanics and related problems in Morse theory. This same index occurs as well in certain asymptotic formulae for solutions of the Schrödinger equations. For a systematic review of the basic vector analysis and geometry and for the physics background, we refer to Arnol'd [4] and M. de Gosson [39].

The Morse index theorem expresses the Morse index of a geodesic by the Maslov index. Later, T. Yoshida [100] and L. Nicolaescu [75, 76] expanded the view by embracing also spectral problems for Dirac type operators on partitioned manifolds and thereby stimulating some quite new research in that direction. For a short review, we refer to our Section 4.1 below.

Weak symplectic forms on Banach manifolds. Early in the 1970s, P. Chernoff, J. Marsden [34] and A. Weinstein [97] called attention to the practical and theoretical importance of symplectic forms on Banach manifolds. See R.C. Swanson [90, 91, 92] for an elaboration of the achievements of that period regarding linear symplectic structures on Banach spaces. It seems, however, that rigorous and operational definitions of the Maslov index of curves of Lagrangian subspaces in spaces of infinite dimension was not obtained until 25 years later. Our [24, Section 3.2] gives an account and compares the various definitions.

At the same place we emphasized a couple of rather serious obstructions (see Figure 0.1) to applying these concepts to arbitrary systems of elliptic differential equations of non-Dirac type: Firstly, some of the key section spaces for studying boundary value problems (the Sobolev space $H^{1/2}(\Sigma; E|_{\Sigma})$ containing the traces over the boundary $\Sigma = \partial M$ of sections over the whole manifold M) are not carrying a strong symplectic structure, but are naturally equipped with a weak structure not admitting the rule $J^2 = -I$. Secondly, in [24] our definition of the Maslov index in weak symplectic spaces requires a symplectic splitting which does not always exist, is not canonical, and therefore, in general, not obtainable in a continuous way for continuously varying symplectic structures. Recall that a *symplectic splitting* of a symplectic Banach space (X, ω) is a decomposition $X = X^- \oplus X^+$ with $-i\omega$ negative, respectively, positive definite on X^{\mp} and vanishing on $X^- \times X^+$. Thirdly, a priori, a symplectic reduction to finite dimensions is not obtainable for weak symplectic structures in the setting of [24].

An additional incitement to investigate weak symplectic structures comes from a stunning observation of E. Witten (explained by M.F. Atiyah in [5] in a heuristic

Data: $A(s): C_0^\infty(M; E) \rightarrow C_0^\infty(M; E)$, $s \in [0, 1]$ curve of symmetric elliptic first order differential operators.

What fixed? $H^1(M; E)$ and $H^{1/2}(\Sigma; E|_\Sigma) \cong H^1(M; E)/H_0^1(M; E)$.

On $L^2(\Sigma; E|_\Sigma)$ **strong** $\omega(s)_{\text{Green}}(x, y) := -\langle J(s)x, y \rangle_{L^2}$.

On $H^{1/2}(\Sigma; E|_\Sigma)$ induced **weak** $\omega(s)(x, y) := \omega(s)_{\text{Green}}(x, y) = -\langle J'(s)x, y \rangle_{H^{1/2}}$ with **compact** $J'(s) = (I + |B|)^{-1/2}J(s)$, B formally self-adjoint elliptic of first order on Σ .

Obstructions:

- $J'(s)^2 \neq -I$, so $H^{1/2} \neq \ker(J'(s) - iI) \oplus \ker(J'(s) + iI)$;
- $\lambda^{\omega(s)\omega(s)} \neq \lambda$ for closed linear subspace λ ; valid for ω -closed subspaces, where the topology is defined by the semi-norms $\rho_y(x) := |\omega(x, y)|$ (R. SCHMID);
- $\text{ind}(\lambda, \mu) \leq 0$ for $(\lambda, \mu) \in \mathcal{FL}$; generally not equal to 0;
- \mathcal{L} i.g. not contractible (SWANSON); $\pi_1(\mathcal{FL}_0(X, \lambda)) \stackrel{?}{=} \mathbb{Z}$ for $\lambda \in \mathcal{L}(X, \omega)$; valid for strong symplectic Hilbert space (X, ω) .

FIGURE 0.1. Why going weak and what obstructions to circumvent?

way). He considered a weak presymplectic form on the loop space $\text{Map}(S^1, M)$ of a finite-dimensional closed orientable Riemannian manifold M and noticed that a (future) thorough understanding of the infinite-dimensional symplectic geometry of that loop space “should lead rather directly to the index theorem for Dirac operators” (l.c., p. 43). Of course, restricting ourselves to the linear case, i.e., to the geometry of Lagrangian subspaces instead of Lagrangian manifolds, we can only marginally contribute to that program in this Memoir.

Symplectic reduction. In their influential paper [64, p. 121], J. Marsden and A. Weinstein describe the purpose of symplectic reduction in the following way:

“... when we have a symplectic manifold on which a group acts symplectically, we can reduce this phase space to another symplectic manifold in which, roughly speaking, the symmetries are divided out.”

and

“When one has a Hamiltonian system on the phase space which is invariant under the group, there is a Hamiltonian system canonically induced on the reduced phase space.”

The basic ideas go back to the work of G. Hamel [53, 54] and C. Carathéodory [32] in dynamical systems at the beginning of the last century, see also J.-M. Souriau [89]. For symplectic reduction in low-dimensional geometry see the monographs by S.K. Donaldson and P.B. Kronheimer, and by D. McDuff and D. Salamon [41, 67].

Our aim is less intricate, but not at all trivial: Following L. Nicolaescu [76] and K. Furutani [17] (joint work with the first author) we are interested in the finite-dimensional reduction of Fredholm pairs of Lagrangian *linear* subspaces in infinite-dimensional Banach space. The general procedure is well understood, see also P. Kirk and M. Lesch in [59, Section 6.3]: let $W \subset X$ be a closed co-isotropic

subspace of a symplectic Banach space (X, ω) . Then W/W^ω inherits a symplectic form from ω such that

$$R_W(\lambda) := \frac{(\lambda + W^\omega) \cap W}{W^\omega} \subset \frac{W}{W^\omega} \text{ isotropic for } \lambda \text{ isotropic.}$$

Here W^ω denotes the annihilator of W with respect to the symplectic form ω (see Definition 1.2.1c).

In general, however, the reduced space $R_W(\lambda)$ does not need to be Lagrangian in W/W^ω even for Lagrangian λ unless we have $W^\omega \subset \lambda \subset W$ (see Proposition 1.4.8). In [76, 17] a closer analysis of the reduction map R_W is given within the setting of strong symplectic structures; with emphasis on the topology of the space of Fredholm pairs of Lagrangians; and for fixed W . Now we drop the restriction to strong symplectic forms; our goal is to define the Maslov index for continuous curves $s \rightarrow (\lambda(s), \mu(s))$ of Fredholm pairs of Lagrangians with respect to continuously varying symplectic forms $\omega(s)$; and, at least locally (for $s \in (t - \varepsilon, t + \varepsilon)$ around $t \in [0, 1]$), we let the pair $(\lambda(t), \mu(t))$ induce the reference space $W(t)$ for the symplectic reduction and the pair $(\lambda(s), \mu(s))$ induce the reduction map $R_{W(t)}^{(s)}$ in a natural way. The key to finding the reference spaces $W(t)$ and defining a suitable reduction map $R_{W(t)}$ is our Proposition 1.3.3. It is on decompositions of symplectic Banach spaces, naturally induced by a given Fredholm pair of Lagrangians of vanishing index. It might be, as well, of independent interest. The assumption of vanishing index is always satisfied for Fredholm pairs of Lagrangian subspaces in strong symplectic Hilbert spaces, and by additional global analysis arguments in our applications as well.

Thus for each path $\{(\lambda(s), \mu(s))\}_{s \in [0, 1]}$ of Fredholm pairs of Lagrangian subspaces of vanishing index, we receive a finite-dimensional symplectic reduction *intrinsically*, i.e., without any other assumption. The reduction transforms the given path into a path of pairs of Lagrangians in finite-dimensional symplectic space. The main part of the Memoir is then to prove the invariance under symplectic reduction and the independence of choices made. That permits us a conservative view in this Memoir. Instead of defining the Maslov index in infinite dimensions via spectral theory of unitary generators of the Lagrangians as we did in [24], we elaborate the concept of the Maslov index in finite dimensions and reduce the infinite-dimensional case to the finite-dimensional case, i.e., we take the symplectic reduction as our beginning for re-defining the Maslov index instead of deploring its missing.

Structure of presentation. This Memoir is divided into four chapters and one appendix. The first three chapters present a rigorous definition of the Maslov index in Banach bundles by symplectic reduction. In Chapter 1, we fix the notation and establish our key technical device, namely the mentioned natural decomposition of a symplectic Banach space into two symplectic spaces, induced by a pair of co-isotropic subspaces with finite codimension of their sum and finite dimension of the intersection of their annihilators. We introduce the symplectic reduction of arbitrary linear subspaces via a fixed co-isotropic subspace W and prove the transitivity of the symplectic reduction when replacing W by a larger co-isotropic subspace W' . For Fredholm pairs of Lagrangian subspaces of vanishing index, that yields an identification of the two naturally defined symplectic reductions. In Chapter 2, we recall and elaborate the Maslov index in strong symplectic Hilbert space,

Levels of arguments:**1 Complex vector spaces**

- Pair (X, Y) with non-degenerate form $\Omega: X \times Y \rightarrow \mathbb{C}$
- (X, ω) with ω symplectic, i.e., sesquilinear, skew-symmetric, non-degenerate

2 Banach spaces

- $\mathcal{S}(X)$ closed linear subspaces of Banach space X
- Gap topology $\hat{\delta}: \mathcal{S}(X) \times \mathcal{S}(X) \rightarrow [0, 1]$
- Weak symplectic Banach spaces
- Banach bundles

3 Hilbert spaces

- Strong symplectic
- Weak symplectic
- Hilbert bundles

4 Global analysis, elliptic problems

- Compact manifold M with smooth boundary Σ
- Partitioned manifold $M = M_- \cup_{\Sigma} M_+$

Levels of application:

- Weak symplectic Sobolev space $H^{1/2}(\Sigma; E|_{\Sigma})$
- Unconstrained variation of elliptic problems

FIGURE 0.2. Natural levels of treatment

particularly in finite dimensions, to prove the invariance of our definition of the Maslov index under different symplectic reductions. In Chapter 3, we investigate the symplectic reduction to finite dimensions for a given path of Fredholm pairs of Lagrangian subspaces in fixed Banach space with varying symplectic structures and define the Maslov index in the general case via finite-dimensional symplectic reduction. In Section 3.3, we show that the Maslov index is invariant under symplectic reduction in the general case. For a first review of the entangled levels of treatment see Figure 0.2.

Chapter 4 is devoted to an application in global analysis. We summarize the predecessor formulae, we prove a wide generalization of the Yoshida-Nicolaescu spectral flow formula, namely the identity Maslov index=spectral flow, both in general terms of Banach bundles and for elliptic differential operators of arbitrary positive order on smooth manifolds with boundary. That involves weak symplectic Hilbert spaces like the Sobolev space $H^{1/2}$ over the boundary. Applying substantially more advanced results we derive a corresponding spectral flow formula in all Sobolev spaces H^{σ} for $\sigma \geq 0$, so in particular in the familiar strong symplectic L^2 .

In the Appendix A on closed subspaces in Banach spaces, we address the continuity of operations of linear subspaces. In gap topology, we prove some sharp estimates which might be of independent interest. E.g., they yield the following basic convergence result for sums and intersections of permutations in the space $\mathcal{S}(X)$ of closed linear subspaces in a Banach space X in Proposition A.3.13 ([72, Lemma 1.5 (1), (2)]): Let $(M'_j)_{j=1,2,\dots}$ be a sequence in $\mathcal{S}(X)$ converging to $M \in \mathcal{S}(X)$ in the gap topology, shortly $M' \rightarrow M$, let similarly $N' \rightarrow N$ and $M + N$ be closed. Then $M' \cap N' \rightarrow M \cap N$ iff $M' + N' \rightarrow M + N$. For each of the three technical main results of the Appendix, some applications are given to the global analysis of elliptic problems on manifolds with boundary.

Relation to our previous results. With this Memoir we conclude a series of our mutually related previous approaches to symplectic geometry, dynamical systems, and global analysis; in chronological order [16, 17, 105, 102, 103, 18, 19, 104, 23, 20, 15, 24].

The model for our various approaches was developed in joint work with K. Furutani and N. Otsuki in [16, 17, 18]. Roughly speaking, there we deal with a *strong* symplectic Hilbert space $(X, \langle \cdot, \cdot \rangle, \omega)$, so that $\omega(x, y) = \langle Jx, y \rangle$ with $J^* = -J$ and $J^2 = -I$, possibly after continuous deformation of the inner product $\langle \cdot, \cdot \rangle$. Then the space $\mathcal{L}(X, \omega)$ of all Lagrangian subspaces is contractible and, for fixed $\lambda \in \mathcal{L}(X, \omega)$, the fundamental group of the Fredholm Lagrangian Grassmannian $\mathcal{FL}(X, \omega, \lambda)$ of all Fredholm pairs (λ, μ) with $\mu \in \mathcal{L}(X, \omega)$ is cyclic, see [17, Section 4] for an elementary proof. By the induced *symplectic splitting* $X = X^+ \oplus X^-$ with $X^\pm := \ker(J \mp iI)$ we obtain

- (i) $\forall \lambda \in \mathcal{L}(X, \omega) \exists U: X^+ \rightarrow X^-$ unitary with $\lambda = \text{graph}(U)$;
- (ii) $(\lambda, \mu) \in \mathcal{FL}(X, \omega) \iff UV^{-1} - I_{X^-} \in \mathcal{F}(X)$; and
- (iii) $\text{Mas}(\lambda(s), \mu(s))_{s \in [0,1]} := \text{sf}_{(0,\infty)}(U_s V_s^{-1})_{s \in [0,1]}$ well defined.

Here $\mathcal{F}(X)$ denotes the space of bounded Fredholm operators on X and $\mathcal{FL}(X, \omega)$ the set of Fredholm pairs of Lagrangian subspaces of (X, ω) (see Definition 1.2.7).

This setting is suitable for the following application in operator theory: Let \mathcal{H} be a complex separable Hilbert space and A a closed symmetric operator. We extend slightly the frame of the Birman-Kreĭn-Vishik theory of self-adjoint extensions of semi-bounded operators (see the review [1] by A. Alonso and B. Simon). Consider the space $\beta(A) := \text{dom}(A^*) / \text{dom}(A)$ of abstract boundary values. It becomes a strong symplectic Hilbert space with

$$\omega(\gamma(x), \gamma(y)) := \langle A^*x, y \rangle - \langle x, A^*y \rangle,$$

and the projection $\gamma: \text{dom}(A^*) \rightarrow \beta(A)$, $x \mapsto [x] := x + \text{dom}(A)$. The inner product $\langle \gamma(x), \gamma(y) \rangle$ is induced by the graph inner product $\langle x, y \rangle_{\mathcal{G}} := \langle x, y \rangle + \langle A^*x, A^*y \rangle$ that makes $\text{dom}(A^*)$ and, consequently, $\beta(A)$ Hilbert spaces. Introduce the abstract *Cauchy data space* $\text{CD}(A) := (\ker(A^*) + \text{dom}(A)) / \text{dom}(A) = \{\gamma(x) \mid x \in \ker A^*\}$. From von Neumann's famous [73] we obtain the correspondence

$$A_D \text{ self-adjoint extension} \iff [D] \subset \beta(A) \text{ Lagrangian,}$$

for $\text{dom}(A) \subset D \subset \text{dom}(A^*)$. Now let A_D be a *self-adjoint Fredholm extension*, $\{C(s)\}_{s \in [0,1]}$ a C^0 curve in $\mathcal{B}^{\text{sa}}(\mathcal{H})$, the space of bounded self-adjoint operators, and assume *weak inner Unique Continuation Property (UCP)*, i.e., $\ker(A^* + C(s) + \varepsilon) \cap \text{dom}(A) = \{0\}$ for small positive ε . Then, [16] shows that

- (i) $\{\text{CD}(A + C(s)), \gamma(D)\}_{s \in [0,1]}$ is a continuous curve of Fredholm pairs of Lagrangians in the gap topology, and
- (ii) $\text{sf}\{(A + C(s))_D\}_{s \in [0,1]} = \text{Mas}\{\text{CD}(A + C(s)), \gamma(D)\}_{s \in [0,1]}$.

On one side, the approach of [16] has considerable strength: It is ideally suited both to Hamiltonian systems of ordinary differential equations of first order over an interval $[0, T]$ with varying lower order coefficients, and to curves of Dirac type operators on a Riemannian partitioned manifold or manifold M with boundary Σ with fixed Clifford multiplication and Clifford module (and so fixed principal symbol), but symmetric bounded perturbation due to varying affine connection (background field). Hence it explains Nicolaescu's Theorem (see below Section 4.1) in purely functional analysis terms and elucidates the decisive role of weak

inner UCP. For such curves of Dirac type operators, the β -space remains fixed and can be described as a subspace of the distribution space $H^{-1/2}(\Sigma)$ with “half” component in $H^{1/2}(\Sigma)$. As shown in [17], the Maslov index constructed in this way is invariant under finite-dimensional symplectic reduction. Moreover, the approach admits varying boundary conditions and varying symplectic forms, as shown in [19, 23] and can be generalized to a spectral flow formula in the common $L^2(\Sigma)$ as shown in [18].

Unfortunately, that approach has severe limitations since it excludes varying maximal domain: there is no β -space when variation of the highest order coefficients is admitted for the curve of elliptic differential operators.

The natural alternative (here for first order operators) is to work with the Hilbert space

$$H^{1/2}(\Sigma; E|_{\Sigma}) \cong H^1(M; E)/H_0^1(M; E)$$

which remains fixed as long as we keep our underlying Hermitian vector bundle $E \rightarrow M$ fixed. So, let $A(s): C_0^\infty(M; E) \rightarrow C_0^\infty(M; E)$, $s \in [0, 1]$ be a curve of symmetric elliptic first order differential operators. Green’s form for $A(s)$ induces on $L^2(\Sigma; E|_{\Sigma})$ a *strong* symplectic form $\omega(s)_{\text{Green}}(x, y) := -\langle J(s)x, y \rangle_{L^2}$. On $H^{1/2}(\Sigma; E|_{\Sigma})$ the induced symplectic form $\omega(s)(x, y) := \omega(s)_{\text{Green}}(x, y) = -\langle J'(s)x, y \rangle_{H^{1/2}}$ is *weak*. To see that, we choose a formally self-adjoint elliptic operator B of first order on Σ to generate the metric on $H^{1/2}$ according to Gårding’s Theorem. Then we find $J'(s) = (I + |B|)^{-1/2}J(s)$, which is a compact operator and so not invertible. This we emphasized already in our [22] where we raised the following questions:

- Q1:** How to define $\text{Mas}(\lambda(s), \mu(s))_{s \in [0, 1]}$ for curves of Fredholm pairs of Lagrangian subspaces?
- Q2:** How to calculate?
- Q3:** What for?
- Q4:** Dispensable? Non-trivial example?

Questions Q3 and Q4 are addressed below in Chapter 4 (see also our [22]). There we point to the necessity to work with the weak symplectic Hilbert space $H^{1/2}(\Sigma)$. Such work is indispensable when we are looking for spectral flow formulae for partitioned manifolds with curves of elliptic operators which are *not* of Dirac type.

To answer questions Q1 and Q2, we recall the following list of obstructions and open problems, partly from [22] (see also Figures 0.1, 0.3). For simplicity, we specify for Hilbert spaces instead of Banach spaces:

Let (X, ω) be a fixed complex Hilbert space with weak symplectic form $\omega(x, y) = \langle Jx, y \rangle$, and $(X(s), \omega(s))$, $s \in [0, 1]$ a curve of weak symplectic Hilbert spaces, parametrized over the interval $[0, 1]$ (other parameter spaces could be dealt with). Then in general we have in difference to strong symplectic forms:

- (I) $J^2 \neq -I$;
- (II) so, in general $X \neq X^- \oplus X^+$ with $X^\pm := \ker(J \mp iI)$; more generally, our Example 2.1.2 shows that there exist strong symplectic Banach spaces that do not admit any symplectic splitting;
- (III) in general, for continuously varying $\omega(s)$ it does not hold that $X^\mp(s)$ is continuously varying;

Examples (blocking direct generalizations *strong* \rightarrow *weak*)

1. **No symplectic splitting:** Let $(X, \omega) := \lambda \oplus \lambda^*$ and $\lambda := \ell^p$ ($p \in (1, +\infty) \setminus \{2\}$). Then X is a strong symplectic Banach space, but there is no splitting $X = X^+ \oplus X^-$ such that $\mp i\omega|_{X^\pm} > 0$, and $\omega(x, y) = 0$ for all $x \in X^+$ and $y \in X^-$. [See Section 2.1]
2. **Annihilator not always involutive on closed subspaces:** Let (X, ω) be a weak symplectic Hilbert space and $\omega(x, y) = \langle Jx, y \rangle$. Let V be a proper closed linear subspace of X such that $V^\perp \cap JX = \{0\}$. Then $V^\omega = J^{-1}V^\perp = \{0\}$ and $V^{\omega\omega} = X \neq V$. [See Section 1.1]
3. **Fredholm pair of Lagrangians with negative index:** Let X be a complex Hilbert space and $X = X_1 \oplus X_2 \oplus X_3$ an orthogonal decomposition with $\dim X_1 = n \in \mathbb{N}$ and $X_2 \simeq X_3$. Then we can find a skew-self-adjoint injective, but not surjective J such that $\omega(x, y) = \langle Jx, y \rangle$ becomes a weak symplectic form on X and $\lambda_\pm = \{(\alpha, \pm\alpha); \alpha \in X_2\}$ becomes a pair of complementary Lagrangian subspaces of $X_2 \oplus X_3$ by identifying X_2 and X_3 , and, in fact, a pair of Lagrangians of X with $\text{ind}(\lambda_+, \lambda_-) = -n$. [See Section 1.2]

FIGURE 0.3. Three counterexamples

- (IV) as shown in our Example 1.2.3, we have $\lambda^{\omega\omega} \supsetneq \lambda$ for some closed linear subspaces λ ; according to our Lemma 1.1.4, the double annihilator, however, is the identity for ω -closed subspaces, where the topology is defined by the semi-norms $p_y(x) := |\omega(x, y)|$ (based on R. Schmid, [84]);
- (V) by Corollary 1.2.9 we have $\text{index}(\lambda, \mu) \leq 0$ for $(\lambda, \mu) \in \mathcal{FL}$; our Example 1.2.11 shows that there exist Fredholm pairs of Lagrangian subspaces with truly negative index; hence, in particular, the concept of the *Maslov cycle* $\mathcal{M}(X, \omega, \lambda_0) := \mathcal{FL}(\lambda_0, \cdot) \setminus \mathcal{FL}^0(\lambda_0, \cdot)$ of a fixed Lagrangian subspace λ_0 (comprising all Lagrangians that form a Fredholm pair with λ_0 but do not intersect λ_0 transversally) is invalidated: we can no longer conclude complementarity of μ and λ_0 from $\mu \cap \lambda_0 = \{0\}$;
- (VI) in general, the space $\mathcal{L}(X, \omega)$ is not contractible and even not connected according to Swanson's arguments for counterexamples [92, Remarks after Theorem 3.6], based on A. Douady, [42];
- (VII) $\pi_1(\mathcal{FL}_0(X, \lambda)) \stackrel{?}{=} \mathbb{Z}$ for $\lambda \in \mathcal{L}(X, \omega)$; valid for strong symplectic Hilbert space (X, ω) .

Limited value of our previous pilot study. Anyway, our previous [24] deals with a continuous family of weak symplectic forms $\omega(s)$ on a curve of Banach spaces $X(s)$, $s \in [0, 1]$. It gives a definition of the Maslov index for a path $(\lambda(s), \mu(s))_{s \in [0, 1]}$ of Fredholm pairs of Lagrangian subspaces of index 0 under the assumption of a continuously varying symplectic splitting $X = X^+(s) \oplus X^-(s)$. The definition is inspired by the careful distinctions of planar intersections in [105, 102, 103, 104]. Then it is shown that all nice properties of the Maslov index are preserved for this general case. However, that approach has four serious drawbacks which render this definition incalculable:

1. In Section 2.1, our Example 2.1.2 provides a strong symplectic Banach space that does not admit a symplectic splitting.
2. Even when a single symplectic splitting is guaranteed, there is no way to establish such splitting for families in a continuous way (see also our obstruction III above).
3. The Maslov index, as defined in [24] becomes independent of the choice of the splitting only for strong symplectic forms.
4. That construction admits finite-dimensional symplectic reduction only for strong symplectic forms.

To us, our [24] is a highly valuable pilot study, but the preceding limitations explain why in this Memoir we begin again from scratch. For that purpose, an encouraging result was obtained in [20] combined with [15]: the continuous variation of the Calderón projection in $L^2(\Sigma)$ for a curve of elliptic differential operators of first order. We shall use this result in our Section 4.5.

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Part 1

Maslov index in symplectic Banach spaces

CHAPTER 1

General theory of symplectic analysis in Banach spaces

We fix the notation and establish our key technical device in Proposition 1.3.3 and Corollary 1.3.4, namely a natural decomposition of a fixed symplectic vector space into two symplectic subspaces induced by a single Fredholm pair of Lagrangians of index 0. Reversing the order of the Fredholm pair, we obtain an alternative symplectic reduction. We establish the transitivity of symplectic reductions in Lemma 1.4.3 and Corollary 1.4.4. In Proposition 1.4.10, we show that the two natural symplectic reductions coincide by establishing Lemma 1.4.6. As we shall see later in Section 3, that yields the symplectic reduction to finite dimensions for a given path of Fredholm pairs of Lagrangian subspaces of index 0 in a fixed Banach space with varying symplectic structures and the invariance of the Maslov index under different symplectic reductions.

Our assumption of vanishing index is trivially satisfied in *strong* symplectic Hilbert space. More interestingly and inspired by and partly reformulating previous work by R. Schmid, and D. Bambusi [84, 9], we obtain in Lemma 1.1.4 a delicate condition for making the annihilator an involution. In Corollary 1.2.9 we show that the index of a Fredholm pair of Lagrangian subspaces can not be positive. In Corollary 1.2.12 we derive a necessary and sufficient condition for its vanishing for *weak* symplectic forms and in the concrete set-up of our global analysis applications in Section 4. In order to emphasize the intricacies of weak symplectic analysis, it seems worthwhile to clarify in Lemma 1.1.4 a potentially misleading formulation in [84, Lemma 7.1], and in Remark 1.2.2, to isolate an unrepairable error in [9, First claim of Lemma 3.2, pp.3387-3388], namely the wrong claim that the double annihilator is the identity on all closed subspaces of reflexive weak symplectic Banach spaces.

To settle some of the ambiguities around weak symplectic forms once and for all, we provide two counterexamples in Examples 1.2.3 and 1.2.11. The first gives a closed subspace where the double annihilator is not itself. The second gives a Fredholm pair of Lagrangians with negative index.

1.1. Dual pairs and double annihilators

Our point of departure is recognizing the difficulties of dealing with both *varying* and *weak* symplectic structures, as explained in our [24]. As shown there, a direct way to define the Maslov index in that context requires a continuously varying symplectic splitting. As mentioned in the Introduction, neither the existence nor a continuous variation of such a splitting is guaranteed. Consequently, that definition is not very helpful for calculations in applications.

To establish an intrinsic alternative, we shall postpone the use of the symplectic structures to later sections and do as much as possible in the rather *neutral* category of linear algebra. A first taste of the use of purely algebraic arguments of linear algebra for settling open questions of symplectic geometry is the making of a kind of annihilator. For the true annihilator concept of symplectic geometry see below Definition 1.2.1.c.

Already here we can explain the need for technical innovations when dealing with weak symplectic structures instead of hard ones. To give a simple example, let us consider a complex symplectic Hilbert space $(X, \langle \cdot, \cdot \rangle, \omega)$ with $w(x, y) = \langle Jx, y \rangle$ for all $x, y \in X$ where $J: X \rightarrow X$ is a bounded, injective and skew-self-adjoint operator (for details see below Section 1.2). Then we get at once $\lambda^\omega = (J\lambda)^\perp$ and $\lambda^{\omega\omega} \supset \bar{\lambda}$ for all linear subspaces $\lambda \subset X$. We denote the orthogonal complement by the common orthogonality exponent \perp and the symplectic annihilator by the exponent ω . Now, if we are in the *strong* symplectic case, we have J surjective and $J^2 = -I$, possibly after a slight deformation of the inner product. In that case, we have immediately

$$\lambda^{\omega\omega} = (J((J\lambda)^\perp))^\perp = (\lambda^\perp)^\perp = \bar{\lambda}.$$

Hence the double annihilator is the identity on the set of closed subspaces in strong symplectic Hilbert space, like in the familiar case of finite-dimensional symplectic analysis. Moreover, from that it follows directly that the index of a Fredholm pair of Lagrangians (see Definition 1.2.7 and Corollaries 1.2.9 and 1.2.12) vanishes in strong symplectic Hilbert space.

The preceding chain of arguments breaks down for the double annihilator in *weak* symplectic analysis, and we are left with two basic technical problems:

- (i) when do we have precisely $\lambda^{\omega\omega} = \bar{\lambda}$, and consequently,
- (ii) when are we guaranteed the vanishing of the index of a Fredholm pair of Lagrangian subspaces?

As mentioned above, we are not the first who try to determine the precise conditions for the annihilator of an annihilator not to become larger than the closure of the original space. We are indebted to the previous work by R. Schmid [84, Arguments of the proof of Lemma 7.1] and D. Bambusi [9, Arguments around Lemmata 2.7 and 3.2]. They suggested to apply a wider setting and address the pair-annihilator concept of linear algebra. We shall follow - and modify - some of their arguments and claims.

DEFINITION 1.1.1. Let X, Y be two complex vector spaces. Denote by \mathbb{R}, \mathbb{C} and \mathbb{Z} the sets of real numbers, complex numbers and integers, respectively. Let $h: \mathbb{C} \rightarrow \mathbb{C}$ be a \mathbb{R} -linear isomorphism. Let $\Omega: X \times Y \rightarrow \mathbb{C}$ be a \mathbb{R} -linear map with $\Omega(ax, by) = ah(b)\Omega(ax, by)$ for all $a, b \in \mathbb{C}$ and $(x, y) \in X \times Y$.

a) For each of the subspaces $\lambda \subset X$ and $\mu \subset Y$, we define the right and left *annihilators* of λ and μ as real linear subspaces of X and Y by

$$(1.1) \quad \lambda^{\Omega, r} := \{y \in Y; \Omega(x, y) = 0, \forall x \in \lambda\},$$

$$(1.2) \quad \mu^{\Omega, l} := \{x \in X; \Omega(x, y) = 0, \forall y \in \mu\}.$$

b) The form Ω is said to be *non-degenerate in X (in Y)* if $X^{\Omega, r} = \{0\}$ ($Y^{\Omega, l} = \{0\}$). The form Ω is said to be just *non-degenerate* if $X^{\Omega, r} = \{0\}$ and $Y^{\Omega, l} = \{0\}$. In that case one says that X, Y form an *algebraic \mathbb{R} -dual pair* (see also Pedersen [77,

2.3.8]).

c) We have the *reduced form*

$$\tilde{\Omega}: X/Y^{\Omega,l} \times Y/X^{\Omega,r} \longrightarrow \mathbb{C}$$

defined by $\tilde{\Omega}(x + Y^{\Omega,l}, y + X^{\Omega,r}) := \Omega(x, y)$ for each $(x, y) \in X \times Y$.

d) The *annihilator map* $\Omega^b: Y \rightarrow \text{Hom}(X, \mathbb{C})$ is the \mathbb{R} -linear map defined by $\Omega^b(y)(x) := \Omega(x, y)$ for all $x \in X$.

NOTE. By definition, the reduced form $\tilde{\Omega}$ is always non-degenerate, since

$$\begin{aligned} (X/Y^{\Omega,l})^{\tilde{\Omega},r} &= \{y + X^{\Omega,r}; \tilde{\Omega}(x + Y^{\Omega,l}, y + X^{\Omega,r}) = \Omega(x, y) = 0 \ \forall x \in X\} \\ &= X^{\Omega,r} = \{0\} \quad \text{in } Y/X^{\Omega,r}, \end{aligned}$$

making the form $\tilde{\Omega}$ non-degenerate in $X/Y^{\Omega,l}$. Similarly, we obtain $(Y/X^{\Omega,r})^{\tilde{\Omega},l} = Y^{\Omega,l}$, making the form $\tilde{\Omega}$ non-degenerate in $Y/X^{\Omega,r}$.

We list a few immediate consequences: First of all, we have $\ker_{\mathbb{R}} \Omega^b = X^{\Omega,r}$, as real vector spaces. Then we have $\lambda + Y^{\Omega,l} \subset (\lambda^{\Omega,r})^{\Omega,l}$, and $\lambda_1^{\Omega,r} \supset \lambda_2^{\Omega,r}$ if $\lambda_1 \subset \lambda_2 \subset X$. From that we get $\lambda^{\Omega,r} \supset ((\lambda^{\Omega,r})^{\Omega,l})^{\Omega,r} \supset \lambda^{\Omega,r}$, hence

$$(1.3) \quad \lambda^{\Omega,r} = ((\lambda^{\Omega,r})^{\Omega,l})^{\Omega,r}.$$

The following lemma generalizes our [24, Lemma 5, Corollary 1]. We shall use it below in the proof of Lemma 1.2.8 to establish the general result that the index of Fredholm pairs of Lagrangians in symplectic Banach space always is non-positive.

LEMMA 1.1.2. (a) If $\dim X < +\infty$ and $X^{\Omega,r} = \{0\}$, we have

$$\dim Y = \dim X/Y^{\Omega,l} \leq \dim X.$$

The equality $\dim X = \dim Y$ holds if and only if $Y^{\Omega,l} = \{0\}$.

(b) Let $\lambda \subset X$ be a linear subspace. If $\dim X/(\lambda + Y^{\Omega,l}) < +\infty$, we have

$$\dim \lambda^{\Omega,r}/X^{\Omega,r} \leq \dim X/(\lambda + Y^{\Omega,l}).$$

The equality holds if and only if $(\lambda^{\Omega,r})^{\Omega,l} = \lambda + Y^{\Omega,l}$.

(c) Let $\lambda \subset X$ be a linear subspace. If $\dim(\lambda + Y^{\Omega,l})/Y^{\Omega,l} < +\infty$, we have

$$\dim(\lambda + Y^{\Omega,l})/Y^{\Omega,l} = \dim Y/\lambda^{\Omega,r} \quad \text{and} \quad \lambda + Y^{\Omega,l} = (\lambda^{\Omega,r})^{\Omega,l}.$$

PROOF. (a): If $\dim X < +\infty$ and $X^{\Omega,r} = \{0\}$, Ω^b is injective. Then we have $2 \dim Y = \dim_{\mathbb{R}} Y \leq \dim_{\mathbb{R}} \text{Hom}(X, \mathbb{C}) = 2 \dim X$. So we have $\dim Y \leq \dim X$.

If Ω is non-degenerate, we have $\dim X \leq \dim Y$ and $\dim X = \dim Y$. Applying the argument for $\tilde{\Omega}$, we have $\dim X/Y^{\Omega,l} = \dim Y$.

If $\dim X = \dim Y$, we have $\dim X = \dim X/Y^{\Omega,l}$ and $Y^{\Omega,l} = \{0\}$.

(b): Define the first λ -reduced form

$$f: X/(\lambda + Y^{\Omega,l}) \times \lambda^{\Omega,r}/X^{\Omega,r} \longrightarrow \mathbb{C}$$

by

$$f(x + \lambda + Y^{\Omega,l}, y + X^{\Omega,r}) := \Omega(x, y), \quad \forall (x, y) \in X \times \lambda^{\Omega,r}.$$

Then we have

$$(X/(\lambda + Y^{\Omega,l}))^{f,r} = \{0\} \quad \text{and} \quad (\lambda^{\Omega,r}/X^{\Omega,r})^{f,l} = (\lambda^{\Omega,r})^{\Omega,l}/(\lambda + Y^{\Omega,l}).$$

By (a), we get our results.

(c): Define the second λ -reduced form

$$g: (\lambda + Y^{\Omega,l})/Y^{\Omega,l} \times Y/\lambda^{\Omega,r} \longrightarrow \mathbb{C}$$

by

$$g(x + Y^{\Omega,l}, y + \lambda^{\Omega,r}) := \Omega(x, y), \quad \forall (x, y) \in (\lambda + Y^{\Omega,l}) \times Y.$$

Then g is non-degenerate. By (a), we have $\dim(\lambda + Y^{\Omega,l})/Y^{\Omega,l} = \dim Y/\lambda^{\Omega,r}$. By (b) we have $\dim Y/\lambda^{\Omega,r} \geq \dim(\lambda^{\Omega,r})^{\Omega,l}/Y^{\Omega,l}$. So we have $\dim(\lambda + Y^{\Omega,l})/Y^{\Omega,l} = \dim(\lambda^{\Omega,r})^{\Omega,l}/Y^{\Omega,l}$. Since $\lambda + Y^{\Omega,l} \subset (\lambda^{\Omega,r})^{\Omega,l}$, we have $\lambda + Y^{\Omega,l} = (\lambda^{\Omega,r})^{\Omega,l}$. \square

Assume that Ω is non-degenerate in Y . Then the family of semi-norms $\mathcal{F} := \{p_y(x) := |\Omega(x, y)|, x \in X\}_{y \in Y}$ is *separating*, i.e., for $x \neq x'$ in X , there is a $y \in Y$ such that $p_y(x - x') \neq 0$. We shall denote the topology on X induced by the family \mathcal{F} by \mathcal{T}_Ω and call it the *weak topology induced by Ω* or shortly the *Ω -topology*. By [77, 1.5.3 and 3.4.2] (X, \mathcal{T}_Ω) becomes a Hausdorff separated, locally convex, topological vector space. The following two lemmata are proved implicitly by [84, Arguments of the proof of Lemma 7.1]. Clearly, we have

LEMMA 1.1.3. *Assume that Ω is non-degenerate in Y . Then the real linear map Ω^b maps Y onto $(X, \mathcal{T}_\Omega)^*$.*

Then the Hahn-Banach Theorem yields

LEMMA 1.1.4 (R. Schmid, 1987). *Assume that Ω is non-degenerate in Y and λ is a closed linear subspace of (X, \mathcal{T}_Ω) . Then we have*

$$(1.4) \quad \lambda = (\lambda^{\Omega,r})^{\Omega,l}.$$

For later use it is worth noting the following extension of Schmid's Lemma which is the weak and corrected version of [9, Lemma 3.2].

LEMMA 1.1.5. *Assume that X, Y, Ω as above and Ω non-degenerate in Y and bounded in X . Assume that X is a reflexive Banach space. Then $\Omega^b(Y)$ is dense in X^* and we have*

$$(1.5) \quad \lambda = (\lambda^{\Omega,r})^{\Omega,l} \quad \text{for any linear and } \omega\text{-closed subspace } \lambda \subset X.$$

1.2. Basic symplectic concepts

Before defining the Maslov index in symplectic Banach space by symplectic reduction to the finite-dimensional case, we recall the basic concepts and properties of symplectic functional analysis.

DEFINITION 1.2.1. Let X be a complex vector space.

(a) A mapping

$$\omega: X \times X \longrightarrow \mathbb{C}$$

is called a *symplectic form* on X , if it is sesquilinear, skew-symmetric, and non-degenerate, i.e.,

- (i) $\omega(x, y)$ is linear in x and conjugate linear in y ;
- (ii) $\omega(y, x) = -\overline{\omega(x, y)}$;
- (iii) $X^\omega := \{x \in X \mid \omega(x, y) = 0 \text{ for all } y \in X\} = \{0\}$.

Then we call (X, ω) a *symplectic vector space*.

(b) Let X be a complex Banach space and (X, ω) a symplectic vector space. (X, ω) is called *(weak) symplectic Banach space*, if ω is bounded, i.e., $|\omega(x, y)| \leq C\|x\|\|y\|$ for all $x, y \in X$.

(c) The *annihilator* of a subspace λ of X is defined by

$$\lambda^\omega := \{y \in X \mid \omega(x, y) = 0 \text{ for all } x \in \lambda\}.$$

(d) A subspace λ is called *symplectic*, *isotropic*, *co-isotropic*, or *Lagrangian* if

$$\lambda \cap \lambda^\omega = \{0\}, \quad \lambda \subset \lambda^\omega, \quad \lambda \supset \lambda^\omega, \quad \lambda = \lambda^\omega,$$

respectively.

(e) The *Lagrangian Grassmannian* $\mathcal{L}(X, \omega)$ consists of all Lagrangian subspaces of (X, ω) .

REMARK 1.2.2. a) Let (X, ω) be a complex weak symplectic Banach space. By definition (see below), the form $\omega: X \times X \rightarrow \mathbb{C}$ is non-degenerate. Then we have three topologies on X : the *norm-topology*, the canonical *weak topology* induced from the family X^* of continuous functionals on X , and the ω -*induced weak topology* \mathcal{T}_ω . The weak topology is weaker than the norm topology; and the ω -induced topology is weaker than the weak topology. So, a closed subset $V \subset X$ is not necessarily weakly closed or closed in (X, \mathcal{T}_ω) : the set V can have more accumulation points in the weak topology and even more in the ω -induced weak topology than in the norm topology. A standard example is the unit sphere that is not weakly closed in infinite dimensions (see, e.g., H. Brezis [26, Example 1, p. 59]. Fortunately, by [26, Theorem 3.7] every norm-closed *linear* subspace is weakly closed. Hence it is natural (but erroneous) to suppose that the difference between the three topologies does not necessarily confine severely the applicability of Schmid's Lemma, namely to linear subspaces.

b) It seems that D. Bambusi in [9, Lemmata 2.7, 3.2] supposed erroneously that in reflexive Banach space all norm-closed subspaces are not only weakly closed but also ω -weakly closed. Rightly, in spaces where that is valid, Schmid's Lemma is applicable (or can be reproved independently).

c) Recall that a Banach space X is *reflexive* if the isometry

$$\iota: X \longrightarrow X^{**}, \text{ given by } \iota(x)(\varphi) := \varphi(x) \text{ for } x \in X, \varphi \in X^*$$

is surjective, i.e., its range is the whole bidual space X^{**} . Typical examples of reflexive spaces are all Hilbert spaces and the L^p -spaces for $1 < p < \infty$, but not L^1 .

d) Unfortunately, in general the claim of [9, Lemma 3.2] (the validity of the idempotence of the double annihilator for closed linear subspaces in complex reflexive symplectic Banach space) is not correct. If it was correct, then, e.g., in (automatically reflexive) weak symplectic Hilbert space $(X, \langle \cdot, \cdot \rangle, \omega)$, the double annihilator $\lambda^{\omega\omega}$ of every closed subspace λ should coincide with λ . However, here is a counterexample: Let $(X, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $J: X \rightarrow X$ a bounded injective skew-self-adjoint operator. Then $\omega: X \times X \rightarrow \mathbb{C}$ defined by $\omega(x, y) := \langle Jx, y \rangle$ is a symplectic form on X . So $\text{im } J$ is dense in X . For $V \subset X$ closed subspace, denote by V^\perp the orthogonal complement of V with respect to the inner product on X , and by V^ω the symplectic complement (i.e., the annihilator) of V . Then we have

$$(1.6) \quad V^\omega = (JV)^\perp = J^{-1}(V^\perp).$$

Now assume that $\text{im } J \neq X$ (like in the weak symplectic Sobolev space $X := H^{1/2}(\Sigma; E|_\Sigma)$, as explained in the Introduction). Let $x \in X \setminus \text{im } J$ and set $V := (\text{span}\{x\})^\perp$. Then we have $J^{-1}(V^\perp) = \{0\}$, hence $V^\omega = \{0\}$ and $V^{\omega\omega} = X \neq V$. That contradicts the first part of Equation (13) in [9, Lemma 3.2].

e) The preceding example contradicts [9, Equation (11)], as well: For any closed subspace $V \subset X$ we have $J((JV)^\perp) \subset V^\perp$. Then Bambusi's Equation (11) is equivalent to

$$\overline{J((JV)^\perp)} = V^\perp.$$

For our concrete example $V := (\text{span}\{x\})^\perp$, however, we obtain

$$J((JV)^\perp) \cap V^\perp = \{0\} \quad \text{and} \quad V^\perp = \text{span}\{x\}.$$

Thus (11) is incorrect.

f) For any Lagrangian subspace λ in a complex symplectic Banach space (X, ω) we have $\lambda^{\omega\omega} = \lambda$ by definition. That follows also directly from the identity (1.3), and, alternatively, from Schmid's Lemma, since a Lagrangian subspace is always ω -closed.

The counterexample of the preceding Remarks d and e can be generalized in the following form.

EXAMPLE 1.2.3 (Closed subspaces different from their double annihilators). Let (X, ω) be a weak symplectic Hilbert space and $\omega(x, y) = \langle Jx, y \rangle$. Let V be a proper closed linear subspace of X such that $V^\perp \cap JX = \{0\}$. Then $V^\omega = J^{-1}V^\perp = \{0\}$ and $V^{\omega\omega} = X \not\subseteq V$.

REMARK 1.2.4. a) By definition, each one-dimensional subspace in real symplectic space is isotropic, and there always exists a Lagrangian subspace in finite-dimensional real symplectic Banach space, namely the maximal isotropic subspace. However, there are complex symplectic Hilbert spaces without any Lagrangian subspace. That is, in particular, the case if $\dim X^+ \neq \dim X^-$ in $\mathbb{N} \cup \{\infty\}$ for a single (and hence for all) symplectic splittings. More generally, we refer to A. Weinstein's Theorem [97] (see also R.C. Swanson, [92, Theorem 2.1 and Corollary]) that relates the existence of complemented Lagrangian subspaces to the generalized Darboux property, recalled below at the end of Subsection 4.2.2.

b) As in the finite-dimensional case, the basic geometric concept in infinite-dimensional symplectic analysis is the Lagrangian subspace, i.e., a linear subspace which is isotropic and co-isotropic at the same time. Contrary to the finite-dimensional case, however, the common definition of a Lagrangian as a *maximal* isotropic space or an isotropic space of *half* dimension becomes inappropriate.

c) In symplectic Banach spaces, the annihilator λ^ω is closed for any linear subspace λ , and we have the trivial inclusion

$$(1.7) \quad \lambda^{\omega\omega} \supset \overline{\lambda}.$$

In particular, all Lagrangian subspaces are closed, and trivially, as emphasized in Remark 1.2.2.f, we have an equality in the preceding (1.7).

If X is a complex Banach space, each symplectic form ω induces a uniquely defined mapping $J: X \rightarrow X^{\text{ad}}$ such that

$$(1.8) \quad \omega(x, y) = (Jx, y) \quad \text{for all } x, y \in X,$$

where we set $(Jx, y) := (Jx)(y)$. The induced mapping J is a bounded, injective mapping $J: X \rightarrow X^{\text{ad}}$ where X^{ad} denotes the (topological) dual space of continuous complex-conjugate linear functionals on X .

DEFINITION 1.2.5. Let (X, ω) be a symplectic Banach space. If J is also surjective (hence with bounded inverse), the pair (X, ω) is called a *strong symplectic Banach space*.

LEMMA 1.2.6. Let (X, ω) be a strong symplectic Banach space, and $\lambda \subset X$ be a linear subspace. Then we have $\lambda^{\omega\omega} = \bar{\lambda}$.

PROOF. Since (X, ω) be a strong symplectic Banach space, J is surjective. So the weak topology of X is the same as the ω -induced weak topology. By [56, Problem III.1.34], $\bar{\lambda}$ is weakly closed. So it is also ω -closed. By Lemma 1.1.4, we have $\lambda^{\omega\omega} = \bar{\lambda}$. \square

We have taken the distinction between *weak* and *strong* symplectic structures from P. Chernoff and J. Marsden [34, Section 1.2, pp. 4-5]. If X is a Hilbert space with symplectic form ω , we identify X and X^* . Then the induced mapping J defined by $\omega(x, y) = \langle Jx, y \rangle$ is a bounded, skew-self-adjoint operator (i.e., $J^* = -J$) on X with $\ker J = \{0\}$. As in the strong symplectic case, we then have that $\lambda \subset X$ is Lagrangian if and only if $\lambda^\perp = J\lambda$. As explained above, in Hilbert space, a main difference between weak and strong is that we can assume $J^2 = -I$ in the strong case (see [24, Lemma 1] for the required smooth deformation of the inner product), but not in the weak case. The importance of such an anti-involution is well-known from symplectic analysis in finite dimensions and exploited in strong symplectic Hilbert spaces, but, in general, it is lacking in weak symplectic analysis.

We recall the key concept to symplectic analysis in infinite dimensions:

DEFINITION 1.2.7. The space of *Fredholm pairs* of Lagrangian subspaces of a symplectic vector space (X, ω) is defined by

$$(1.9) \quad \mathcal{FL}(X) := \{(\lambda, \mu) \in \mathcal{L}(X) \times \mathcal{L}(X) \mid \dim(\lambda \cap \mu) < +\infty \text{ and } \dim X/(\lambda + \mu) < +\infty\}$$

with

$$(1.10) \quad \text{index}(\lambda, \mu) := \dim(\lambda \cap \mu) - \dim X/(\lambda + \mu).$$

For $k \in \mathbb{Z}$ we define

$$(1.11) \quad \mathcal{FL}_k(X) := \{(\lambda, \mu) \in \mathcal{FL}(X) \mid \text{index}(\lambda, \mu) = k\}.$$

For $k \in \mathbb{Z}$ and $\mu \in \mathcal{L}(X)$ we define

$$(1.12) \quad \mathcal{FL}(X, \mu) := \{\lambda \in \mathcal{L}(X); (\lambda, \mu) \in \mathcal{FL}(X)\},$$

$$(1.13) \quad \mathcal{FL}_k(X, \mu) := \{\lambda \in \mathcal{L}(X); (\lambda, \mu) \in \mathcal{FL}_k(X)\},$$

$$(1.14) \quad \mathcal{FL}_0^k(X, \mu) := \{\lambda \in \mathcal{FL}_0(X, \mu); \dim(\lambda \cap \mu) = k\}.$$

What do we know about the index of Fredholm pairs of Lagrangian subspaces in the weak symplectic case? Here we give another proof for the fact (proved before in our [24, Proposition 1]) that Fredholm pairs of Lagrangian subspaces in symplectic vector spaces never can have positive index.

LEMMA 1.2.8. *Let (X, ω) be a symplectic vector space and $\lambda_1, \dots, \lambda_k$ linear subspaces of X . Assume that $\dim X / (\sum_{j=1}^k \lambda_j) < +\infty$. Then the following holds.*

(a) *We have*

$$(1.15) \quad \dim\left(\bigcap_{j=1}^k \lambda_j^\omega\right) \leq \dim X / \left(\sum_{j=1}^k \lambda_j\right).$$

The equality holds if and only if $\sum_{j=1}^k \lambda_j = (\sum_{j=1}^k \lambda_j)^\omega$.

(b) *If λ_j is isotropic for each j , we have*

$$(1.16) \quad \dim\left(\bigcap_{j=1}^k \lambda_j\right) \leq \dim X / \left(\sum_{j=1}^k \lambda_j\right).$$

The equality holds if and only if $\bigcap_{j=1}^k \lambda_j = \bigcap_{j=1}^k \lambda_j^\omega$ and $\sum_{j=1}^k \lambda_j = (\sum_{j=1}^k \lambda_j)^\omega$.

PROOF. (a) Since $\bigcap_{j=1}^k \lambda_j^\omega = (\sum_{j=1}^k \lambda_j)^\omega$, our result follows from Lemma 1.1.2.b.

(b) By (a) and $\bigcap_{j=1}^k \lambda_j \subset \bigcap_{j=1}^k \lambda_j^\omega$. \square

COROLLARY 1.2.9 (Fredholm index never positive). *a) Let X be a complex vector space with symplectic form ω . Then each Fredholm pair (λ, μ) of Lagrangian subspaces of (X, ω) has negative index or is of index 0.*

b) If (X, ω) is a strong symplectic Banach space, then we have

$$(1.17) \quad V^{\omega\omega} = V \quad \text{for each closed subspace } V \subset X, \text{ and}$$

$$(1.18) \quad \text{index}(\lambda, \mu) = 0 \quad \text{for each } (\lambda, \mu) \in \mathcal{FL}(X, \omega).$$

PROOF. (a) is immediate from the Lemma. To derive (b) from the Lemma, we shall summarize a couple of elementary concepts and identities about symplectic Banach spaces:

For (1.17) we recall from (1.8) that any symplectic form ω on a complex Banach space X induces a uniquely defined bounded, injective mapping $J: X \rightarrow X^{\text{ad}}$ such that $\omega(x, y) = (Jx)(y)$ for all $x, y \in X$. Here X^{ad} denotes the space $\mathcal{B}^{\text{ad}}(X, \mathbb{C})$ of all continuous complex-conjugate functionals on X . For linear subspaces $W \subset X$ and $Z \subset X^{\text{ad}}$, we set $W^\perp := \{e \in X^{\text{ad}}; e(x) = 0 \text{ for all } x \in W\}$ and $Z^\perp := \{x \in X; e(x) = 0 \text{ for all } e \in Z\}$, as usual. By the Hahn-Banach extension theorem, we have

$$(1.19) \quad W^{\perp\perp} = \overline{W} \quad \text{and} \quad Z^{\perp\perp} = \overline{Z}.$$

Moreover, we have the following elementary identities

$$(1.20) \quad W^\omega = (JW)^\perp = J^{-1}(W^\perp).$$

They correspond exactly to the identities of (1.6), given there only for X symplectic Hilbert space.

Recall that we call ω strong, if J is surjective, i.e., an isomorphism. That we assume now. Then we have

$$(1.21) \quad V^{\omega\omega} \stackrel{i}{=} J^{-1}(V^{\omega\perp}) \stackrel{ii}{=} J^{-1}((JV)^{\perp\perp}) \stackrel{iii}{=} J^{-1}(JV) \stackrel{iv}{=} V.$$

The identities (i) and (ii) follow from (1.20) and are valid also in the weak case, while we for identity (iii) need that J is bounded and surjective, hence JV is closed by the Open Mapping Theorem. Identity (iv) is a trivial consequence of the injectivity

of J and so valid also in the weak case. That proves (1.21). In particular, we have $(\lambda + \mu)^{\omega\omega} = \lambda + \mu$ and so by (b) of the Lemma $\dim(\lambda \cap \mu) = \dim X/(\lambda + \mu)$. In general, i.e., for weak symplectic form, we have $\dim(\lambda \cap \mu) = \dim X/(\lambda^\omega + \mu^\omega)$ which does not suffice to prove the vanishing of the index. \square

REMARK 1.2.10. (a) The Corollary has a wider validity. Let (λ, μ) be a Fredholm pair of isotropic subspaces. Then we have by Lemma 1.2.8.b $\text{index}(\lambda, \mu) \leq 0$. If $\text{index}(\lambda, \mu) = 0$, λ and μ are Lagrangians (see [24, Corollary 1 and Proposition 1]).

(b) To obtain $\text{index}(\lambda, \mu) = 0$ from Lemma 1.2.8.b for strong symplectic Banach spaces, it was crucial that we have

$$(\lambda + \mu)^{\omega\omega} = \lambda + \mu \quad \text{and} \quad \lambda = \lambda^\omega \quad \text{and} \quad \mu = \mu^\omega.$$

For Lagrangian subspaces the last two equations are satisfied by definition, and the first is our (1.17), valid for strong symplectic ω . More generally, by Lemma 1.1.4, the first equation is satisfied if the space $\lambda + \mu$ is ω -closed, i.e., closed in the weak topology \mathcal{T}_ω (see above). In a symplectic Banach space (X, ω) all Lagrangian subspaces are norm-closed, weakly closed and ω -weakly closed at the same time, as emphasized in Remark 1.2.2. Since λ, μ are norm-closed and $\dim X/(\lambda + \mu) < +\infty$, $\lambda + \mu$ is norm-closed by [17, Remark A.1] and [57, Problem 4.4.7]. However, that does not suffice to prove that $\lambda + \mu$ is ω -closed, see Remark 1.2.4.c.

(c) Our (1.18) is well known for strong symplectic Hilbert spaces (follow, e.g., the arguments of [16, Corollary 3.7]). Below, in Example 1.2.11 we give a Fredholm pair of Lagrangian subspaces in a weak symplectic Hilbert space with negative index. Hence, we can not take the vanishing of the index for granted for weak symplectic forms, neither in Hilbert spaces - contrary to the well established vanishing of the index of closed (not necessarily bounded) self-adjoint Fredholm operators in Hilbert space ([12, p. 43]). That may appear a bit strange: Below in Section 4.2, we shall consider closed operators as special instances of closed linear relations. Then, e.g., a closed self-adjoint Fredholm operator A in a Hilbert space $(X, \langle \cdot, \cdot \rangle)$ is a self-adjoint Fredholm relation, i.e., the pair $(\text{graph } A, X \times \{0\})$ is a Fredholm pair of Lagrangian subspaces of the Hilbert space $X \times X$ with the canonical strong symplectic structure

$$(1.22) \quad \omega_{\text{can}}: (X \times X) \times (X \times X) \longrightarrow \mathbb{C}, \quad (x_1, y_1, x_2, y_2) \mapsto \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle.$$

That yields an alternative, namely symplectic proof of the vanishing of the index of a closed self-adjoint Fredholm operator in Hilbert space, since $\text{index } A = \text{index}(\text{graph } A, X \times \{0\})$ by (4.6) and $(\text{graph } A, X \times \{0\})$ a Fredholm pair of Lagrangian subspaces of $(X \times X, \omega_{\text{can}})$. The preceding arguments generalize immediately for any closed self-adjoint Fredholm operator $A: X \rightarrow X^*$ with $\text{dom } A \subset X$ and X reflexive complex Banach space. We only need to reformulate the canonical strong symplectic form in (1.22) on the Banach space $X \times X^*$, replacing $X \times X$ by $X \times X^*$ and $\langle x, y \rangle$ by $y(x)$. That yields a strong symplectic form if and only if X is reflexive. For examples of self-adjoint Fredholm operators in “non-Hilbertable” Banach spaces we refer to self-adjoint extensions of the Laplacian in L^p -spaces appearing with convex Hamiltonian systems in I. Ekeland [44, p. 108]. Later in Section 4.2.2, for our applications we shall introduce a new (and weak) concept of a Fredholm operator $A: X \rightarrow Y$ in Banach spaces X, Y that is “self-adjoint” relative to a weak symplectic structure on $X \times Y$ induced by a non-degenerate sesquilinear form $\Omega: X \times Y \rightarrow \mathbb{C}$. A priori, we can not exclude .

(d) In view of our Example 1.2.11, we shall need special assumptions below in Chapter 4 to exclude intractable complications with index calculations for arbitrary Fredholm relations and “self-adjoint” Fredholm operators (e.g., see the assumptions of Proposition 4.3.1, Assumption 4.3.3 (iv), and Assumption 4.4.1 (iv)).

(e) In our applications, we shall deal only with Fredholm pairs of Lagrangians where the vanishing of the index is granted by arguments of global analysis or simply because the underlying form is strong symplectic.

Here is an example which shows that the index of a Fredholm pair of Lagrangian subspaces in weak symplectic Banach space need not vanish.

EXAMPLE 1.2.11 (Fredholm pairs of Lagrangians with negative index). Let X be a complex Hilbert space and $X = X_1 \oplus X_2 \oplus X_3$ an orthogonal decomposition with $\dim X_1 = n \in \mathbb{N}$ and $X_2 \simeq X_3$. Then we can find a bounded skew-self-adjoint injective, but not surjective $J: X \rightarrow X$ such that $\omega(x, y) = \langle Jx, y \rangle$ becomes a weak symplectic form on X . Let J be of the form

$$J = i \begin{pmatrix} A_{11} & A_{12} & \bar{k}A_{12} \\ A_{21} & A_{22} & 0 \\ kA_{21} & 0 & -A_{22} \end{pmatrix},$$

where $k \in \mathbb{C}, k \neq \pm 1$, $\text{im } A_{21} \cap \text{im } A_{22} = \{0\}$ and $\ker A_{21} = \ker A_{22} = \{0\}$.

Set $V = X_2 \oplus X_3$. We identify the vectors in X_2 and X_3 . Then the pair (λ_+, λ_-) with $\lambda_{\pm} := \{(\alpha, \pm\alpha); \alpha \in X_2\}$ becomes a Fredholm pair of Lagrangian subspaces of $(V, \omega|_V)$ with $\lambda_+ \cap \lambda_- = \{0\}$ and

$$V = \lambda_+ \oplus \lambda_-.$$

We claim that $J^{-1}(X_1 \oplus \lambda_{\pm}) \subset V$. In fact, let $(x_1, x_2, x_3) \in J^{-1}(X_1 \oplus \lambda_{\pm})$. Then there is an $\alpha \in X_2$ such that $A_{21}x_1 + A_{22}x_2 = \alpha$ and $kA_{21}x_1 - A_{22}x_3 = \pm\alpha$. So $(1 \mp k)A_{21}x_1 + A_{22}(x_2 \pm x_3) = 0$. Since $\text{im } A_{21} \cap \text{im } A_{22} = 0$ and $\ker A_{21} = 0$, we have $x_1 = 0$.

Note that $\lambda_{\pm}^{\perp} = X_1 \oplus \lambda_{\mp}$ and $\lambda_{\pm}^{\omega} \cap V = \lambda^{\pm}$. Then we have $\lambda_{\pm}^{\omega} = J^{-1}(X_1 \oplus \lambda_{\mp}) \subset V$ and $\lambda_{\pm}^{\omega} = \lambda_{\pm}^{\omega} \cap V = \lambda^{\pm}$. So λ_{\pm} are Lagrangian subspaces of (X, ω) . Then, by definition of J they form a Fredholm pair of Lagrangians of X with $\text{index}(\lambda_+, \lambda_-) = -n$.

COROLLARY 1.2.12. *Let (X, ω) be a symplectic vector space and λ, μ two linear subspaces. Assume that*

$$\dim X/(\lambda + \mu) < +\infty \text{ and } \dim X/(\lambda^{\omega} + \mu^{\omega}) < +\infty.$$

Then the following holds.

(a) (λ, μ) and $(\lambda^{\omega}, \mu^{\omega})$ are Fredholm pairs, and we have

$$(1.23) \quad \text{index}(\lambda, \mu) + \text{index}(\lambda^{\omega}, \mu^{\omega}) \leq 0.$$

(b) *The equality holds in (1.23) if and only if $\lambda + \mu = (\lambda + \mu)^{\omega\omega}$, $\lambda^{\omega} + \mu^{\omega} = (\lambda^{\omega} + \mu^{\omega})^{\omega\omega}$, and $\lambda \cap \mu = \lambda^{\omega\omega} \cap \mu^{\omega\omega}$.*

PROOF. (a) By Lemma 1.2.8, we have

$$(1.24) \quad \dim(\lambda^{\omega} \cap \mu^{\omega}) \leq \dim X/(\lambda + \mu) < +\infty,$$

$$(1.25) \quad \dim(\lambda \cap \mu) \leq \dim(\lambda^{\omega\omega} \cap \mu^{\omega\omega}) \leq \dim X/(\lambda^{\omega} + \mu^{\omega}) < +\infty.$$

Then (λ, μ) and $(\lambda^\omega, \mu^\omega)$ are Fredholm pairs, and we have

$$\begin{aligned} \text{index}(\lambda, \mu) + \text{index}(\lambda^\omega, \mu^\omega) &= \dim(\lambda \cap \mu) - \dim X/(\lambda + \mu) \\ &\quad + \dim(\lambda^\omega \cap \mu^\omega) - \dim X/(\lambda^\omega + \mu^\omega) \\ &= \dim(\lambda \cap \mu) - \dim X/(\lambda^\omega + \mu^\omega) \\ &\quad + \dim(\lambda^\omega \cap \mu^\omega) - \dim X/(\lambda + \mu) \leq 0. \end{aligned}$$

(b) By the proof of (a), the equality in (1.23) holds if and only if $\dim(\lambda^\omega \cap \mu^\omega) = \dim X/(\lambda + \mu)$ and $\dim(\lambda \cap \mu) = \dim(\lambda^{\omega\omega} \cap \mu^{\omega\omega}) = \dim X/(\lambda^\omega + \mu^\omega)$. Since $\lambda \cap \mu \subset \lambda^{\omega\omega} \cap \mu^{\omega\omega}$, by Lemma 1.2.8, the equality in (1.23) holds if and only if $\lambda + \mu = (\lambda + \mu)^{\omega\omega}$, $\lambda^\omega + \mu^\omega = (\lambda^\omega + \mu^\omega)^{\omega\omega}$, and $\lambda \cap \mu = \lambda^{\omega\omega} \cap \mu^{\omega\omega}$. \square

1.3. Natural decomposition of X induced by a Fredholm pair of Lagrangian subspaces with vanishing index

The following lemmata are the key to the definition of the Maslov index in symplectic Banach spaces by symplectic reduction to the finite-dimensional case. For technical reasons, in this section, Fredholm pairs of Lagrangians are always assumed to be of index 0.

We begin with some general facts.

LEMMA 1.3.1. *Let (X, ω) be a symplectic vector space and X_0, X_1 two linear subspaces with $X = X_0 + X_1$. Assume that $X_0 \subset X_1^\omega$. Then we have $X_0 = X_1^\omega$, $X_1 = X_0^\omega$, $X = X_0 \oplus X_1$, and X_0, X_1 are symplectic.*

PROOF. Since $X_0 \subset X_1^\omega$, we have $X_1 \subset X_1^{\omega\omega} \subset X_0^\omega$. Since $X = X_0 + X_1$, there holds

$$X_1 \cap X_1^\omega \subset X_0^\omega \cap X_1^\omega = (X_0 + X_1)^\omega = \{0\}.$$

So X_1 is symplectic, and we have $X_1^\omega = X_1^\omega \cap (X_0 + X_1) = X_0 + X_1^\omega \cap X_1 = X_0$ and $X_1 \cap X_0 = X_1 \cap X_1^\omega = \{0\}$. Hence we have $X = X_0 \oplus X_1$. Since $X_1 \subset X_0^\omega$ and $X = X_0 + X_1$, we have $X_1 = X_0^\omega$ and X_0 is symplectic. \square

LEMMA 1.3.2. *Let (X, ω) be a symplectic vector space and λ, V two linear subspaces. Assume that $\dim V < +\infty$. Then we have*

$$(1.26) \quad \dim \lambda/(\lambda \cap V^\omega) \leq \dim V.$$

The equality holds if and only if $\lambda + V^\omega = X$. In this case we have $\lambda^\omega \cap V = \{0\}$.

PROOF. By [24, Corollary 1], we have $\dim X/V^\omega = \dim V$. Hence we have

$$\dim \lambda/(\lambda \cap V^\omega) = \dim(\lambda + V^\omega)/V^\omega \leq \dim X/V^\omega = \dim V.$$

The equality holds if and only if $\lambda + V^\omega = X$. In this case we have $\lambda^\omega \cap V = (\lambda + V^\omega)^\omega = \{0\}$. \square

Now we turn to our key observation.

PROPOSITION 1.3.3. *Let (X, ω) be a symplectic vector space. Let (λ, μ) be a pair of co-isotropic subspaces with $\dim \lambda_0 = \dim X/(\lambda + \mu) < +\infty$, where $\lambda_0 = \lambda^\omega \cap \mu^\omega$. Let V be a linear subspace of X with $X = V \oplus (\lambda + \mu)$. Let $\lambda_1 = V^\omega \cap \lambda$ and $\mu_1 = V^\omega \cap \mu$. Let $X_0 = \lambda_0 + V$ and $X_1 = \lambda_1 + \mu_1$. Then the following holds.*

(a) $V^\omega + \lambda_0 = X$.

(b) $X_0 = \lambda_0 \oplus V$, $\lambda = \lambda_0 \oplus \lambda_1$ and $\mu = \lambda_0 \oplus \mu_1$. $X_1 = \lambda_1 \oplus \mu_1$ if λ and μ are

Lagrangian subspaces of X .

(c) $\lambda_1 = \lambda \cap X_1$, $\mu_1 = \mu \cap X_1$ and $\lambda + \mu = \lambda_0 + X_1$.

(d) $X_1 = X_0^\omega = V^\omega \cap (\lambda + \mu)$, $X_0 = X_1^\omega$, $X = X_0 \oplus X_1$, and X_0 and X_1 are symplectic.

(e) The subspace λ_0 is a Lagrangian subspace of X_0 . λ_1, μ_1 are Lagrangian subspaces of X_1 if λ and μ are Lagrangian subspaces of X .

PROOF. (a) Since $X = V \oplus (\lambda + \mu)$, we have $V \cap \lambda_0 = \{0\}$ and $V^\omega \cap \lambda_0 = \{0\}$. By [24, Corollary 1], we have $\dim X/V^\omega = \dim V = \dim \lambda_0$. So we have $X = V^\omega + \lambda_0$.
 (b) Note that

$$\begin{aligned} \dim \lambda_0 &\leq \dim \lambda / (V^\omega \cap \lambda) = \dim (V^\omega + \lambda) / V^\omega \\ &\leq \dim X / V^\omega \leq \dim V = \dim \lambda_0. \end{aligned}$$

We have $\lambda_1 \cap \mu_1 = V^\omega \cap \lambda \cap \mu = V^\omega \cap \lambda_0 = \{0\}$ if λ and μ are Lagrangian subspaces of X . So (b) holds.

(c) Since $X_1 = \lambda_1 + \mu_1 \subset V^\omega$, we have $\lambda \cap X_1 \subset \lambda_1 \subset \lambda \cap X_1$. So $\lambda_1 = \lambda \cap X_1$ holds. Similarly we have $\mu_1 = \mu \cap X_1$. By (b) we have

$$\lambda + \mu = \lambda_0 + \lambda_1 + \lambda_0 + \mu_1 = \lambda_0 + \lambda_1 + \mu_1 = \lambda_0 + X_1.$$

(d) Since $X = X_0 + X_1$, our claim follows from Lemma 1.3.1 and the fact

$$X_0^\omega = V^\omega \cap \lambda_0^\omega \supset V^\omega \cap (\lambda + \mu) \supset X_1.$$

(e) By definition, λ_0 is isotropic. Moreover, $\dim \lambda_0 = \frac{1}{2} \dim X_0$. So λ_0 is Lagrangian in X_0 .

Now assume that λ and μ are Lagrangian subspaces of X . Note that λ_1 and μ_1 are isotropic. Since $X_1 = \lambda_1 \oplus \mu_1$, by [24, Lemma 4], λ_1 and μ_1 are Lagrangian subspaces of X_1 . \square

COROLLARY 1.3.4. *Let (X, ω) be a symplectic vector space. Let (λ, μ) be a Fredholm pair of Lagrangian subspaces of index 0. Then there exists a Lagrangian subspace $\tilde{\mu} \subset X$ such that $X = \lambda \oplus \tilde{\mu}$ and $\dim \mu / (\mu \cap \tilde{\mu}) = \dim \tilde{\mu} / (\mu \cap \tilde{\mu}) = \dim (\lambda \cap \mu)$.*

PROOF. By Proposition 1.3.3, X_0 is symplectic and λ_0 is a Lagrangian subspace of X_0 . Choose a Lagrangian \tilde{V} of X_0 with $X_0 = \lambda_0 \oplus \tilde{V}$. Then set $\tilde{\mu} := \tilde{V} \oplus \mu_1$. \square

LEMMA 1.3.5. *Let (X, ω) be a symplectic vector space and λ an isotropic subspace of X . Assume that $\dim \lambda = n < +\infty$. Then there exists a $2n$ dimensional symplectic subspace X_0 such that λ is a Lagrangian subspace of X_0 , $X_0 = X_0^{\omega\omega}$ and $X = X_0 \oplus X_0^\omega$.*

PROOF. Since $\dim \lambda = n < +\infty$, by [24, Corollary 1] we have $\lambda^{\omega\omega} = \lambda$ and $\dim X/\lambda^\omega = n$. Take an n dimensional linear subspace V of X such that $X = V \oplus \lambda^\omega$. Since $\lambda \subset \lambda^\omega$, we have

$$\lambda^\omega \cap (\lambda + V) = \lambda + \lambda^\omega \cap V = \lambda.$$

Since $\dim V = n < +\infty$, by [24, Corollary 1] we have $V^{\omega\omega} = V$ and $\dim X/V^\omega = n$. Set $X_0 := \lambda + V$. Then we have

$$X_0 \cap X_0^\omega = (\lambda + V) \cap \lambda^\omega \cap V^\omega = \lambda \cap V^\omega = (\lambda^\omega + V)^\omega = \{0\}.$$

By [24, Corollary 1], $\dim X/X_0^\omega = \dim X_0 = 2n$ and $X_0^{\omega\omega} = X_0$. So we have $X = X_0 \oplus X_0^\omega$. Since $\dim \lambda = n$ and λ is isotropic, λ is a Lagrangian subspace of X_0 . \square

COROLLARY 1.3.6. *Let ε be a positive number. Let $(X, \omega(s))$, $s \in (-\varepsilon, \varepsilon)$ be a family of symplectic Banach space with continuously varying $\omega(s)$. Let $X_0(s)$, $s \in (-\varepsilon, \varepsilon)$ be a continuous family of linear subspaces of dimension $2n < +\infty$ such that $(X_0(0), \omega(0)|_{X_0(0)})$ is symplectic. Let $\lambda(0)$ be a Lagrangian subspace of $(X_0(0), \omega(0)|_{X_0(0)})$. Then there exist a $\delta \in (0, \varepsilon)$ and a continuous family of linear subspaces $\lambda(s)$, $s \in (-\delta, \delta)$ such that $(X_0(s), \omega(s)|_{X_0(s)})$ is symplectic and $\lambda(s)$ is a Lagrangian subspace of $(X_0(s), \omega(s)|_{X_0(s)})$ for each $s \in (-\delta, \delta)$.*

PROOF. Since $\dim X_0(s) = 2n < +\infty$ and $X_0(0)$ is symplectic, we have $X = X_0(0) \oplus X_0(0)^{\omega(0)}$. By Appendix A.3, there exists a $\delta_1 \in (0, \varepsilon)$ such that $X = X_0(s) \oplus X_0(0)^{\omega(s)}$ for each $s \in (-\delta_1, \delta_1)$.

By the proof of [57, Lemma III.1.40], there exists a closed subspace X_1 such that $X = X_0(0) \oplus X_1$. Then there exists a $\delta_2 \in (0, \delta_1)$ such that $X = X_0(s) \oplus X_1$ for each $s \in (-\delta_2, \delta_2)$. By [57, Lemma I.4.10], we can work on a finite-dimensional symplectic vector space $X_0(0)$ with continuously varying symplectic structure.

We give $X_0(0)$ an inner product $\langle \cdot, \cdot \rangle$. Let $J_0(s) \in \text{GL}(X_0(0))$ be the operators that define symplectic structures on $X_0(0)$. Since $\lambda(0)$ is a Lagrangian subspace of $(X_0(0), \omega(0))$, $\text{sign}(iJ_0(s)) = 0$. Then there exists a continuous family $T(s) \in \text{GL}(X_0(0))$, $s \in (-\delta, \delta)$ with $\delta \in (0, \delta_2)$ such that $T(s)^* J_0(s) T(s) = J_{2n}$, where

$$J_{2n} := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Set $\lambda(s) := T(s)\lambda(0)$ and our result follows. \square

1.4. Symplectic reduction of Fredholm pairs

We recall the general definition of symplectic reduction.

DEFINITION 1.4.1. Let (X, ω) be a symplectic vector space and W a co-isotropic subspace.

(a) The space W/W^ω is a symplectic vector space with induced symplectic structure

$$(1.27) \quad \tilde{\omega}(x + W^\omega, y + W^\omega) := \omega(x, y) \text{ for all } x, y \in W.$$

We call $(W/W^\omega, \tilde{\omega})$ the *symplectic reduction* of X via W .

(b) Let λ be a linear subspace of X . The *symplectic reduction* of λ via W is defined by

$$(1.28) \quad R_W(\lambda) = R_W^\omega(\lambda) := ((\lambda + W^\omega) \cap W)/W^\omega = (\lambda \cap W + W^\omega)/W^\omega.$$

Clearly, $R_W(\lambda)$ is isotropic if λ is isotropic. If $W^\omega \subset \lambda \subset W$ and λ is Lagrangian, $R_W(\lambda)$ is Lagrangian. We have the following lemma.

LEMMA 1.4.2. *Let (X, ω) be a symplectic vector space with isotropic subspace W_0 . Let $\lambda \supset W_0$ be a linear subspace. Then λ is a Lagrangian subspace of X if and only if $W_0^{\omega\omega} \subset \lambda \subset W_0^\omega$ and $R_{W_0^\omega}(\lambda)$ is a Lagrangian subspace of $W_0^\omega/W_0^{\omega\omega}$.*

PROOF. By (1.3) we have $W_0^{\omega\omega\omega} = W_0^\omega$. Since $W_0 \subset W_0^\omega$, $W_0^{\omega\omega} \subset W_0^\omega$.

If $\lambda \in \mathcal{L}(X)$ and $\lambda \supset W_0$, we have $\lambda \subset W_0^\omega$ and $W_0^{\omega\omega} \subset \lambda$. Then we get $R_{W_0^\omega}(\lambda) = \lambda/W_0^{\omega\omega}$ and $(\lambda/W_0^{\omega\omega})^\omega = (\lambda^\omega \cap W_0^\omega)/W_0^{\omega\omega} = \lambda/W_0^{\omega\omega}$, i.e., $R_{W_0^\omega}(\lambda) \in \mathcal{L}(W_0^\omega/W_0^{\omega\omega})$.

Assume that $W_0^{\omega\omega} \subset \lambda \subset W_0^\omega$, we have $W_0^{\omega\omega} \subset \lambda^\omega \subset W_0^\omega$. If $R_{W_0^\omega}(\lambda) \in \mathcal{L}(W_0^\omega/W_0^{\omega\omega})$, we have

$$\lambda/W_0^{\omega\omega} = (\lambda/W_0^{\omega\omega})^{\tilde{\omega}} = (\lambda^\omega \cap W_0^\omega)/W_0^{\omega\omega} = \lambda^\omega/W_0^{\omega\omega}.$$

So we get $\lambda = \lambda^\omega$, i.e., $\lambda \in \mathcal{L}(X)$. \square

LEMMA 1.4.3 (Transitivity of symplectic reduction). *Let (X, ω) be a symplectic vector space with two co-isotropic subspaces $W_1 \subset W_2$, hence clearly $W_1/W_2^\omega \subset W_2/W_2^\omega$ with $(W_1/W_2^\omega)^{\omega_2} = W_1^\omega/W_2^\omega$, where ω_2 denotes the symplectic form on W_2/W_2^ω induced by ω . Then the following holds.*

(a) *Denote by $K_{W_1, W_2}: W_1/W_2^\omega \rightarrow W_1/W_1^\omega$ the map induced by I_{W_1} , where I_W denotes the identity map on a space W . Then K_{W_1, W_2} induces a symplectic isomorphism*

$$(1.29) \quad \tilde{K}_{W_1, W_2}: (W_1/W_2^\omega)/(W_1^\omega/W_2^\omega) \rightarrow W_1/W_1^\omega,$$

such that the following diagram becomes commutative:

$$(1.30) \quad \begin{array}{ccccc} W_1 & \xrightarrow{[\cdot + W_2^\omega]} & W_1/W_2^\omega & \xrightarrow{[\cdot + W_1^\omega/W_2^\omega]} & (W_1/W_2^\omega)/(W_1^\omega/W_2^\omega) \\ \downarrow I_{W_1} & & \downarrow K_{W_1, W_2} & & \nwarrow \tilde{K}_{W_1, W_2} \\ W_1 & \xrightarrow{[\cdot + W_1^\omega]} & W_1/W_1^\omega & & \end{array}$$

(b) *For a linear subspace of λ of X , we have*

$$(1.31) \quad R_{W_1/W_2^\omega}(R_{W_2}(\lambda)) = \tilde{K}_{W_1, W_2}^{-1}(R_{W_1}(\lambda)).$$

Differently put, the following diagram is commutative:

$$(1.32) \quad \begin{array}{ccc} \text{Lin}(X) & \xrightarrow{R_{W_1}} & \text{Lin}(W_1/W_1^\omega) \\ R_{W_2} \downarrow & & \downarrow (\tilde{K}_{W_1, W_2})^{-1} \\ \text{Lin}(W_2/W_2^\omega) & \xrightarrow{R_{W_1/W_2^\omega}} & \text{Lin}((W_1/W_2^\omega)/(W_1^\omega/W_2^\omega)) \end{array}$$

Here $\text{Lin}(X)$ denotes the set of linear subspaces of the vector space X .

PROOF. (a) Since $W_1 \subset W_2$ and they are co-isotropic, we have $W_2^\omega \subset W_1^\omega \subset W_1 \subset W_2$. So K_{W_1, W_2} is well-defined. Since $\ker K_{W_1, W_2} = W_1^\omega/W_2^\omega$, \tilde{K}_{W_1, W_2} is a linear isomorphism. By Definition 1.4.1, \tilde{K}_{W_1, W_2} is a symplectic isomorphism.

(b) Note that

$$\begin{aligned} R_{W_2}(\lambda) \cap (W_1/W_2^\omega) + W_1^\omega/W_2^\omega &= ((\lambda \cap W_2 + W_2^\omega) \cap W_1 + W_1^\omega)/W_2^\omega \\ &= (\lambda \cap W_1 + W_2^\omega + W_1^\omega)/W_2^\omega = (\lambda \cap W_1 + W_1^\omega)/W_2^\omega. \end{aligned}$$

So (1.31) holds. \square

COROLLARY 1.4.4. *Let (X, ω) be a symplectic vector space with a co-isotropic subspace W , a Lagrangian subspace μ and two linear spaces V, λ . Assume that $\dim W^\omega \cap \mu = \dim X/(W + \mu) = \dim V < +\infty$, $X = V \oplus (W + \mu)$ and $W^\omega \cap \mu \subset \lambda \subset W + \mu$. Set $X_0 := W^\omega \cap \mu + V$ and $X_1 := V^\omega \cap W + V^\omega \cap \mu$. Denote by $P_1: X \rightarrow X_1$ defined by $X = X_0 \oplus X_1$ (see Proposition 1.3.3). Then the following holds.*

(a) $W \cap X_1 = V^\omega \cap W$, $W^\omega \cap X_1 = W^\omega \cap V^\omega$, $\mu \cap X_1 = V^\omega \cap \mu$, $\lambda = W^\omega \cap \mu + \lambda \cap X_1$, and $(W \cap X_1)^\omega = W^\omega + V = X_0 + W^\omega \cap X_1$.

(b) P_1 induces a symplectic isomorphism

$$\tilde{P}_1: (W + \mu)/(W^\omega \cap \mu) \longrightarrow X_1 \quad \text{and} \quad \tilde{P}_1(R_{W+\mu}(\lambda)) = \lambda \cap X_1.$$

(c) Denote by $R_{V^\omega \cap W}^{X_1}(\lambda \cap X_1)$ the symplectic reduction of $\lambda \cap X_1$ in X_1 via $V^\omega \cap W$. Define $\tilde{L}_{W, W+\mu}: (W \cap X_1)/(W^\omega \cap X_1) \rightarrow W/W^\omega$ by $\tilde{L}_{W, W+\mu}(x + W^\omega \cap X_1) = x + W^\omega$ for all $x \in W \cap X_1$. Then the following diagram is commutative

$$(1.33) \quad \begin{array}{ccc} \text{Lin}_{W, \mu}(X) & \xrightarrow{R_W} & \text{Lin}(W/W^\omega) \\ \cap X_1 \downarrow & & \cong \downarrow (\tilde{L}_{W, W+\mu})^{-1} \\ \text{Lin}(X_1) & \xrightarrow{R_{V^\omega \cap W}^{X_1}} & \text{Lin}((W \cap X_1)/(W^\omega \cap X_1)) \end{array}$$

and, in particular, we have

$$(1.34) \quad R_{V^\omega \cap W}^{X_1}(\lambda \cap X_1) = \tilde{L}_{W, W+\mu}^{-1}(R_W(\lambda)).$$

Here $\text{Lin}_{W, \mu}(X) := \{\lambda \in \text{Lin}(X) \mid W^\omega \cap \mu \subset \lambda \subset W + \mu\}$.

(d) W is complemented (see Remark 1.4.5) in X if and only if $W \cap X_1$ is complemented in X_1 . In the case of a Banach space we require all the appeared subspaces to be closed.

(e) W^ω is complemented in W if and only if $W^\omega \cap X_1$ is complemented in $W \cap X_1$. In the case of a Banach space we require all the appeared subspaces to be closed.

PROOF. (a) By Proposition 1.3.3, we have $W \cap X_1 = V^\omega \cap W$, $\mu \cap X_1 = V^\omega \cap \mu$, $W + \mu = W^\omega \cap \mu + X_1$, and $X = V^\omega + W = V^\omega + \mu$. Since $W^\omega \cap \mu \subset \lambda \subset W + \mu$, we have

$$\lambda = \lambda \cap (W + \mu) = \lambda \cap (W^\omega \cap \mu + X_1) = W^\omega \cap \mu + \lambda \cap X_1.$$

Note that $W = W^\omega \cap \mu + W^\omega \cap X_1$. By Lemma 1.3.2 we have $\dim(W^\omega \cap X_1)^\omega/W^\omega \leq \dim(W^\omega \cap \mu) = \dim V$. Moreover, we have

$$(W \cap X_1)^\omega \supset W^\omega + X_1^\omega = W^\omega + X_0 = W^\omega + V.$$

Since $W^\omega \cap V = (W + V^\omega)^\omega = \{0\}$, we have

$$(W \cap X_1)^\omega = W^\omega + V = X_0 + W^\omega \cap X_1.$$

(b) Since $W^\omega \cap \mu$ is isotropic and $W + \mu = W^\omega \cap \mu + X_1$, P_1 induces a symplectic isomorphism $\tilde{P}_1: (W + \mu)/(W^\omega \cap \mu) \rightarrow X_1$. Since $W^\omega \cap \mu \subset \lambda \subset W + \mu$, we have $R_{W+\mu}(\lambda) = \lambda/(W^\omega \cap \mu)$. So it holds that $\tilde{P}_1(R_{W+\mu}(\lambda)) = \lambda \cap X_1$.

(c) Let $\tilde{K}_{W, W+\mu}$ denote the symplectic isomorphism defined by (1.29). Note that $\tilde{K}_{W, W+\mu} = \tilde{L}_{W, W+\mu}$ under the symplectic isomorphism \tilde{P}_1 . So (1.34) follows from (b) and Lemma 1.4.3.

(d) If $W \cap X_1$ is complemented in X_1 , there exists a linear subspace M_1 such that $X_1 = W \cap X_1 \oplus M_1$. Since $\dim X_0 < +\infty$, there exists a linear subspace M_0 such that $X_0 = W^\omega \cap \mu \oplus M_0$. Take $M = M_0 \oplus M_1$ and we have $X = W \oplus M$.

Conversely, if W is complemented in X , there exists a linear subspace of M such that $X = W \oplus M$. By (a), we have $W = W \cap X_1 \oplus W^\omega \cap \mu$. So we have

$$X_1 = X_1 \cap (W \cap X_1 + W^\omega \cap \mu + M) = W \cap X_1 \oplus X_1 \cap (W^\omega \cap \mu + M).$$

(e) If $W^\omega \cap X_1$ is complemented in $W \cap X_1$, there exists a linear subspace N_1 such that $W \cap X_1 = W^\omega \cap X_1 \oplus N_1$. Then we have $W = W^\omega \cap \mu \oplus W^\omega \cap X_1 \oplus N_1 = W^\omega \oplus N_1$.

Conversely, if W^ω is complemented in W , there exists a linear subspace of N such that $W = W^\omega \oplus N$. By (a), we have

$$\begin{aligned} W \cap X_1 &= (W \cap X_1 \oplus W^\omega \cap \mu) \cap X_1 \\ &= (W^\omega \cap X_1 \oplus N \oplus W^\omega \cap \mu) \cap X_1 \\ &= (W^\omega \cap X_1) \oplus (N \oplus W^\omega \cap \mu) \cap X_1. \end{aligned} \quad \square$$

REMARK 1.4.5. A linear subspace M of a vector space X is called *complemented* in X if there exists another linear subspace N of X such that $X = M \oplus N$. In Banach space we require M, N to be closed and write $M \in \mathcal{S}^c(X)$. Note that any linear subspace in a vector space is complemented by Zorn's lemma. Our Corollary 1.4.4 (d), (e) is not trivial if either X is a Banach space or one does not want to use Zorn's lemma.

To ensure that symplectic reduction does not lead us out of our class of pairs of Fredholm Lagrangian subspaces of index 0, we prove Proposition 1.4.8 further below.

LEMMA 1.4.6. *Let X be a vector space and $W_1 \subset W_2$, λ, μ four linear subspaces of X . For each linear subspace V , set $R(V) := (V \cap W_2 + W_1)/W_1$. Assume that $W_1 \subset \lambda \subset W_2$. Then (λ, μ) is a Fredholm pair of subspaces of X if and only if $(R(\lambda), R(\mu))$ is a Fredholm pair of subspaces of W_2/W_1 , $\dim(\mu \cap W_1) < +\infty$ and $\dim X/(W_2 + \mu) < +\infty$. In this case it holds that*

$$\begin{aligned} \dim(R(\lambda) \cap R(\mu)) &= \dim(\lambda \cap \mu) - \dim(\mu \cap W_1), \\ \dim(W_2/W_1)/(R(\lambda) + R(\mu)) &= \dim X/(\lambda + \mu) - \dim X/(W_2 + \mu), \\ \text{index}(R(\lambda), R(\mu)) &= \text{index}(\lambda, \mu) \\ &\quad - \dim(\mu \cap W_1) + \dim X/(W_2 + \mu). \end{aligned}$$

PROOF. Since $W_1 \subset \lambda \subset W_2$, we have

$$\begin{aligned} R(\lambda) \cap R(\mu) &= (\lambda/W_1) \cap ((\mu + W_1) \cap W_2)/W_1 = (\lambda \cap \mu + W_1)/W_1 \\ &\cong (\lambda \cap \mu)/(\lambda \cap \mu \cap W_1), \end{aligned}$$

and

$$\begin{aligned} (W_2/W_1)/(R(\lambda) + R(\mu)) &\cong W_2/(\lambda + \mu \cap W_2) = W_2/((\lambda + \mu) \cap W_2) \\ &= (W_2 + \lambda + \mu)/(\lambda + \mu) = (W_2 + \mu)/(\lambda + \mu) \\ &\cong (X/(\lambda + \mu))/(X/(W_2 + \mu)). \end{aligned}$$

So our lemma follows. \square

Now we can prove the basic calculation rule of symplectic reduction:

PROPOSITION 1.4.7 (Symplectic quotient rule). *Let (X, ω) be a symplectic vector space and λ, μ, W subspaces. Assume that $\lambda \subset W$, $\mu = \mu^\omega$ and*

$$(1.35) \quad \text{index}(\lambda, \mu) + \text{index}(\lambda^\omega, \mu) = 0.$$

Then we have $\dim(W^\omega \cap \mu) = \dim X/(W + \mu) < +\infty$ and we have $W + \mu = W^{\omega\omega} + \mu$.

PROOF. Since $\lambda \subset W$, we have $W^\omega \subset \lambda^\omega$. Since $\mu = \mu^\omega$, we have $(W + \mu)^\omega = W^\omega \cap \mu \subset W + \mu$. Denote by $\tilde{\omega}$ the symplectic structure on $(W + \mu)/(W^\omega \cap \mu)$. Then we have

$$\begin{aligned}\lambda \cap (W + \mu) &= \lambda, & \lambda^\omega + W^\omega \cap \mu &= \lambda^\omega, \\ (\lambda + W^\omega \cap \mu)^\omega \cap (W + \mu) &= \lambda^\omega \cap (W + \mu), \\ R_{W+\mu}(\lambda) &= (\lambda + W^\omega \cap \mu)/(W^\omega \cap \mu), \\ R_{W+\mu}(\lambda^\omega) &= (\lambda^\omega \cap (W + \mu))/(W^\omega \cap \mu) = (R_{W+\mu}(\lambda))^{\tilde{\omega}}, \\ R_{W+\mu}(\mu) &= \mu/(W^\omega \cap \mu) = (R_{W+\mu}(\mu))^{\tilde{\omega}}.\end{aligned}$$

By Lemma 1.4.6 and (1.35) we have

$$\begin{aligned}\text{index}(\lambda, \mu) &= \text{index}(R_{W+\mu}(\lambda), R_{W+\mu}(\mu)) \\ &\quad + \dim(\lambda \cap W^\omega \cap \mu) - \dim X/(W + \mu), \\ \text{index}(\lambda^\omega, \mu) &= \text{index}(R_{W+\mu}(\lambda^\omega), R_{W+\mu}(\mu)) \\ &\quad + \dim(W^\omega \cap \mu) - \dim X/(\lambda^\omega + W + \mu).\end{aligned}$$

Note that $(\lambda^\omega + W + \mu)^\omega = \lambda^{\omega\omega} \cap W^\omega \cap \mu \supset \lambda \cap W^\omega \cap \mu$. By Lemma 1.2.8 and Corollary 1.2.12 we have

$$\begin{aligned}\text{index}(R_{W+\mu}(\lambda), R_{W+\mu}(\mu)) + \text{index}(R_{W+\mu}(\lambda^\omega), R_{W+\mu}(\mu)) &\leq 0, \\ \dim(\lambda \cap W^\omega \cap \mu) &\leq \dim X/(\lambda^\omega + W + \mu), \\ \dim(W^\omega \cap \mu) &\leq \dim X/(W + \mu).\end{aligned}$$

By (1.35), the above three inequalities take equalities.

By (1.3), we have $W^{\omega\omega\omega} = W^\omega$. Apply the above result to $W^{\omega\omega}$, we have $\dim(W^\omega \cap \mu) = \dim X/(W^{\omega\omega} + \mu)$. Since $W \subset W^{\omega\omega}$, we have $W + \mu = W^{\omega\omega} + \mu$. \square

The following proposition is inspired by [16, Proposition 3.5]. It gives a natural sufficient condition for preserving the Lagrangian property under symplectic reduction.

PROPOSITION 1.4.8. *Let (X, ω) be a symplectic vector space with a co-isotropic subspace W . Let (λ, μ) be a Fredholm pair of Lagrangian subspaces of X with index 0. Assume that $W^\omega \subset \lambda \subset W$. Then we have $\dim(W^\omega \cap \mu) = \dim X/(W + \mu) < +\infty$, $W + \mu = W^{\omega\omega} + \mu$, and $(R_W(\lambda), R_W(\mu))$ is a Fredholm pair of Lagrangian subspaces of W/W^ω with index 0.*

PROOF. By Proposition 1.4.7 we have $\dim(W^\omega \cap \mu) = \dim X/(W + \mu) < +\infty$ and $W + \mu = W^{\omega\omega} + \mu$.

By Lemma 1.4.6, $(R_W(\lambda), R_W(\mu))$ is a Fredholm pair of subspaces of W/W^ω , $\dim(W^\omega \cap \mu) < +\infty$, and $\dim X/(W + \mu) < +\infty$. Since λ and μ are Lagrangian subspaces of X , $R_W(\lambda)$ and $R_W(\mu)$ are isotropic subspaces of W/W^ω . By Lemma 1.2.8, we have $\dim(W^\omega \cap \mu) \leq \dim X/(W + \mu)$ and $\text{index}(R_W(\lambda), R_W(\mu)) \leq 0$. By Lemma 1.4.6, we have $\dim(W^\omega \cap \mu) = \dim X/(W + \mu)$ and $\text{index}(R_W(\lambda), R_W(\mu)) = 0$. By [24, Proposition 1], $R_W(\lambda)$ and $R_W(\mu)$ are Lagrangian subspaces of W/W^ω . \square

COROLLARY 1.4.9. *Let (X, ω) be a symplectic vector space with a finite-dimensional linear subspace V . Let (λ, μ) be a Fredholm pair of Lagrangian subspaces of X with index 0. Assume that $V + \lambda + \mu = X$ and $V \cap \lambda = \{0\}$. Then we have $V^\omega + \lambda = X$.*

PROOF. Set $W := V + \lambda$. Then $W^\omega = V^\omega \cap \lambda$. By Lemma 1.3.2, we have $\dim \lambda/W^\omega \leq \dim V$. Since $V \cap \lambda = \{0\}$, we have $\dim W/W^\omega = \dim V + \dim \lambda/W^\omega$.

By Proposition 1.4.8, $R_W(\lambda) = \lambda/W^\omega$ is a Lagrangian subspace of W/W^ω . Then we have

$$\dim \lambda/W^\omega = \frac{1}{2} \dim W/W^\omega = \dim V.$$

By Lemma 1.3.2 we have $V^\omega + \lambda = X$. \square

The following proposition gives us a new understanding of the symplectic reduction.

PROPOSITION 1.4.10. *Let (X, ω) be a symplectic vector space and λ_0, V linear subspaces. Let λ and μ be Lagrangian subspaces. Set $\lambda_1 := V^\omega \cap \lambda$, $\mu_1 := V^\omega \cap \mu$, $X_0 := \lambda_0 + V$ and $X_1 := \lambda_1 + \mu_1$. Assume that*

$$(1.36) \quad X = \lambda_0 \oplus V \oplus \lambda_1 \oplus \mu_1 = \lambda \oplus (V + \mu_1) = \mu \oplus (V + \lambda_1).$$

Denote by $P_0: X \rightarrow X_0$ the projection defined by $X = X_0 \oplus X_1$. Then the following holds.

(a) *There exist $A_1 \in \text{Hom}(\lambda_0, V)$, $A_2 \in \text{Hom}(\lambda_0, \mu_1)$, $B_1 \in \text{Hom}(\lambda_0, V)$ and $B_2 \in \text{Hom}(\lambda_0, \lambda_1)$ such that*

$$(1.37) \quad \lambda = \{x_0 + A_1 x_0 + x_1 + A_2 x_0; x_0 \in \lambda_0, x_1 \in \lambda_1\},$$

$$(1.38) \quad \mu = \{y_0 + B_1 y_0 + B_2 y_0 + y_1; y_0 \in \lambda_0, y_1 \in \mu_1\},$$

where $\text{Hom}(X, Y)$ denotes the linear maps from X to Y .

(b) *The linear maps $P_0|_{(V+\lambda)}$ and $P_0|_{(V+\mu)}$ induce linear isomorphisms $T_l: (V + \lambda)/\lambda_1 \rightarrow X_0$ and $T_r: (V + \mu)/\mu_1 \rightarrow X_0$ respectively, and*

$$(1.39) \quad \dim(\lambda \cap \mu) = \dim(P_0(\lambda) \cap \dim P_0(\mu)).$$

(c) *We have*

$$(1.40) \quad T_l(R_{V+\lambda}(\lambda)) = T_r(R_{V+\mu}(\lambda)) = P_0(\lambda),$$

$$(1.41) \quad T_l(R_{V+\lambda}(\mu)) = T_r(R_{V+\mu}(\mu)) = P_0(\mu).$$

(d) *Denote by ω_l the symplectic structure of X_0 induced by T_l from $(V + \lambda)/\lambda_1$ and ω_r the symplectic structure of X_0 induced by T_r from $(V + \mu)/\mu_1$. Then we have*

$$\begin{aligned} \omega_l(x_0 + v, x'_0 + v') &= \omega(x_0 + v, x'_0 + v') - \omega(x_0 + A_1 x_0, x'_0 + A_1 x'_0) \\ &= \omega_r(x_0 + v, x'_0 + v') = \omega(x_0 + v, x'_0 + v') - \omega(x_0 + B_1 x_0, x'_0 + B_1 x'_0) \end{aligned}$$

for all $x_0, x'_0 \in \lambda_0$ and $v, v' \in V$. If either $\lambda_0 \subset \lambda_1^\omega$ or $\lambda_0 \subset \mu_1^\omega$, we have $\omega_l = \omega_r = \omega|_{X_0}$.

(e) *Assume that V is isotropic.*

- (i) *The sesquilinear form $Q(x_0, x'_0) := \omega(x_0, (A_1 - B_1)x'_0)$ on λ_0 is a quadratic form. We call the form Q the intersection form of (λ, μ) on λ_0 at V . If $\lambda_0 = \mu$ and V is a Lagrangian subspace W of X , we set $Q(\mu, W; \lambda) := Q$ (see [43, (2.4)]).*

(ii) Assume that there is another pair $(\tilde{\lambda}, \tilde{V})$ that satisfies the conditions for the pair (λ_0, V) , $\lambda_0 = \lambda \cap \mu \subset \tilde{\lambda}$ and $V \subset \tilde{V}$. Then we have

$$(1.42) \quad \omega(x_0, A_1 x'_0) = \overline{\omega(x'_0, A_1 x_0)} = \omega(x_0, \tilde{A}_1 x'_0),$$

$$(1.43) \quad \omega(x_0, B_1 x'_0) = \overline{\omega(x'_0, B_1 x_0)} = \omega(x_0, \tilde{B}_1 x'_0)$$

for all $x_0, x'_0 \in \lambda_0$.

(f) We have $V + \lambda + \mu = X$.

PROOF. (a) Note that $\lambda_1 = V^\omega \cap \lambda \subset \lambda$ and $\mu_1 = V^\omega \cap \mu \subset \mu$. Our claim follows from the assumptions.

(b) By (a) we have

$$V + \lambda = \{x_0 + v + x_1 + A_2 x_0; x_0 \in \lambda_0, v \in V, x_1 \in \lambda_1\}.$$

So $P_0|_{(V+\lambda)}$ induces a linear map $T_l: (V + \lambda)/\lambda_l \rightarrow X_0$. Clearly, $\ker T_l = \{0\}$. By Corollary A.1.2, T_l is surjective. Thus T_l is a linear isomorphism. Similarly we get that the map $P_0|_{(V+\mu)}$ induces a linear isomorphism $T_r: (V + \mu)/\mu_1 \rightarrow X_0$. The equation (1.39) follows from Lemma 1.4.6.

(c) By (a) and (b) we have $T_l(R_{V+\lambda}(\lambda)) = P_0(\lambda)$. Note that

$$\mu \cap (V + \lambda) = \{x_0 + B_1 x_0 + B_2 x_0 + A_2 x_0; x_0 \in \lambda_0\}.$$

By (a) and (b) we have $T_l(R_{V+\lambda}(\mu)) = P_0(\mu)$. Similarly we get the result for T_r .

(d) Since $\lambda_1 = (V + \lambda)^\omega$ and $\mu_1 = (V + \mu)^\omega$, $(V + \lambda)/\lambda_1$ and $(V + \mu)/\mu_1$ are symplectic vector spaces. Let $x_0, x'_0 \in \lambda_0$ and $v, v' \in V$ be vectors in X . By (a) and (b), we have

$$\begin{aligned} \omega_l(x_0 + v, x'_0 + v') &= \omega(x_0 + v + A_2 x_0, x'_0 + v' + A_2 x'_0) \\ &= \omega(x_0 + v, x'_0 + v') + \omega(x_0 + v, A_2 x'_0) \\ &\quad + \omega(A_2 x_0, x'_0 + v') \\ (1.44) \quad &= \omega(x_0 + v, x'_0 + v') + \omega(x_0, A_2 x'_0) + \omega(A_2 x_0, x'_0). \end{aligned}$$

So we have $\omega_l = \omega|_{X_0}$ if $X_0 = X_1^\omega$. Note that $A_1 x_0, A_1 x'_0 \in V$. Then we have

$$\begin{aligned} 0 &= \omega(x_0 + A_1 x_0 + A_2 x_0, x'_0 + A_1 x'_0 + A_2 x'_0) \\ &= \omega(x_0 + A_1 x_0, x'_0 + A_1 x'_0) + \omega(x_0, A_2 x'_0) + \omega(A_2 x_0, x'_0). \end{aligned}$$

Thus it holds

$$\omega_l(x_0 + v, x'_0 + v') = \omega(x_0 + v, x'_0 + v') - \omega(x_0 + A_1 x_0, x'_0 + A_1 x'_0).$$

Similarly we get the expression for ω_r . Since $P_0(\mu) = T_l(R_{V+\lambda}(\mu))$ is isotropic in (X_0, ω_l) , we have

$$\omega(x_0 + B_1 x_0, x'_0 + B_1 x'_0) = \omega(x_0 + A_1 x_0, x'_0 + A_1 x'_0)$$

for all $x_0, x'_0 \in \lambda_0$ and $v, v' \in V$. So we have $\omega_l = \omega_r$.

If $\lambda_0 \subset \mu_1^\omega$, by (1.44) we have $\omega_l = \omega_r = \omega|_{X_0}$. Similarly, we have $\omega_l = \omega_r = \omega|_{X_0}$ if $\lambda_0 \subset \lambda_1^\omega$.

(e) (i) By (d).

(ii) We have

$$\begin{aligned}
 0 &= \omega(x_0 + A_1x_0 + A_2x_0, x'_0 + \tilde{A}_1x'_0 + \tilde{A}_2x'_0) \\
 &= \omega(A_1x_0, x'_0 + \tilde{A}_2x'_0) + \omega(x_0 + A_2x_0, \tilde{A}_1x'_0) \\
 &= \omega(A_1x_0, x'_0) + \omega(x_0, \tilde{A}_1x'_0)
 \end{aligned}$$

for all $x_0, x'_0 \in \lambda_0$. By taking $\tilde{\lambda}_0 = \lambda_0$ and $\tilde{V} = V$, we have

$$0 = \omega(A_1x_0, x'_0) + \omega(x_0, A_1x'_0) = -\overline{\omega(x'_0, A_1x_0)} + \omega(x_0, A_1x'_0).$$

Then we obtain (1.42). Similarly we have (1.43).

(f) Since $V + \lambda + \mu \supset X_1$ and $X_0 \supset P_0(V + \lambda) \supset P_0(V + \lambda) = X_0$, by Corollary A.1.2 we have $V + \lambda + \mu = X$. \square

CHAPTER 2

The Maslov index in strong symplectic Hilbert space

As explained in the Introduction, the goal of this Memoir is to provide a calculable definition of the Maslov index in weak symplectic Banach (or Hilbert) spaces. Later in Chapter 3 we shall achieve that in an intrinsic way, namely by providing a natural symplectic reduction to the finite-dimensional case, based on the novel decomposition and reduction techniques introduced in the preceding Chapter 1. To get through with that plan, we have to bring the - in principle - well understood definition and calculation of the Maslov index in finite dimensions (or, similarly, in strong symplectic Hilbert space) into a form suitable to receive the symplectic reduction from the weak infinite-dimensional setting. That is what this chapter is about.

2.1. The Maslov index via unitary generators

In [16] K. Furutani, jointly with the first author of this Memoir, explained how the Maslov index of a curve of Fredholm pairs of Lagrangian subspaces in strong symplectic Hilbert space can be defined and calculated as the spectral flow of a corresponding curve of unitary operators through a control point on the unit sphere. In this section we give a slight reformulation and simplification, adapted to our application. Moreover, we show why this approach can *not* be generalized to weak symplectic Banach spaces nor to weak symplectic Hilbert spaces immediately.

Let $p: \mathbb{X} \rightarrow [0, 1]$ be a Hilbert bundle with fibers $X(s) := p^{-1}(s)$ for each $s \in [0, 1]$. Let $(X(s), \omega(s))$, $s \in [0, 1]$ be a family of strong symplectic Hilbert spaces with continuously varying Hilbert inner product $\langle \cdot, \cdot \rangle_s$ and continuously varying symplectic form $\omega(s)$. For a rigorous definition of the terms *Hilbert bundle* and *continuous variation* we refer to our Appendix A.5. As usual, we assume that we can write $\omega(s)(x, y) = \langle J(s)x, y \rangle_s$ with invertible $J(s): X(s) \rightarrow X(s)$ and $J(s)^* = -J(s)$. The fiber bundle \mathbb{X} is always trivial. So we can actually assume that $X(s) \equiv X$. By [57, Lemma I.4.10] and Lemma A.4.5, the set of closed subspaces is a Hilbert manifold and can be identified locally with bounded invertible linear maps of X .

NOTE. Let $N \subset M \subset X$ be closed linear subspaces. Note that we then have the useful rules $M/N \cong N^{\perp_M} = N^{\perp} \cap M$.

Denote by $X^{\mp}(s)$ the positive (negative) eigenspace of $iJ(s)$. Together they yield a spectral decomposition of X . Then the quadratic form $-i\omega(s)$ is negative definite, respectively, positive definite on the subspaces $X^{\mp}(s)$ and we have a symplectic splitting $X = X^-(s) \oplus X^+(s)$.

DEFINITION 2.1.1 (Oriented Maslov index in strong symplectic Hilbert space). Let $\{\lambda(s), \mu(s)\}_{s \in [0,1]}$ be a path of Fredholm pairs of Lagrangian subspaces of (X, ω_s) . Let $U(s), V(s): X_{-,s} \rightarrow X_{+,s}$ be generators for $(\lambda(s), \mu(s))$, i.e., $\lambda(s) = \text{graph}(U(s))$ and $\mu(s) = \text{graph}(V(s))$ (see [24, Proposition 2]). Then $U(s)V(s)^{-1}$ is a continuous family of unitary operators on continuous families of Hilbert spaces $X^+(s)$ with Hilbert structure $-i\omega(s)|_{X^+(s)}$, and $U(s)V(s)^{-1} - I_{X^+(s)}$ is a family of Fredholm operators with index 0. Denote by ℓ_{\pm} the curve $(1 - \varepsilon, 1 + \varepsilon)$ with real $\varepsilon \in (0, 1)$ and with upward (downward) co-orientation. The *oriented Maslov index* $\text{Mas}_{\pm}\{\lambda(s), \mu(s)\}$ of the path $(\lambda(s), \mu(s))$, $s \in [0, 1]$ is defined by

$$(2.1) \quad \text{Mas}\{\lambda(s), \mu(s)\} = \text{Mas}_+\{\lambda(s), \mu(s)\} = -\text{sf}_{\ell_-}\{U(s)V(s)^{-1}\},$$

$$(2.2) \quad \text{Mas}_-\{\lambda(s), \mu(s)\} = \text{sf}_{\ell_+}\{U(s)V(s)^{-1}\}.$$

Here we refer to [103, Definition 2.1] and [24, Definition 13] for the definition of the spectral flow sf_{ℓ} .

The following simple example shows that the preceding definition of the Maslov index can not be generalized literally to symplectic Banach spaces or weak symplectic Hilbert spaces. It shows that there exist strong symplectic Banach spaces that do not admit a symplectic splitting in the preceding sense. That may seem to contradict Zorn's Lemma. However, in a symplectic Banach space (X, ω) Zorn's Lemma can only provide the existence of a maximal subspace X^+ where the form $-i\omega$ is positive definite. Then $-i\omega$ is negative definite on $X^- := (X^+)^{\omega}$ and vanishing on $X^+ \times X^-$. However, one can not show that $X = X^+ \oplus X^-$. Denote by $V := X^+ \oplus X^-$, then $V^{\omega} = \{0\}$. We see from it that $\overline{V}^{\mathcal{T}} = X$, where \mathcal{T} denotes the locally convex topology defined by ω .

EXAMPLE 2.1.2 (Symplectic splittings do not always exist). Let $(X, \omega) := \lambda \oplus \lambda^*$ and $\lambda := \ell^p$ with $p \in (1, +\infty)$ and $p \neq 2$. Then X is a strong symplectic Banach space, but there is no splitting $X = X^+ \oplus X^-$ such that $\mp i\omega|_{X^{\pm}} > 0$, and $\omega(x, y) = 0$ for all $x \in X^+$ and $y \in X^-$. Otherwise we could establish an inner product on X that makes X a Hilbert space.

Moreover, even when a symplectic splitting exists, there is no way to establish such splitting for families of symplectic Banach spaces in a continuous way, as emphasized in the Introduction.

2.2. The Maslov index in finite dimensions

Consider the special case $\dim X = 2n < +\infty$. Note that the eigenvalues of $U(s)V(s)^{-1}$ are on the unit circle S^1 . Recall that each map in $C([0, 1], S^1)$ can be lifted to a map $C([0, 1], \mathbb{R})$. By [57, Theorem II.5.2], there are n continuous functions $\theta_1, \dots, \theta_n \in C([0, 1], \mathbb{R})$ such that the eigenvalues of the operator $U(s)V(s)^{-1}$ for each $s \in [0, 1]$ (counting algebraic multiplicities) have the form

$$e^{i\theta_j(s)}, \quad j = 1, \dots, n.$$

Denote by $[a]$ the integer part of $a \in \mathbb{R}$. Define

$$(2.3) \quad E(a) := \begin{cases} a, & a \in \mathbb{Z} \\ [a] + 1, & a \notin \mathbb{Z}. \end{cases}$$

In this case, we have

$$(2.4) \quad \text{Mas}_+ \{ \lambda(s), \mu(s); s \in [0, 1] \} = \sum_{j=1}^n \left(E\left(\frac{\theta_j(1)}{2\pi}\right) - E\left(\frac{\theta_j(0)}{2\pi}\right) \right),$$

$$(2.5) \quad \text{Mas}_- \{ \lambda(s), \mu(s); s \in [0, 1] \} = \sum_{j=1}^n \left(\left[\frac{\theta_j(1)}{2\pi} \right] - \left[\frac{\theta_j(0)}{2\pi} \right] \right).$$

By definition, $\text{Mas}_\pm \{ \lambda(s), \mu(s); s \in [0, 1] \}$ is an integer that does not depend on the choices of the arguments $\theta_j(s)$. By [24, Proposition 6], it does not depend on the particular choice of the paths of the symplectic splittings.

2.3. Properties of the Maslov index in strong symplectic Hilbert space

From the properties of the spectral flow, we get all the basic properties of the Maslov index for strong symplectic Hilbert spaces (see S. E. Cappell, R. Lee, and E. Y. Miller [30, Section 1] for a more comprehensive list). Be aware that the proof of Proposition 2.3.1.d is less trivial (see [104, Corollary 4.1]).

The properties of the following list will first be used for establishing a rigorous and calculable concept of the Maslov index in weak symplectic Banach space. For the Maslov index defined in that way by symplectic finite-dimensional reduction, we shall later recover the full list of valid properties in Theorem 3.1.4 for the general case.

PROPOSITION 2.3.1 (Basic properties of the Maslov index). *Let $p: \mathbb{X} \rightarrow [0, 1]$, $p_i: \mathbb{X}_i \rightarrow [0, 1], i = 1, 2$, be Hilbert bundles with continuously varying strong symplectic forms $\omega(s)$ (respectively, $\omega_i(s)$) on $X(s) := p^{-1}(s)$ (respectively $X_i(s) := p_i^{-1}(s)$). Let $(\lambda, \mu), (\lambda_1, \mu_1), (\lambda_2, \mu_2)$ be curves of Fredholm pairs of Lagrangians in $\mathbb{X}, \mathbb{X}_1, \mathbb{X}_2$. Then we have:*

(a) *The Maslov index is invariant under homotopies of curves of Fredholm pairs of Lagrangian subspaces with fixed endpoints. In particular, the Maslov index is invariant under re-parametrization of paths.*

(b) *The Maslov index is additive under catenation, i.e.,*

$$\text{Mas}_\pm \{ \lambda, \mu \} = \text{Mas}_\pm \{ \lambda|_{[0,a]}, \mu|_{[0,a]} \} + \text{Mas}_\pm \{ \lambda|_{[a,1]}, \mu|_{[a,1]} \},$$

for any $a \in [0, 1]$.

(c) *The Maslov index is additive under direct sum, i.e.,*

$$\text{Mas}_\pm \{ \lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2 \} = \text{Mas}_\pm \{ \lambda_1, \mu_1 \} + \text{Mas}_\pm \{ \lambda_2, \mu_2 \},$$

where $\{ \lambda_j(s) \}, \{ \mu_j(s) \}$ are paths of Lagrangian subspaces in $(X_j, \omega_j(s))$, $j = 1, 2$ and $\lambda_1 \oplus \lambda_2$ is a path of subspaces in $(X_1 \oplus X_2, \omega_1(s) \oplus \omega_2(s))$.

(d) *The Maslov index is natural under symplectic action: given a second Hilbert bundle $\mathbb{X}' = \{ X'(s) \}$, a path of symplectic structures $\omega'(s)$ on $X'(s)$, and a path of bundle isomorphisms $\{ L(s) \in \mathcal{B}(X(s), X'(s)) \}$ such that $L(s)^*(\omega'(s)) = \omega(s)$, then we have*

$$\text{Mas}_\pm \{ \lambda(s), \mu(s); \omega(s) \} = \text{Mas}_\pm \{ L(s)\lambda(s), L(s)\mu(s); \omega'(s) \}.$$

(e) *The Maslov index vanishes, if $\dim(\lambda(s) \cap \mu(s))$ constant for all $s \in [0, 1]$.*

(f) Flipping. We have

$$\begin{aligned} \text{Mas}_+ \{\lambda(s), \mu(s)\} + \text{Mas}_+ \{\mu(s), \lambda(s)\} \\ &= \text{Mas}_+ \{\lambda(s), \mu(s)\} - \text{Mas}_- \{\lambda(s), \mu(s)\} \\ &= \dim(\lambda(0) \cap \mu(0)) - \dim(\lambda(1) \cap \mu(1)), \end{aligned}$$

and $\text{Mas}_\pm \{\lambda(s), \mu(s); \omega(s)\} = \text{Mas}_\pm \{\mu(s), \lambda(s); -\omega(s)\}$.

(g) Local range. There exists an $\varepsilon > 0$ such that

$$0 \leq \text{Mas}_+ \{\lambda(s), \mu(s); \omega(s)\} \leq \dim(\lambda(0) \cap \mu(0)) - \dim(\lambda(1) \cap \mu(1)),$$

if the variation of the curves λ, μ of Lagrangians and the variation of the symplectic forms $\omega(s)$ is sufficiently small, namely if in the notations of Appendix A.2

$$\hat{\delta}(\lambda(s), \lambda(0)), \hat{\delta}(\mu(s), \mu(0)), \|\omega(s) - \omega(0)\| < \varepsilon, \text{ for all } s \in [0, 1].$$

NOTE. To preserve the preceding basic properties for the weak symplectic case, we have to make the explicit assumption of the vanishing of the index of the relevant Fredholm pairs of Lagrangian subspaces in properties (a) and (g), see Assumption 3.1.1 and Theorem 3.1.4.

We have the following lemma (see J. Robbin and D. Salamon, [82, Theorem 2.3, Localization] for the constant symplectic structure case).

LEMMA 2.3.2. Let $(X, \omega(s))$ be a continuous family of $2n$ dimensional symplectic vector spaces with Lagrangian subspaces λ_0, μ_0 such that $X = \lambda_0 \oplus \mu_0$. Let $A(s) \in \text{Hom}(\lambda_0, \mu_0)$, $s \in [0, 1]$ be a path of linear maps such that $\lambda(s) = \text{graph}(A(s))$ is a Lagrangian subspace of $(\mathbb{C}^{2n}, \omega(s))$ for each $s \in [0, 1]$. Define $Q(s)(x, y) = \omega(s)(x, A(s)y)$ for all $s \in [0, 1]$, $x \in \lambda_0$ and $y \in \mu_0$. Then $Q(s)$ is a quadratic form on λ_0 and we have

$$(2.6) \quad \text{Mas}_+ \{\lambda(s), \lambda_0; s \in [0, 1]\} = m^+(Q(1)) - m^+(Q(0)),$$

$$(2.7) \quad \text{Mas}_- \{\lambda(s), \lambda_0; s \in [0, 1]\} = m^-(Q(0)) - m^-(Q(1)),$$

where $m^\pm(Q)$, $m^0(Q)$ denote the positive (negative) Morse index and nullity of Q respectively for a quadratic form Q .

PROOF. Clearly, $\lambda(s)$ is Lagrangian if and only if $Q(s)$ is quadratic. By choosing a frame, we can assume that $X = \mathbb{C}^{2n}$, $\lambda_0 = \mathbb{C}^n \times \{0\}$ and $\mu_0 = \{0\} \times \mathbb{C}^n$. Let $J(s)$ be defined by $\omega(s)(x, y) = \langle J(s)x, y \rangle$ for each $s \in [0, 1]$. Then we have $J(s) = \begin{pmatrix} 0 & -K(s)^* \\ K(s) & 0 \end{pmatrix}$ for some $K(s) \in \text{GL}(n, \mathbb{C})$. Set $T(s) := \text{diag}(K(s)^{-1}, I_n)$. Then we have $T(s) * J(s)T(s) = J_{2n}$. By Proposition 2.3.1.d, we can assume that $J(s) = J_{2n}$. Then we have $X^\pm = \{(x, \mp ix); x \in \mathbb{C}^n\}$. The generator of $\lambda(s)$ is the map $(x, ix) \mapsto (U(s)x, -iU(s)x)$, $x \in \mathbb{C}^n$. So $U(s) = (I_n + iA(s))(I_n - iA(s))^{-1}$. We have $U(s) = 0$ if $A(s) = 0$. Note that $A(s)$ is a continuous family of self-adjoint operators. By the definition of the spectral flow we have

$$\begin{aligned} \text{Mas}_+ \{\lambda(s), \lambda_0\} &= \text{sf}_{\ell_-} \{U(s)\} = -\text{sf} \{-A(s)\} \\ &= m^+(A(1)) - m^+(A(0)) = m^+(Q(1)) - m^+(Q(0)). \end{aligned}$$

Similarly we have (2.7). □

The following proposition is a slight generalization of [17, Theorem 4.2 and Remark 5.1], where it was shown for the first time that the Maslov index is preserved under certain symplectic reductions. It was that result that inspired us to base our new definition of the Maslov index in weak symplectic infinite dimensional spaces on the concept of symplectic reduction. From a technical point of view, the following very general proposition for strong symplectic structures together with its modifications for weak symplectic structures in Section 3.3 is one of the main achievements in this Memoir. Note that the arguments depend on our novel intrinsic decomposition techniques of Section 1.3 in the preceding chapter.

PROPOSITION 2.3.3 (Invariance of symplectic reduction in strong symplectic Hilbert space). *Let $(X, \omega(s))$, $s \in (-\varepsilon, \varepsilon)$ be a family of strong symplectic Hilbert spaces with continuously varying symplectic form $\omega(s)$, where $\varepsilon > 0$. Let $(\lambda(s), \mu(s))$, $s \in (-\varepsilon, \varepsilon)$ be a path of Fredholm pairs of Lagrangian subspaces of $(X, \omega(s))$. Let $V(s)$ be a path of finite-dimensional subspaces of X with $X = V(0) \oplus (\lambda(0) + \mu(0))$. Then there exists a $\delta \in (0, \varepsilon)$ such that*

$$X = V(0) + \lambda(s) + \mu(s) = V(s)^{\omega(s)} + \lambda(s) = V(s)^{\omega(s)} + \mu(s)$$

for all $s \in (-\delta, \delta)$, and

$$(2.8) \quad \text{Mas}_{\pm}\{\lambda(s), \mu(s); s \in [s_1, s_2]\} \\ = \text{Mas}_{\pm}\{R_{V(s)+\lambda(s)}^{\omega(s)}(\lambda(s)), R_{V(s)+\lambda(s)}^{\omega(s)}(\mu(s)); s \in [s_1, s_2]\}$$

$$(2.9) \quad = \text{Mas}_{\pm}\{R_{V(s)+\mu(s)}^{\omega(s)}(\lambda(s)), R_{V(s)+\mu(s)}^{\omega(s)}(\mu(s)); s \in [s_1, s_2]\}$$

for all $[s_1, s_2] \subset (-\delta, \delta)$.

PROOF. Set $\lambda_0(0) := \lambda(0) \cap \mu(0)$, $\lambda_1(s) := V(s)^{\omega(s)} \cap \lambda(s)$, $\mu_1(s) := V(s)^{\omega(s)} \cap \mu(s)$, $X_1(s) := \lambda_1(s) + \mu_1(s)$, and $X_0(s) := X_1(s)^{\omega(s)}$. By Proposition 1.3.3 we have

$$X = \lambda_0(0) \oplus V(0) \oplus \lambda_1(0) \oplus \mu_1(0),$$

$X_0(0) = \lambda_0(0) + V(0)$, and $X_1(0) = X_0(0)^{\omega}$.

By Appendix A.3 and Corollary 1.4.9, there exists a $\delta_1 \in (0, \varepsilon)$ such that

$$X = V(s) + \lambda(s) + \mu(s) = V(s)^{\omega(s)} + \lambda(s) = V(s)^{\omega(s)} + \mu(s) \\ = \lambda(s) \oplus (V(s) + \mu_1(s)) = \mu \oplus (V(s) + \lambda_1(s)),$$

$X_1(s) = \lambda_1(s) \oplus \mu_1(s)$, and $X = X_0(s) \oplus X_1(s)$ for all $s \in (-\delta, \delta)$. Set $X_0(s) := X_1(s)^{\omega(s)}$. Then we have $V(s) \subset X_0(s)$. Since $X_0(s)$ is a Hilbert space, there exists a path $\lambda_0(s)$, $s \in (-\delta, \delta)$ such that $X_0(s) = \lambda_0(s) \oplus V(s)$.

Denote by $P_0(s): X \rightarrow X_0(s)$ the projection defined by $X = X_0(s) \oplus X_1(s)$. By Proposition 1.3.3.c,d and Proposition 2.3.1.c,d,e, we have

$$\text{Mas}_{\pm}\{\lambda(s), \mu(s); s \in [s_1, s_2]\} \\ = \text{Mas}_{\pm}\{P_0(s)(\lambda(s)), P_0(s)(\mu(s)); s \in [s_1, s_2]\} \\ + \text{Mas}_{\pm}\{\lambda_1(s), \mu_1(s); s \in [s_1, s_2]\} \\ = \text{Mas}_{\pm}\{T_l(R_{V(s)+\lambda(s)}^{\omega(s)}(\lambda(s))), T_l(R_{V(s)+\lambda(s)}^{\omega(s)}(\mu(s))); s \in [s_1, s_2]\} \\ = \text{Mas}_{\pm}\{(R_{V(s)+\lambda(s)}^{\omega(s)}(\lambda(s))), (R_{V(s)+\lambda(s)}^{\omega(s)}(\mu(s))); s \in [s_1, s_2]\}.$$

Note that by Proposition 1.3.3 and Appendix A.3, the Maslov indices in the above calculations are well-defined. The equality (2.9) follows similarly. \square

CHAPTER 3

The Maslov index in Banach bundles over a closed interval

3.1. The Maslov index by symplectic reduction to a finite-dimensional subspace

For this section, we fix some data and notations and make the following assumption:

ASSUMPTION 3.1.1. Let $p: \mathbb{X} \rightarrow [0, 1]$ be a Banach bundle. Denote by $X(s) := p^{-1}(s)$ the fiber of p at $s \in [0, 1]$. Let $\{\omega(s)\}_{s \in [0, 1]}$ be a continuous family of symplectic structures with $\omega(s)$ acting on $X(s) \times X(s)$. Let $\{(\lambda(s), \mu(s))\}_{s \in [0, 1]}$ be a path of Fredholm pairs of Lagrangian subspaces of $(X(s), \omega(s))$ of index 0.

Here for a fiber bundle $p: \mathbb{X} \rightarrow [0, 1]$, a path $c(s)$, $s \in [0, 1]$ of \mathbb{X} is a continuous map $c: [0, 1] \rightarrow \mathbb{X}$ such that $c(s) \in p^{-1}(s)$ for each $s \in [0, 1]$. We refer to [101] for the concept of Banach bundles; see also our summary in the Appendix A.5. The fiber bundle \mathbb{X} is always trivial. So we can actually assume that $X(s) \equiv X$. By [57, Lemma I.4.10] and Lemma A.4.5, the set of complemented closed subspaces is a Banach manifold and can be identified locally with the general linear group $\mathcal{B}^\times(X)$ of bounded invertible operators of X .

As shown in Example 1.2.11, the assumption of vanishing index is a restriction for Fredholm pairs of Lagrangian subspaces in weak symplectic structures, even when the fibres are Hilbert spaces.

To define the Maslov index via finite-dimensional symplectic reduction, we begin with a purely formal definition.

We make Assumption 3.1.1 and the following choices and notations.

CHOICES AND NOTATIONS 3.1.2. By the definition of Fredholm pairs, for each $t \in [0, 1]$, there exists $V(t) \subset X(t)$ such that $V(t) \oplus (\lambda(t) + \mu(t)) = X(t)$. Set $\lambda_0(t) := \lambda(t) \cap \mu(t)$ and $X_0(t) := \lambda_0(t) \oplus V(t)$. Then there exists for each t a $\delta(t) > 0$ such that

- (i) there exists a local frame $L(t, s): X(t) \rightarrow X(s)$, $s \in (t - \delta(t), t + \delta(t)) \cap [0, 1]$ of the bundle \mathbb{X} ,
- (ii) $X(s) = L(t, s)V(t) + \lambda(s) + \mu(s) = (L(t, s)V(t))^{\omega(s)} + \lambda(s)$ for all $s \in (t - \delta(t), t + \delta(t)) \cap [0, 1]$, and
- (iii) we have

$$\begin{aligned}
 (3.1) \quad X(s) &= L(t, s)X_0(t) \oplus \lambda_1(t, s) \oplus \mu_1(t, s) \\
 &= \lambda(s) \oplus (L(t, s)V(t) + \mu_1(t, s)) \\
 &= \mu(s) \oplus (L(t, s)V(t) + \lambda_1(t, s))
 \end{aligned}$$

for all $s \in (t - \delta(t), t + \delta(t)) \cap [0, 1]$, where $\lambda_1(t, s) := (L(t, s)V(t))^{\omega(s)} \cap \lambda(s)$ and $\mu_1(t, s) := (L(t, s)V(t))^{\omega(s)} \cap \mu(s)$.

Denote by $X_1(t, s) := \lambda_1(t, s) + \mu_1(t, s)$. Denote by $P_0(t, s): X(s) \rightarrow X_0(s)$ the projection defined by $X(s) = L(t, s)X_0(t) \oplus X_1(t, s)$. Denote by $\omega_l(t, s) = \omega_r(t, s)$ the symplectic structure defined by Proposition 1.4.10.d. We have the finite-dimensional vector space $\{(X_0(t), L(t, s)^*(\omega_l(t, s)))\}_{s \in [0, 1]}$ with continuously varying symplectic structure for fixed $t \in [0, 1]$.

DEFINITION 3.1.3 (Maslov index by symplectic reduction). With the notations and choices above, let $0 = a_0 < a_1 < \dots < a_n = 1$ be a partition with $[a_k, a_{k+1}] \subset (t_k - \delta(t_k), t_k + \delta(t_k))$ for some $t_k \in [0, 1]$, $k = 0, \dots, n-1$. Define

$$(3.2) \quad \text{Mas}_{\pm} \{ \lambda(s), \mu(s); s \in [0, 1] \} := \sum_{k=0}^{n-1} \text{Mas}_{\pm}^{\omega_l(t_k, s)} \{ L(t_k, s)^{-1} P_0(t_k, s)(\lambda(s)), L(t_k, s)^{-1} P_0(t_k, s)(\mu(s)); s \in [a_k, a_{k+1}] \},$$

where $\text{Mas}_{\pm}^{\omega_l(t_k, s)} \{ \dots \}$ denotes the oriented Maslov index for the specified Fredholm pair of symplectically reduced Lagrangian subspaces in the finite-dimensional complex vector space $X_0(t_k)$ with continuously varying induced symplectic structures $\omega_l(t_k, s)$. That oriented Maslov index was introduced in Definition 2.1.1. We call Mas_{\pm} the *positive (negative) Maslov index*. We call the positive Maslov index $\text{Mas} := \text{Mas}_+$ the *Maslov index*.

To lift the formal concepts of Definition 3.1.3 to a useful definition of the Maslov index in Banach spaces, we prove the following theorem:

THEOREM 3.1.4 (Main Theorem). *Under Assumption 3.1.1, the mappings Mas_{\pm} are well-defined (i.e., independent of the choices) and the common properties of the Maslov index (listed in Proposition 2.3.1) are preserved.*

REMARK 3.1.5. By the definition of the spectral flow, our definition coincides with that in Definition 2.1.1, and more generally, [24, Definition 7] in their special cases. Our definition of the Maslov index generalizes the ideas in [17, 90].

We firstly show that Theorem 3.1.4 is true in the local case. For sufficiently small parameter variation, that follows from the homotopy invariance of the Maslov index. This property was established in Proposition 2.3.1.a for strong symplectic Hilbert spaces, and so for finite dimensions, i.e., it is valid in our case after symplectic reduction. Besides the application of the established homotopy invariance, the point of the following lemma is the intrinsic decomposition introduced in Section 1.3: Roughly speaking, for curves of Fredholm pairs of Lagrangian subspaces of index 0, the technique of intrinsic decomposition permits to extend the choice of a single complementary space $V(s_0)$ for $\lambda(s_0) + \mu(s_0)$ in $X(s_0)$ to a continuous decomposition of the induced finite-dimensional subspace $X_0(s_0) := (\lambda(s_0) \cap \mu(s_0)) \oplus V(s_0)$ of $X(s_0)$. While we can identify all Banach spaces $X(s)$ ($s \in [0, 1]$) with one fixed Banach space X , we can not identify the subspaces $X_0(s)$ ($s \in [0, 1]$) with one finite-dimensional subspace $X_0 \subset X$, in general and even not locally. The reason is that the dimension of the intersection $\lambda(s) \cap \mu(s)$ is upper semi-continuous in any $s_0 \in [0, 1]$, and so must be the codimension of the sum $\lambda(s) + \mu(s)$, having the jumps at the same parameters like the intersection dimension due to the vanishing index.

The following lemma shows how we can get around that difficulty *locally*, namely assuming (3.3)-(3.4).

LEMMA 3.1.6 (Two-parameter Maslov index). *We make Assumption 3.1.1 with $X(s) = X$. We assume that there exists a finite-dimensional subspace V of X that is a supplement in X (not necessarily transversal) to all sums $\lambda(s) + \mu(s)$. More precisely, we assume that*

$$(3.3) \quad X = X_0 \oplus \lambda_1(s) \oplus \mu_1(s)$$

$$(3.4) \quad = \lambda(s) \oplus (V + \mu_1(s)) = \mu(s) \oplus (V + \lambda_1(s))$$

for all $s \in [0, 1]$, where $X_0 := V + \lambda_0$, λ_0 is a finite dimensional subspace of V , $\lambda_1(s) := V^{\omega(s)} \cap \lambda(s)$ and $\mu_1(s) := V^{\omega(s)} \cap \mu(s)$.

Denote by $X_1(s) := \lambda_1(s) + \mu_1(s)$. Denote by $P_0(s): X \rightarrow X_0$ the projection defined by $X = X_0 \oplus X_1(s)$. Denote by $\omega_l(s) = \omega_r(s)$ the symplectic structure defined by Proposition 1.4.10.d. By definition of Fredholm pairs, for each $t \in [0, 1]$, there exists $V(t)$ such that $V(t) \oplus (\lambda(t) + \mu(t)) = X$. Then there exists a $\delta(t) > 0$ for each $t \in [0, 1]$ such that, for all $[s_1, s_2] \subset (t - \delta(t), t + \delta(t)) \cap [0, 1]$,
a) the properties (ii) and (iii) in Choices and Notations 3.1.2 are satisfied with $L(t, s) = I$, and
b) we have the following two-parameter formula:

$$(3.5) \quad \begin{aligned} & \text{Mas}_{\pm} \{P_0(s)(\lambda(s)), P_0(s)(\mu(s)); \omega_l(s); s \in [s_1, s_2]\} \\ & = \text{Mas}_{\pm} \{P_0(t, s)(\lambda(s)), P_0(t, s)(\mu(s)); \omega_l(t, s); s \in [s_1, s_2]\}, \end{aligned}$$

where $P_0(t, s)$ and $\omega_l(t, s)$ are given by Definition 3.1.3.

PROOF. Let $s \in [0, 1]$. By Proposition 1.4.10.f, we have $X = V + \lambda(s) + \mu(s)$. By Corollary 1.4.9, we have $X = V^{\omega(s)} + \lambda(s) = V^{\omega(s)} + \mu(s)$. Set $W_l(s) := V + \lambda(s)$. Then we have $W_l(s)^{\omega(s)} = \lambda_1(s)$. By Proposition 1.4.10.b, we have a linear isomorphism $T_l(s): W_l(s)/\lambda_1(s) \rightarrow X_0$ induced by P_0 . So X_0 is symplectic. Denote by $\tilde{\omega}(s)$ the induced symplectic structure on $W_l(s)/\lambda_1(s)$. By Proposition 1.4.10.d, the symplectic structure on X_0 induced from $\tilde{\omega}(s)$ by $T_l(s)$ is given by $\omega_l(s)$.

Let $t \in [0, 1]$. Since $X = V + \lambda(t) + \mu(t)$, there exists a linear subspace $V(t)'$ of V such that $V(t)' \oplus (\lambda(t) + \mu(t)) = X$ for each $t \in [0, 1]$. Recall that $\lambda_0(t) = \lambda(t) \cap \mu(t)$. For $s \in [0, 1]$ with small $|s - t|$, we set

$$\begin{aligned} \lambda_1(t, s)' &:= (V(t)')^{\omega(s)} \cap \lambda(s), & \mu_1(t, s)' &:= (V(t)')^{\omega(s)} \cap \mu(s), \\ X_0(t)' &:= V(t)' \oplus \lambda_0(t), & X_1(t, s)' &:= \lambda_1(t, s)' \oplus \mu_1(t, s)'. \end{aligned}$$

Denote by $P_0(t, s)'$ the projection onto $X_0(t)'$ defined by

$$(3.6) \quad X = X_0(t)' \oplus X_1(t, s).$$

We denote by $\omega_l(t, s)'$ the symplectic structure on $X_0(t)$ defined by (3.6) and Proposition 1.4.10.d.

Set

$$\begin{aligned}
\tilde{W}(t, s) &:= R_{V+\lambda(s)}^{\omega(s)}(V(t)' + \lambda(s)) = \frac{V(t)' + \lambda(s)}{\lambda_1(s)}, \\
\tilde{V}(t, s) &:= R_{V+\lambda(s)}^{\omega(s)}(V(t)') = \frac{V(t)' + \lambda_1(s)}{\lambda_1(s)}, \\
\tilde{\lambda}(s) &:= R_{V+\lambda(s)}^{\omega(s)}(\lambda(s)) = \frac{\lambda(s)}{\lambda_1(s)}, \\
\tilde{\mu}(s) &:= R_{V+\lambda(s)}^{\omega(s)}(\mu(s)) = \frac{\mu(s) \cap (V + \lambda(s)) + \lambda_1(s)}{\lambda_1(s)}.
\end{aligned}$$

Since $V(t)' \subset V$, by Lemma A.1.1 we have

$$\begin{aligned}
\tilde{V}(t, t) + \tilde{\lambda}(t) + \tilde{\mu}(t) &= \frac{V(t)' + \lambda(t) + \mu(t) \cap (V + \lambda(t))}{\lambda_1(t)} \\
&= \frac{(V(t)' + \lambda(t) + \mu(t)) \cap (V + \lambda(t))}{\lambda_1(t)} \\
&= \frac{X \cap (V + \lambda(t))}{\lambda_1(t)} = \frac{V + \lambda(t)}{\lambda_1(t)}.
\end{aligned}$$

By Proposition 1.4.10, Lemma 1.4.3 and Proposition 2.3.3, there exists a $\delta_1(t) > 0$ for each t such that, for $[s_1, s_2] \subset (t - \delta_1(t), t + \delta_1(t)) \cap [0, 1]$, we have

$$\begin{aligned}
&\text{Mas}_{\pm}\{P_0(s)(\lambda(s)), P_0(s)(\mu(s)); \omega_l(s); s \in [s_1, s_2]\} \\
&= \text{Mas}_{\pm}\{\tilde{\lambda}(s), \tilde{\mu}(s); \tilde{\omega}(s); s \in [s_1, s_2]\} \\
&= \text{Mas}_{\pm}\{R_{\tilde{W}(t,s)}^{\tilde{\omega}(s)}(\tilde{\lambda}(s)), R_{\tilde{W}(t,s)}^{\tilde{\omega}(s)}(\tilde{\mu}(s)); s \in [s_1, s_2]\} \\
&= \text{Mas}_{\pm}\{R_{V(t)'+\lambda_1(s)}^{\omega(s)}(\lambda(s)), R_{V(t)'+\lambda_1(s)}^{\omega(s)}(\mu(s)); s \in [s_1, s_2]\} \\
&= \text{Mas}_{\pm}\{P_0(t, s)'(\lambda(s)), P_0(t, s)'(\mu(s)); \omega_l(t, s)'; s \in [s_1, s_2]\}.
\end{aligned}$$

For any closed subspace $N \subset X$ we denote by $G(X, N)$ the (possibly empty) set of closed subspaces of X that are transversal and complementary to N in X . By Lemma A.4.5, it is an open affine space. Hence in our case there exists a path $f(t, \cdot): [0, 1] \rightarrow G(X, \lambda(t) + \mu(t))$ with $f(t, 0) = V(t)$ and $f(t, 1) = V(t)'$ for each $t \in [0, 1]$. For $t, a \in [0, 1]$ and s “close” to t (to be specified at once), set $\lambda_1(t, a, s) := f(t, a)^{\omega(s)} \cap \lambda(s)$, and $\mu_1(t, a, s) := f(t, a)^{\omega(s)} \cap \mu(s)$. By Proposition 1.3.3 and Appendix A.3, there exists $\delta(t) \in (0, \delta_1(t))$ for each t such that

$$\begin{aligned}
X &= \lambda_0(t) \oplus f(t, a) \oplus \lambda_1(t, a, s) \oplus \mu_1(t, a, s) \\
&= \lambda(s) \oplus (f(t, a) + \mu_1(t, a, s)) = \mu(s) \oplus (f(t, a) + \lambda_1(t, a, s))
\end{aligned}$$

for all $s \in (t - \delta(t), t + \delta(t)) \cap [0, 1]$. That proves Properties (ii) and (iii) of our Choices and Notations 3.1.2, i.e., our claim a.

To b, we observe that in our case the symplectic reduction does not change the dimension of the intersection of Lagrangian subspaces. By Lemma A.4.5, we can find a path connecting λ_0 and λ'_0 in $G(X, V \oplus X_1(t))$. By Proposition 2.3.1.a, the left hand side of (3.5) remains unchanged in a small interval $[s_1, s_2]$ if we replace $V(t)$ by $V(t)'$ and λ_0 by λ'_0 . Then (3.5) holds. \square

NOTE. We emphasize that for fixed t , $\{f(t, a)\}_{a \in [0, 1]}$ is a path of finite-dimensional subspaces of X . For each $t, a \in [0, 1]$, $f(t, a)$ satisfies that $X = f(t, a) \oplus (\lambda(t) + \mu(t))$,

$f(t, 0) = V(t)'$, and $f(t, 1) = V(t)$. So by the homotopy invariance and the vanishing of the Maslov index (in the finite-dimensional case), the Maslov index is unchanged along the path $f(t, a)$ (fix t). The difference between $V(t)$ and $V(t)'$ is that $V(t)' \subset V$, while $V(t)$ can vary widely.

PROOF OF THEOREM 3.1.4. By taking a common refinement of the partitions, the first part of the Theorem follows from Lemma 3.1.6. The second part of the Theorem is a repetition of the list of properties given in Proposition 2.3.1 for the case of a strong symplectic Hilbert space. The validity in the general case follows from the proposition and our definition of the Maslov index. \square

We have the following lemma from ([24, Lemma 8]):

LEMMA 3.1.7. *Let (X, ω) be a symplectic vector space. Let Δ denote the diagonal (i.e., the canonical Lagrangian) in the product symplectic space $(X \oplus X, (-\omega) \oplus \omega)$, and λ, μ are linear subspaces of (X, ω) . Then*

$$(\lambda, \mu) \in \mathcal{FL}(X) \iff (\lambda \oplus \mu, \Delta) \in \mathcal{FL}(X \oplus X)$$

and

$$\text{index}(\lambda, \mu) = \text{index}(\lambda \oplus \mu, \Delta),$$

where $\lambda \oplus \mu := \{(x, y); x \in \lambda, y \in \mu\}$.

The following proposition generalizes [24, Proposition 4 (b)].

PROPOSITION 3.1.8. *Denote by $\Delta(s)$ the diagonal of $X(s) \times X(s)$. Under Assumption 3.1.1, we have*

$$(3.7) \quad \text{Mas}\{\lambda(s) \oplus \mu(s), \Delta(s); \omega(s) \oplus (-\omega(s))\} = \text{Mas}\{\lambda(s), \mu(s); \omega(s)\}$$

$$(3.8) \quad = \text{Mas}\{\mu(s), \lambda(s); -\omega(s)\}$$

$$(3.9) \quad = \text{Mas}\{\Delta(s), \lambda(s) \oplus \mu(s); (-\omega(s)) \oplus \omega(s)\}.$$

PROOF. By [24, Proposition 4 (b)], our results hold in the finite-dimensional case. The general case follows from the definition of the Maslov index. \square

3.2. Calculation of the Maslov index

We start with the general case. We fix our data and make some choices.

DATA. Let $\varepsilon > 0$ be a positive number. Let X be a (complex) Banach space with continuously varying symplectic structure $\omega(s)$, $s \in (-\varepsilon, \varepsilon)$. Let $(\lambda(s), \mu(s))$ be a path of Fredholm pairs of Lagrangian subspaces of $(X, \omega(s))$ of index 0. Let $V(s)$ and $\lambda_0(s)$ be a path of finite-dimensional subspaces of X with $\lambda_0(0) = \lambda(0) \cap \mu(0)$ and $V(0) \oplus (\lambda(0) + \mu(0)) = X$. Set $\lambda_1(s) := V(s)^{\omega(s)} \cap \lambda(s)$, $\mu_1(s) := V(s)^{\omega(s)} \cap \mu(s)$, $X_0(s) := \lambda_0(s) + V(s)$, $X_1(s) := \lambda_1(s) + \mu_1(s)$.

THEOREM 3.2.1 (Intrinsic decompositions and representations). *There exists a $\delta > 0$ such that for each $s \in (-\delta, \delta)$ and each subinterval $[s_1, s_2] \subset (-\delta, \delta)$ we have the following intrinsic decompositions, representations, and formulae:*

$$(a) \quad X = V(s) + \lambda(s) + \mu(s) = V(s)^{\omega(s)} + \lambda(s) = V(s)^{\omega(s)} + \mu(s);$$

$$(b) \quad X = \lambda_0(s) \oplus V(s) \oplus \lambda_1(s) \oplus \mu_1(s) = V(s)^{\omega(s)} \oplus \lambda_0(s);$$

(c) $\lambda(s)$ and $\mu(s)$ are expressed by

$$(3.10) \quad \lambda(s) = \text{graph} \left(\begin{pmatrix} A_1(s) & 0 \\ A_2(s) & 0 \end{pmatrix} : \lambda_0(s) \oplus \lambda_1(s) \rightarrow V(s) \oplus \mu_1(s) \right),$$

$$(3.11) \quad \mu(s) = \text{graph} \left(\begin{pmatrix} B_1(s) & 0 \\ B_2(s) & 0 \end{pmatrix} : \lambda_0(s) \oplus \mu_1(s) \rightarrow V(s) \oplus \lambda_1(s) \right)$$

with $A_1(0) = A_2(0) = B_1(0) = B_2(0) = 0$;

(d) Setting

$$(3.12) \quad \omega_l(s) := \omega(s)|_{X_0(s)} - \begin{pmatrix} I & 0 \\ A_1(s) & 0 \end{pmatrix}^* (\omega(s)|_{X_0(s)}), \text{ and}$$

$$(3.13) \quad \omega_r(s) := \omega(s)|_{X_0(s)} - \begin{pmatrix} I & 0 \\ B_1(s) & 0 \end{pmatrix}^* (\omega(s)|_{X_0(s)}),$$

we obtain $\omega_l(s) = \omega_r(s)$; moreover, the images $P_0(s)(\mu(s))$ are Lagrangian subspaces of the symplectic vector space $(X_0(s), \omega_l(s))$, where $P_0(s) : X \rightarrow X_0(s)$ denotes the projection defined by $X = X_0(s) \oplus X_1(s)$ like before.

(e) The following equalities hold for the segmental Maslov indices and intersection dimensions:

$$(3.14) \quad \begin{aligned} \text{Mas}_\pm \{ \lambda(s), \mu(s); s \in [s_1, s_2] \} \\ = \text{Mas}_\pm \{ P_0(s)(\lambda(s)), P_0(s)(\mu(s)); \omega_l(s); s \in [s_1, s_2] \}, \end{aligned}$$

and

$$(3.15) \quad \dim(\lambda(s) \cap \mu(s)) = \dim(P_0(s)(\lambda(s)) \cap P_0(s)(\mu(s))).$$

PROOF. (a) By Appendix A.3.

(b), (c), (d) By (a), Proposition 1.4.10 and Appendix A.3.

(e) If $V(s) \equiv V(0)$, our result follows from the definition of the Maslov index and Proposition 1.4.10.c.

In the general case, by [57, Lemma I.4.10] there exists a path of bounded invertible map $L(s) \in \mathcal{B}^\times(X)$ such that $L(s)X_0 = X_0(s)$ with $L(0) = I$. By Proposition 2.3.1.d we have

$$(3.16) \quad \begin{aligned} & \text{Mas}_\pm \{ P_0(s)(\lambda(s)), P_0(s)(\mu(s)); \omega_l(s); s \in [s_1, s_2] \} \\ &= \text{Mas}_\pm \{ L(s)^{-1}P_0(s)(\lambda(s)), L(s)^{-1}P_0(s)(\mu(s)); L(s)^*\omega_l(s); s \in [s_1, s_2] \}. \end{aligned}$$

Note that $L(s)$, $P_0(s)$ and $\omega_l(s)$ depend continuously on $\lambda_0(s)$, $V(s)$, $\lambda(s)$, $\mu(s)$ and $\omega(s)$. Replacing $\lambda_0(s)$ by $\lambda_0(ts)$ and $V(s)$ by $V(ts)$ for $t \in [0, 1]$, we get a homotopy of the right hand side of (3.16). Note that in our case $\dim(P_0(s)(\lambda(s)) \cap P_0(s)(\mu(s))) = \dim(\lambda(s) \cap \mu(s))$. Then our result follows from the special case and Proposition 2.3.1. \square

By Corollary 1.3.4 and [72, Lemma 0.2], we have a path $\lambda_0(s) \subset \mu(s)$ with $\lambda_0(0) = \lambda(0) \cap \mu(0)$. By Corollary 1.3.6 and Proposition 1.3.3.d, we have a Lagrangian path $V(s) \in X_0(s)$ of $X_0(s)$. We have the following corollary.

COROLLARY 3.2.2. Assume that $\lambda_0(s) \subset \mu(s)$ as in the data before Theorem 3.2.1 and let $\delta > 0$ be found correspondingly. Then for each $s \in (-\delta, \delta)$ and

$[s_1, s_2] \subset (-\delta, \delta)$, we have $\omega_l(s) = \omega(s)|_{X_0(s)}$, and

$$(3.17) \quad \begin{aligned} & \text{Mas}_\pm\{\lambda(s), \mu(s); s \in [s_1, s_2]\} \\ &= \text{Mas}_\pm\{P_0(s)(\lambda(s)), \lambda_0(s); \omega(s)|_{X_0(s)}; s \in [s_1, s_2]\}. \end{aligned}$$

PROOF. In this case we have $B_1(s) = B_2(s) = 0$, $P_0(s)(\mu(s)) = \lambda_0(s)$ and $\omega_l(s) = \omega(s)|_{X_0(s)}$. By Theorem 3.2.1, our results follow. \square

PROPOSITION 3.2.3. *Assume that $V(s)$ is isotropic in Theorem 3.2.1. Let δ be as given there. Then for each $s \in (-\delta, \delta)$ and $[s_1, s_2] \subset (-\delta, \delta)$ we have a quadratic form $Q(s)$ such that*

$$(3.18) \quad \text{Mas}_+\{\lambda(s), \mu(s); s \in [s_1, s_2]\} = m^+(Q(s_2)) - m^+(Q(s_1)),$$

$$(3.19) \quad \text{Mas}_-\{\lambda(s), \mu(s); s \in [s_1, s_2]\} = m^-(Q(s_1)) - m^-(Q(s_2)),$$

$$(3.20) \quad \dim(\lambda(s) \cap \mu(s)) = m^0(Q(s)),$$

where $Q(s)(x, y) := \omega(s)(x, (A_1(s) - B_1(s))y)$ for all $x, y \in \lambda_0(s)$ and $s \in (-\delta, \delta)$.

PROOF. Since $V(s) \subset X_0(s)$ is isotropic, $P_0(\mu(s))$ and $V(s)$ is Lagrangian in $(X_0(s), \omega_l(s))$. We have $X_0(s) = P_0(\mu(s)) \oplus V(s)$ and $Q(s)$ is a quadratic form. For each $x \in \lambda_0(s)$, we have

$$\begin{aligned} x + A_1(s)x &= x + B_1(s)x + (A_1(s) - B_1(s))x, \text{ and} \\ \omega_l(s)(x + A_1(s)x, (A_1(s) - B_1(s))x) \\ &= \omega(s)(x + A_1(s)x, (A_1(s) - B_1(s))x) = Q(s)(x, x). \end{aligned}$$

By Theorem 3.2.1, Lemma 2.3.2 and Proposition 2.3.1.b, our results follow. \square

We now calculate $Q(s)$.

LEMMA 3.2.4. *Let (X, ω) be a symplectic vector space with Lagrangian subspaces λ , μ , isotropic spaces α_0 , V and a linear subspace λ_0 . Assume that $\dim \alpha_0 = \dim \lambda_0 = \dim V < +\infty$. Set $\lambda_1 := V^\omega \oplus \lambda$ and $\mu_1 := \mu \oplus \mu$. Let $\alpha_1, \beta_1 \subset V^\omega$ be isotropic subspaces. Assume that*

$$X = \alpha_0 \oplus V \oplus \alpha_1 \oplus \beta_1 = \lambda_0 \oplus V \oplus \lambda_1 \oplus \mu_1.$$

Assume that $\lambda = \text{graph}(A) = \text{graph} \tilde{A}$ and $\mu = \text{graph}(B)$, where

$$(3.21) \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : \alpha_0 \oplus \alpha_1 \rightarrow V \oplus \beta_1,$$

$$(3.22) \quad \tilde{A} = \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix} : \lambda_0 \oplus \lambda_1 \rightarrow V \oplus \mu_1,$$

$$(3.23) \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} : \alpha_0 \oplus \beta_1 \rightarrow V \oplus \alpha_1.$$

Then the following holds.

- (a) $V^\omega = V \oplus \alpha_1 \oplus \beta_1$ and $\lambda_1 = \text{im}(A_{12} + I_{\alpha_1} + A_{22})$.
- (b) If $\mu = \alpha_0 \oplus \beta_1$ and $\lambda_0 = \alpha_0$ hold, we have $\mu_1 = \beta_1$, $A_1 = A_{11}$ and $A_2 = A_{22}$.
- (c) Set

$$f := I_{\alpha_0} + B_{11} + B_{21} : \alpha_0 \rightarrow X, \quad g := B_{12} + B_{22} + I_{\beta_1} : \beta_1 \rightarrow X.$$

Assume that $\lambda_0 = f(\alpha_0)$ and $I_{\beta_1} - A_{22}B_{22}$ is invertible. Then we have

$$(3.24) \quad A_1 f = A_{11} - B_{11} + A_{12}B_{21} - (B_{12} - A_{12}B_{22})$$

$$(I_{\beta_1} - A_{22}B_{22})^{-1}(A_{21} + A_{22}B_{21}),$$

$$(3.25) \quad A_2 f = g(I_{\beta_1} - A_{22}B_{22})^{-1}(A_{21} + A_{22}B_{21}).$$

PROOF. (a), (b) by definition.

(c) Let $x \in \alpha_0$. Set $\tilde{x} := f(x)$ and $w := g^{-1}(A_2 \tilde{x})$. Then we have $\tilde{x} + A_1 \tilde{x} + A_2 \tilde{x} \in \lambda$. Since $\lambda = \text{graph}(A)$, we have

$$\begin{pmatrix} B_{11}x + A_1 \tilde{x} + B_{12}w \\ w \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x \\ B_{21}x + B_{22}w \end{pmatrix}.$$

By direct calculations we get (3.24) and (3.25). \square

We generalize the notion of crossing forms of [82] to the case of C^1 varying symplectic structures.

Let (X, ω) be a symplectic Banach space and let $\lambda = \{\lambda(s)\}_{s \in [0,1]}$ be a C^1 curve of Lagrangian subspaces. Assume that $\lambda(t)$ is complemented. Let W be a fixed Lagrangian complement of $\lambda(t)$. The form

$$(3.26) \quad Q(\lambda, t) := Q(\lambda, W, t) = \frac{d}{ds} \Big|_{s=t} Q(\lambda(t), W; \lambda(s))$$

on $\lambda(t)$ is independent of the choice of W , where $Q(\alpha, \beta; \gamma)$ is defined by Proposition 1.4.10.e (see [43, (2.3)]).

If $X = \alpha \oplus W = \lambda(s) \oplus W$ for two Lagrangian subspaces α and W and $|s - t| \ll 1$, then, by Lemma A.4.5, there exists a path $A(s) \in \mathcal{B}(\alpha, W)$ with $\lambda(s) = \{x + A(s)x; x \in \alpha\}$. By definition we have

$$(3.27) \quad Q(\lambda, W, t)(x + A(t)x, y + A(t)y) = \frac{d}{ds} \Big|_{s=t} Q(\alpha, W; \lambda(s))(x, y).$$

LEMMA 3.2.5 (Crossing form independence). *Let $(X, \omega(s))$ $s \in (-\varepsilon, \varepsilon)$, be a C^1 path of symplectic Banach spaces with two C^1 families of Lagrangian subspaces $\alpha(s), \beta(s)$. Assume that $X = \alpha(s) \oplus \beta(s)$. Let $x(s), y(s) \in \alpha(s)$ be two C^1 paths. Let $A(s), B(s), C(s) \in \mathcal{B}(\alpha(s), \beta(s))$ and $D(s) \in \mathcal{B}(\beta(s), \alpha(s))$ are C^1 families of bounded linear maps with $A(0) = B(0) = C(0) = 0$. Set $\lambda(s) := \text{graph}(A(s))$, $\mu(s) := \text{graph}(B(s))$, $\tilde{\alpha}(s) := \text{graph}(C(s))$ and $\tilde{\beta}(s) := \text{graph}(D(s))$. Then the following holds.*

(a) *There exists a $\delta \in (0, \varepsilon)$ such that for all $s \in (-\delta, \delta)$, we have*

$$X = \tilde{\alpha}(s) \oplus \tilde{\beta}(s) = \lambda(s) \oplus \tilde{\beta}(s) = \mu(s) \oplus \tilde{\beta}(s).$$

(b) *For $s \in (-\delta, \delta)$, let $u(s), v(s) \in \beta(s)$ be such that*

$$y(s) + C(s)y(s) + w_1(s) \in \lambda(s), y(s) + C(s)y(s) + w_2(s) \in \lambda(s),$$

where $w_1(s) := u(s) + D(s)u(s)$, $w_2(s) := v(s) + D(s)v(s)$. Then we have

$$(3.28) \quad \Gamma(\lambda, \mu, 0)(x(0), y(0)) : \\ = \frac{d}{ds} \Big|_{s=0} \omega(s)(x(s) + C(s)x(s), w_1(s) - w_2(s))$$

$$(3.29) \quad = \frac{d}{ds} \Big|_{s=0} \omega(0)(x(0), (A(s) - B(s))y(s)).$$

(c) The form $\Gamma(\lambda, \mu, 0)$ is a quadratic form if $\lambda(s)$ and $\mu(s)$ are Lagrangian subspaces of $(X, \omega(s))$.

PROOF. (a) By the continuity of the given families.

(b) By the definitions we have $y(s) + D(s)u(s) + u(s) + C(s)y(s) \in \lambda(s)$. Then we have $u(s) + C(s)y(s) = A(s)(y(s) + D(s)u(s))$, and

$$u(s) = (I_{\beta(s)} - A(s)D(s))^{-1}(A(s) - C(s))y(s).$$

Since $\alpha(s)$ and $\beta(s)$ are Lagrangian subspaces, we have

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \omega(s)(x(s) + C(s)x(s), w_1(s)) \\ &= \frac{d}{ds} \Big|_{s=0} \omega(s)(x(s), u(s)) \\ &= \frac{d}{ds} \Big|_{s=0} \omega(s)(x(s), (A(s) - C(s))y(s)) \\ (3.30) \quad &= \frac{d}{ds} \Big|_{s=0} \omega(0)(x(0), (A(s) - C(s))y(s)). \end{aligned}$$

Applying (3.30) for $\mu(s)$, we obtain (3.29).

(c) If $\lambda(s)$ and $\mu(s)$ are Lagrangian subspaces of $(X, \omega(s))$, then the two forms $\omega(s)(x(s), A(s)y(s))$ and $\omega(s)(x(s), B(s)y(s))$ are bilinear and the form $\Gamma(\lambda, \mu, 0)$ becomes quadratic. \square

LEMMA 3.2.6 (Crossing form calculation). *We use the data of Theorem 3.2.1. Assume that $\omega(s)$, $V(s)$, $\lambda(s)$, $\mu(s)$ with $s \in (-\delta, \delta)$ are C^1 families and $\lambda_0(s) = \lambda_0 = \lambda(0) \cap \mu(0)$. Then the following holds.*

- (a) *The families $V(s)^{\omega(s)}$, $\lambda_1(s)$, $\mu_1(s)$ are C^1 families.*
(b) *The form*

$$(3.31) \quad \Gamma(\lambda, \mu, \lambda_0, V, 0)(x(0), y(0)) := \frac{d}{ds} \Big|_{s=0} \omega(s)(x(s), (A_1(s) - B_1(s))y(s))$$

on $\lambda_0(0)$ is quadratic, where $x(s), y(s) \in \lambda_0(s)$ are two C^1 paths.

(c) *The form $\Gamma(\lambda, \mu, 0) := \Gamma(\lambda, \mu, \lambda_0, V, 0)$ does not depend on the choices of paths λ_0 and V . In particular,*

- (i) *if $V(s)$ is a C^1 isotropic path and $Q(s)$ is defined as in Proposition 3.2.3, we have*

$$(3.32) \quad \Gamma(\lambda, \mu, 0)(x(0), y(0)) = \frac{d}{ds} \Big|_{s=0} Q(s)(x(s), y(s));$$

- (ii) *for $A_{11}(s)$ and $B_{11}(s)$ as defined in Lemma 3.2.4, we have*

$$(3.33) \quad \Gamma(\lambda, \mu, 0)(x(0), y(0)) = \frac{d}{ds} \Big|_{s=0} \omega(s)(x(s), (A_{11}(s) - B_{11}(s))y(s));$$

- (iii) *if $\omega(s) \equiv \omega$ is fixed, we have*

$$(3.34) \quad \Gamma(\lambda, \mu, 0) = (Q(\lambda, 0) - Q(\mu, 0))|_{\lambda(0) \cap \mu(0)}.$$

PROOF. (a) By Lemma A.4.9.

(b) By (a), the families $A_1(s)$, $A_2(s)$, $B_1(s)$, $B_2(s)$ are of class C^k . Note that $A_1(0) =$

$A_2(0) = B_1(0) = B_2(0) = 0$. Then we have

$$\begin{aligned} 0 &= \frac{d}{ds} \Big|_{s=0} \omega_l(s)(x(s) + B_1(s)x(s), y(s) + B_1(s)y(s)) \\ &= \frac{d}{ds} \Big|_{s=0} \omega(s)((B_1(s) - A_1(s))x(s), y(s)) \\ &\quad + \frac{d}{ds} \Big|_{s=0} \omega(s)(x(s), (B_1(s) - A_1(s))y(s)). \end{aligned}$$

Since $\omega(s)$ is symplectic, we get our result.

(c) Fix a C^1 path $\tilde{\lambda}_0(s) \subset \mu(s)$ with $\tilde{\lambda}_0(0) = \lambda_0(0)$. Consider the C^1 symplectic space $(V(s) + \mu(s))/\mu_1(s)$. By Lemma 3.2.5, we have

$$(3.35) \quad \Gamma(\lambda, \mu, \lambda_0, V, 0) = \Gamma(\lambda, \mu, \tilde{\lambda}_0, V, 0).$$

Take a C^1 path of Lagrangian subspaces $\tilde{V}(s)$ of symplectic subspaces $\tilde{X}_0(s) := \tilde{\lambda}_0(s) + V(s)$. Then we have $\omega_l(s) = \omega(s)|_{\tilde{X}_0(s)}$. By Lemma 3.2.5, we have

$$(3.36) \quad \Gamma(\lambda, \mu, \tilde{\lambda}_0, V, 0) = \Gamma(\lambda, \mu, \tilde{\lambda}_0, \tilde{V}, 0).$$

Fix a C^1 isotropic path $\bar{V}(s)$ with $X(0) = \bar{V}(0) \oplus (\lambda(0) + \mu(0))$. Fix C^1 paths $\tilde{x}(s), \tilde{y}(s) \in \tilde{\lambda}_0(s)$. Let $\tilde{A}_1(s), \tilde{A}_2(s)$ and $\bar{A}_1(s), \bar{A}_2(s)$ be C^1 paths defined by Theorem 3.2.1 for $(\lambda, \mu, \tilde{\lambda}_0, \tilde{V})$ and $(\lambda, \mu, \tilde{\lambda}_0, \bar{V})$. Since $\tilde{x}(s) + \tilde{A}_1(s)\tilde{x}(s) + \tilde{A}_2(s)\tilde{x}(s), \tilde{y}(s) + \bar{A}_1(s)\tilde{y}(s) + \bar{A}_2(s)\tilde{y}(s) \in \lambda(s)$, we have

$$\begin{aligned} 0 &= \frac{d}{ds} \Big|_{s=0} \omega(s) \\ &\quad (\tilde{x}(s) + \tilde{A}_1(s)\tilde{x}(s) + \tilde{A}_2(s)\tilde{x}(s), \tilde{y}(s) + \bar{A}_1(s)\tilde{y}(s) + \bar{A}_2(s)\tilde{y}(s)) \\ &= \frac{d}{ds} \Big|_{s=0} \omega(s)(\tilde{x}(s), \tilde{y}(s) + \bar{A}_1(s)\tilde{y}(s) + \bar{A}_2(s)\tilde{y}(s)) \\ &\quad + \frac{d}{ds} \Big|_{s=0} \omega(s)(\tilde{x}(s) + \tilde{A}_1(s)\tilde{x}(s) + \tilde{A}_2(s)\tilde{x}(s), \tilde{y}(s)) \\ &= \frac{d}{ds} \Big|_{s=0} \left(\omega(s)(\tilde{x}(s), \bar{A}_1(s)\tilde{y}(s) + \bar{A}_2(s)\tilde{y}(s)) + \omega(s)(\tilde{A}_1(s)\tilde{x}(s), \tilde{y}(s)) \right) \\ &= \Gamma(\lambda, \mu, \tilde{\lambda}_0, \bar{V})(x(0), y(0)) - \Gamma(\lambda, \mu, \tilde{\lambda}_0, \tilde{V})(x(0), y(0)). \end{aligned}$$

By (3.35) and (3.36), we obtain

$$(3.37) \quad \Gamma(\lambda, \mu, \lambda_0, V, 0) = \Gamma(\lambda, \mu, \tilde{\lambda}_0, \bar{V}, 0).$$

For the special cases, (i) is clear, (ii) by taking $\lambda_0(s) = \alpha_0(s)$, and (iii) by taking $\lambda_0(s) \equiv \lambda_0(0)$ and $V(s) \equiv V(0)$ to be an isotropic subspace. \square

Let $p: \mathbb{X} \rightarrow [0, 1]$ be a C^1 Banach bundle with $p^{-1}(s) = X(s)$. Let $\{(\lambda(s), \mu(s))\}$, $0 \leq s \leq 1$ be a curve of Fredholm pairs of Lagrangian subspaces of C^1 family $(X(s), \omega(s))$ of index 0. By Corollary 1.3.4, $\lambda(s)$ and $\mu(s)$ are complemented. For $t \in [0, 1]$, the *crossing form* $\Gamma(\lambda, \mu, t)$ is a quadratic form on $\lambda(t) \cap \mu(t)$ defined by Lemma 3.2.6.

A *crossing* is a time $t \in [0, 1]$ such that $\lambda(t) \cap \mu(t) \neq \{0\}$. A crossing is called *regular* if $\Gamma(\lambda, \mu, t)$ is non-degenerate. It is called *simple* if it is regular and $\lambda(t) \cap \mu(t)$ is one-dimensional. As before, we shall denote by m^+, m^-, m^0 the dimensions of the subspaces where the form is positive-definite, negative-definite, or vanishing, respectively.

Now we give a method for using the crossing form to calculate Maslov indices (see [82] for the fixed finite-dimensional symplectic vector space case, [16, Theorem 2.1] and [104, Proposition 4.1] for the fixed strong symplectic Hilbert space case).

PROPOSITION 3.2.7 (Calculation of the Maslov index). *Let $(X(s), \omega(s))$ be a C^1 family of symplectic Banach space and $\{(\lambda(s), \mu(s))\}$, $0 \leq s \leq 1$ be a C^1 curve of Fredholm pairs of Lagrangian subspaces of X of index 0 with only regular crossings. Then we have*

$$(3.38) \quad \text{Mas}\{\lambda, \mu\} = m^+(\Gamma(\lambda, \mu, 0)) - m^-(\Gamma(\lambda, \mu, 1)) + \sum_{0 < t < 1} \text{sign}(\Gamma(\lambda, \mu, t)).$$

PROOF. For each crossing $t \in [0, 1]$, we consider the path $(\lambda(s+t), \mu(s+t))$ for $|s-t| \ll 1$. By Proposition 1.3.3.c, we can take an isotropic V with $X = V \oplus (\lambda(t) + \mu(t))$. Then the assumptions of Lemma 3.2.4 can be satisfied. Let $Q(s)$ be defined in Proposition 3.2.3. By Lemma 3.2.6.c we have

$$\frac{d}{ds}\bigg|_{s=0} Q(s) = \Gamma(\lambda, \mu, t).$$

Since the crossing t is regular, for $0 < |s| \ll 1$, by (3.32) $Q(s)$ and $s\Gamma(\lambda, \mu, t)$ are non-degenerate and they have the same positive (negative) Morse index. Thus the set of crossings is discrete (and then finite, for $[0, 1]$ is compact). By Proposition 3.2.3 and Proposition 2.3.1.b, our results hold. \square

We recall (see [25, Definition 3.1] for the finite-dimensional case).

DEFINITION 3.2.8. Let (X, ω) be a symplectic Banach space and let $\lambda := \{\lambda(s)\}_{s \in [0, 1]}$, be a C^1 curve of complemented Lagrangian subspaces. We call the curve λ (semi-)positive at $t \in [0, 1]$, if $Q(\lambda, t)$ is positive definite, respectively semi-positive definite. The curve $\{\lambda(s)\}$ is called (semi-)positive if it is (semi-)positive at all $t \in [0, 1]$, respectively.

LEMMA 3.2.9. *Let (X, ω) be a symplectic Banach space and $\{\lambda(s)\}_{s \in [0, 1]}$ a C^1 curve with a Lagrangian complement W . Then $\{\lambda(s)\}$ is (semi-) positive if and only if the path of quadratic forms $Q(\lambda(0), W; \lambda(s))$ is strictly increasing (respectively, increasing).*

PROOF. By (3.27). \square

LEMMA 3.2.10 (Morse-positive path). *Let X be a finite-dimensional Hilbert space and*

$$A: (-\varepsilon, \varepsilon) \longrightarrow \mathcal{B}^{\text{sa}}(X)$$

a family of self-adjoint operators. Assume that $A(s_1) \leq A(s_2)$ for all $-\varepsilon < s_1 \leq s_2 < \varepsilon$. Then the following holds.

(a) *There exists a $\delta \in (-\varepsilon, \varepsilon)$ such that the functions $m^{\pm, 0}(A(s))$ are constant for $s \in (-\delta, 0)$ or $s \in (0, \delta)$.*

(b) *Assume that $A(s)$ is continuous at s . Then we have*

$$(3.39) \quad m^+(A(s)) = m^+(A_0), m^-(A(s)) - m^-(A(0)) = m^0(A(0)) - m^0(A(s))$$

for $s \in (-\delta, 0)$, and

$$(3.40) \quad m^-(A(s)) = m^-(A(0)), m^+(A(s)) - m^+(A(0)) = m^0(A(0)) - m^0(A(s))$$

for $s \in (0, \delta)$.

PROOF. (a) For all linear subspace V of X , and $-\varepsilon < s_1 \leq s_2 < \varepsilon$, we have that $A(s_1)|_V > 0$ implies $A(s_2) > 0$. So $m^+(A(s))$ is an increasing function. Similarly, $m^-(A(s))$ is a decreasing function. Since the two functions are bounded integer valued, they have finitely many discontinuous points. Since $m^0(A(s)) = \dim X - m^+(A(s)) - m^-(A(s))$, the same result holds for $m^0(A(s))$. So we obtain (a).

(b) Since $A(s)$ is continuous at s , we have $m^\pm(0) \leq m^\pm(s)$. Note that $m^+(A(s))$ is an increasing function and $m^-(A(s))$ is a decreasing function. Then the first equalities of (3.39) and (3.40) follow. The second equalities of (3.39) and (3.40) follow from the first ones. \square

PROPOSITION 3.2.11 (Counting rule for (semi-)positive paths). *Let (X, ω) be a symplectic Banach space and let $\{(\lambda(s), \mu)\}$, $0 \leq s \leq 1$ be a C^1 curve of Fredholm pairs of Lagrangian subspaces of X of index 0 with a semi-positive path λ and constant path μ . Then $\dim(\lambda(s) \cap \mu)$ is locally constant except for finitely many points $s \in [0, 1]$, and we have*

$$(3.41) \quad \text{Mas}\{\lambda, \mu\} = \sum_{0 < t \leq 1} (\dim(\lambda(t) \cap \mu) - \lim_{s \rightarrow t^-} \dim(\lambda(s) \cap \mu)).$$

PROOF. Let $t \in [0, 1]$ and consider the path $(\lambda(s+t), \mu)$ for $|s| \ll 1$. By Proposition 1.3.3, there exists an isotropic V such that $X = V \oplus (\lambda(s) + \mu)$. Set $\lambda_1(t+s) = V^\omega \cap \lambda(t+s)$, $\mu_1 := V^\omega \cap \mu$ and $W := V + \lambda_1$. Then W is a Lagrangian subspace and $X = \lambda(t+s) \oplus W$. Let $Q(s)$ be defined by Proposition 3.2.3 in our case. Then we have

$$Q(s) = Q(\lambda(t), W; \lambda(t+s))|_{\lambda(t) \cap \mu}.$$

By Lemma 3.2.9, the family of forms $Q(s)$ is increasing. By Proposition 3.2.3 and Lemma 3.2.10, we obtain our results. \square

The following theorem axiomatizes the well-known technique of identifying the Maslov indices of two one-parameter curves of Fredholm pairs of Lagrangian subspaces by suitable two-parameter extensions to topological trivial families with vanishing *top* edge and calculable *side* edges (see [16, Section 5]).

THEOREM 3.2.12 (Comparing two-parameter families). *Let $p: \mathbb{X} \rightarrow [0, 1]$ and $\tilde{p}: \tilde{\mathbb{X}} \rightarrow [0, 1]$ be two Banach bundles with $X(s) := p^{-1}(s)$, $\tilde{X}(s) := \tilde{p}^{-1}(s)$ for each $s \in [0, 1]$. Let $\{\omega(s)\}, \{\tilde{\omega}(s)\}$ be paths of symplectic forms for $X(s)$, respectively, $\tilde{X}(s)$, $0 \leq s \leq 1$. For $0 \leq a \leq \delta$, $\delta > 0$, we are given continuous two-parameter families*

$$(3.42) \quad \{(\lambda(s, a), \mu(s)) \in \mathcal{L}^2(X(s), \omega(s))\} \text{ and } \{(\tilde{\lambda}(s, a), \tilde{\mu}(s)) \in \mathcal{L}^2(\tilde{X}(s), \tilde{\omega}(s))\}.$$

We assume that

$$(3.43) \quad (\lambda(s, 0), \mu(s)) \in \mathcal{FL}_0(X(s)) \text{ and } (\tilde{\lambda}(s, 0), \tilde{\mu}(s)) \in \mathcal{FL}_0(\tilde{X}(s)),$$

$$(3.44) \quad \{\lambda(s, a)\} \text{ differentiable in } a \text{ and semi-positive for fixed } s,$$

$$(3.45) \quad \{\tilde{\lambda}(s, a)\} \text{ differentiable in } a \text{ and positive for fixed } s,$$

$$(3.46) \quad \dim(\lambda(s, a) \cap \mu(s)) - \dim(\tilde{\lambda}(s, a) \cap \tilde{\mu}(s)) = c(s).$$

Then we have

$$(3.47) \quad \text{Mas}\{\lambda(s, 0), \mu(s); \omega(s)\} = \text{Mas}\{\tilde{\lambda}(s, 0), \tilde{\mu}(s); \tilde{\omega}(s)\}.$$

PROOF. Since $[0, 1]$ is compact, after making δ smaller, we may assume that the given two families (3.42) are families of Fredholm pairs of index 0.

Fix $t \in [0, 1]$. Since $\tilde{\lambda}(t, a)$ is differentiable in a and positive, by Proposition 3.2.7 there exists a $\delta(t) \in (0, \delta)$ such that $\tilde{\lambda}(t, a) \cap \tilde{\mu}(t) = \{0\}$ for $a \in (0, \delta(t)]$. From the continuity of our family $(\tilde{\lambda}(s, a), \tilde{\mu}(s))$, there exists an $\varepsilon(t) > 0$ such that

$$(3.48) \quad \tilde{\lambda}(s, \delta(t)) \cap \tilde{\mu}(s) = \{0\} \text{ for } s \in (t - \varepsilon(t), t + \varepsilon(t)) \cap [0, 1].$$

By compactness of $[0, 1]$, there exists a partition $0 = s_0 < s_1 < \dots < s_n = 1$ of $[0, 1]$ and $t_1, \dots, t_n \in [0, 1]$ with $s_{k-1}, s_k \in (t_k - \varepsilon(t_k), t_k + \varepsilon(t_k))$ for $k = 1, \dots, n$.

We now prove the formula (3.47) for a small interval $[s_{k-1}, s_k]$. We consider the two-parameter families (3.42) for $s \in [s_{k-1}, s_k]$ and $a \in [0, \delta(t_k)]$. Because of the homotopy invariance of Maslov index, both integers $\text{Mas}\{\lambda(s, a), \mu(s)\}$ and $\text{Mas}\{\tilde{\lambda}(s, a), \tilde{\mu}(s)\}$ must vanish for the boundary loop going counter clockwise around the rectangular domain from the corner point $(s_{k-1}, 0)$ via the corner points $(s_k, 0)$, $(s_k, \delta(t_k))$, and $(s_{k-1}, \delta(t_k))$ back to $(s_{k-1}, 0)$.

Moreover, by (3.46) and (3.48), for all $s \in [s_{k-1}, s_k]$ we have

$$\dim(\lambda(s, \delta(t_k)) \cap \mu(s)) = c_{\delta(t_k)} \text{ and } \tilde{\lambda}(s, \delta(t_k)) \cap \tilde{\mu}(s) = \{0\}.$$

Hence, our two Maslov indices must vanish on the top segment of our box.

Finally, by Proposition 3.2.11 and (3.46) we have

$$\begin{aligned} & \text{Mas}\{\lambda(s_{k-1}, a), \mu(s_{k-1}); a \in [0, \delta(t_k)]\} \\ & - \text{Mas}\{\tilde{\lambda}(s_{k-1}, a), \tilde{\mu}(s_{k-1}); a \in [0, \delta(t_k)]\} \\ & = \sum_{0 < a \leq \delta(t_k)} (\dim(\lambda(s_{k-1}, a) \cap \mu(s_{k-1})) - \lim_{b \rightarrow a^-} \dim(\lambda(s_{k-1}, b) \cap \mu(s_{k-1}))) \\ & - \sum_{0 < a \leq \delta(t_k)} \dim(\tilde{\lambda}(s_{k-1}, a) \cap \tilde{\mu}(s_{k-1})) \\ & = \sum_{0 < a \leq \delta(t_k)} (c_a - \lim_{b \rightarrow a^-} c_b) \\ & = \sum_{0 < a \leq \delta(t_k)} (\dim(\lambda(s_k, a) \cap \mu(s_k)) - \lim_{b \rightarrow a^-} \dim(\lambda(s_k, b) \cap \mu(s_k))) \\ & - \sum_{0 < a \leq \delta(t_k)} \dim(\tilde{\lambda}(s_k, a) \cap \tilde{\mu}(s_k)) \\ & = \text{Mas}\{\lambda(s_k, a), \mu(s_k); a \in [0, \delta(t_k)]\} \\ & - \text{Mas}\{\tilde{\lambda}(s_k, a), \tilde{\mu}(s_k); a \in [0, \delta(t_k)]\}. \end{aligned}$$

By additivity under catenation, the formula (3.47) holds for the small interval $[s_{k-1}, s_k]$. Again by additivity under catenation, the formula (3.47) holds for the whole interval $[0, 1]$. \square

3.3. Invariance of the Maslov index under symplectic operations

In this section we show that the Maslov index is invariant under symplectic reduction and symplectic embedding under natural conditions.

The first theorem generalizes our previous set-up of Choices and Notations 3.1.2 and our previous Definition 3.1.3. We begin with a lemma. It transfers the purely algebraic arguments of Definition 1.4.1 to the case of Banach spaces and gives a sufficient condition for the symplectic reduction becoming a symplectic Banach space.

LEMMA 3.3.1. *Let (X, ω) be a symplectic Banach space and $W \subset X$ a co-isotropic subspace. Assume that W is a Banach space (not necessarily induced by the norm on X) such that the injection $j: W \hookrightarrow X$ is bounded. Then the symplectic reduction $(W/W^\omega, \tilde{\omega})$ is a symplectic Banach space.*

PROOF. Note that W^ω is closed in X . Then by Lemma A.6.1, W^ω is closed in W , so the quotient W/W^ω with the norm induced by W is a Banach space. Since j is bounded, the induced form $\tilde{\omega}$ is bounded on W/W^ω . Then $(W/W^\omega)^{\tilde{\omega}} = W^\omega/W^\omega = \{0\}$. So $\tilde{\omega}$ is non-degenerate. Since W is a Banach space, $(W/W^\omega, \tilde{\omega})$ is a symplectic Banach space. \square

In the following we shall parameterize the symplectic reduction.

ASSUMPTION 3.3.2. We make the following assumptions.

(i) We are given Banach bundles $q_0: \mathbb{W}_0 \rightarrow [0, 1]$, $q: \mathbb{W} \rightarrow [0, 1]$, $\tilde{q}: \widetilde{\mathbb{W}} \rightarrow [0, 1]$, and $p: \mathbb{X} \rightarrow [0, 1]$ with fibers $q_0^{-1}(s) := W_0(s)$, $q^{-1}(s) := W(s)$, $\tilde{q}^{-1}(s) := \widetilde{W}(s)$ and $p^{-1}(s) := X(s)$ for each $s \in [0, 1]$, respectively. Assume that we have Banach subbundle maps

$$\mathbb{W}_0 \hookrightarrow \mathbb{W}, \quad \mathbb{W} \hookrightarrow \widetilde{\mathbb{W}}, \quad \widetilde{\mathbb{W}} \hookrightarrow \mathbb{X}.$$

(ii) We are given a path of symplectic structures $\{\omega(s)\}_{s \in [0, 1]}$ on $X(s)$.
 (iii) We have a path $\{(\lambda(s), \mu(s))\}_{s \in [0, 1]}$ of Fredholm pairs of Lagrangian subspaces of $(X(s), \omega(s))$, all of index 0.

(iv) We assume that

- $W_0(s) = W(s)^{\omega(s)}$,
- $W(s)^{\omega(s)} \subset \lambda(s) \subset W(s)$,
- $\dim(W(s)^{\omega(s)} \cap \mu(s)) = k$ for each $s \in [0, 1]$, and
- $\{W(s) + \mu(s)\}_{s \in [0, 1]}$ is a path of closed subspaces of $X(s)$ (it holds automatically if $W(s)$ is closed in $X(s)$).

(v) For each $s \in [0, 1]$ the space $W(s)$ is closed in $\widetilde{W}(s)$, and we have $\widetilde{W}(s) + \mu(s) = X(s)$ (there are three special cases:

- $\widetilde{W}(s)$ is dense in $X(s)$,
- $k = 0$, and
- $W(s)$ is closed in $X(s)$

that yield the existence of such an intermediate space $\widetilde{W}(s)$).

THEOREM 3.3.3 (Invariance under symplectic reduction). *Under Assumption 3.3.2, we have the following:*

(a) For each $s \in [0, 1]$, we have $\dim X(s)/(W(s) + \mu(s)) = k$ and $W(s) + \mu(s) = W(s)^{\omega(s)\omega(s)} + \mu(s)$.

(b) The family

$$\{(R_W(s)^{\omega(s)}(\lambda(s)), R_W(s)^{\omega(s)}(\mu(s)))\}_{s \in [0, 1]}$$

is a path of Fredholm pairs of Lagrangian subspaces of

$$(W(s)/W(s)^{\omega(s)}, \tilde{\omega}(s))$$

of index 0.

(c) We have

$$(3.49) \quad \text{Mas}_{\pm}\{\lambda(s), \mu(s)\} = \text{Mas}_{\pm}\{R_{W(s)}^{\omega(s)}(\lambda(s)), R_{W(s)}^{\omega(s)}(\mu(s))\}.$$

PROOF. We divide the proof into three steps.

Step 1. By Proposition 1.4.8, (a) and (b) hold.

By Lemma A.6.1, $\lambda(s)$, $\mu(s) \cap W(s)$, and $W(s)^{\omega(s)}$ are closed in $W(s)$ for each $s \in [0, 1]$. Since $R_{W(s)}^{\omega(s)}(\mu(s))$ is closed in $W(s)/W(s)^{\omega(s)}$, $\mu(s) \cap W(s) + W(s)^{\omega(s)}$ is closed in $W(s)$ for each $s \in [0, 1]$. By Corollary A.6.3, $\{\lambda(s)\}_{s \in [0, 1]}$ is a path in $\mathcal{S}(W(s))$. Here we use the symbol $\mathcal{S}(X)$ to denote the set of all closed subspaces of a Banach space X , equipped with the gap topology (see our Appendix A.2). By Lemma A.3.1, $\{R_{W(s)}^{\omega(s)}(\lambda(s))\}_{s \in [0, 1]}$ is a path in $\mathcal{S}(W(s)/W(s)^{\omega(s)})$.

Since $W(s) + \mu(s)$ and $\tilde{W}(s) + \mu(s)$ are finite-dimensional extensions of the closed subspace $\lambda(s) + \mu(s) \in X(s)$, by Proposition A.2.2 they are closed in $X(s)$. Note that we have $\tilde{W}(s) + \mu(s) = X(s)$ (if $\tilde{W}(s)$ is dense in $X(s)$ or $k = 0$).

Since $W(s) \subset \tilde{W}(s)$, we have

$$W(s) + \mu(s) \cap \tilde{W}(s) = (W(s) + \mu(s)) \cap \tilde{W}(s).$$

By Lemma A.6.1, the spaces $\mu(s) \cap \tilde{W}(s)$ and $(W(s) + \mu(s)) \cap \tilde{W}(s)$ are closed in $\tilde{W}(s)$. By Corollary A.3.14,

$$\{\mu(s) \cap W(s)\}_{s \in [0, 1]}, \quad \{\mu(s) \cap W(s)^{\omega(s)}\}_{s \in [0, 1]}, \quad \text{and} \quad \{\mu(s) \cap W(s) + W(s)^{\omega(s)}\}_{s \in [0, 1]}$$

are paths of closed subspaces of $W(s)$. By Lemma A.3.1, $\{R_{W(s)}^{\omega(s)}(\mu(s))\}_{s \in [0, 1]}$ is a path in $\mathcal{S}(W(s)/W(s)^{\omega(s)})$. Therefore the Maslov index on the right hand side of (3.49) is well-defined.

Step 2. Reduce to the case of $W(s) + \mu(s) = X(s)$.

Since $W(s) + \mu(s)$, $s \in [0, 1]$ is a path of closed subspaces of $X(s)$ of finite codimension, we have $W(s) + \mu(s) \in \mathcal{S}^c(X(s))$. Recall from Remark 1.4.5 that we denote the space of complemented subspaces of a Banach space X by $\mathcal{S}^c(X)$. By Lemma A.4.8 (see also [72, Lemma 0.2]), $\bigcup_{s \in [0, 1]} (W(s) + \mu(s))$ is a Banach bundle over $[0, 1]$, and there exists a finite-dimensional Banach subbundle $\bigcup_{s \in [0, 1]} V(s)$ of \mathbb{X} such that $V(s) \oplus (W(s) + \mu(s)) = X(s)$. Note that we can take $\tilde{W}(s) := W(s) + V(s)$ if $W(s)$ is closed in $X(s)$.

We use the notations from Corollary 1.4.4. Set $\mathbb{X}_j := \bigcup_{s \in [0, 1]} X_j(s)$, $j = 0, 1$. By Step 1, $\{W(s)^{\omega(s)}\}_{s \in [0, 1]}$ is a path of $\mathcal{S}^c(\tilde{W}(s))$. By Corollary A.6.3, it is a path of $\mathcal{S}^c(X(s))$. So $\{X_0(s) := W(s)^{\omega(s)} + V(s)\}_{s \in [0, 1]}$ is a path of $\mathcal{S}^c(X(s))$, and \mathbb{X}_0 is a Banach subbundle of \mathbb{X} . By Proposition 1.3.3, we have

$$X_1(s) := V(s)^{\omega(s)} \cap W(s) + V(s)^{\omega(s)} \cap \mu(s) = V(s)^{\omega(s)} \cap (W(s) + \mu(s)).$$

Note that $X(s) = X_0(s) \oplus X_1(s)$. Then $X_1(s)$ is a path of $\mathcal{S}^c(X(s))$. By Lemma A.4.8 (see also [72, Lemma 0.2]), \mathbb{X}_1 is a Banach subbundle of \mathbb{X} . Set $W_{01}(s) := W(s)^{\omega(s)} \cap V(s)$, $W_1(s) := V(s)^{\omega(s)} \cap W(s)$, and $\mathbb{W}_j(s) := \bigcup_{s \in [0, 1]} W_j(s)$ for $j = 0, 1$. By Lemma A.4.9, \mathbb{W}_{01} is a Banach subbundle of \mathbb{W}_0 , \mathbb{W}_1 is a Banach

subbundle of \mathbb{W} , and \mathbb{W}_{01} is a Banach subbundle of \mathbb{W}_1 . Then we can replace $X_1(s)$ for $X(s)$.

Set $\lambda_0(s) = \mu_0(s) := W(s)^{\omega(s)} \cap \mu(s)$, $\lambda_1(s) := \lambda(s) \cap X_1(s)$ and $\mu_1(s) := \mu \cap X_1(s)$. By Proposition 2.3.1, for a local path $s \in [s_1, s_2] \subset (t - \delta(t), t + \delta(t))$ we have

$$\begin{aligned} \text{Mas}_{\pm}\{\lambda(s), \mu(s)\} &= \text{Mas}_{\pm}\{\lambda_0(s), \mu_0(s)\} + \text{Mas}_{\pm}\{\lambda_1(s), \mu_1(s)\} \\ &= \text{Mas}_{\pm}\{\lambda_1(s), \mu_1(s)\}. \end{aligned}$$

Then our result follows from the compactness of $[0, 1]$ and Definition 3.1.3.

Step 3. The case of $W(s) + \mu(s) = X(s)$.

Fix $t \in [0, 1]$. Let $V_1(t) \subset W(t)$ be a linear subspace such that $X(s) = V_1(t) \oplus (\lambda(t) + \mu(t))$. Let $L(t, s): W(t) \rightarrow W(s)$ be the local frame of the bundle \mathbb{W} . Set $V_1(t, s) := L(t, s)V_1(t) \subset W(s)$. By Lemma 1.4.3 and Theorem 3.2.1, for a local path $s \in [s_1, s_2] \subset (t - \delta(t), t + \delta(t))$ we have

$$\begin{aligned} \text{Mas}_{\pm}\{\lambda(s), \mu(s)\} &= \text{Mas}_{\pm}\{R_{V_1(t, s) + \lambda(s)}(\lambda(s)), R_{V_1(t, s) + \lambda(s)}(\mu(s))\} \\ &= \text{Mas}_{\pm}\{R_{W(s)}(\lambda(s)), R_{W(s)}(\mu(s))\}. \end{aligned}$$

Then our result follows from the compactness of $[0, 1]$ and Definition 3.1.3. \square

We have the following side results:

COROLLARY 3.3.4. *The equation (3.49) holds always if just $\{W(s)\}_{s \in [0, 1]}$ is a path in $\mathcal{S}^c(X(s))$ and $\{W(s)^{\omega(s)}\}_{s \in [0, 1]}$ a path in $\mathcal{S}^c(W(s))$.*

PROOF. Set $\mathbb{W} := \bigcup_{s \in [0, 1]} W(s)$ and $\mathbb{W}_0 := \bigcup_{s \in [0, 1]} W(s)^{\omega(s)}$. By Lemma A.4.8 (see also [72, Lemma 0.2]), \mathbb{W} is a Banach subbundle of \mathbb{X} , and \mathbb{W}_0 is a Banach subbundle of \mathbb{W} . By Theorem 3.3.3, our result follows. \square

COROLLARY 3.3.5. *The equation (3.49) holds for $W(s) = V(s) \oplus \lambda(s)$ if $\{V(s)\}_{s \in [0, 1]}$ is a path of finite-dimensional linear subspaces of $X(s)$ and it holds that*

$$(3.50) \quad X(s) = V(s) + \lambda(s) + \mu(s).$$

PROOF. Since $V(s) \cap \lambda(s) = \{0\}$ and since Equation (3.50) holds, by Corollary 1.4.9 we have $V(s)^{\omega(s)} + \lambda = X(s)$. Note that $X(s) = W(s) + \mu(s)$ and $V(s) \cap \lambda(s) = \{0\}$. By Lemma 1.3.2, we have

$$\dim W(s)/\lambda(s) = \dim \lambda(s)/W(s)^{\omega(s)} = \dim V(s).$$

By Proposition A.2.2, $W(s) \in \mathcal{S}^c(X)$. Clearly, $W(s)^{\omega(s)} \in \mathcal{S}^c(W(s))$ since it is closed and of finite codimension. Since $\{V(s)^{\omega(s)}\}_{s \in [0, 1]}$ and $\{\lambda(s)\}_{s \in [0, 1]}$ are paths and $V(s)^{\omega(s)} + \lambda(s) = X$, by Proposition A.3.13, $\{V(s)^{\omega(s)} \cap \lambda(s)\}_{s \in [0, 1]}$ is also a path. By Theorem 3.3.3, our result follows. \square

REMARK 3.3.6. In Section 1.4 we established the invariance and transitivity of symplectic reduction in purely algebraic terms. That was more general - and simpler than our present situation in Section 3.1 and in this section. Here topological considerations come into play.

a) Proposition 1.4.10 is the model for the symplectic reductions in Theorem 3.1.4, based on the Choices and Notations 3.1.2 and Definition 3.1.3. We have the uniform decomposition $X(s) = L(t, s)X_0(t) \oplus X_1(t, s)$ of Equation (3.1) and make a reduction to the finite-dimensional spaces $X_0(s)$. To consider $R_W(\cdot)$ we have two choices of W : We may take $W_l := L(t, s)V(t) \oplus \lambda(s)$ and $W_r := L(t, s)V(t) \oplus \mu(s)$.

In both cases we have $W/W^{\omega(t)} = L(t, s)X_0(t)$. Their symplectic structures are the same (see Proposition 1.4.10.d).

b) The $W(s)$ of the present section is just W_l . In the preceding Corollary 3.3.5, however, we impose an alternative condition: $\dim W(s)/W(s)^{\omega(s)}$ is finite. Note that the corresponding Condition (3.50) is weaker than the former conditions (3.3)-(3.4).

Our second theorem (Theorem 3.3.9) shows the invariance of the Maslov index under symplectic embedding. It generalizes [24, Lemma 12]. We make some preparations for it.

LEMMA 3.3.7. *Let (X, ω) be a symplectic vector space and X_0, X_1 two symplectic subspaces with $X = X_0 \oplus X_1$ and $X_0 = X_1^\omega$. Let $\lambda \subset X$ be a Lagrangian subspace of X . Assume that $\lambda \cap X_0$ is a Lagrangian subspace of X_0 . Then $\lambda \cap X_1$ is a Lagrangian subspace of X_1 , and we have*

$$(3.51) \quad \lambda = \lambda \cap X_0 \oplus \lambda \cap X_1.$$

PROOF. By Lemma 1.3.1, we have $X_1 = X_0^\omega$. So there holds $(\lambda \cap X_0)^\omega \supset X_0^\omega = X_1$. Since $X = X_0 \oplus X_1$, By Corollary A.1.2 we have

$$(\lambda \cap X_0)^\omega = (\lambda \cap X_0)^\omega \cap X_0 + X_1.$$

Since $\lambda \cap X_0$ is a Lagrangian subspace of X_0 , we have $(\lambda \cap X_0)^\omega \cap X_0 = \lambda \cap X_0 \subset \lambda$. Then we have $\lambda = \lambda^\omega \subset (\lambda \cap X_0)^\omega = \lambda \cap X_0 + X_1$. Thus there holds

$$\begin{aligned} \lambda &= \lambda \cap (\lambda \cap X_0 + X_1) = \lambda \cap X_0 + \lambda \cap X_1, \text{ and} \\ \lambda^\omega &= (\lambda \cap X_0)^\omega \cap X_0 + (\lambda \cap X_1)^\omega \cap X_1. \end{aligned}$$

Consequently, $(\lambda \cap X_1)^\omega \cap X_1 = \lambda \cap X_1$ and $\lambda \cap X_1$ is a Lagrangian subspace of X_1 . \square

PROPOSITION 3.3.8 (Decomposition of the Maslov index into parts). *Let $p: \mathbb{X} \rightarrow [0, 1]$ be a Banach bundle with $X(s) := p^{-1}(s)$ for each $s \in [0, 1]$. Let $\omega(s)$ be a path of symplectic structures on $X(s)$. Let $(\lambda(s), \mu(s))$ be a path of Fredholm pairs of Lagrangian subspaces of $(X, \omega(s))$ of index 0. Let $p_j: \mathbb{X}_j \rightarrow [0, 1]$ be two Banach subbundles of $p: \mathbb{X} \rightarrow [0, 1]$ with $X_j(s) := p_j^{-1}(s)$, $s \in [0, 1]$, $j = 1, 2$. We assume that*

- (i) $\omega(s)|_{X_j(s)}$ are continuously varying for $j = 0, 1$,
- (ii) $X(s) = X_0(s) \oplus X_1(s)$ and $X_0(s) = X_1(s)^{\omega(s)}$, and
- (iii) $\{(\lambda(s) \cap X_0(s), \mu(s) \cap X_0(s))\}_{s \in [0, 1]}$ is a path of pairs of Lagrangian subspaces in $(X_0(s), \omega(s)|_{X_0(s)})$.

Then $(\lambda(s) \cap X_j(s), \mu(s) \cap X_j(s))$ is a path of Fredholm pairs of Lagrangian subspaces in $(X_j(s), \omega(s)|_{X_j(s)})$ of index 0, $j = 0, 1$, and

$$(3.52) \quad \begin{aligned} \text{Mas}_\pm\{\lambda(s), \mu(s)\} &= \text{Mas}_\pm\{\lambda(s) \cap X_0(s), \mu(s) \cap X_0(s)\} \\ &\quad + \text{Mas}_\pm\{\lambda(s) \cap X_1(s), \mu(s) \cap X_1(s)\}. \end{aligned}$$

PROOF. By (ii), we have an injective continuous map

$$f(s): \mathcal{S}(X_0(s)) \times \mathcal{S}(X_1(s)) \longrightarrow \mathcal{S}(X(s)), \quad \text{defined by } f(M, N) := M + N,$$

for all $s \in [0, 1]$, and $f(s)$ is a homeomorphism onto its image. By Lemma 3.3.7, $\{(\lambda(s) \cap X_j(s), \mu(s) \cap X_j(s))\}_{s \in [0, 1]}$ is a path of Fredholm pairs of Lagrangian subspaces in $(Y(s), \omega(s)|_{X_j(s)})$, $j = 0, 1$, and

$$\begin{aligned} \text{index}(\lambda(s), \mu(s)) &= \text{index}(\lambda(s) \cap X_0(s), \mu(s) \cap X_0(s)) \\ &\quad + \text{index}(\lambda(s) \cap X_1(s), \mu(s) \cap X_1(s)) \end{aligned}$$

for all $s \in [0, 1]$. By Lemma 1.2.8, we have $\text{index}(\lambda(s) \cap X_j(s), \mu(s) \cap X_j(s)) \leq 0$, $j = 0, 1$. So we have $\text{index}(\lambda(s) \cap X_j(s), \mu(s) \cap X_j(s)) = 0$, $j = 0, 1$. Then the Equation (3.52) follows from Proposition 2.3.1.c. \square

The following theorem is the second major result of this section. It strengthens the claim of the preceding proposition in the following sense: The second term on the right hand side of Equation 3.52 is vanishing, if the intersection dimensions of the Fredholm pairs diminishes by a constant under the decomposition. In general, the reduced intersection dimensions will *not* become constant. Otherwise the Maslov index must vanish by Proposition 2.3.1.e and its analogue Theorem 3.1.4.

Intuitively, the claim of Theorem 3.3.9 seems unquestionable. As often in infinite-dimensional analysis, however, intuition can not be trusted. As a matter of fact, the following rigorous proof of the theorem may appear quite involved and is definitely not straight forward.

THEOREM 3.3.9 (Invariance under symplectic embedding). *Let $p: \mathbb{X} \rightarrow [0, 1]$ be a Banach bundle. Denote by $X(s) := p^{-1}(s)$ the fiber of p at $s \in [0, 1]$. Let $\omega(s)$ be a path of symplectic structures on $X(s)$. Let $(\lambda(s), \mu(s))$ be a path of Fredholm pairs of Lagrangian subspaces of $(X, \omega(s))$ of index 0. Let $p_1: \mathbb{Y} \rightarrow [0, 1]$ be a second Banach bundle which is a linear subbundle of $p: \mathbb{X} \rightarrow [0, 1]$ (in general the inclusion $Y(s) \hookrightarrow X(s)$ is neither continuous nor dense), where $Y(s) := p_1^{-1}(s)$. We assume that*

- $\omega(s)|_{Y(s)}$ is continuously varying,
- $\{(\lambda(s) \cap Y(s), \mu(s) \cap Y(s))\}_{s \in [0, 1]}$ is a path of Fredholm pairs of Lagrangian subspaces in $(Y(s), \omega(s)|_{Y(s)})$ of index 0, and
- $\dim(\lambda(s) \cap \mu(s)) - \dim(\lambda(s) \cap \mu(s) \cap Y(s))$ is a constant k .

Then we have

$$(3.53) \quad \text{Mas}_{\pm}\{\lambda(s), \mu(s)\} = \text{Mas}_{\pm}\{\lambda(s) \cap Y(s), \mu(s) \cap Y(s)\}.$$

PROOF. Since $[0, 1]$ is compact, by Definition 3.1.3 we need only consider the local case. In this case the bundles \mathbb{X} and \mathbb{Y} are both trivial, i.e., we can assume that $X(s) = X$ and $Y(s) = Y$.

Fix $t \in [0, 1]$. Set $\lambda_Y(s) := \lambda(s) \cap Y$, $\mu_Y(s) := \mu(s) \cap Y$, and $\omega_Y(s) := \omega(s)|_{Y \times Y}$ for all $s \in [0, 1]$. By the Fredholm properties, there exist finite-dimensional linear subspaces $V_1 \subset Y$ and $V_2 \subset X$ such that

$$Y = V_1 \oplus (\lambda(t) \cap Y + \mu(t) \cap Y), \quad X = V_2 \oplus (Y + \lambda(t) + \mu(t)).$$

Set $V := V_1 \oplus V_2$. Then we have

$$V + \lambda(t) + \mu(t) = V_1 + V_2 + \lambda(t) + \mu(t) = V_2 + Y + \lambda(t) + \mu(t) = X.$$

Note that

$$\begin{aligned} V \cap \lambda(t) &= (V_1 + V_2) \cap (V_1 + \lambda(t)) \cap \lambda(t) \\ &= (V_1 + V_2 \cap (V_1 + \lambda(t))) \cap \lambda(t) \\ &= V_1 \cap \lambda(t) = V_1 \cap Y \cap \lambda(t) = \{0\}. \end{aligned}$$

By Appendix A.3, there exists a $\delta > 0$ such that for $s \in (-\delta, \delta) \cap [0, 1]$, we have $V + \lambda(s) + \mu(s) = X$, $V_1 + \lambda_Y(s) \cap Y + \mu(s) \cap Y = Y$, and $V \cap \lambda(s) = V_1 \cap \lambda_Y(s) = \{0\}$. By Corollary 3.3.5, for all paths $[s_1, s_2] \subset (-\delta, \delta) \cap [0, 1]$ we have

$$\begin{aligned} \text{Mas}_\pm\{\lambda(s), \mu(s)\} &= \text{Mas}_\pm\{R_{V+\lambda(s)}^{\omega(s)}(\lambda(s)), R_{V+\lambda(s)}^{\omega(s)}(\mu(s))\}, \\ \text{Mas}_\pm\{\lambda_Y(s), \mu_Y(s)\} &= \text{Mas}_\pm\{R_{V_1+\lambda_Y(s)}^{\omega_Y(s)}(\lambda(s)), R_{V_1+\lambda_Y(s)}^{\omega_Y(s)}(\mu(s))\}. \end{aligned}$$

Consider the symplectic linear maps

$$(3.54) \quad \frac{V_1 + \lambda_Y(s)}{V_1^{\omega_Y(s)} \cap \lambda(s)} \xrightarrow{f(s)} \frac{V_1 + \lambda(s)}{V_1^{\omega(s)} \cap \lambda(s)} \xrightarrow{g(s)} \frac{V + \lambda(s)}{V^{\omega(s)} \cap \lambda(s)},$$

where $f(s)$ is induced by the embedding $\lambda_Y(s) \hookrightarrow \lambda(s)$ and $g(s)$ is induced by the embedding $V_1 \hookrightarrow V$. Note that a symplectic linear map is an injection. By comparing dimensions, each $f(s)$ is an isomorphism and each $g(s)$ is an injection. For any linear subspace M of X , we have

$$(3.55) \quad f(s)(R_{V_1+\lambda_Y(s)}^{\omega_Y(s)}(M \cap Y)) \subset R_{V_1+\lambda(s)}^{\omega(s)}(M).$$

If $M = \lambda(s)$ or $M = \mu(s)$, then

- $R_{V_1+\lambda_Y(s)}^{\omega_Y(s)}(M \cap Y)$ is a Lagrangian subspace in the reduced space $(V_1 + \lambda_Y(s))/(V_1^{\omega_Y(s)} \cap \lambda(s))$, and
- $R_{V_1+\lambda(s)}^{\omega(s)}(M)$ is a Lagrangian subspace in $(V_1 + \lambda(s))/(V_1^{\omega(s)} \cap \lambda(s))$.

So (3.55) is an equality of two Banach bundles of finite fiber dimension for each $s \in [0, 1]$. Then we can apply Lemma 1.4.3, Lemma 1.4.6 and Proposition 1.4.8. Our problem is then reduced to a path of symplectic embeddings $g(s) \circ f(s)$, which replace the linear embedding of the bundles.

Now we are in the finite-dimensional case, i.e., we can assume that $\dim X < +\infty$. In this case, the embedding is always continuous, and $X = Y(s) \oplus Y(s)^{\omega(s)}$. By Proposition 3.3.8 and Proposition 2.3.1.e we have

$$\begin{aligned} \text{Mas}_\pm\{\lambda(s), \mu(s)\} &= \text{Mas}_\pm\{\lambda(s) \cap Y(s), \mu(s) \cap Y(s)\} \\ &\quad + \text{Mas}_\pm\{\lambda(s) \cap Y(s)^{\omega(s)}, \mu(s) \cap Y(s)^{\omega(s)}\} \\ &= \text{Mas}_\pm\{\lambda(s) \cap Y(s), \mu(s) \cap Y(s)\}. \end{aligned}$$

Our result is then proved. \square

3.4. The Hörmander index

In this section we fix the symplectic Banach space (X, ω) . Firstly we give some topological and calculatory preparations. Recall from Definition 1.2.7: for $k, m \in \mathbb{Z}$

and $\mu \in \mathcal{L}(X)$, we define

$$\begin{aligned}\mathcal{FL}_k(X) &:= \{(\lambda, \mu) \in \mathcal{FL}(X) \mid \text{index}(\lambda, \mu) = k\}, \\ \mathcal{FL}_k(X, \mu) &:= \{\lambda \in \mathcal{L}(X); (\lambda, \mu) \in \mathcal{FL}_k(X)\}, \\ \mathcal{FL}_0^m(X, \mu) &:= \{\lambda \in \mathcal{FL}_0(X, \mu); \dim(\lambda \cap \mu) = m\}.\end{aligned}$$

LEMMA 3.4.1. *Let $\mu \in \mathcal{L}(X)$. Then we have that*

- (a) $\mathcal{FL}_0^0(\mu)$ is an affine space (hence contractible),
- (b) $\mathcal{FL}_0^0(\mu)$ is dense in $\mathcal{FL}_0(\mu)$ and $\mathcal{FL}_0(\mu)$ is path connected.

PROOF. (a) Let $\lambda \in \mathcal{FL}_0^0(\mu)$. By Lemma A.4.5, we have

$$\begin{aligned}\mathcal{FL}_0^0(X, \mu) &= \{\text{graph}(A) \in \mathcal{L}(X); A \in \mathcal{B}(\lambda, \mu)\} \\ &= \{\text{graph}(A); A \in \mathcal{B}(\lambda, \mu), \omega(x, Ay) + \omega(y, Ax) = 0, \forall x, y \in \lambda\}.\end{aligned}$$

So (a) is proved.

(b) Let $\lambda \in \mathcal{FL}_0(X, \mu)$. By Proposition 1.3.3, we have $X = X_0 \oplus X_1$, where $X_0 := V \oplus \lambda_0$, $X_1 := \lambda_1 \oplus \mu_1$, $\lambda_0 := \lambda \cap \mu$, $\lambda_1 = V^\omega \cap \lambda$, $\mu_1 = \mu^\omega \cap \mu$, and V is chosen to be isotropic. We have $X_0 = X_1^\omega$ is of finite dimension, and X_0, X_1 are symplectic. Note that $V, \lambda_0 \in \mathcal{L}(X_0)$ and $\lambda_1, \mu_1 \in \mathcal{L}(X_1)$. Let $A: \lambda_0 \rightarrow V$ be a linear isomorphism with $\omega(x, Ay) + \omega(y, Ax) = 0, \forall x, y \in \lambda_0$. Set $c_1(s) := \text{graph}(sA)$, $s \in [0, 1]$ and $c(s) = c_1(s) \oplus \lambda_1$. The $c(0) = \lambda$ and $c(s) \in \mathcal{FL}_0^0(X, \mu)$. By (a), we get (b). \square

LEMMA 3.4.2. *Let $\lambda, \mu \in \mathcal{L}(X)$ and $X = \lambda \oplus \mu$. Then we have*

- (a) $\mathcal{FL}_0^0(X, \mu) \cap \mathcal{CP}_0(X, \lambda)$ is an affine space (hence contractible),
- (b) $\mathcal{FL}_0^0(X, \mu) \cap \mathcal{CP}_0(X, \lambda)$ is dense in $\mathcal{FL}_0(X, \mu) \cap \mathcal{CP}_0(X, \lambda)$, and $\mathcal{FL}_0(X, \mu) \cap \mathcal{CP}_0(X, \lambda)$ is path connected.

As explained in the Appendix (Corollary A.7.7), the set $\mathcal{CP}_0(X, \lambda)$ consists of all complemented finite changes of λ in X of vanishing relative index. For the notion of *finite change* and *relative index* see Definition A.7.3.

PROOF. The proof of Lemma 3.4.2 is similar to that of Corollary A.7.7 and we omit it. \square

COROLLARY 3.4.3. *Let $\mu_1, \mu_2 \in \mathcal{L}(X)$ such that $\mu_1 \sim^c \mu_2$ and $[\mu_1 - \mu_2] = 0$. Let $\{\lambda_j(s)\}_{s \in [0, 1]}$ be two paths in $\mathcal{FL}_0(X, \mu_1)$ with the same endpoints for $j = 1, 2$. Then $\{\lambda_j(s)\}_{s \in [0, 1]}$ is a path in $\mathcal{FL}_0(X, \mu_2)$ and we have*

$$(3.56) \quad \text{Mas}\{\lambda_1, \mu_2\} - \text{Mas}\{\lambda_1, \mu_1\} = \text{Mas}\{\lambda_2, \mu_2\} - \text{Mas}\{\lambda_2, \mu_1\}.$$

PROOF. By Lemma 3.4.2, there is a path $\mu(s)$ with $\mu(0) = \mu_1$ and $\mu(1) = \mu_2$, $\mu(s) \sim^c \mu_2$ and $[\mu_1 - \mu(s)] = 0$. By Proposition A.7.6.g, we have $\mathcal{F}_{0, \mu_1}(X) = \mathcal{F}_{0, \mu_2}(X)$ and $\mathcal{FL}_0(X, \mu_1) = \mathcal{FL}_0(X, \mu(s))$. Then we have $(\lambda_j(s), \mu(s)) \in \mathcal{FL}_0(X)$. Then we have two homotopies $(\lambda_j(s), \mu(t)) \in \mathcal{FL}_0(X)$, $(s, t) \in [0, 1]$ for $j = 1, 2$. By Proposition 2.3.1 we have

$$\begin{aligned}\text{Mas}\{\lambda_1, \mu_2\} - \text{Mas}\{\lambda_1, \mu_1\} &= \text{Mas}\{\lambda_1(1), \mu\} - \text{Mas}\{\lambda_1(0), \mu\} \\ &= \text{Mas}\{\lambda_2(1), \mu\} - \text{Mas}\{\lambda_2(0), \mu\} \\ &= \text{Mas}\{\lambda_2, \mu_2\} - \text{Mas}\{\lambda_2, \mu_1\}.\end{aligned} \quad \square$$

Now we are in the position of defining the Hörmander index for a quadruple of Lagrangian subspaces in symplectic Banach space, so in particular in *weak* symplectic Hilbert space. Formally, our definition reminds the definition given by

K. Furutani and the first author in [17, Definition 5.2] for strong symplectic Hilbert space. The novelty of the following definition is that we need two additional conditions for the weak symplectic case, namely

$$\text{index}(\lambda_1, \mu_1) = \text{index}(\lambda_2, \mu_1) = 0 \text{ and } [\mu_1 - \mu_2] = 0.$$

Note that these conditions are always satisfied in the strong symplectic case.

DEFINITION 3.4.4. Let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathcal{L}(X)$ be Lagrangian subspaces of X . Assume that $\lambda_1, \lambda_2 \in \mathcal{FL}_0(X, \mu_1)$, $\mu_1 \sim^c \mu_2$ and $[\mu_1 - \mu_2] = 0$. By Lemma 3.4.1, there is a path $\lambda: [0, 1] \rightarrow \mathcal{FL}_0(X, \mu_1)$ with $\lambda(0) = \lambda_1$, $\lambda(1) = \lambda_2$. By Lemma 3.4.3, we can define the *Hörmander index* $\sigma(\lambda_1, \lambda_2; \mu_1, \mu_2)$ by

$$(3.57) \quad \sigma(\lambda_1, \lambda_2; \mu_1, \mu_2) = \text{Mas}\{\lambda, \mu_2\} - \text{Mas}\{\lambda, \mu_1\}.$$

We note that the condition $[\mu_1 - \mu_2] = 0$ is automatically satisfied under the assumptions of the following lemma.

LEMMA 3.4.5. *Let (X, ω) be a symplectic Banach space with three closed subspaces λ, μ_1, μ_2 . Let $(\lambda, \mu_1), (\lambda, \mu_2) \in \mathcal{FL}(X)$. Assume that $\mu_1 \sim^c \mu_2$. If $\text{index}(\lambda, \mu_1) = \text{index}(\lambda, \mu_2) = 0$, we have $[\mu_1 - \mu_2] = 0$.*

Note that by Proposition A.7.6.g,

$$(\lambda, \mu_1) \in \mathcal{FL}(X) \text{ and } \mu_1 \sim^c \mu_2 \implies (\lambda, \mu_2) \in \mathcal{FL}(X).$$

PROOF. The Lemma is just a special case of Proposition A.7.6.g. □

REMARK 3.4.6. In [40], M. de Gosson gave a very elegant definition of the Hörmander index in finite dimensions in great generality. His definition differs slightly from ours. By admitting half-integer indices, it yields more simple proofs, but may be more difficult to apply in concrete applications in Morse theory.

Part 2

Applications in global analysis

CHAPTER 4

The desuspension spectral flow formula

In this section, we study self-adjoint Fredholm extensions of symmetric operators, and prove a general spectral flow formula under the assumption of a certain weak inner unique continuation property (wiUCP).

4.1. Short account of predecessor formulae

To begin with, we describe the topological and analytic background of our applications.

4.1.1. The spectral flow. In various branches of mathematics one is interested in the calculation of the spectral flow of a continuous family of closed densely defined (not necessarily bounded) self-adjoint Fredholm operators in a fixed Hilbert space. Roughly speaking, the spectral flow is an intersection number between the spectrum and the real line and counts the net number of eigenvalues changing from the negative real half axis to the nonnegative one.

The spectral flow for a one parameter family of linear self-adjoint Fredholm operators was introduced by M. Atiyah, V. Patodi, and I. Singer [7] in their study of index theory on manifolds with boundary. Since then, other significant applications have been found; many of them were inspired by C. Vafa and E. Witten's use of the spectral flow to estimate uniform bounds for the spectral gap of Dirac operators in [93]. The spectral flow was implicit already in Atiyah and Singer [8] as the isomorphism from the fundamental group of the non-trivial connected component of bounded self-adjoint Fredholm operators in complex Hilbert space onto the integers. Later this notion was made rigorous for not necessarily closed curves of bounded self-adjoint Fredholm operators in J. Phillips [78] and for gap-continuous curves of self-adjoint (generally unbounded) Fredholm operators in Hilbert spaces in [19] by the Cayley transform. The notion was generalized to the higher dimensional case in X. Dai and W. Zhang [37] for Riesz-continuous families, and to more general operators by K.P. Wojciechowski and C. Zhu and Y. Long in [98, 103, 105].

4.1.2. Switch between symmetric and symplectic category. In this section we derive spectral flow formulae in the following sense. We are given a continuous curve of self-adjoint Fredholm operators, or more generally, a continuous curve of self-adjoint Fredholm relations. Such curves arise typically from a family of elliptic operators over a compact manifold with boundary with smoothly varying coefficients and smoothly varying regular boundary conditions. Then we consider two mutually related invariants: within the *symmetric* category, we have the number of negative eigenvalues or, more generally, the spectral flow; that is our first invariant. Basically, it is an intersection number of the spectral lines with the real axis. It is well defined, but, being a spectral invariant, difficult to determine in general. To define the second invariant, we switch from the symmetric category to

the *symplectic* category. We notice that self-adjoint extensions are characterized by Lagrangian subspaces in corresponding symplectic Hilbert spaces coming from the domains, i.e., from the boundary values. That consideration yields another intersection number, the Maslov index. The Maslov index does not arise from the spectrum, but can be calculated directly from the boundary values of the solutions. Speaking roughly, the Maslov index counts the changes of the intersection dimensions of two curves of Lagrangians. In our case, the one curve is made of the continuously varying Lagrangians coming from the Fredholm domains. The other curve is made of the Cauchy data spaces, which also form Lagrangians and vary continuously under suitable assumptions. Then the type of spectral flow formulae we are interested in are formulae where the spectral flow of a given curve of self-adjoint Fredholm relations or operators is expressed by a related Maslov index. Here the point is that the calculation of the Maslov index is different from the calculation of the spectral flow, and, in general, easier.

4.1.3. Origin and applications in Morse theory. The first spectral flow formula was the classical Morse index theorem (cf. M. Morse [68]) for geodesics on Riemannian manifolds. It was extended by W. Ambrose [2] in 1961 to more general boundary conditions, which allowed the two endpoints of the geodesics to vary in two submanifolds of the manifold. In 1976, J.J. Duistermaat [43] completely solved the problem of calculating the Morse index for the one-dimensional variational problems, where the positivity of the second order terms was required. In 2000-2002, P. Piccione and D.V. Tausk [79, 80] were able to prove the Morse index theorem for semi-Riemannian manifolds for the same boundary conditions as in [2], and certain non-degeneracy conditions were needed. In 2001, the second author [103] was able to solve the general problem for the calculation of the Morse index of index forms for regular Lagrangian systems. See also the work of M. Musso, J. Pejsachowicz, and A. Portaluri on a Morse index theorem for perturbed geodesics on semi-Riemannian manifolds in [70] which has in particular lead N. Waterstraat to a K -theoretic proof of the Morse Index Theorem in [96].

4.1.4. From ordinary to partial differential equations. In 1988, A. Floer [46] emphasized that the notion of a Morse index of a function on a finite-dimensional manifold cannot be generalized directly to the symplectic action function α on the loop space of a manifold. He defined for any pair of critical points of α a relative Morse index, which corresponds to the difference of the two Morse indices in finite dimensions. It is based on the spectral flow of the Hessian of α . That paper opened another line of studying spectral flow formulae, namely for partial differential operators:

Let $\{A(s) : C^\infty(M; E) \rightarrow C^\infty(M; E)\}_{s \in [0,1]}$ be a family of continuously varying formally self-adjoint linear elliptic differential operators of first order over a smooth compact Riemannian manifold M with boundary Σ , acting on sections of a Hermitian vector bundle E over M . Fixing a unitary bundle isomorphism between the original bundle and a product bundle in a collar neighborhood N of the boundary, the operators $A(s)$ can be written in the form

$$(4.1) \quad A(s)|_N = J_{s,t} \left(\frac{\partial}{\partial t} + B_{s,t} \right)$$

with skew-self-adjoint bundle isomorphisms $J_{s,t}$ and first order elliptic differential operators $B_{s,t}$ on Σ . Here t denotes the inward normal coordinate in N . For details see B. Booß-Bavnbek, M. Lesch and C. Zhu [20, Section 1].

Following another seminal paper by A. Floer [45], in 1991 T. Yoshida [100] elaborated on Floer homology of 3-manifolds by studying a curve $\{A(s)\}$ of Dirac operators with invertible ends, such that over the boundary, the bundle isomorphisms $J_{s,t} = J(s)$ are unitary and the tangential operators $B_{s,t} = B(s)$ symmetric in the preceding notation. In 1995, L. Nicolaescu [75] generalized Yoshida's results to arbitrary $\dim M$. One year later, S.E. Cappell, R. Lee, and E.Y. Miller [31, Theorem G] found a somewhat intricate spectral flow formula for curves of arbitrary elliptic operators of first order under the conditions of constant coefficients close to the boundary in normal direction and symmetric induced tangential operators. In 2000, M. Daniel [38] removed the nondegenerate conditions in [75]. In 1998-2001, the first author, jointly with K. Furutani and N. Otsuki [17, 18] proved the case that the $A(s)$ differ by 0th order operators, and the boundary condition is fixed. In 2001, P. Kirk and M. Lesch [59, Theorem 7.5] proved the case that $A(s)$ is of Dirac type, $J_{s,t}$ is fixed unitary, and $B_{s,t} = B(s)$ symmetric. Later in this section we shall only assume that each $A(s)$ satisfies weak inner unique continuation property (wiUCP), i.e., $\ker A(s)|_{H_0^1(M;E)} = \{0\}$.

The formulae are of varying generality: Some deal with a fixed (elliptic) differential operator with varying self-adjoint extensions (i.e., varying boundary conditions); others keep the boundary condition fixed and let the operator vary. An example for a path of operators with fixed principal symbol is a curve of Dirac operators on a manifold with fixed Riemannian metric and Clifford multiplication but varying defining connection (varying background field which is a zero-order perturbation and as such does not inflict the principal symbol). See also the results by the present authors in [23] for varying operator and varying boundary conditions but fixed maximal domain. Recently, M. Prokhorova [81] considered a path of Dirac operators on a two-dimensional disk with a finite number of holes subjected to local elliptic boundary conditions of chiral bag type. She obtained a beautiful explicit formula for the spectral flow (respectively, the Maslov index) which recently was re-proved and generalized by M. Katsnelson, V. Nazaikinskii, A. Gorokhovskiy, and M. Lesch in [58, 50].

4.1.5. Our contribution in this Memoir. In this Memoir we have substantially expanded and settled the validity range of the predecessor formulae. Roughly speaking, we have achieved the following results:

- (i) In the language of Banach bundles we present the list of assumptions on operator families that yield an abstract general spectral flow formula. The list can be found in Assumption 4.4.1 and the obtained formulae in Equations (4.18) and (4.19) of Theorem 4.4.2. This result expands substantially the validity of the functional analytic spectral flow formula of [16]. The novelty of the approach is the replacement of a fixed strong symplectic Hilbert space (the β -space of the quotients of maximal and minimal domain of a closed symmetric operator) by the quotient of a fixable intermediate domain with the minimal domain, equipped with varying weak symplectic forms.
- (ii) In Section 4.5 we turn to the geometric setting. We consider a smooth family of formally self-adjoint elliptic differential operators of fixed order

acting on sections of varying vector bundles over varying manifolds with boundary and impose varying well-posed self-adjoint boundary conditions. Under a technical condition that generalizes weak inner UCP, we obtain an array of general spectral flow formulae in Equations (4.27) and (4.28) of Theorem 4.5.4 in all Sobolev spaces over the boundaries of non-negative order. This result removes the restriction of previous formulae to curves of Dirac type operators or curves with only lower order variation.

- (iii) In Theorem 4.5.6 we give the conditions for the validity of two formulae for the spectral flow of a curve of formally self-adjoint elliptic differential operators over a curve of closed partitioned manifolds $M(s) = M(s)^+ \cup_{\Sigma(s)} M(s)^-$ with separating hypersurfaces $\Sigma(s)$, $s \in [0, 1]$. The first Formula (4.31) expresses the spectral flow over the whole manifold(s) in terms of a spectral flow of a canonically associated curve of well posed boundary problems over one part. The second Formula (4.32) expresses the spectral flow over the whole manifold(s) by the Maslov index of the corresponding Cauchy data spaces from both sides along the separating hypersurface(s). This result generalizes the splitting formulae of T. Yoshida [100] and L. Nicolaescu [75] and determines the limits of their validity.

4.1.6. Spectral flow formulae also for higher order operators. Usually one is only interested in the spectral flow of elliptic differential operators of first order. Typically, elliptic differential operators of second order like the various Laplacians are essentially positive (or essentially negative). For such operators the spectral flow of loops must vanish and the spectral flow of curves is just the difference between the number of negative (respectively positive) eigenvalues at the endpoints, hence trivial. However, the formula $\text{sf} = \text{Mas}$ is not trivial and can give radically new insight also in the case of second order operators. Therefore, we have not restricted our treatment to operators of order one.

For first order operators, in most applications the formula $\text{sf} = \text{Mas}$ will be read as a *desuspension*-type formula, namely expressing the spectral flow (a kind of quantum variable arising from the spectrum) over a manifold by the Maslov index (a kind of classical variable arising from solution spaces) over a submanifold of codimension 1. Then for essentially positive second order elliptic differential operators, in the applications we have in mind (e.g., a higher order Morse index theorem) the formula $\text{sf} = \text{Mas}$ should be read as a *suspension*-type formula, namely expressing the a-priori unknown Maslov index by the in that case trivial spectral flow via the introduction of an additional parameter.

4.1.7. Partitioned manifolds in topology, geometry, and analysis. In topology, the interest in partitioned manifolds is connected to the name of P. Heegaard who in his dissertation [55] introduced ways of splitting 3-manifolds and gaining corresponding graphs for algebraic investigation. In that way he could point to essential differences between homology and homotopy theory that had been missed by H. Poincaré (see, e.g., the elementary presentation by M. Scharlemann in [83]). Later his ideas were lavishly generalized in the concepts of cobordism, surgery, and cutting and pasting of the 1950-60s, see C.T.C. Wall [95]. In spite of the great expectations, the concept of partitioned manifolds has not proved valuable for proving Poincaré's Conjecture. Years before G. Perelman's final proof of the Conjecture,

A. Floer expressed in [45] his expectation that the approach via Heegaard splittings or more general decompositions most probably would not solve the Poincaré Conjecture but would support the complementary topological program, namely to determine all groups that can show up as fundamental groups of 3-manifolds.

In geometry, the interest in partitioned manifolds is connected both to the concept of coarse geometry and to the geometry of singular spaces. In the first case one separates arduous, but topologically uninteresting parts out of complete (non-compact) manifolds, e.g., in the relative index theorems of M. Gromov and H.B. Lawson [51]. In the second case one focuses on the geometry around singularities by separating them out.

In analysis, the interest in partitioned manifolds is connected with the Riemann-Hilbert Problem of complex analysis. Classically, one looks for pairs of functions where one is holomorphic inside, and the other outside the disc and that are linearly conjugated by a transmission condition along the circle, see, e.g., N.I. Muskhelishvili [69]. In [13] B. Bojarski conjectured the general validity of a Riemann-Hilbert type index formula for elliptic operators on even-dimensional closed partitioned manifolds in terms of the index of the Fredholm pair of Cauchy data spaces along the separating hypersurface. That Bojarski Conjecture was proved by K.P. Wojciechowski and the first author in [21, Chapter 24]. We missed the odd-dimensional case which was then treated by L. Nicolaescu in [75]. While his result is restricted to Dirac type operators it served as the model for the present treatise.

There is a remarkable difference between the topological and the analytic approach to invariants of partitioned manifolds. The topological approach is characterized by the ease of achieving additivity formulae for topological invariants like the Euler characteristic or the signature solely by means of singular homology. Deriving the same results by analytic means, e.g., via the Atiyah-Singer Index Theorem is much more demanding. For finer topological invariants and spectral or differential invariants, homology theory may not suffice and harder means are demanded either from homotopy theory or, after all, from analysis. On the analysis level, there is clearly no recognizable splitting of the spectrum of a Dirac or Laplace operator on a partitioned manifold in its components from the parts. For the index (the chiral multiplicity of the zero-eigenvalues) we have both topological and analytical splitting formulae ([21, Chapters 23-25]). For the analytic torsion we have a topological splitting formula by W. Lück in [63]. For the η -invariant we have an analytic splitting formula by K.P. Wojciechowski in [99]. Similarly, our Theorem 4.5.6 should be considered an analytic splitting formula for the spectral flow.

4.1.8. Wider perspectives. In this section, we focus solely on the intertwining of the symmetric category (here the spectral flow) and the anti-symmetric category (here symplectic analysis). Clearly, each side deserves independent investigations and poses puzzles of their own.

Symplectic error terms in global analysis of singular manifolds One such puzzle is to find the correct place of symplectic invariants (like the Maslov index and the Hörmander index) in the hierarchy of invariants in global analysis, compared with the index, the η -invariant, and the ζ -regularized determinant. We meet the *index* of Fredholm operators as the index of elliptic problems on closed manifolds, on manifolds with boundary, and on manifolds with singularities. From the viewpoint of global analysis, however, the index of elliptic problems on *closed* manifolds is

distinguished because there the index can be expressed by an integral over an integrand that is locally expressed by the coefficients of the operator. The η -invariant arises in *boundary value problems*. It is not given by an integral, not by a local formula. It depends, however, only on finitely many terms of the symbol of the resolvent and will not change when one changes or removes a finite number of eigenvalues. Its derivative is local.

Keeping this difference in mind, we meet a question repeatedly put forward by I.M. Gelfand: “what comes next?” To this, I.M. Singer remarked in personal communication [88]: “Just as η arises in boundary value problems for smooth boundaries, I think the next level will come from corner contributions when the boundary has corners.” Indeed, when the boundary has corners, a third term, the *Hörmander index* of symplectic analysis appears, see C.T.C. Wall [94]. He noticed the non-additivity of the signature for the Hopf bundle with fibre D^2 over S^2 having signature ± 1 depending on the choices of sign: this is the union of the induced bundles over the upper and lower hemispheres of S^2 , each of which (being contractible) has signature zero. Wall’s observation was in striking contradiction to the common wisdom in topology, first observed by S.P. Novikov: If two manifolds are glued by an orientation-preserving diffeomorphism of their boundaries, then the signature of their union is the sum of their signatures. So, Wall found that this additivity property does not hold for the more general situation where one glues two $4k$ -manifolds Y_{\pm} along a common submanifold X_0 of the boundaries, which itself has a boundary Z . That yields an abstract Zaremba problem (see also B.-W. Schulze, C.-C. Chang, and N. Habal [33]). Wall determined the non-additivity term as the Hörmander index of three associated Lagrangian subspaces of an induced finite-dimensional symplectic vector space. His result was extended to the gluing of η -invariants by U. Bunke in [28, 29].

That supports the claim of a *hierarchy of asymmetry invariants*, placing the index of elliptic problems on closed manifolds at the bottom; placing the *eta*-invariant a little higher, namely as an error term for smooth boundary value problems; and placing symplectic invariants even higher, namely as error terms for boundary value problems with corners. In this Memoir, we shall not follow that line of thoughts any further and content with placing the Maslov index on the level of smoothly partitioned manifolds for now.

4.1.9. Other approaches to the spectral flow. It may be worth mentioning that there is a multitude of *other* formulae involving spectral flow, e.g., as error term under cutting and pasting of the index (see the first author with K.P. Wojciechowski [21, Chapter 25]) or under pasting of the eta-invariant as in [59]. Whereas these formulae typically relate the spectral flow of a family on a closed manifold of dimension $n - 1$ to the index or the eta-invariant of a single operator on a manifold of dimension n , this Memoir addresses the opposite direction, namely how to express the spectral flow of a family over a manifold of dimension n by objects (here by the Maslov index) defined on a hypersurface of dimension $n - 1$. See also our Subsection 4.1.6 above.

4.2. Spectral flow for closed self-adjoint Fredholm relations

In this section, we show that one can easily obtain a formula expressing the spectral flow for curves of linear self-adjoint Fredholm relations in Hilbert spaces by the Maslov index on a very basic and abstract level (Proposition 4.2.4). This leads

us to a general definition of the spectral flow for curves of closed linear self-adjoint Fredholm relations of index 0 in Banach spaces (Definition 4.2.5).

4.2.1. Basic facts and notions of linear relations. First we have to explain the terms *linear relation*, *closed linear relation*, *self-adjoint linear relation*, and *Fredholm relation*.

Recall that a *linear relation* A between two linear spaces X and Y is a linear subspace of $X \times Y$. We use the notions of linear relations and spectral flow in [24, Appendix A.2, A.3]. For additional details on linear relations, see Cross [36]. In this Memoir, however, we are not directly interested in the many applications of the concept of linear relations in the fields of multi-valued functions and singular spaces. We rather employ the concept of linear relations to clarify the *algebraic character* of geometric incidence numbers like the Maslov index and the spectral flow.

To begin with, we summarize the basic algebraic notions of linear relations: As usual, we define the *domain*, the *range*, the *kernel* and the *indeterminant part* of A by

$$\begin{aligned} \text{dom}(A) &:= \{x \in X; \text{ there exists } y \in Y \text{ such that } (x, y) \in A\}, \\ \text{im } A &:= \{y \in Y; \text{ there exists } x \in X \text{ such that } (x, y) \in A\}, \\ \ker A &:= \{x \in X; (x, 0) \in A\}, \\ A(0) &:= \{y \in Y; (0, y) \in A\}, \end{aligned}$$

respectively.

Then the *sum* $A + B$ and the *composition* $C \cdot A$ are defined by

$$(4.2) \quad A + B := \{(x, y + z) \in X \times Y; (x, y) \in A, (x, z) \in B\},$$

$$(4.3) \quad C \cdot A := \{(x, z) \in X \times Z; \exists y \in Y \text{ such that } (x, y) \in A, (y, z) \in C\}.$$

Here X, Y, Z are three vector spaces, A, B are linear relations between X and Y , and C is a linear relation between Y and Z .

The *inverse* A^{-1} of a linear relation A is always defined. It is the linear relation between Y, X defined by

$$(4.4) \quad A^{-1} = \{(y, x) \in Y \times X; (x, y) \in A\}.$$

A linear relation A is (the graph of) an *operator* if and only if $A(0) = \{0\}$. In that case we identify A and the graph of A .

We get more interesting relations by incorporating topological aspects. Let X, Y be two Banach spaces. A *closed linear relation* between X, Y is an element of $\mathcal{S}(X \times Y)$, i.e., a closed linear subspace of $X \times Y$. It is called *bounded invertible*, if A^{-1} is the graph of a bounded operator from Y to X , shortly: $A^{-1} \in \mathcal{B}(Y, X)$. Then the *resolvent set* $\rho(A)$ of a closed linear relation A consists of all $z \in \mathbb{C}$ where $A - z$ is bounded invertible. As usual, we define the *spectrum* by $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

By definition, the graph of a closed operator is a closed linear relation.

A closed linear relation $A \in \mathcal{S}(X \times Y)$ is called *Fredholm*, if $\dim \ker A < +\infty$, $\text{im } A$ is closed in Y and $\dim(Y/\text{im } A) < +\infty$. In this case, we define the *index* of A to be

$$(4.5) \quad \text{index } A := \dim \ker A - \dim(Y/\text{im } A).$$

By [24, Lemma 16 (a)], a closed linear relation A is Fredholm if and only if $(A, X \times \{0\})$ is a Fredholm pair of elements of $\mathcal{S}(X \times Y)$. In that case, we have

$$(4.6) \quad \text{index } A = \text{index}(A, X \times \{0\}),$$

where the index on the left is that defined in (4.5) and the index on the right that defined in (1.10). Moreover, a closed linear relation A between X and Y is bounded invertible, if and only if $X \times Y$ is the direct sum of A and $X \times \{0\}$. These two results follow from the fact that

$$(4.7) \quad A \cap (X \times \{0\}) = \ker A \times \{0\}, \quad \text{and} \\ A + (X \times \{0\}) = (\{0\} \times \text{im } A) + (X \times \{0\}).$$

We define a purely *algebraic Fredholm relation* by dropping the requirement that A is closed and that $\text{im } A$ is closed. Then $\text{index } A$ is still well defined in (4.5) and the equations (4.6) and (4.7) remain valid.

4.2.2. Induced symplectic forms on product spaces. Our next goal is a purely algebraic, respectively, symplectic characterization of symmetric and self-adjoint relations. Inspired by C. Bennewitz [10] and I. Ekeland [44], we define

DEFINITION 4.2.1. Let X, Y be two complex vector spaces and $\Omega: X \times Y \rightarrow \mathbb{C}$ be a non-degenerate sesquilinear map. Set $Z := X \times Y$ and

$$(4.8) \quad \omega((x_1, y_1), (x_2, y_2)) := \Omega(x_1, y_2) - \overline{\Omega(x_2, y_1)}.$$

Then (Z, ω) is a symplectic vector space with two canonical Lagrangian subspaces $X \times \{0\}$ and $\{0\} \times Y$. We call ω on Z the *symplectic structure induced by Ω* . The *adjoint* of a linear relation $A \subset Z$ is defined to be the annihilator A^ω . A linear relation $A \subset Z$ is called *symmetric*, respectively *self-adjoint*, if $A \subset (Z, \omega)$ is isotropic, respectively Lagrangian. Note that we admit that Y is different of X . If A is symmetric, we define a form $Q(A)$ on $\text{dom}(A)$ by $Q(A)(x_1, x_2) := \Omega(x_1, z)$ for all $x_1, x_2 \in \text{dom}(A)$ and $z \in A(x_2)$. Since A is an isotropic subspace of $X \times Y$, the value $\Omega(x_1, z)$ is independent of the choice of $z \in A(x_2)$. More generally, the form $Q(A)$ is a well-defined quadratic form. We call the form $Q(A)$ the *quadratic form associated to A* .

Note that Ω corresponds to a conjugate-linear injection $\tau: Y \rightarrow X^*$ such that $\bigcap_{y \in Y} \ker \tau(y) = \{0\}$ by

$$(4.9) \quad \Omega(x, y) = (\tau(y))(x), \quad \text{for all } x \in X, y \in Y.$$

We consider the case when X is a Banach space and $\tau: Y \rightarrow X^*$ is an \mathbb{R} -linear isomorphism. Note that so far, Y is only a vector space. So Y is identified with X^* by τ . In the real case we set $Y := X^*$ and $\tau = I_Y$. If X is a complex Hilbert space, we set $Y := X$ and $\tau(y)(x) := \langle x, y \rangle$. The space Y becomes a Banach space with the norm $\|y\|_Y := \|\tau(y)\|_{X^*}$. Then $(X \times Y, \omega)$ is a symplectic Banach space for

$$(4.10) \quad \omega((x_1, y_1), (x_2, y_2)) := (\tau(y_2))(x_1) - \overline{(\tau(y_1))(x_2)}.$$

Note that $X \times \{0\}$ and $\{0\} \times Y$ are two natural Lagrangian subspaces of $X \times Y$. The symplectic structure ω is strong if and only if X is reflexive (see R.C. Swanson, [92]). In this case we call $(X \times Y, \omega)$ *Darboux*, following A. Weinstein, [97].

REMARK 4.2.2. a) There is an alternative and fully natural way to introduce the adjoint relation $A^* \in \mathcal{S}(Y^* \times X^*)$ of a closed linear relation $A \in \mathcal{S}(X \times Y)$, where X, Y are Banach spaces and X^*, Y^* denote the norm-dual spaces, see [36, Chapter III.1]. Contrary to that definition, our construction of the adjoint relation as the annihilator A^ω stays within $\mathcal{S}(X \times Y)$. It is less general, though, since it depends of the choice of Ω inducing ω , respectively, the existence of a conjugate-linear injection $\tau : Y \rightarrow X^*$. Of course, such τ is naturally given for $X = Y$ a Hilbert space. In that case, we have $A^* = A^{\omega_{\text{can}}}$ with the canonical strong symplectic form ω_{can} defined in (1.22). In this Memoir, our applications deal with elliptic operators on manifolds with boundary which induce other symplectic forms than the strong form ω_{can} for dealing with adjoints. In so far, defining adjoint relations by A^ω is more flexible than the standard definition A^* .

b) As explained in Section 1.1, Example 1.2.3 the double adjoint $A^{\omega\omega}$ of a closed linear relation A is not necessarily the original A , unless the form ω is strong symplectic.

c) For strong symplectic ω , the adjoint A^ω of a Fredholm relation A is again a Fredholm relation, and we have $\text{index } A + \text{index } A^\omega = 0$. For weak symplectic forms radically new features appear: (i) The adjoint A^ω of a Fredholm relation A is not necessarily a Fredholm relation. (ii) Even when it is a Fredholm relation, we have not necessarily $\text{index } A + \text{index } A^\omega = 0$ and, as explained in Section 1.2, Example 1.2.11 (iii) the index of a Fredholm operator or relation that is self-adjoint relative to a weak form ω does not necessarily vanish.

4.2.3. Natural coincidence of spectral flow and Maslov index. Let X be a Hilbert space. Clearly, a closed operator $A : X \supset \text{dom}(A) \rightarrow X$ is symmetric, respectively self-adjoint, if and only if $\text{graph}(A)$ is a symmetric, respectively self-adjoint closed linear relation. Inspired by [19, Theorem 1.1b and Remark 1.4] we have the following results about the Cayley image of various types of closed relations in Hilbert space.

LEMMA 4.2.3. *Let X be a Hilbert space and $A \in \mathcal{S}(X \times X)$ a symmetric relation. Then we have:*

(a) *The Cayley transform*

$$\kappa(A) := (A - i)(A + i)^{-1}$$

is a well-defined relation; it is the graph of a partial isometry on X . Moreover, we have $\sigma(\kappa(A)) = \kappa(\sigma(A))$, where $\kappa : z \mapsto \frac{z-i}{z+i}$.

(b) *If A is self-adjoint, then $\kappa(A) \in \mathcal{B}(X)$ and it is unitary.*

(c) *Moreover, in that case we have $\sigma(A) \subset \mathbb{R}$.*

(d) *The number 1 is discrete in $\sigma(A) \cup \{1\}$ if A is self-adjoint Fredholm.*

PROOF. To (a), (b): We have a symplectic decomposition $X \times X = X^- \oplus X^+$ with

$$X^\mp := \{(x, \pm ix); x \in X\}.$$

Define the relation $U := \{(y, z) \in X \times X; (y, iy) + (z, -iz) \in A\}$. By [24, Lemma 3], \tilde{U} is a partial isometry. If A is self-adjoint, $\text{dom}(U) = \text{im } U = X$ and $U \in \mathcal{B}(X)$ is unitary.

Given $(x, x') \in A$, $x, x' \in X$, we decompose

$$(x, x') = (y, iy) + (z, -iz) \text{ with suitable } y, z \in X.$$

We obtain at once $y = \frac{x-ix'}{2}$ and $z = \frac{x+ix'}{2}$, or, equivalently in the language of linear relations,

$$(4.11) \quad y \in \frac{I - iA}{2} \cdot x \text{ and } z \in \frac{I + iA}{2} \cdot x.$$

Inverting the y -formula in (4.11) yields

$$(4.12) \quad z \in (I - iA)^{-1}(I + iA) \cdot y.$$

Conversely, if (4.12) holds, we get $(x, x') \in A$. Then we have $U = (I - iA)^{-1}(I + iA) = -\kappa(A)$. By functional calculus we have $\sigma(\kappa(A)) = \kappa(\sigma(A))$.

To (c), (d): By (a), (b). \square

We have the following basic coincidence of spectral flow and Maslov index.

PROPOSITION 4.2.4. *Let $\mathbb{X} \rightarrow [0, 1]$ be a Hilbert bundle with $X(s) := p^{-1}(s)$, continuous varying inner product $\langle \cdot, \cdot \rangle_s$ on $X(s)$, and $\{s \mapsto A(s) \in \mathcal{S}^c(X(s) \times X(s))\}_{s \in [0, 1]}$ a continuous curve of self-adjoint Fredholm relations of $X(s)$. Then we have*

$$\text{sf}\{A(s)\} = \text{Mas}_- \{A(s), X(s) \times \{0\}; \omega_{\text{can}}(s)\},$$

where $\omega_{\text{can}}(s)$ denotes the canonical strong symplectic form on $X(s) \times X(s)$.

PROOF. Note that $X(s) \times \{0\} = \text{graph}(0)$. By Lemma 4.2.3 and Definition 2.1.1 we have

$$\begin{aligned} \text{sf}\{A(s)\} &= \text{sf}_{\ell_+}(\kappa(A(s))) = \text{sf}_{\ell_+}(-\kappa(A(s))(-\kappa(0))^{-1}) \\ &= \text{Mas}_- \{A(s), X(s) \times \{0\}; \omega_{\text{can}}(s)\}. \end{aligned} \quad \square$$

The proposition leads to the following definition.

DEFINITION 4.2.5. Let $p: \mathbb{X} \rightarrow [0, 1]$, $q: \mathbb{Y} \rightarrow [0, 1]$ be Banach bundles with fibers $X(s) := p^{-1}(s)$, $Y(s) := q^{-1}(s)$ for each $s \in [0, 1]$ respectively. Let $\Omega(s): X(s) \times Y(s) \rightarrow \mathbb{C}$ be a path of bounded non-degenerate sesquilinear forms, and let $\omega(s)$ denote the weak symplectic structure on $Z(s) := X(s) \times Y(s)$ induced by $\Omega(s)$. Let $A(s)$, $s \in [0, 1]$ be a path of linear self-adjoint Fredholm relations of index 0. By [24, Lemma 16], we have $\text{index}(A(s), X(s) \times \{0\}) = 0$. The *spectral flow* of $A(s)$ is defined by

$$(4.13) \quad \text{sf}\{A(s)\} := \text{Mas}_- \{A(s), X(s) \times \{0\}; \omega(s)\}.$$

REMARK 4.2.6 (Reversing the order between spectral flow and Maslov index).

a) In modern times, any rigorous definition of the spectral flow begins with the careful partitioning and local hedging of eigenvalues of J. Phillips [78]. From that, in [16] and followers, a rigorous definition of the Maslov index is derived for curves of Fredholm pairs of Lagrangian subspaces in strong symplectic Hilbert space via the spectral flow of an associated curve of unitary generators. That also is the path we chose in Chapter 2, followed by our definition of the Maslov index in weak symplectic Banach spaces in Chapter 3 via symplectic reduction to the finite-dimensional strong case. So far, we followed the usual view which considers the concept of the spectral flow for more fundamental than the concept of the Maslov index which is derived from it.

However, with the preceding definition we reverse the order: Now the spectral flow for curves of self-adjoint Fredholm relations is defined via the general Maslov index as it was introduced in Chapter 3. In this way it turns out at the bottom

line that the concept of the Maslov index now becomes more fundamental than the spectral flow which has to be defined via the Maslov index.

b) It is informative to compare the present Definition 4.2.5 of the spectral flow with its intricate definition in [24, Appendix]. In one respect, our present definition is more general than our previous one since we dealt in [24, Appendix] only with relations in $X \times X$ while we admit now relations in $X \times Y$ with possibly $Y \neq X$. In another perhaps more relevant respect, our present definition is much less general since we restrict ourselves to self-adjoint relations relative to the choice of a curve $\Omega(s)$ of non-degenerate sesquilinear forms. This restriction explains why we here can avoid the intricate considerations regarding the spectral admissibility of the curves of Fredholm relations of our previous paper.

As shown in Proposition 4.2.4, the two definitions coincide for curves of self-adjoint Fredholm relations in complex Hilbert space.

4.3. Symplectic analysis of operators and relations

We are now ready to follow the famous von Neumann program [73] of investigating all self-adjoint extensions for our special case of Fredholm relations. We are inspired by the Birman-Kreĭn-Vishik theory of self-adjoint extensions of semi-bounded operators (see the review [1] by A. Alonso and B. Simon), modified in [16, 17, 18] for the spectral theory of curves of self-adjoint Fredholm extensions of symmetric operators on a Hilbert space. As explained in the Introduction (see also our Remark 4.3.2 below), we need a slightly broader setting for our applications, based on our concept of ω -symmetric linear relations.

To begin with, we shall determine a few preconditions for our calculus with adjoint Fredholm relations in a weak symplectic setting. Let X, Y be two complex vector spaces and let $\Omega: X \times Y \rightarrow \mathbb{C}$ be a non-degenerate sesquilinear map. Set $Z := X \times Y$. Let ω denote the symplectic form induced by Ω and defined by (4.8). As observed there, (Z, ω) is a symplectic vector space with $X \times \{0\}$ and $\{0\} \times Y$ Lagrangian subspaces. We shall determine conditions to transfer the classical dualities of kernel and cokernel from adjoint Fredholm operators to adjoint linear relations. As emphasized above (see also Remark 1.2.10), contrary to the easy calculus with strong symplectic forms, for a weak symplectic form ω the double adjoint $A^{\omega\omega}$ of a closed linear relation A is not necessarily the original A , and the adjoint A^ω of a Fredholm relation A is not necessarily a Fredholm relation. Even when it is a Fredholm relation, in the weak symplectic case we have not necessarily $\text{index } A + \text{index } A^\omega = 0$ and the index of a self-adjoint Fredholm relation does not necessarily vanish. Finally, we must recall from Remark 1.2.10.c, that the index of a Fredholm operator in Banach space that is self-adjoint relative to a weak form ω does neither necessarily vanish.

That explains why we need special assumptions to exclude intractable complications with index calculations for self-adjoint Fredholm relations. Fortunately it turns out that these special assumptions are naturally satisfied in our applications. That is what this section is about.

PROPOSITION 4.3.1 (Calculus with adjoint Fredholm relations). *Let $A \subset W \subset Z$ be two linear relations. Assume that A, A^ω are algebraic Fredholm relations with $\text{index } A + \text{index } A^\omega = 0$. Then we have $\dim \ker W^\omega = \dim Y / \dim W$ and $\dim W = \dim W^{\omega\omega}$.*

PROOF. We apply Proposition 1.4.7, taking our Z as the underlying symplectic vector space and setting $\lambda := A$ and $\mu := X \times \{0\}$. Then we have $\dim(W^\omega \cap X \times \{0\}) = \dim(Z/(W + X \times \{0\}))$ and $W + X \times \{0\} = W^{\omega\omega} + X \times \{0\}$. By (4.6) and (4.7), our results follow. \square

Let X, Y be two complex Banach spaces and $\Omega: X \times Y \rightarrow \mathbb{C}$ a bounded non-degenerate sesquilinear map with induced symplectic form ω on $X \times Y$. Let $\mathcal{C}(X, Y)$ denote the space of closed linear operators from X to Y and let $A_m \in \mathcal{C}(X, Y)$ with $(\text{dom}(A_m))^{\Omega, r} = \{0\}$ with this right annihilator defined in 1.1. (If $X = Y$ a Hilbert space with $\Omega := \langle \cdot, \cdot \rangle$, the vanishing of the right annihilator of the domain of a closed linear operator A_m means just that A_m is densely defined. We shall come back to this condition in our Assumption 4.3.3 (ii) for a revised von-Neumann setting of abstract boundary value problems and Assumption 4.4.1 (ii) for the proof of our abstract spectral flow formula.) We consider A_m as a closed linear relation. Then the adjoint relation A_m^ω is (the graph of) a closed operator. We assume that A_m is symmetric, i.e., $\text{graph}(A_m)$ is an isotropic subspace of the symplectic product space $(X \times Y, \omega)$; in operator notation that means shortly $A_m^\omega \supset A_m$. Then A_m^ω is a closed co-isotropic subspace of $(X \times Y, \omega)$ and by Lemma 3.3.1, the quotient space $(\text{graph}(A_m^\omega)/\text{graph}(A_m^{\omega\omega}); \tilde{\omega})$ is a naturally symplectic Banach space with the reduced form $\tilde{\omega}$ induced by ω . We denote the domains of A_m by D_m (the *minimal* domain) and of A_m^ω by D_{\max} (the *maximal* domain). For these data, we have, as in [16, 17]:

- (i) The space D_{\max} is a Banach space with the graph norm
- $$(4.14) \quad \|x\|_{\mathfrak{G}} := \|x\|_X + \|A_m^\omega x\|_Y \quad \text{for } x \in D_{\max}.$$
- (ii) The space D_m is a closed subspace in the graph norm and the quotient space D_{\max}/D_m is a Banach space with the minus Green's form
- $$(4.15) \quad -\omega_{\text{Green}}(x + D_m, y + D_m) := \Omega(x, A_m^\omega y) - \overline{\Omega(y, A_m^\omega x)} \quad \text{for } x, y \in D_{\max},$$
- where both A_m and Ω enter into the definition. The form is symplectic if and only if $A_m^{\omega\omega} = A_m$.
- (iii) Let $B \supset A_m$ be an extension of A_m . By Lemma 1.4.2, the operator B is self-adjoint if and only if $A_m^{\omega\omega} \subset B \subset A_m^\omega$, and for the symplectic reduction of $\text{graph}(B) \subset X \times Y$ via the co-isotropic $\text{graph}(A_m^\omega)$, there holds

$$R_{\text{graph}(A_m^\omega)}(\text{graph}(B)) \in \mathcal{L}(\text{graph}(A_m^\omega)/\text{graph}(A_m^{\omega\omega}); \tilde{\omega}).$$

- (iv) We denote by γ the natural projection

$$\gamma: D_{\max} \longrightarrow D_{\max}/D_m.$$

For any linear subspace $D \subset X$, we set

$$\gamma(D) := (D \cap D_{\max} + D_m)/D_m.$$

REMARK 4.3.2. In our applications, we consider families of self-adjoint Fredholm operators with *varying domain* and *varying maximal domain*. To us, there is no natural way to identify the different symplectic spaces and to define continuity of Lagrangian subspaces and continuity of symplectic forms in these varying symplectic spaces. Fortunately, in most applications the minimal domain is fixed and also an intermediate Hilbert space D_M , typically the Sobolev space H^d for elliptic differential operators of order d .

We shall show that meaningful modifications of the preceding statements can be obtained when we replace D_{\max} by this intermediate space D_M under the following assumptions.

ASSUMPTIONS 4.3.3 (Revised von-Neumann setting for abstract boundary value problems). (i) As in the preceding statements, we let X, Y be two Banach spaces and $\Omega: X \times Y \rightarrow \mathbb{C}$ a bounded non-degenerate sesquilinear map. We set $Z := X \times Y$ and let ω be defined by (4.8).

(ii) Our data are now four Banach spaces with continuous inclusions

$$D_m \hookrightarrow D_M \hookrightarrow D_{\max} \hookrightarrow X,$$

where the Banach space structure is given on D_{\max} and D_m by the graph inner product of a fixed closed symmetric operator $A_m \in \mathcal{C}(X, Y)$ with $\text{dom}(A_m) = D_m$. Assume that $(D_m)^{\Omega, r} = \{0\}$.

(iii) We assume that the $(A_M)^\omega = A_m$, where $A_M := A_m^\omega|_{D_M}$.

(iv) Finally, we assume that there exists a self-adjoint Fredholm extension A_D of A_m of index 0 with domain $D_m \subset D \subset D_M$.

Assumption 4.3.3 (ii) implies

$$(4.16) \quad \|x\|_{\mathfrak{G}} = \|x\|_X + \|A_M x\|_X \leq C_1 \|x\|_{D_M} \text{ for all } x \in D_M.$$

In particular, it follows that $A_M: D_M \rightarrow X$ is bounded.

We have an injection $j: D_M \rightarrow D_{\max}$ with $D_m \subset D_M$ and D_m closed in D_{\max} . Therefore, by Proposition A.6.2.a, D_m is closed in D_M , and on D_m the graph norm and the norm induced by the Banach space D_M are equivalent. Then we have the opposite estimate to (4.16), namely a Gårding type inequality, known from the study of elliptic regularity:

$$(4.17) \quad \|x\|_{D_M} \leq C_2 (\|x\|_X + \|A_M x\|_X) = C_2 \|x\|_{\mathfrak{G}} \text{ for all } x \in D_m.$$

By (1.3), Assumption 4.3.3 (iii) implies $A_m^{\omega} = A_m$.

LEMMA 4.3.4. *Under Assumptions 4.3.3 (i), (ii), (iii) the quotient space D_M/D_m is a weak symplectic Banach space with the symplectic form ω_{Green} induced by the minus Green's form on D_{\max} defined in (4.15).*

PROOF. By Lemma 3.3.1. □

The lemma shows that any intermediate space D_M satisfying Assumptions 4.3.3 (i), (ii), (iii) is big enough to permit a meaningful symplectic analysis on the reduced quotient space D_M/D_m . The point of this construction is that the norm in D_M/D_m does not come from the graph norm in D_{\max} but from the norm of D_M . Therefore, it can be kept fixed even when our operator varies. The *symplectic structure* of D_M/D_m , however, is induced by the minus Green's form and therefore will change with varying operators.

In [16, Proposition 3.5], in the spirit of the classical von-Neumann program, self-adjoint Fredholm extensions were characterized by the property that their domains, projected down into the strong symplectic space $\beta(A_m) := D_{\max}/D_m$ of abstract boundary values, make Fredholm pairs of Lagrangian subspaces with the abstract, reduced Cauchy data space $(\ker A_m^* + D_m)/D_m$. Immediately, this does not help for operator families with varying maximal domain. Surprisingly, however, the arguments generalize to the weak symplectic space intrinsically, i.e., without additional topological conditions.

For the following Lemma 4.3.5 and Proposition 4.3.6 we exploit our Assumption 4.3.3 (iv), i.e., the existence of a self-adjoint Fredholm extension A_D .

LEMMA 4.3.5. *Denote by $P_X: Z = X \times Y \rightarrow X$ the projection onto the first component. Set $W := \text{graph}(A_M)$, $\lambda := \text{graph}(A_D)$ and $\mu := X \times \{0\}$. Then P_X induces a symplectic Banach isomorphism $\tilde{P}_X: (W/W^\omega, \tilde{\omega}) \rightarrow (D_M/D_m, \omega_{\text{Green}})$, and we have*

$$\tilde{P}_X(R_W(\lambda)) = \gamma(D), \quad \tilde{P}_X(R_W(\mu)) = \gamma(\ker A_M).$$

Here ω denotes the symplectic structure on $X \times Y$ induced by the given non-degenerate sesquilinear form $\Omega: X \times Y \rightarrow \mathbb{C}$ and defined in (4.8), $\tilde{\omega}$ denotes the corresponding symplectic structure on the reduced space, and ω_{Green} denotes the form defined in Equation (4.15) and established as symplectic form on D_M/D_m in Lemma 4.3.4.

PROOF. By definition and direct calculation. \square

PROPOSITION 4.3.6 (Abstract regularity). *Under the Assumptions 4.3.3, the quotient space D/D_m and the reduced Cauchy data space $(\ker A_M + D_m)/D_m$ form a Fredholm pair of Lagrangian subspaces of the (weak) symplectic Banach space $(D_M/D_m, \omega_{\text{Green}})$ with index 0, and $\dim \ker A_m = \dim Y/(\dim A_M)$. Moreover, it follows that $\text{im } A_M = \text{im } A_m^\omega$.*

PROOF. Take Z as the symplectic vector space, $W := \text{graph}(A_M)$, $\lambda := \text{graph}(A_D)$ and $\mu := X \times \{0\}$. By Lemma 4.3.5 and Proposition 1.4.8, our results follow. \square

REMARK 4.3.7. a) By definition it is clear that $\text{im } A_M \subset \text{im } A_m^\omega$. The point of the preceding Proposition is the opposite inclusion. That inclusion is well known for any elliptic differential operator A acting from sections of a bundle E over a compact manifold M to sections of a bundle F over M , say of order $d = 1$: Let us denote the maximal extension of A on $L^2(X; E)$ by A_{\max} .

Then, as one aspect of the Lifting Jack (the existence of parametrices) for elliptic operators, for any $w \in \text{im } A_{\max}$ we have by definition a $v \in D_{\max}$ and therefore by elliptic regularity a $v' \in D_M$ such that $A_{\max}v = A_Mv' = w$. In general, there will be many different $v \in D_{\max}$ with $A_{\max}v = w$, and not all such v belong to D_M , but some of them will. In the classical theory of well-posed elliptic boundary value problems, those are typically elements that satisfy an extra condition at the boundary (as specified in [21, Theorem 19.6] for elliptic operators of first order and in [47, Theorem 2.2.1] for elliptic operators of higher order).

b) If we remove the topological requirements in Assumptions 4.3.3, the algebraic results of Lemma 4.3.4, Lemma 4.3.5 and Proposition 4.3.6 still hold.

4.4. Proof of the abstract spectral flow formula

In this section we prove an abstract spectral flow formula by Theorem 3.3.3. We shall make the following new assumptions. They are all natural in our applications, as we shall see later in Section 4.5.

ASSUMPTION 4.4.1. (i) Let $r_0: \mathbb{G}_0 \rightarrow [0, 1]$, $r: \mathbb{G} \rightarrow [0, 1]$, $p: \mathbb{X} \rightarrow [0, 1]$, and $q: \mathbb{Y} \rightarrow [0, 1]$ be Banach bundles with fibers $r_0^{-1}(s) := D_m(s)$, $r^{-1}(s) := D_M(s)$, $p^{-1}(s) := X(s)$ and $q^{-1}(s) := Y(s)$ for each $s \in [0, 1]$ respectively. Assume that we

have Banach subbundle maps $\mathbb{G}_0 \rightarrow \mathbb{G}$, $\mathbb{G} \rightarrow \mathbb{X}$.

(ii) Let

$$\{\Omega(s): X(s) \times Y(s) \longrightarrow \mathbb{C}\}_{s \in [0,1]}$$

be a path of bounded non-degenerate sesquilinear forms, and denote by $\omega(s)$ the weak symplectic structure on $Z(s) := X(s) \times Y(s)$ induced by $\Omega(s)$ and defined by (4.8). Assume that $(D_m(s))^{\Omega, r} = \{0\}$, which implies that the adjoint of an operator with domain $D_m(s)$ is an operator.

(iii) Let

$$\{A_m(s): X(s) \supset D_m(s) \longrightarrow Y(s)\}_{s \in [0,1]}$$

be a family of closed symmetric operators such that the norm on $D_m(s)$ is equivalent to the graph norm of $D_m(s)$ defined by $A_m(s)$. Assume that there exists a constant integer k such that $\dim \ker A_m(s) = k$, i.e., we assume weak inner unique continuation property (wiUCP) up to a finite constant dimension. Set $A_M(s) := A_m(s)^{\omega(s)}|_{D_M(s)}$. Assume that $(A_M(s))^{\omega(s)} = A_m(s)$, and

$$\{A_M(s) \in \mathcal{B}(D_M(s), Y(s))\}_{s \in [0,1]}$$

is a path of bounded operators.

(iv) Let $\{D(s) \in \mathcal{S}(D_M(s))\}_{s \in [0,1]}$ be a path of closed subspaces with $D_m(s) \subset D(s) \subset D_M(s)$. Assume that $\{A(s, D(s)) := A_M(s)|_{D(s)}\}_{s \in [0,1]}$ is a family of self-adjoint Fredholm operators of index 0.

THEOREM 4.4.2 (Abstract spectral flow formula). *Under Assumption 4.4.1, we have the following.*

(a) *We have $\operatorname{im} A_M(s) = \operatorname{im}(A_m(s))^{\omega(s)}$ and $\dim Y(s)/(\operatorname{im} A_M(s)) = k$ holds for each $s \in [0, 1]$.*

(b) *The family*

$$\{(\gamma(D(s)), \gamma(\ker A_M(s)))\}_{s \in [0,1]}$$

is a path of Fredholm pairs of Lagrangian subspaces of the symplectic Banach space $(D_M(s)/D_m(s), \omega_{\text{Green}}(s))$ of index 0.

(c) *We have*

$$(4.18) \quad \operatorname{sf}\{A(s, D(s))\} = -\operatorname{Mas}\{\gamma(D(s)), \gamma(\ker A_M(s)); \omega_{\text{Green}}(s)\}$$

$$(4.19) \quad = -\operatorname{Mas}\{\gamma(\ker A_M(s)), \gamma(D(s)); -\omega_{\text{Green}}(s)\}.$$

PROOF. Let $s \in [0, 1]$. By [24, Lemma 16], the pair

$$(\operatorname{graph}(A(s, D(s))), X(s) \times \{0\})$$

is a Fredholm pair of Lagrangian subspaces of the symplectic Banach space $Z(s)$ with the form $\omega(s)$. By Lemma 4.3.4, the quotient space $D_M(s)/D_m(s)$ is a weak symplectic Banach space with the symplectic form induced by the minus Green's form on $D_{\max}(s)$. By Proposition 4.3.6, (a) holds and the pair

$$(\gamma(D(s)), \gamma(\ker A_M(s)))$$

is a Fredholm pair of Lagrangian subspaces of the symplectic Banach space

$$(D_M(s)/D_m(s), \omega_{\text{Green}}(s)) \text{ of index 0 for each } s \in [0, 1].$$

By Proposition A.6.2, the norm on $D_m(s)$ is uniformly equivalent to the graph norm of $D_m(s)$ defined by $A_m(s)$.

By Lemma A.3.1, $\{\gamma(D(s)) \subset \mathcal{S}(D_M(s)/D_m(s))\}_{s \in [0,1]}$ is a continuous path.

By Corollary A.6.4, $\{A(s, D(s)) \in \mathcal{C}(X(s), Y(s))\}_{s \in [0,1]}$ is a continuous family.

We shall use the following notations:

$$\begin{aligned} W_0(s) &:= \text{graph}(A_m(s)), \quad W(s) := \text{graph}(A_M(s)), \quad W_1(s) := D_m(s) \times Y(s) \\ \widetilde{W}(s) &:= D_W(s) \times Y(s), \quad \mu(s) := X(s) \times \{0\}, \\ \mathbb{W}_0 &:= \bigcup_{s \in [0,1]} W_0(s), \quad \mathbb{W} := \bigcup_{s \in [0,1]} W(s), \quad \mathbb{W}_1 := \bigcup_{s \in [0,1]} W_1(s), \\ \widetilde{\mathbb{W}} &:= \bigcup_{s \in [0,1]} \widetilde{W}(s), \quad \mathbb{Z} := \bigcup_{s \in [0,1]} Z(s). \end{aligned}$$

Then we have $W_0(s)$ is closed and complemented in $W_1(s)$, in short, $W_0(s) \in \mathcal{S}^c(W_1(s))$, and similarly $W(s) \in \mathcal{S}^c(\widetilde{W}(s))$. By Lemma A.4.8.b (see also [72, Lemma 0.2]), \mathbb{W}_0 is a subbundle of \mathbb{W}_1 , and \mathbb{W} is a subbundle of $\widetilde{\mathbb{W}}$. The bundle $\widetilde{\mathbb{W}}$ is a subbundle of \mathbb{Z} , and we have $\widetilde{W}(s) + \mu(s) = Z(s)$. By Definition 4.2.5, Theorem 3.3.3, Lemma 4.3.5 and Proposition 2.3.1.d,f, we have

$$\begin{aligned} \text{sf}\{A(s, D(s))\} &= \text{Mas}_- \{\text{graph}(A(s, D(s)), X(s) \times \{0\})\} \\ &= \text{Mas}_- \{R_W(s)^{\omega(s)}(\text{graph}(A(s, D(s))), R_W(s)^{\omega(s)}(X(s) \times \{0\}))\} \\ &= \text{Mas}_- \{\gamma(D(s)), \gamma(\ker A_M(s)); -\omega_{\text{Green}}(s)\} \\ &= -\text{Mas}\{\gamma(\ker A_M(s)), \gamma(D(s)); -\omega_{\text{Green}}(s)\} \\ &= -\text{Mas}\{\gamma(D(s)), \gamma(\ker A_M(s)); \omega_{\text{Green}}(s)\}. \end{aligned} \quad \square$$

REMARK 4.4.3. Consider the special case when $X(s)$ is a Hilbert space with Hilbert structure $\Omega(s)$ for each s . If we assume that $\dim \ker A_m(s) = 0$ for each s , the method in the proof of [22, Theorem 2.13], [23, Theorem 1.3] works here. The method is not applicable in the case that $\dim \ker A_m(s) = k > 0$ for each s .

4.5. An application: A general desuspension formula for the spectral flow of families of elliptic boundary value problems

Having expanded weak symplectic linear algebra and analysis to some length and detail in the two preceding sections, we shall turn to the geometric setting and the geometric applications.

4.5.1. Parametrization of vector bundles over manifolds with boundary and domains in Sobolev chains. Consider a (big) Hermitian vector bundle \mathbb{E} over a (big) compact Hausdorff space \mathbb{M} . We assume that \mathbb{M} itself is a fiber bundle over the interval $[0, 1]$ such that \mathbb{M} is a continuous family of compact smooth Riemannian manifolds $j(s): M(s) \hookrightarrow \mathbb{M}$ with boundary $\Sigma(s)$, $s \in [0, 1]$. We require that the vector bundle structure is compatible with the boundary part. More precisely, we shall have a trivialization

$$\varphi: M(0) \times [0, 1] \simeq \mathbb{M}$$

such that $\pi \circ \varphi^{-1} \circ j(s): (M(s), \Sigma(s)) \rightarrow (M(0), \Sigma(0))$ is a diffeomorphism. Here π denotes the natural projection $\pi: M(0) \times [0, 1] \rightarrow M(0)$. We do not assume that $M(s)$ or $\Sigma(s)$ are connected. Note that the trivialization defines smooth structures on $\text{im } \varphi|_{(M(0) \setminus \Sigma(0)) \times [0, 1]}$ and so on \mathbb{M} and \mathbb{E} .

Let $E(s) \rightarrow M(s)$ be the induced bundle, i.e., the pull back $j(s)^*(\mathbb{E})$. Denote by $C_0^\infty(M(s); E(s))$ the space of smooth sections with support in the interior $M(s)^0 :=$

$M(s) \setminus \Sigma(s)$ of $M(s)$. Assume that $d > 0$ is a positive integer and $\sigma \geq 0$ a non-positive real (on manifolds with boundary, Sobolev spaces of negative order are a nuisance and shall be avoided here). We define the Hilbert (before choices rather “Hilbertable”) space

$$H_0^\sigma(M(s); E(s)) := \overline{C_0^\infty(M(s); E(s))}^{H^\sigma(M(s); E(s))}.$$

Here $H^\sigma(M(s); E(s))$ denotes the Sobolev space of order σ defined in [21, Chapter 11] as the restrictions to $M(s)$ of sections belonging to $H^\sigma(\widetilde{M(s)}; \widetilde{E(s)})$, where $\widetilde{E(s)} \rightarrow \widetilde{M(s)}$ is a smooth extension of the given vector bundle $E(s) \rightarrow M(s)$ over a smooth closed extension $\widetilde{M(s)}$ of $M(s)$, e.g., the closed double. C. Frey [47, p. 14] has shown that these definitions of $H^\sigma(M(s); E(s))$, $H_0^\sigma(M(s); E(s))$ coincide with the definitions given in J.-L. Lions and E. Magenes [62, Chapter 9]. The inner product is given by the Sobolev inner product. Set

$$D_m(s; \sigma) := H_0^{\sigma+d}(M(s); E(s)), \quad D_M(s; \sigma) := H^{\sigma+d}(M(s); E(s)),$$

$$X(s) = Y(s) := L^2(M(s); E(s)), \quad S(s; \sigma) := \sum_{j=1}^d H^{\sigma+d-j}(\Sigma(s); E(s)|_{\Sigma(s)}),$$

$$\mathbb{G}_0 := \bigcup_{s \in [0,1]} D_m(0, s), \quad \mathbb{G} := \bigcup_{s \in [0,1]} D_M(0, s), \quad \mathbb{X} := \bigcup_{s \in [0,1]} X(s).$$

Then \mathbb{G}_0 , \mathbb{G} , \mathbb{X} have Banach bundle structures over $[0, 1]$, and the natural inclusions $\mathbb{G}_0 \rightarrow \mathbb{G}$, $\mathbb{G} \rightarrow \mathbb{X}$ are Banach subbundle maps. So Assumption 4.4.1 (i) holds.

By the trace theorem, for $\sigma > -1/2$ we have

$$(4.20) \quad S(s; \sigma + 1/2) \cong D_M(s; \sigma) / D_m(s; \sigma).$$

Let $\Omega(s): X(s) \times X(s) \rightarrow \mathbb{C}$ denote the L^2 inner product, and let $\omega(s)$ be the strong symplectic structure on $Z(s) := X(s) \times X(s)$ induced by $\Omega(s)$. Since $D_m(0, s)$ is dense in $X(s)$, we have $(D_m(0, s))^{\Omega, r} = (D_m(0, s))^\perp = \{0\}$. So Assumption 4.4.1 (ii) holds. For any closed operator $T \in \mathcal{C}(X(s))$, we have $T^* = T^{\omega(s)}$.

4.5.2. Curves of elliptic differential operators and their Calderón projections and Cauchy data spaces. We consider a smooth linear differential operator $\mathbb{A}: C^\infty(\mathbb{M}; \mathbb{E}) \rightarrow C^\infty(\mathbb{M}; \mathbb{E})$ which induces a smooth family of elliptic differential operators $A(s)$ of order $d > 0$

$$(4.21) \quad A(s): C_0^\infty(M(s); E(s)) \longrightarrow C^\infty(M(s); E(s)).$$

For each $s \in [0, 1]$ and $\sigma \geq 0$, the operator $A(s)$ extends to a bounded operator

$$(4.22) \quad A_m(s; \sigma): D_m(s; \sigma) \rightarrow H^\sigma(M(s); E(s)),$$

$$(4.23) \quad A_M(s; \sigma): D_M(s; \sigma) \rightarrow H^\sigma(M(s); E(s)).$$

For each $s \in [0, 1]$, by the interior elliptic estimate for $A(s)$, the operator

$$A_m(s; 0): X(s) \supset D_m(s; 0) \longrightarrow X(s)$$

is a closed operator, and the graph norm on $D_m(s; 0)$ defined by $A_m(s; 0)$ is equivalent to the Sobolev norm (see, e.g., [21, Proposition 20.7] for the first order case and [47, Proposition 1.1.1] in the higher order case). The family

$$\{A_M(s; 0) \in \mathcal{B}(D_M(s; 0), X(s))\}_{s \in [0,1]}$$

is a path of bounded operators.

We have the following [47, Proposition 1.1.2].

PROPOSITION 4.5.1. *Denote by ν the inner normal vector field on $\Sigma(s)$, ν^\flat the dual of ν , and $\hat{A}(s)$ the principal symbol of $A(s)$. Then the minus Green's form on $S(s; \sigma)$ is given by*

$$(4.24) \quad -\omega_{\text{Green}}(s)(u, v) = \langle J(s)u, v \rangle_{L^2(\Sigma(s); E(s)^d|_{\Sigma(s)})},$$

where $J(s)$ is a matrix of differential operators $J_{k,j}(s)$ of order $d + 1 - k - j$, $k, j = 1, \dots, d$. Moreover,

$$(4.25) \quad J_{k,j}(s) = \begin{cases} i^d (-1)^{d+1-k} \hat{A}(s)(\nu^\flat), & \text{if } k + j = d + 1, \\ 0, & \text{if } k + j > d + 1. \end{cases}$$

On $S(s; \sigma)$ we have the family of the minus Green's form $\{-\omega_{\text{Green}}(s)\}_{s \in [0,1]}$ which is a continuous path of bounded non-degenerate sesquilinear forms for $\sigma \geq \frac{1-d}{2}$. The forms are invertible for $\sigma = \frac{1-d}{2}$, and they are not well-defined for $\sigma < \frac{1-d}{2}$. We have $(A_M(s; 0))^* = A_m^t(s; 0)$, where $A^t(s)$ denotes the formal adjoint of $A(s)$.

Let $Q(s; \sigma): S(s; \sigma) \rightarrow S(s; \sigma)$ denote the projection defined by the orthogonal pseudo-differential Calderón projection $Q(s)$ belonging to the operator $A(s)$. For the construction of the Calderón projection (first depending on choices and then orthogonalized), we refer to R.T. Seeley [86, Section 4], [87, Theorem 1]. There it is shown that it is a pseudo-differential idempotent with the Cauchy data space as its range. Recall that the Cauchy data space of a differential operator A of order d consists of the closure in $S(\sigma)$ of the array of all derivatives of sections in $\ker A(s)$ up to order $d - 1$ in normal direction along the boundary. For operators of Dirac type, Seeley's definition was worked out and made canonical in [21, Chapter 12], see also [47, Section 2.3] for elliptic differential operators of arbitrary order.

PROPOSITION 4.5.2 (Continuity of Calderón projection). *Assume that*

$$\dim \ker A_m(s; 0) = k \text{ and } \dim \ker A_m^t(s; 0) = l$$

are independent of s and $\sigma \geq 1/2 - d$. Then the family $\{\text{im } Q(s; \sigma)\}_{s \in [0,1]}$ is continuous.

REMARK 4.5.3. a) If $d = 1$, $A(s)$ is of Dirac type and $M(s)$ has product structure near $\Sigma(s)$ compatible with the fiber structure for each s , then we have weak Unique Continuation Property (UCP, see [21, Chapter 8]) and, by the construction, the projectors $Q(s)$ form a continuous family of pseudo-differential projectors and the family $\{\text{im } Q(s; \sigma)\}_{s \in [0,1]}$ is continuous for all real σ .

b) In [85] L. Schwartz conjectured $l = k$. As a student, the first author gave arguments in support for the Schwartz Conjecture in [14]. A rigorous proof (or a striking counterexample), though, is still missing.

PROOF. By (4.20), for $\sigma > 0$ we have

$$(4.26) \quad \text{im } Q(s; \sigma) = \gamma(\ker A_M(s; \sigma - 1/2)).$$

Let $\sigma \geq 1/2$. Since $A(s)$ is elliptic,

$$\ker A_m(s; \sigma - 1/2) = \ker A_m(s; 0) \text{ and } \ker A_m^t(s; \sigma - 1/2) = \ker A_m^t(s; 0)$$

consist of smooth sections. Since $\dim \ker A_m^t(s; 0) = l$ is constant, by Corollary A.3.16, $\{\ker A_M(s; \sigma - 1/2)\}_{s \in [0,1]}$ is a continuous family. Since $\text{im } Q(s; \sigma)$ is closed,

the subspace $\ker A_M(s; \sigma - 1/2) + D_m(s; \sigma - 1/2)$ is closed in $D_M(s; \sigma - 1/2)$. Since $\dim \ker A_m(s; 0) = k$ is constant, by Corollary A.3.15, the family $\{\operatorname{im} Q(s; \sigma)\}_{s \in [0,1]}$ is continuous.

By our [15, Theorem 5.3], based on our [20, Theorem 7.2b], the Cauchy data family $\{\operatorname{im} Q(s; \sigma)\}_{s \in [0,1]}$ is continuous for $\sigma \in [1/2 - d, 1/2]$. Then $\{\operatorname{im} Q(s; \sigma)\}_{s \in [0,1]}$ is a continuous family for all $\sigma \geq 1/2 - d$. Note that the continuity is only proved there for the case when $d = 1$ and $k = l = 0$. The proof can be somewhat simplified by G. Grubb [52], and the proof can be easily transferred to the general case. \square

We assume that all $A(s)$ are formally self-adjoint, i.e., $A_m(s; 0) \subset (A_m(s; 0))^*$. Note that we make no assumptions about product structures near the boundary $\Sigma(s)$. Then $(A_M(s; 0))^* = A_m(s; 0)$. Assume that there exists a constant integer k such that $\dim \ker A_m(s; 0) = k$. Then Assumption 4.4.1 (iii) holds.

4.5.3. Desuspension spectral flow formula for curves of self-adjoint well-posed elliptic boundary value problems. For each s we choose a well-posed self-adjoint boundary condition $P(s) \in \operatorname{Grass}_{\text{sa}}(A(s))$ in the sense of R.T. Seeley [87, Definition 3 and Theorem 7], worked out in our [21, Definition 18.1 and Proposition 20.3] and further expanded by J. Brüning and M. Lesch [27] for the first order case and by C. Frey [47, Definition 1.2.5] for $d \geq 1$). It is a self-adjoint projection $P(s; \sigma): S(s; \sigma) \rightarrow S(s; \sigma)$ defined by a pseudo-differential projection $P(s)$ satisfying a certain conjugacy condition between $I - P(s)$ and $P(s)$ such that it yields a self-adjoint Fredholm extension $A(s, P(s))$ in $X(s)$ with

$$\operatorname{dom} A(s, P(s)) = D(s) := \{x \in D_M(s; 0); P(s; 1/2)(\gamma(x)) = 0\}.$$

Fix $\sigma \geq \frac{1-d}{2}$. We assume $\{P(s; 1/2)\}_{s \in [0,1]}$ and $\{P(s; \sigma)\}_{s \in [0,1]}$ are continuous families. By Lemma A.3.1, $\{D(s)\}_{s \in [0,1]}$ is a continuous family. Then Assumption 4.4.1 (iv) holds.

We then have the following spectral flow formula.

THEOREM 4.5.4 (Desuspension spectral flow formula). *Under the above assumptions, we have the following.*

(a) *We have that*

$$\operatorname{im} A_M(s; 0) = \operatorname{im} (A_m(s; 0))^* \quad \text{and} \quad \dim X(s) / (\operatorname{im} A_M(s; 0)) = k$$

holds for each $s \in [0, 1]$.

(b) *The family*

$$\{(\ker P(s; \sigma), \operatorname{im} Q(s; \sigma))\}_{s \in [0,1]}$$

is a path of Fredholm pairs of Lagrangian subspaces of the symplectic Banach space $(S(s; \sigma), \omega_{\text{Green}}(s))$ of index 0.

(c) *We have*

$$(4.27) \quad \operatorname{sf}\{A(s, D(s))\} = -\operatorname{Mas}\{\ker P(s; \sigma), \operatorname{im} Q(s; \sigma); \omega_{\text{Green}}(s)\}$$

$$(4.28) \quad = -\operatorname{Mas}\{\operatorname{im} Q(s; \sigma), \ker P(s; \sigma); -\omega_{\text{Green}}(s)\}.$$

REMARK 4.5.5. To us, the preceding Formulae (4.27) and (4.28) are most natural for $\sigma = 1/2$, i.e., when we evaluate the Maslov index on the right side of the formulae in the weak symplectic quotient spaces $S(s; 1/2) = H^d(M(s); E(s)) / H_0^d(M(s); E(s))$. In that case the arguments are most easily derived from the abstract spectral flow formula in the preceding section. Note, however, that the two formulae remain valid for all $\sigma \geq \frac{1-d}{2}$, so, in particular also for $\sigma = \frac{1-d}{2}$, i.e., for

calculating the Maslov index in the continuous family of the common strong symplectic Hilbert spaces $S(s; \frac{1-d}{2})$. The arguments are getting more involved, though, as indicated by the double continuity requirement for the boundary projections $P(s)$, namely requiring continuity both in $S(s; 1/2)$ and $S(s; \sigma)$.

PROOF. Since $X(s)$ is a Hilbert space and $A(s, P(s))$ is a self-adjoint Fredholm operator, $\text{index } A(s, P(s)) = 0$ and $P(s)$ is a well-posed boundary value condition for $A(s)$ in the sense of [47, Definition 1.2.5]. By Theorem 4.4.2, (b), (c) hold for $\sigma = 1/2$ and (a) holds.

By [47, Theorem 2.1.4] and the regularity theory for elliptic operators, we have

$$(4.29) \quad \dim(\ker P(s; \sigma) \cap \text{im } Q(s; \sigma)) = \dim(\ker P(s; 1/2) \cap \text{im } Q(s; 1/2))$$

and

$$(4.30) \quad \begin{aligned} \dim S(s; \sigma) / (\ker P(s; \sigma) \cap \text{im } Q(s; \sigma)) \\ = \dim S(s; 1/2) / (\ker P(s; 1/2) \cap \text{im } Q(s; 1/2)). \end{aligned}$$

Then the pair $(\ker P(s; \sigma), \text{im } Q(s; \sigma))$ is a Fredholm pair of isotropic subspaces of the symplectic Banach space $S(s; \sigma)$ of index 0, so it is a Lagrangian pair by [24, Proposition 1]. Then (b) holds.

Note that we have $S(s; \sigma) \subset S(s; 1/2)$ for $\sigma \geq 1/2$ and $S(s; \sigma) \supset S(s; 1/2)$ for $\sigma \in [\frac{1-d}{2}, 1/2]$. By (4.29) and (4.30), we can apply Theorem 3.3.9 and obtain (c). \square

4.5.4. General spectral splitting formula on partitioned manifolds.

Now we assume that the manifold $M(s) = M(s)^+ \cup_{\Sigma(s)} M(s)^-$ is a partitioned closed manifold with a hypersurface $\Sigma(s)$. Let $\sigma \geq \frac{1-d}{2}$. We denote the restrictions of $A(s)$ to the parts by $A(s)^\pm$. Note that we now have a pair of Calderón projections $(Q(s)^+, Q(s)^-)$ for each $s \in [0, 1]$ with $\text{im } Q(s; \sigma)^\pm$ Lagrangian subspaces in $S(s; \sigma)$ with symplectic form again defined by the minus Green's form $-\omega_{\text{Green}}(s)$. Then we have the following generalization of the Yoshida-Nicolaescu splitting formula for the spectral flow:

THEOREM 4.5.6 (General Yoshida-Nicolaescu Splitting Formula). *For the partitioned case we assume that $\sigma \geq \frac{1-d}{2}$ and*

$$\dim \ker A_m^\pm(s; 0) = k^\pm.$$

Then we have

$$(4.31) \quad \text{sf}\{A(s)\} = \text{sf}\{A^-(s, I - Q^+(s))\}$$

$$(4.32) \quad = -\text{Mas}\{\text{im } Q^-(s, \sigma), \text{im } Q^+(s, \sigma); \omega_{\text{Green}}(s)\}.$$

PROOF. Let $M^\sharp(s)$ denote the compact manifold

$$M^+(s) \sqcup M^-(s) = (M(s) \setminus \Sigma(s)) \cup ((\Sigma(s) \sqcup (-\Sigma(s)))$$

with boundary

$$\partial M^\sharp(s) = \partial M^+(s) \sqcup \partial M^-(s) = \Sigma(s) \sqcup (-\Sigma(s)) =: \Sigma^\sharp(s)$$

and $E^\sharp(s) \rightarrow M^\sharp(s)$ the corresponding Hermitian bundle. Over $M(s)$, $M^a(s)$ and $\Sigma^a(s)$ with $a \in \{\pm, \sharp\}$ we have specified section spaces with the notations $X(s)$,

$X^a(s)$, $D_M(s; \sigma)$, $D_M^a(s; \sigma)$, $D_m(s; \sigma)$, $D_m^a(s; \sigma)$, $S(s; \sigma)$, and $S^\sharp(s; \sigma)$. Fixing $\Sigma(s)$ induces a decomposition

$$X(s) \cong X^+(s) \oplus X^-(s) = X^\sharp(s),$$

and for the Sobolev space

$$D_M^+(s; \sigma) \oplus D_M^-(s; \sigma) = D_M^\sharp(s; \sigma), \quad D_m^+(s; \sigma) \oplus D_m^-(s; \sigma) = D_m^\sharp(s; \sigma).$$

We have the symplectic decomposition

$$(S^\sharp(s; \sigma), \omega_{\text{Green}}^\sharp(s)) = (S(s; \sigma) \times S(s, \sigma), \omega_{\text{Green}}(s) \oplus (-\omega_{\text{Green}}(s))).$$

Correspondingly, we obtain an operator $A(s)^\sharp$ for each $s \in [0, 1]$ which is a formally self-adjoint elliptic differential operator of order d according to the assumptions made for Theorem 4.5.6.

For the Calderón projection of A^\sharp we have

$$(4.33) \quad \text{im } Q^\sharp(s) = \text{im } Q^+(s) \oplus \text{im } Q^-(s).$$

Let $\Delta(s; \sigma)$ denote the diagonal in $S(s; \sigma) \times S(s, \sigma)$. By Lemma 3.1.7, for each $s \in [0, 1]$, the diagonal $\Delta(s; \sigma)$ is a Lagrangian subspace of $S^\sharp(s, \sigma)$ with respect to $\omega_{\text{Green}}^\sharp(s)$ and makes a Fredholm pair with each $\text{im } Q^\sharp(s)$. By [47, Theorem 2.1.4], the projection of $S^\sharp(s)$ onto $\Delta(s)$ is well-posed for $A^\sharp(s)$ in the sense of [47, Definition 1.2.5] (even if it is not a pseudo-differential operator over the manifold $\Sigma^\sharp(s)$, as noticed in [59, Section 5] in the $d = 1$ case).

Consequently, we have on the manifold $M^\sharp(s)$ a natural self-adjoint elliptic boundary condition (in the sense of our Theorem 4.5.4) defined for $A^\sharp(s)$ by the *pasting* domain

$$(4.34) \quad D^\sharp(s) := \{(x, y) \in D_M^\sharp(s; 0); (\gamma^+(s))(x) = (\gamma^-(s))(y)\}$$

$$(4.35) \quad = \{(x, y) \in D_M^\sharp(s; 0); (\gamma^\sharp(s))(x, y) \in \Delta(s; 1/2)\},$$

where $\gamma^a(s): D_M^\sharp(s; 0) \rightarrow S^\sharp(s, 1/2)$ denotes the trace maps for $s \in [0, 1]$ and $a = \pm, \sharp$. Let $A^\sharp(s, D^\sharp(s))$ denote the operator which acts like $A^\sharp(s)$ and has domain $D^\sharp(s)$.

By these definitions and applying Proposition 3.1.8.b and Theorem 4.5.4 to the operator family $\{A^\sharp(s, D^\sharp(s))\}$ we obtain

$$\begin{aligned} \text{sf}\{A(s)\} &= \text{sf}\{A^\sharp(s, D^\sharp(s))\} \\ &\stackrel{\text{Th. 4.5.4}}{=} -\text{Mas}\{\Delta(s; \sigma), \text{im } Q^+(s; \sigma) \oplus \text{im } Q^-(s; \sigma); \omega_{\text{Green}}(s) \oplus (-\omega_{\text{Green}}(s))\} \\ &\stackrel{(3.9)}{=} -\text{Mas}\{\text{im } Q^-(s; \sigma), \text{im } Q^+(s; \sigma); \omega_{\text{Green}}(s)\} \\ &\stackrel{(3.8)}{=} -\text{Mas}\{\text{im } Q^+(s; \sigma), \text{im } Q^-(s; \sigma); -\omega_{\text{Green}}(s)\} \\ &\stackrel{\text{Th. 4.5.4}}{=} \text{sf}\{A^-(s, I - Q^+(s))\}. \end{aligned} \quad \square$$

REMARK 4.5.7. If one is only interested in the equality (4.31), one need not argue with the Maslov index, as we do, but can find a direct proof in [59, Corollary 5.6] based solely on the homotopy invariance of the spectral flow of a related two-parameter family.

APPENDIX A

Perturbation of closed subspaces in Banach spaces

This appendix serves as an introduction to the topology of closed linear subspaces in Banach spaces with applications to families of closed operators with nested domains and perturbations of Fredholm pairs. Denote by $\mathcal{S}(X)$ ($\mathcal{S}^c(X)$) the set of all (complemented) closed linear subspaces of a Banach space X . Denote by $\mathcal{B}(X, Y)$ ($\mathcal{C}(X, Y)$) the set of all bounded operators (closed, not necessarily bounded operators) between Banach spaces X and Y . Let $\mathcal{S}^c(X), \mathcal{S}^c(X)$ be equipped with the gap topology (see below Section A.2). Then, we shall solve the following problems:

- (I) Under what conditions do the elementary linear operations (intersection, sum and making quotients) become continuous for pairs of closed subspaces?
- (II) Under what conditions do we obtain a continuous mapping $(A, D) \mapsto A_D$, where the operator A varies continuously in $\mathcal{B}(X, Y)$, the domain D varies continuously in $\mathcal{S}(X)$, and A_D denotes the restriction of A to the domain D and varies in $\mathcal{C}(X, Y)$?
- (III) How can we control changes of a space of Fredholm pairs under finite or compact perturbation of one factor?

Question (I) will be answered in Propositions A.3.5 and A.3.13. Question (II) will be answered in Corollary A.6.4. Question (III) will be answered in Proposition A.7.6.

These results will be formulated and proved in general terms. We shall emphasize, however, the various applications to solving variational problems of the global analysis of elliptic operators on manifolds with boundary. Problem (I) has two immediate applications: The first application is the local stability of weak inner UCP, see Corollary A.3.9. The second application is the continuous variation of the Cauchy data spaces under variation of the operator under the assumption of weak inner UCP (or fixed dimension of the inner solution spaces), see Corollary A.3.16.

Problem (II) settles the intricate delicacies of independent variation of operator and boundary condition, yielding continuous variation of the induced Fredholm extension.

Problem (III) addresses the changes, roughly speaking, when we replace one boundary condition by another one under *small* perturbation. Here *small* means by finite or compact change of the domain, to be defined rigorously below. To give an idea of what kind of changes we are dealing with, we refer to the Grassmannian of pseudo-differential projections with the same principal symbols, that define large classes of well-posed and mutually intimately related boundary problems, as in [21].

This program requires rather detailed investigations of the topology of graphs and domains of closed operators. Our topological approach is based on the *gap*

$\hat{\delta}: \mathcal{S}(X) \times \mathcal{S}(X) \rightarrow \mathbb{R}_+$ and the *angular distance* $\hat{\gamma}: \mathcal{S}(X) \times \mathcal{S}(X) \rightarrow [0, 1]$ (also called *minimum gap*), see Definition A.2.1 below. According to E. Berkson in [11], the concept of *opening* (as the *gap* was called in the 1940s and 1950s) was first introduced in Hilbert space in 1947 by M. G. Krein and M.A. Krasnosel'ski in [60]. The definition was one year later extended to arbitrary Banach spaces in [61] by M.G. Krein, M.A. Krasnosel'ski, and D.P. Mil'man. Ten years later, it was supplemented by the definition of the *minimum gap/angular distance* $\hat{\gamma}$ in [49] by I. Gohberg and A.S. Markus.

We shall use T. Kato's [57, Chapter IV] as our general reference. We shall apply considerable diligence to the estimates to guarantee the sharpest versions of our invariance results. Some of the results, often in different and weaker form, can be found in the quoted original papers and the classical treatises [35, 48, 66, 71, 72, 74] by H.O. Cordes and J.-P. Labrousse, I. Gohberg and M.G. Krein, J.L. Massera and J.J. Schäffer, G. Neubauer, and J.D. Newburgh.

A.1. Some linear algebra facts

We have the following elementary fact of linear algebra.

LEMMA A.1.1. *Let X be a vector space and V_1, V_2, V_3 three linear subspaces. If $V_1 \subset V_3$, we have*

$$(A.1) \quad (V_1 + V_2) \cap V_3 = V_1 + V_2 \cap V_3.$$

COROLLARY A.1.2. *Let X be a vector space and V, X_0, X_1 three linear subspaces with $X = X_0 \oplus X_1$. Denote by $P_0: X \rightarrow X_0$ the projection defined by the decomposition $X = X_0 \oplus X_1$. Assume that $V \supset X_1$. Then we have $V = P_0V + X_1$ and $P_0V = V \cap X_0$. In particular, we have $V = X$ if $P_0V = X_0$.*

PROOF. Since $V \supset X_1$, by Lemma A.1.1 we have $V = V \cap (X_0 + X_1) = V \cap X_0 + X_1$. So we have $P_0V = V \cap X_0$, and $V = P_0V + X_1$. If $P_0V = X_0$, we have $V = X$. \square

A.2. The gap topology

Let X be a Banach space. Let M, N be two closed linear subspaces of X , i.e., $M, N \in \mathcal{S}(X)$. Denote by S_M the unit sphere of M . We recall three common definitions of distances in $\mathcal{S}(X)$ (see also [57, Sections IV.2.1 and IV.4.1]):

- the *Hausdorff metric* \hat{d} ;
- the *aperture (gap distance)* $\hat{\delta}$, that is not a metric since it does not in general satisfy the triangle inequality, but defines the same topology as the metric \hat{d} , called *gap topology*, and is easier to estimate than \hat{d} ; and
- the *angular distance (minimum gap)* $\hat{\gamma}$, that is useful in our estimates, though not defining any suitable topology.

DEFINITION A.2.1 (The gap between subspaces). (a) We set

$$d(M, N) = d(S_M, S_N)$$

$$:= \begin{cases} \max \left\{ \sup_{u \in S_M} \text{dist}(u, S_N), \sup_{u \in S_N} \text{dist}(u, S_M) \right\}, & \text{if both } M \neq 0 \text{ and } N \neq 0, \\ 0, & \text{if } M = N = 0, \\ 2, & \text{if either } M = 0 \text{ and } N \neq 0 \text{ or vice versa.} \end{cases}$$

(b) We set

$$\begin{aligned}\delta(M, N) &:= \begin{cases} \sup_{u \in S_M} \text{dist}(u, N), & \text{if } M \neq \{0\}, \\ 0, & \text{if } M = \{0\}, \end{cases} \\ \hat{\delta}(M, N) &:= \max\{\delta(M, N), \delta(N, M)\}.\end{aligned}$$

$\hat{\delta}(M, N)$ is called the *gap* between M and N .

(c) We set

$$\begin{aligned}\gamma(M, N) &:= \begin{cases} \inf_{u \in M \setminus N} \frac{\text{dist}(u, N)}{\text{dist}(u, M \cap N)} (\leq 1), & \text{if } M \not\subset N, \\ 1, & \text{if } M \subset N, \end{cases} \\ \hat{\gamma}(M, N) &:= \min\{\gamma(M, N), \gamma(N, M)\}.\end{aligned}$$

$\hat{\gamma}(M, N)$ is called the *minimum gap* between M and N . If $M \cap N = \{0\}$, we have

$$\gamma(M, N) = \inf_{u \in S_M} \text{dist}(u, N).$$

In this Memoir we shall impose the gap topology on the space $\mathcal{S}(X)$ of all closed linear subspaces of a Banach space X and its subset $\mathcal{S}^c(X)$ of complemented subspaces.

We recall the following two results on finite-dimensional variation. For the second see [26, Proposition 11.4]. For the first see ([57, Lemma III.1.9]).

PROPOSITION A.2.2 (Finite extension). *Let X be a Banach space and M be a closed subspace of X . Let $M' \supset M$ be a linear subspace of X with $\dim M'/M < +\infty$. Then we have*

- (a) M' is closed, and
- (b) $M' \in \mathcal{S}^c(X)$ if and only if $M \in \mathcal{S}^c(X)$.

DEFINITION A.2.3. (a) The space of (algebraic) *Fredholm pairs* of linear subspaces of a vector space X is defined by

$$(A.2) \quad \mathcal{F}_{\text{alg}}^2(X) := \{(M, N) \mid \dim(M \cap N) < +\infty \text{ and } \dim X/(M + N) < +\infty\}$$

with

$$(A.3) \quad \text{index}(M, N) := \dim(M \cap N) - \dim X/(M + N).$$

(b) In a Banach space X , the space of (topological) *Fredholm pairs* is defined by

$$(A.4) \quad \mathcal{F}^2(X) := \{(M, N) \in \mathcal{F}_{\text{alg}}^2(X) \mid M, N, \text{ and } M + N \subset X \text{ closed}\}.$$

A pair (M, N) of closed subspaces is called *semi-Fredholm* if $M + N$ is closed, and at least one of $\dim(M \cap N)$ and $\dim(X/(M + N))$ is finite.

(c) Let X be a Banach space, $M \in \mathcal{S}(X)$ and $k \in \mathbb{Z}$. We define

$$(A.5) \quad \mathcal{F}_M(X) := \{N \in \mathcal{S}(X); (M, N) \in \mathcal{F}^2(X)\},$$

$$(A.6) \quad \mathcal{F}_{k,M}(X) := \{N \in \mathcal{S}(X); (M, N) \in \mathcal{F}^2(X), \text{index}(M, N) = k\}.$$

REMARK A.2.4. Actually, in Banach space the closedness of $\lambda + \mu$ follows from its finite codimension in X in combination with the closedness of λ, μ (see [17, Remark A.1] and [57, Problem 4.4.7]).

The following lemma is from [57, Problem IV.4.6].

LEMMA A.2.5. *Let X be a vector space and M', M, N be linear subspaces. Assume that $M' \supset M$ and $\dim M'/M = n < +\infty$. Then we have $\text{index}(M', N) = \text{index}(M, N) + n$.*

We give the following elementary fact.

LEMMA A.2.6. *Let X be a Banach space and $(M, N) \in \mathcal{F}^2(X)$. Then we have $M, N \in \mathcal{S}^c(X)$.*

PROOF. Since $(M, N) \in \mathcal{F}^2(X)$, there exist closed linear subspaces $M_1 \subset M$, $N_1 \subset N$ and a finite-dimensional linear subspace $V \subset X$ such that

$$M = M \cap N \oplus M_1, N = M \cap N \oplus N_1, X = V \oplus (M + N).$$

Then we have $N_1 \cap M = N_1 \cap M \cap N = \{0\}$, and

$$(A.7) \quad X = M \cap N \oplus M_1 \oplus N_1 \oplus V.$$

So $M, N \in \mathcal{S}^c(X)$ holds. \square

A.3. Continuity of operations of linear subspaces

We study the continuity of M/L , $M \cap N$ and $M + N$ for varying closed subspaces M and N and fixed closed subspace of a Banach space X .

For the quotient space, we have the following lemma.

LEMMA A.3.1. *Let X be a Banach space with closed subspaces $M, N, L \in \mathcal{S}(X)$ such that $M, N \supset L$. Denote by p the natural map $p: X \rightarrow X/L$. Then we have*

- (a) $d(p(u), p(M)) = d(u, M)$ for $u \in X$,
- (b) $\gamma(p(M), p(N)) = \gamma(M, N)$,
- (c) $d(u, N) \leq d(u, L)\delta(M, N)$ for $u \in M$, and
- (d) $\delta(M, N) = \delta(p(M), p(N))$.

PROOF. (a), (b) By the last paragraph of the proof of [57, Theorem IV.4.2].
(c) Let $\varepsilon \in (0, 1)$. By [57, Lemma III.1.12], for any $u \in M$, there exists a $v \in L$ such that $d(u, L) \geq (1 - \varepsilon)\|u - v\|$. Since $L \subset N$, we have

$$d(u, N) = d(u - v, N) \leq \|u - v\|\delta(M, N) \leq (1 - \varepsilon)^{-1}d(u, L)\delta(M, N).$$

Let $\varepsilon \rightarrow 0$, and we have $d(u, N) \leq d(u, L)\delta(M, N)$.

(d) If $M = L$, we have $\delta(M, N) = \delta(p(M), p(N)) = 0$. Assume that $M \neq L$. By definition and the first equality we have

$$\begin{aligned} \delta(M, N) &= \max\{d(u, N); u \in S_M\} \\ &\leq \max\{d(u, N); u \in M, d(u, L) \leq 1\} = \delta(p(M), p(N)). \end{aligned}$$

By (a) and (c) we have

$$\delta(p(M), p(N)) = \max\{d(u, N); u \in M, d(u, L) = 1\} \leq \delta(M, N).$$

Thus we obtain (d). \square

Firstly, we consider the case of $\dim(M \cap N) < +\infty$. We need the following uniform estimate of the given Banach norm by the coefficients with regard to a basis for finite-dimensional subspaces.

LEMMA A.3.2. *Let X be a complex Banach space and $u_1, \dots, u_n \in S_X$. Set*

$$V_k := \begin{cases} \{0\}, & \text{for } k = 0, \\ \text{span}\{u_1, \dots, u_k\}, & \text{for } k = 1, \dots, n. \end{cases}$$

Assume that $\text{dist}(u_k, V_{k-1}) \geq \delta$ for $k = 1, \dots, n-1$ and $\delta > 0$. Then we have $\delta \leq 1$, $\dim V_k = k$, and

$$\frac{1}{n} \left(\frac{\delta}{1+\delta} \right)^{n-1} \sum_{k=1}^n |a_k| \leq \left\| \sum_{k=1}^n a_k u_k \right\| \leq \sum_{k=1}^n |a_k|$$

for all $a_1, \dots, a_n \in \mathbb{C}$.

PROOF. Only the left inequality needs a proof. It is a Banach space variant of Bessel's Inequality of harmonic analysis (see, e.g., [57, Section I.6.3]). Certainly, our precise version of the left inequality will be well known in functional analysis. For the convenience of the reader we give, however, an elementary proof.

Clearly we have $1 = \|u_1\| = \text{dist}(u_1, V_0) \geq \delta$. Since $\delta > 0$, we have $u_k \notin V_{k-1}$, and by induction we have $\dim V_k = k$.

Also by induction: $\|a_1 u_1\| = |a_1|$, and so

$$\begin{aligned} \|a_1 u_1 + a_2 u_2\| &\geq \max\{\delta |a_2|, |a_1| - |a_2|\} \\ &\geq \max\left\{ \frac{\delta}{1+\delta} |a_1|, \delta |a_2| \right\}, \dots, \\ \|a_1 u_1 + \dots + a_n u_n\| &\geq \max\left\{ \left(\frac{\delta}{1+\delta} \right)^{n-1} |a_1|, \left(\frac{\delta}{1+\delta} \right)^{n-k} \delta |a_k|; \right. \\ &\quad \left. k = 2, \dots, n \right\} \\ &\geq \frac{1}{n} \left(\frac{\delta}{1+\delta} \right)^{n-1} \sum_{k=1}^n |a_k|. \end{aligned}$$

Since $u_1, \dots, u_n \in S_X$, we have $\left\| \sum_{k=1}^n a_k u_k \right\| \leq \sum_{k=1}^n |a_k|$. \square

In general, the distances (M, N) and (N, M) can be very different and, even worse, behave very differently under small perturbations. However, for finite-dimensional subspaces of the same dimension in a Hilbert space we can estimate (M, N) by (N, M) in a uniform way. We can give the following generalization of [24, Lemma 14], which is different from [72, Lemma 1.7]:

LEMMA A.3.3. *Let X be a Banach space and M, N be two linear subspaces with $\dim M = \dim N = n$. Then we have*

$$\delta(M, N) \leq \frac{2^{n-1} n \delta(N, M)}{(1 - \delta(N, M))^n},$$

if $1 - \delta(N, M) > 0$.

PROOF. Take $\varepsilon \in (0, 1 - \delta(N, M))$. By induction and [57, Lemma IV.2.3], there exist $v_1, \dots, v_n \in S_N$ and $u_1, \dots, u_n \in M$ such that

$$\text{dist}(v_k, V_{k-1}) = 1 \text{ and } \|u_k - v_k\| \leq \delta(N, M) + \varepsilon$$

for

$$V_k := \begin{cases} \{0\}, & \text{for } k = 0, \\ \text{span}\{u_1, \dots, u_k\}, & \text{for } k = 1, \dots, n. \end{cases}$$

Then $1 - \delta(N, M) - \varepsilon \leq \|u_k\| \leq 1 + \delta(N, M) + \varepsilon$ and $\text{dist}(u_k, V_{k-1}) \geq 1 - \delta(N, M) - \varepsilon$. By Lemma A.3.2, $V_n = M$. For any $u \in S_M$, there exist $a_1, \dots, a_n \in \mathbb{C}$ with $u = \sum_{k=1}^n a_k u_k$. By Lemma A.3.2, we also have

$$\begin{aligned} 1 &= \left\| \sum_{k=1}^n a_k u_k \right\| \geq \frac{1}{n} \left(\frac{1 - \delta(N, M) - \varepsilon}{2} \right)^{n-1} \sum_{k=1}^n |a_k| \|u_k\| \\ &\geq \frac{(1 - \delta(N, M) - \varepsilon)^n}{2^{n-1} n} \sum_{k=1}^n |a_k|. \end{aligned}$$

Set $v := \sum_{k=1}^n a_k v_k$. Then we have:

$$\begin{aligned} \|u - v\| &= \left\| \sum_{k=1}^n a_k (u_k - v_k) \right\| \leq \sum_{k=1}^n |a_k| \delta(N, M) \\ &\leq \frac{2^{n-1} n \delta(N, M)}{(1 - \delta(N, M) - \varepsilon)^n}. \end{aligned}$$

So $\delta(M, N) \leq \frac{2^{n-1} n \delta(N, M)}{(1 - \delta(N, M) - \varepsilon)^n}$. Let $\varepsilon \rightarrow 0$, then we have $\delta(M, N) \leq \frac{2^{n-1} n \delta(N, M)}{(1 - \delta(N, M))^n}$. \square

The diligence with the preceding estimates pays back with the following Proposition A.3.5 that confines possible changes of the dimensions of intersections and the co-dimensions of sums of pairs of closed linear subspaces under variation. For that, we shall use the concepts of approximate nullity (approximate deficiency) defined by [57, §IV.4]:

DEFINITION A.3.4. Let M, N be closed linear manifolds (i.e., closed subspaces) of a Banach space Z .

a) We define the *approximate nullity* of the pair M, N , denoted by $\text{nul}'(M, N)$, as the least upper bound of the set of integers m ($m = +\infty$ being permitted) with the property that, for any $\varepsilon > 0$, there is an m -dimensional closed linear subspace $M_\varepsilon \subset M$ with $\delta(M_\varepsilon, N) < \varepsilon$.

b) We define the *approximate deficiency* of the pair M, N , denoted by $\text{def}'(M, N)$, by $\text{def}'(M, N) := \text{nul}'(M^\perp, N^\perp)$.

NOTE. While $\text{nul}(M, N) := \dim M \cap N$ and $\text{def}(M, N) := \dim Z / (M + N)$ are defined in a purely algebraic fashion, the definition of $\text{nul}'(M, N)$ and $\text{def}'(M, N)$ depends on the topology of the underlying space Z . Moreover, it is easy to show (see l.c., Theorems 4.18 and 4.19) that

$$\begin{aligned} \text{nul}'(M, N) &= \begin{cases} \text{nul}(M, N), & \text{for } M + N \text{ closed,} \\ +\infty, & \text{else,} \end{cases} \quad \text{and} \\ \text{def}'(M, N) &= \begin{cases} \text{def}(M, N), & \text{for } M + N \text{ closed,} \\ +\infty, & \text{else.} \end{cases} \end{aligned}$$

We are now ready for the first main result of this appendix:

PROPOSITION A.3.5. *Let Z be a Banach space and M, N, M', N' be closed linear subspaces. Assume that $M + N$ is closed. Then $\gamma(M, N) > 0$ by [57, Theorem IV.4.2], and we have*

- (a) $\delta(M' \cap N', M \cap N) \leq \frac{2}{\gamma(M, N)}(\delta(M', M) + \delta(N', N)),$
- (b) $\dim(M' \cap N') \leq \text{nul}'(M', N') \leq \dim(M \cap N)$
if $\delta(M', M)(1 + \gamma(M, N)) + \delta(N', N) < \gamma(M, N),$
- (c) $\dim Z/(M' + N') \leq \text{def}'(M', N') \leq \dim Z/(M + N)$
if $\delta(M, M') + \delta(N, N')(1 + \gamma(M, N)) < \gamma(M, N),$ and
- (d) $M' \cap N' \rightarrow M \cap N$ if $\dim(M' \cap N') = \dim(M \cap N) < +\infty$ and $\delta(M', M) + \delta(N', N) \rightarrow 0.$

PROOF. (a): If $M' \cap N' = \{0\}$, we have

$$\delta(M' \cap N', M \cap N) = 0 \leq \frac{2}{\gamma(M, N)}(\delta(M', M) + \delta(N', N)).$$

If $M' \cap N' \neq \{0\}$, (a) follows from [57, Lemma IV.4.4].

(b) and (c): Similar to the proof of [57, Theorem IV.4.24].

(d): By Lemma A.3.3. □

We give a first application of the preceding proposition.

ASSUMPTION A.3.6. Assume that the following data are given:

- a compact smooth Riemannian manifold (M, g) with smooth boundary $\Sigma := \partial M$,
- Hermitian vector bundles (E, h^E) and (F, h^F) over M ,
- an order $d > 0$ elliptic differential operator

$$(A.8) \quad A: C^\infty(M; E) \longrightarrow C^\infty(M; F),$$

- A^t denotes the formal adjoint of A with respect to the metrics g, h^E, h^F .
- Let $\sigma \geq 0$. Then $A_{m, \sigma}$ denotes the operator $A: H_0^{d+\sigma}(M; E) \rightarrow H^\sigma(M; E)$ (see Subsection 4.5.1 for the definition of the Sobolev spaces on the manifold with boundary), and $A_{M, \sigma}$ denotes the operator $A: H^{d+\sigma}(M; E) \rightarrow H^\sigma(M; E)$.

The following lemma is standard in elliptic operator theory.

LEMMA A.3.7. *Let A satisfy Assumption A.3.6. Then $A_{m, \sigma}$ and $A_{M, \sigma}$ are semi-Fredholm operators, $\ker A_{m, \sigma} = \ker A_{m, 0}$ consists of smooth sections, and we have $\dim(H^\sigma(M; E))/(\text{im } A_{M, \sigma}) = \dim \ker A_{m, 0}^t$.*

PROOF. By [47, Proposition A.1.4] and Gårding's inequality, $A_{m, \sigma}$ is left-Fredholm, i.e., $\dim \ker A_{m, \sigma} < +\infty$ and $\text{im } A_{m, \sigma}$ is closed in $H^\sigma(M; E)$. By the regularity, $\ker A_{m, \sigma}$ consists of smooth sections and hence $\ker A_{m, \sigma} = \ker A_{m, 0}$.

Denote by $C_+(A)$ the Calderón projection of A . Denote by γ the trace map. Set

$$D_\sigma := \{u \in H^{d+\sigma}(M; E); C_+(A)(\gamma(u)) = 0\}.$$

Denote by A_{D_σ} the operator $A: D_\sigma \rightarrow H^\sigma(M; E)$. Then A_{D_σ} is a Fredholm operator. Since $\text{im } A_{M, \sigma} \supset \text{im } A_{D_\sigma}$, the space $\text{im } A_{D_\sigma}$ is closed and we have $\dim(H^\sigma(M; E))/(\text{im } A_{D_\sigma}) < +\infty$. Then we have

$$\dim(H^\sigma(M; E))/(\text{im } A_{M, \sigma}) = \dim \ker A_{m, \sigma}^t = \dim \ker A_{m, 0}^t. \quad \square$$

DEFINITION A.3.8. Let A satisfy Assumption A.3.6. The elliptic operator A is said to have *weak inner unique continuation property (UCP)* if $\ker A_{m,0} = \{0\}$.

COROLLARY A.3.9 (Local stability of weak inner UCP). *Let X, Y be Banach spaces and $A \in \mathcal{B}(X, Y)$ a bounded operator. Assume that $\ker A = \{0\}$ and $\text{im } A$ is closed in Y . Then there exists a $\delta > 0$ such that for all $A' \in \mathcal{B}(X, Y)$ and $\|A' - A\| < \delta$, we have $\ker A = \{0\}$.*

PROOF. Set $Z := X \times Y$, $M := \text{graph}(A)$, $M' := \text{graph}(A')$ and $N = N' := X \times \{0\}$. By Proposition A.3.5.b and the proof of [24, Lemma 16], our result follows. \square

Now we refine our estimates to investigate the deformation behavior a bit further.

LEMMA A.3.10. *Let X be a Banach space and M, N be closed subspaces of X . Assume that $M \not\subseteq N$. Then for any $\varepsilon \in (0, 1)$ and $u \in M \setminus N$, there exists a $u_0 \in M \setminus N$ such that $\text{dist}(u_0, N) = \text{dist}(u, N)$ and $\text{dist}(u_0, M \cap N) = \text{dist}(u, M \cap N) \geq (1 - \varepsilon)\|u_0\|$.*

PROOF. There exists $v \in M \cap N$ such that $\text{dist}(u, M \cap N) \leq (1 - \varepsilon)\|u - v\|$. Set $u_0 := u - v$. \square

We have the following estimate. See [72, (1.4.2)] for a different estimate.

LEMMA A.3.11. *Let X be a Banach space and M, N, M', N' be closed linear subspaces. Assume that $\frac{1 - \delta(M' \cap N', M \cap N)}{1 + \delta(M' \cap N', M \cap N)} > \delta(M, M')$. Then we have*

$$(A.9) \quad \gamma(M', N') \leq \frac{(1 + \delta(N, N'))\gamma(M, N) + \delta(M, M') + \delta(N, N')}{\frac{1 - \delta(M' \cap N', M \cap N)}{1 + \delta(M' \cap N', M \cap N)} - \delta(M, M')}.$$

PROOF. 1. If $M \subset N$, we have $\gamma(M', N') \leq 1 = \gamma(M, N)$. So (A.9) holds.
2. Assume that $M \not\subseteq N$. Then for any $\varepsilon > 0$ and $u \in M \setminus N$, there exists $u' \in M'$ such that $\|u - u'\| \leq \|u\|(\delta(M, M') + \varepsilon)$. By [57, Lemma IV.2.2] we have

$$\begin{aligned} \text{dist}(u', M' \cap N') &\geq \text{dist}(u, M' \cap N') - \|u - u'\| \\ &\geq \frac{\text{dist}(u, M \cap N) - \|u\|\delta(M' \cap N', M \cap N)}{1 + \delta(M' \cap N', M \cap N)} \\ &\quad - \|u\|(\delta(M, M') + \varepsilon). \end{aligned}$$

If the right side of the inequality is larger than 0, we have $u' \in M' \setminus N'$. By [57, Lemma IV.2.2] we also have

$$\begin{aligned} \gamma(M', N') &\leq \frac{\text{dist}(u', N')}{\text{dist}(u', M' \cap N')} \\ &\leq \frac{(1 + \delta(N, N'))\text{dist}(u, N) + \|u\|\delta(N, N') + \|u\|(\delta(M, M') + \varepsilon)}{\frac{\text{dist}(u, M \cap N) - \|u\|\delta(M' \cap N', M \cap N)}{1 + \delta(M' \cap N', M \cap N)} - \|u\|(\delta(M, M') + \varepsilon)}. \end{aligned}$$

Let $\varepsilon \rightarrow 0$, and we have

$$\gamma(M', N') \leq \frac{(1 + \delta(N, N'))\text{dist}(u, N) + \|u\|(\delta(M, M') + \delta(N, N'))}{\frac{\text{dist}(u, M \cap N) - \|u\|\delta(M' \cap N', M \cap N)}{1 + \delta(M' \cap N', M \cap N)} - \|u\|\delta(M, M')}.$$

By Lemma A.3.10, for any

$$\varepsilon_1 \in (0, 1 - \delta(M' \cap N', M \cap N) - \delta(M, M')(1 + \delta(M' \cap N', M \cap N))),$$

there exists $v \in M$ such that

$$\text{dist}(v, N) \leq (\gamma(M, N) + \varepsilon_1) \text{dist}(v, M \cap N) \leq \|v\|(\gamma(M, N) + \varepsilon_1)$$

and $\text{dist}(v, M \cap N) \geq (1 - \varepsilon_1)\|v\|$. Then we have

$$\begin{aligned} & \gamma(M', N') \\ & \leq \frac{(1 + \delta(N, N'))\|v\|(\gamma(M, N) + \varepsilon_1) + \|v\|(\delta(M, M') + \delta(N, N'))}{\frac{\|v\|(1 - \varepsilon_1) - \|v\|\delta(M' \cap N', M \cap N)}{1 + \delta(M' \cap N', M \cap N)} - \|v\|\delta(M, M')} \\ & = \frac{(1 + \delta(N, N'))(\gamma(M, N) + \varepsilon_1) + \delta(M, M') + \delta(N, N')}{\frac{1 - \varepsilon_1 - \delta(M' \cap N', M \cap N)}{1 + \delta(M' \cap N', M \cap N)} - \delta(M, M')}. \end{aligned}$$

Let $\varepsilon_1 \rightarrow 0$, and we have

$$\gamma(M', N') \leq \frac{(1 + \delta(N, N'))\gamma(M, N) + \delta(M, M') + \delta(N, N')}{\frac{1 - \delta(M' \cap N', M \cap N)}{1 + \delta(M' \cap N', M \cap N)} - \delta(M, M')}. \quad \square$$

By [57, Theorem IV.4.2 and Lemma IV.4.4], we have

COROLLARY A.3.12. *Let X be a Banach space and M, N be closed linear subspaces of X . Then we have*

- (a) $\limsup_{M' \rightarrow M, N' \rightarrow N} \gamma(M', N') \leq \gamma(M, N)$ if $M + N$ is closed, and
- (b) $\lim_{M' \rightarrow M, N' \rightarrow N, M' \cap N' \rightarrow M \cap N} \gamma(M', N') = \gamma(M, N)$.

Now we are ready to investigate the deformation behavior, following some lines of [72, Lemma 1.5 (1), (2)]:

PROPOSITION A.3.13. *Let $(M'_j)_{j=1,2,\dots}$ be a sequence in $\mathcal{S}(X)$ converging to $M \in \mathcal{S}(X)$ in the gap topology, shortly $M' \rightarrow M$, let similarly $N' \rightarrow N$ and let $M + N$ be closed. Then $M' \cap N' \rightarrow M \cap N$ if and only if $M' + N' \rightarrow M + N$. Differently put, we prove in the gap topology that $\cap: \mathcal{S}(X)_{\text{cl}}^2 \rightarrow \mathcal{S}(X)$ is continuous if and only if $+: \mathcal{S}(X)_{\text{cl}}^2 \rightarrow \mathcal{S}(X)$ is continuous, where $\mathcal{S}(X)_{\text{cl}}^2$ denotes the set $\{(M, N) \in \mathcal{S}(X) \times \mathcal{S}(X); M + N \text{ closed}\}$.*

PROOF. By Theorem [57, IV.4.8], we have $\gamma(N^\perp, M^\perp) = \gamma(M, N)$. Here $N^\perp, M^\perp \subset X^*$ denote the annihilators in the dual space X^* . By [57, Theorem IV.4.8], $M^\perp + N^\perp$ is closed. So the proposition follows from the above lemma. \square

COROLLARY A.3.14. *Let $M' \rightarrow M$, $N' \rightarrow N$ and let $M + N$ be closed. Assume that $\dim(M' \cap N') = \dim(M \cap N) < +\infty$ or $\dim X/(M' + N') = \dim X/(M + N) < +\infty$. Then we have $M' \cap N' \rightarrow M \cap N$ and $M' + N' \rightarrow M + N$.*

PROOF. If $\dim(M' \cap N') = \dim(M \cap N) < +\infty$, by Proposition A.3.5.d we have $M' \cap N' \rightarrow M \cap N$. By Proposition A.3.13 we have $M' + N' \rightarrow M + N$. If $\dim X/(M' + N') = \dim X/(M + N) < +\infty$, by [57, Section IV.4.11] we have $\dim((M')^\perp \cap (N')^\perp) = \dim(M^\perp \cap N^\perp) < +\infty$. By [57, Theorem 4.2.9, Theorem 4.4.8] and the above arguments we have $(M')^\perp \cap (N')^\perp \rightarrow M^\perp \cap N^\perp$ and $(M')^\perp + (N')^\perp \rightarrow M^\perp + N^\perp$. By [57, Theorem 4.2.9, Theorem 4.4.8] we have $M' \cap N' \rightarrow M \cap N$ and $M' + N' \rightarrow M + N$. \square

COROLLARY A.3.15. *Let X be a Banach space with closed subspaces $M, M', L \in \mathcal{S}(X)$. Denote by p the natural map $p: X \rightarrow X/L$. Let $(M'_j)_{j=1,2,\dots}$ be a sequence in $\mathcal{S}(X)$ converging to $M \in \mathcal{S}(X)$ in the gap topology, shortly $M' \rightarrow M$. Assume that $M + L$ is closed, and $M' \rightarrow M$. If $M' \cap L \rightarrow M \cap L$, or $\dim(M' \cap L) = \dim(M \cap L) < +\infty$, or $\dim X/(M' + L) = \dim X/(M + L) < +\infty$, we have $p(M') \rightarrow p(M)$.*

PROOF. By Proposition A.3.13 and Corollary A.3.14, our condition implies that $M' + L \rightarrow M + L$. By Lemma A.3.1 we have $p(M') = p(M' + L) \rightarrow p(M + L) = p(M)$. \square

Combined with the preceding corollary, the following corollary generalizes [16, Theorem 3.8] and yields Proposition 4.5.2.

COROLLARY A.3.16 (Continuity of the family of the inner solution spaces and the solution spaces). *Let X, Y be Banach spaces and $A', A \in \mathcal{S}(X \times Y)$ be closed linear relations with $A' \rightarrow A$ and $\text{im } A$ closed. If $\dim \ker A' = \dim \ker A < +\infty$ or $\dim Y / \text{im } A' = \dim Y / \text{im } A < +\infty$, we have $\ker A' \rightarrow \ker A$ and $\text{im } A \rightarrow \text{im } A'$.*

PROOF. We have

$$\ker A \times \{0\} = A \cap (X \times \{0\}), \quad X \times \text{im } A = A + X \times \{0\}.$$

By Corollary A.3.14, our results follows. \square

Similar to the proof in [57, Section IV.4.5], we have (see [57, Remark IV.4.31] for discussions):

PROPOSITION A.3.17. *Let X be a Banach space and let (M, N) be a (semi-)Fredholm pair. Then there is a $\delta > 0$ such that $\hat{\delta}(M', M) + \hat{\delta}(N', N) < \delta$ implies that (M', N') is a (semi-)Fredholm pair and*

$$\text{index}(M', N') = \text{index}(M, N).$$

A.4. Smooth family of closed subspaces in Banach spaces

We begin with the definition.

DEFINITION A.4.1. Let X be a Banach space and B a C^k manifold, k is a nonnegative integer or $+\infty$ or ω . A map $f: B \rightarrow \mathcal{S}(X)$ is called C^k at $b_0 \in B$ if there exist a neighborhood U of b_0 and a C^k map $L: U \rightarrow \mathcal{B}(X)$ such that $L(b)$ is invertible and $L(b)f(b_0) = f(b)$ for each $b \in U$. f is called a C^k map if and only if f is C^k at each point $b \in B$. For the C^0 case we need B to be a topological space only.

By the definition we have

LEMMA A.4.2. *Let X be a Banach space and B a topological space. Let $f: B \rightarrow \mathcal{S}(X)$ be a map. If f is C^0 at $b_0 \in B$, f is continuous at b_0 .*

The converse is not true in general (see [72, Lemma 0.2]).

Recall from Remark 1.4.5 that $\mathcal{S}^c(X)$ denotes the set of complemented subspaces of a Banach space X . We omit the proof of the following standard facts.

LEMMA A.4.3. *Let X be a Banach space and $M \in \mathcal{S}(X)$. We have*

- (a) $M \in \mathcal{S}^c(X)$ if and only if there exists a $P \in \mathcal{B}(X)$ such that $P^2 = P$ and $\text{im } P = M$, and
- (b) $M \in \mathcal{S}^c(X)$ if either $\dim M < +\infty$ or $\dim X/M < +\infty$.

LEMMA A.4.4. *Let X be a Banach space. Let $n \geq 0$ be an integer. Set $G(n, X) := \{V \in \mathcal{S}(X); \dim V = n\}$. Then the set $G(n, X)$ is open and path connected in $\mathcal{S}(X)$.*

PROOF. By [57, Corollary IV.2.6], the set $G(n, X)$ is open. Let V_1 and V_2 be in $G(n, X)$. Since $G(n, V_1 + V_2)$ is path connected, V_1 and V_2 can be joined by a path in $G(n, V_1 + V_2)$. So our result follows. \square

LEMMA A.4.5. *Let X be a Banach space with a closed linear subspace X_1 . Set $G(X, X_1) := \{M \in \mathcal{S}(X); X = M \oplus X_1\}$ (can be an empty set). Then the set $G(X, X_1)$ is an open affine subspace of $\mathcal{S}(X)$.*

PROOF. If $G(X, X_1) = \emptyset$, our results hold. Now we assume that $H(X, X_1) \neq \emptyset$. By [57, Lemma IV.4.29], the set $G(X, X_1)$ is open. Let $X_0 \in G(X, X_1)$. Denote by $\text{graph}(A) := \{x + Ax; x \in X_0\}$ for all bounded operators $A \in \mathcal{B}(X_0, X_1)$. By the closed graph theorem, we have

$$(A.10) \quad G(X, X_1) = \{\text{graph}(A); A \in \mathcal{B}(X_0, X_1)\}.$$

Since the topology of $G(X, X_1)$ coincides with that of $\mathcal{B}(X_0, X_1)$, our results follow. \square

COROLLARY A.4.6. *The set $\mathcal{S}^c(X)$ is a Banach manifold. The local chart at $X_0 \in \mathcal{S}^c(X)$ is defined by the equation (A.10).*

COROLLARY A.4.7. *Let X be a Banach space with a closed linear subspace X_1 . Then the set $G(X, X_1)$ is dense in $\mathcal{F}_{0, X_1}(X)$, and the set $\mathcal{F}_{0, X_1}(X)$ is path connected.*

PROOF. Let $M \in \mathcal{F}_{0, X_1}(X)$. Then there exist closed subspaces M_1, X_2 and a finite-dimensional subspace V such that $M = M \cap X_1 \oplus M_1$, $X_1 = M \cap X_1 \oplus X_2$, and $X = V \oplus (M + X_1)$. By (A.7) we have

$$X = M \cap X_1 \oplus M_1 \oplus X_2 \oplus V.$$

Since $\text{index}(M, X_1) = 0$, we have $\dim M \cap X_1 = \dim V$. Let $A: M \cap X_1 \rightarrow V$ be a linear isomorphism. Set $c_1(s) := \text{graph}(sA)$ for $s \in [0, 1]$. Then the path $c_1: [0, 1] \rightarrow \mathcal{S}(M \cap X_1 \oplus V)$ satisfies that $c_1(0) = M \cap X_1$ and $M \cap X_1 \oplus V = M \cap X_1 \oplus c_1(s)$ for each $s \in (0, 1]$. Set $c(s) := c_1(s) \oplus M_1$. Then we have $c(0) = M$ and $c(s) \in G(X, X_1)$ for $s \in (0, 1]$. So the set $G(X, X_1)$ is dense in $\mathcal{F}_{0, X_1}(X)$. By Lemma A.4.5, the set $\mathcal{F}_{0, X_1}(X)$ is path connected. \square

The "if" part of the following Lemma A.4.8.b is [72, Lemma 0.2].

LEMMA A.4.8. *Let X be a Banach space and B a C^k manifold. For the C^0 case we need B to be a topological space only. Let $f: B \rightarrow \mathcal{S}(X)$ be a map. Let $b_0 \in B$ be a point. Assume that $f(b_0)$ is complemented in X . Then*

- (a) *f is C^k at b_0 if and only if there exist a neighborhood U of b_0 and a C^k map $P: U \rightarrow \mathcal{B}(X)$ such that $P(b)^2 = P(b)$ and $\text{im } P(b) = f(b)$ for each $b \in U$, and*
- (b) *f is C^0 at b_0 if and only if f is continuous at b_0 .*

PROOF. (a) Since $f(b_0)$ is complemented in X , there exists a projection $P_0 \in \mathcal{B}(X)$ such that $f(b_0) = \text{im } P_0$. If f is C^k at b_0 , there exist a neighborhood U of b_0 and a C^k map $L: U \rightarrow \mathcal{B}(X)$ such that $L(b)$ is invertible and $L(b)f(b_0) = f(b)$ for each $b \in U$. Define $P(b) := L(b)P_0L(b)^{-1}$. Then $P: U \rightarrow \mathcal{B}(X)$ is of class C^k and $\text{im } P(b) = L(b)\text{im } P_0 = f(b)$ for each $b \in U$.

Conversely, if there exists a neighborhood U of b_0 and a C^k map $P: U \rightarrow \mathcal{B}(X)$ such that $P(b)^2 = P(b)$ and $\text{im } P(b) = f(b)$ for each $b \in U$, there exists a neighborhood $U_1 \subset U$ of b_0 such that $\|P(b) - P(b_0)\| < 1$. By [57, Lemma I.4.10],

there exist a C^k map $L: U_1 \rightarrow \mathcal{B}(X)$ such that $L(b)$ is invertible, $L(b_0) = I$, and $L(b)P(b_0) = P(b)L(b)$ for each $b \in U_1$. So for $b \in U_1$, we have $L(b)f(b_0) = f(b)$.
 (b) By Lemma A.4.2 and [72, Lemma 0.2]. \square

LEMMA A.4.9. *Let X be a Banach space and B a C^k manifold. For the C^0 case we need B to be a topological space only. Let $f: B \rightarrow \mathcal{S}(X^*)$ be a C^k map. Assume that $\dim f(b) = n < +\infty$ for each $b \in B$. Then the map $b \mapsto f(b)^\perp$ is of class C^k .*

PROOF. Fix $b_0 \in B$. Let x_1^*, \dots, x_n^* be a base of $f(b_0)$. Since f is C^k , there exist a neighborhood U of b_0 and a C^k map $L: U \rightarrow \mathcal{B}(X)$ such that $L(b)$ is invertible and $L(b)f(b_0) = f(b)$. Set $x_k^*(b) := L(b)x_k^*$ for $k = 1, \dots, n$ and $b \in U$. Then $x_k^*: U \rightarrow X^*$ is a C^k map for each $k = 1, \dots, n$.

Since $\dim f(b_0) = n$, there exist $x_1, \dots, x_n \in X$ such that the matrix $M(b_0)$ is invertible, where $M(b) := ((x_j^*(b))(x_k))_{j,k=1,\dots,n}$. The map $M: U \rightarrow \text{gl}(n, \mathbb{C})$ is of class C^k . Then there exists a neighborhood $U_1 \subset U$ of b_0 such that $\det M(b) \neq 0$. Set $N(b, x) := ((x_k^*(b))(x)x_j)_{j,k=1,\dots,n}$. Define $P(b) \in \mathcal{B}(X)$ by $P(b)x = x - M(b)^{-1}N(b, x)$. Then $P: U_1 \rightarrow \mathcal{B}(X)$ is C^k , $P(b)^2 = P(b)$, and $f(b)^\perp = \text{im } P(b)$ for each $b \in U_1$. By Lemma A.4.8, the map $b \mapsto f(b)^\perp$ is of class C^k . \square

A.5. Basic facts about symplectic Banach bundles

The central concept of this Memoir is the continuous variation (i.e., the parametrization) of Fredholm pairs of Lagrangian subspaces in varying Hilbert or Banach spaces with varying symplectic forms, see the preceding Sections 2.1 (dealing with strong symplectic Hilbert bundles), 3.1 and 3.2 (dealing with symplectic Banach bundles), 4.4 (proving the abstract desuspension spectral flow formula), and 4.5 (dealing with curves of well-posed elliptic boundary value problems). The concept of *symplectic Banach bundles* is a natural generalization of the familiar concept of vector bundles. It provides a suitable frame for making the notion of *continuous variation* rigorous. We summarize the essential properties in the following list. We refer to [101] for more details regarding the concept of Banach bundles.

NOTE. Typically in this Memoir the base space of the considered Banach bundles is the interval, hence contractible. Then the total space can be written as a trivial product, i.e., the fibres can always be identified.

PROPERTIES A.5.1 (Basic properties of Banach bundles). 1. *Let B be a topological space and $p: \mathbb{X} \rightarrow B$ a Banach bundle with fibers $p^{-1}(b) = X(b)$ for each $b \in B$. For simplicity, we shall restrict ourselves to the case $B = [0, 1]$ and write shortly $\{X(s)\}_{s \in [0, 1]}$ instead of $p: \mathbb{X} \rightarrow [0, 1]$. This means that there exists an open covering $\{I(t)\}_{t \in A}$ of $[0, 1]$, with $t \in I(t)$, and A is a given subset of $[0, 1]$, such that there exists a Banach isomorphism $\varphi(t, s): X(t) \rightarrow X(s)$ for each $s \in I(t)$. It is called of class C^k if $\varphi(t_1, s)\varphi(t_2, s)^{-1}$ is of class C^k for $t \in I(t_1) \cap I(t_2)$.*

2. *A family of forms $\{\omega(s)\}_{s \in [0, 1]}$ on the fibers $X(s)$ is called continuous (C^k) iff all the families $\{\varphi(t, s)^*(\omega(s))\}_{s \in I(t)}$ are continuous (C^k) for each $t \in A$.*

3. *A family of closed subspaces $\{M(s)\}_{s \in [0, 1]}$ of $X(s)$ is called continuous (C^k) iff all the families $\{\varphi(t, s)^{-1}(M(s))\}_{s \in I(t)}$ are continuous (C^k) for each $t \in A$.*

4. *For simplicity we identify all the fibers $X(s)$ of \mathbb{X} with one fixed Banach space X . Let $\{X = X(s)^+ \oplus X(s)^-\}_{s \in [0, 1]}$ be a family of splittings. It is called continuous iff the family of projections $\{P(s)\}_{s \in [0, 1]}$ from X onto $X(s)^+$ along*

$X(s)^-$ is continuous. Then for all $s, t \in [0, 1]$ with $\|P(s) - P(t)\| < 1$, there exists an invertible operator $U(t, s)$, such that $P(s)U(t, s) = U(t, s)P(t)$. Then the family $\{X, P(s)\}_{s \in [0, 1]}$ forms a bundle. So we may fix $X(s)^\pm =: X^\pm$ and $P(s) =: P$ locally.

5. Let $\{M(s)\}_{s \in [0, 1]}$ be a family of closed subspaces of a fixed Banach space X , as in Property 4. For all s , let $M(s)$ be the graph of a suitable $U(s): X^+ \rightarrow X^-$. Then the family $\{M(s)\}$ is continuous in the gap topology iff the operator family $\{U(s)\}$ is continuous in the graph norm (by definition). If all $U(s)$ are bounded linear maps, then the family $\{U(s)\}$ is continuous in the graph norm iff the family $\{U(s)\}$ is continuous in the operator norm.

REMARK A.5.2. a. The preceding list becomes very simple in the special case of a Hilbert bundle $p: \mathbb{H} \rightarrow [0, 1]$, considered above in Section 2.1. We identify the underlying vector spaces of the fibers $p^{-1}(s) = H(s) =: H$ of Hilbert spaces for all $s \in [0, 1]$ and require that the bounded invertible operators $A_{s,0}$ defined by

$$\langle x, y \rangle_s = \langle A_{s,0}x, y \rangle_0, \text{ for all } x, y \in H$$

form a continuous family. That is the reformulation of Property 1. It explains what we mean by $\{H, \langle \cdot, \cdot \rangle_s\}_{s \in [0, 1]}$ being a continuous family of Hilbert spaces.

b. Similarly, we can reformulate Property 2 in that case: A family of symplectic forms $\{\omega(s)(x, y) = \langle J(s)x, y \rangle\}$ is continuous iff the family of injective operators $\{J(s)\}$ is continuous in the operator norm (in the case of strong symplectic forms) or in the gap topology (in the case of weak symplectic forms). Actually, that definition generalizes to symplectic forms in Banach bundles, namely requiring that the family of injective operators $\{J(s): X(s) \rightarrow X(s)^*\}$ given by $\omega(x, y) = (J(s)(x))(y)$ is continuous.

c. For strong symplectic Hilbert bundles, the continuity of the canonical splitting $\{H = \ker(J(s) - iI) \oplus \ker(J(s) + iI)\}$ is just Property 3.

d. Properties 4 and 5 explain the equality of the two natural topologies of the Fredholm Lagrangian Grassmannian in the presence of a symplectic splitting (as canonically given in strong symplectic Hilbert spaces and assumed in [24]): a curve of Lagrangian subspaces is continuous in the gap topology iff the curve of the unitary generators of the Lagrangians is continuous.

A.6. Embedding Banach spaces

Let $j: W \rightarrow X$ be a Banach space embedding. In this subsection we study the continuous family of closed subspaces in W which is also closed in X .

LEMMA A.6.1. *Let W, X be topological spaces and $j: W \rightarrow X$ a continuous injective map. Let $A \in X$ be a closed subset. Then $j^{-1}(A)$ is closed in W .*

PROOF. Since $j: W \rightarrow X$ is an injective map, we have $j^{-1}(X \setminus A) = W \setminus j^{-1}(A)$. Since $A \in X$ is closed and j is continuous, we have $X \setminus A$ is open, $W \setminus j^{-1}(A)$ is open and $j^{-1}(A)$ is closed. \square

PROPOSITION A.6.2. *Let W, X be Banach spaces. Let $j \in \mathcal{B}(W, X)$ be an injective bounded linear map. Let $M \subset W$ be such that $j(M) \in \mathcal{S}(X)$. Then the following hold.*

- (a) M is closed in W , and the linear map $(j|_M)^{-1}: j(M) \rightarrow M$ is bounded.
- (b) Denote by $C(M) = \|(j|_M)^{-1}\|$. Let $N \subset \mathcal{S}(W)$ be a closed linear subspace.

Assume that $M \neq \{0\}$ and $\delta(N, M) < (1 + \|j\|C(M))^{-1}$. Then we have $j(N) \in \mathcal{S}(X)$ and

$$(A.11) \quad C(N) \leq C(M)(1 - (1 + \|j\|C(M))\delta(N, M))^{-1}.$$

(c) Under the assumptions of (b), we have

$$(A.12) \quad \hat{\delta}(j(M), j(N)) \leq C(M)\|j\|(1 - (1 + \|j\|C(M))\delta(N, M))^{-1}\hat{\delta}(M, N).$$

PROOF. (a) By Lemma A.6.1, $M = j^{-1}j(M)$ is closed. By the closed graph theorem, the linear map $(j|_M)^{-1}: j(M) \rightarrow M$ is bounded.

(b) Since $M \neq \{0\}$, we have $C(M) > 0$. Let $x \in N$ and $\varepsilon \in (0, 1 - (1 + \|j\|C(M))\delta(N, M))$. Then there exists a $y \in M$ such that

$$(1 - \varepsilon)(\|x\|_W - \|y\|_W) \leq (1 - \varepsilon)\|x - y\|_W \leq d(x, M) \leq \delta(N, M)\|x\|_W.$$

So we have

$$(A.13) \quad \|x\|_W \leq \frac{(1 - \varepsilon)\|y\|_W}{1 - \varepsilon - \delta(N, M)}.$$

Note that

$$\begin{aligned} C(M)^{-1}\|y\|_W - \|j(x)\|_X &\leq \|j(y)\|_X - \|j(x)\|_X \\ &\leq \|j(x) - j(y)\|_X \leq \|j\|\|x - y\|_W. \end{aligned}$$

By (A.13) we have

$$\begin{aligned} \|j(x)\|_X &\geq C(M)^{-1}\|y\|_W - \|j\|\|x - y\|_W \\ &\geq \frac{C(M)^{-1}(1 - \varepsilon - \delta(N, M))\|x\|_W}{1 - \varepsilon} - \frac{\|j\|\delta(N, M)\|x\|_W}{1 - \varepsilon} \\ &= C(M)^{-1}\|x\|_W \left(1 - \frac{(1 + \|j\|C(M))\delta(N, M)}{1 - \varepsilon}\right). \end{aligned}$$

Let $\varepsilon \rightarrow 0$, and we have

$$\|j(x)\|_X \geq C(M)^{-1}\|x\|_W (1 - (1 + \|j\|C(M))\delta(N, M)).$$

Since N is closed in Banach space W , we have $j(N) \in \mathcal{S}(X)$ and the equation (A.11) holds.

(c) By the definition of the gap we have

$$(A.14) \quad \delta(j(M), j(N)) \leq C(M)\|j\|\delta(M, N),$$

$$(A.15) \quad \delta(j(N), j(M)) \leq C(N)\|j\|\delta(N, M).$$

Then we have

$$(A.16) \quad \hat{\delta}(j(M), j(N)) \leq \max\{C(M), C(N)\}\|j\|\hat{\delta}(M, N).$$

By (b), our result follows. \square

COROLLARY A.6.3. Let B be a topological space. Let $q: F \rightarrow B$, $\tilde{q}: \tilde{F} \rightarrow B$, and $p: E \rightarrow B$ be Banach bundles with fibers $q^{-1}(b) := W(b)$, $\tilde{q}^{-1}(b) := W(b)$ and $p^{-1}(b) := X(b)$ for each $b \in B$ respectively. Assume that we have Banach subbundle maps $F \rightarrow \tilde{F}$, $\tilde{F} \rightarrow E$, and there is a family $M(b) \in \mathcal{S}(X(b))$, $b \in B$ such that $M(b) \subset W(b)$ for each $b \in B$, and the family $M(b) \in \mathcal{S}(W(b))$, $b \in B$ is continuous. Then the family $M(b) \in \mathcal{S}(X(b))$, $b \in B$ is continuous.

PROOF. By Proposition A.6.2.a, we have $M(b) \in \mathcal{S}(W(b))$ and $M(b) \in \mathcal{S}(\widetilde{W}(b))$ for each $b \in B$. By Proposition A.6.2.c, the families $M(b) \in \mathcal{S}(\widetilde{W}(b))$ and $M(b) \in X(b)$, $b \in B$ are continuous. \square

The following corollary is the second main result of this appendix. It generalizes [76, Proposition B.1] and [20, Theorem 7.16].

COROLLARY A.6.4 (Continuity of the operator family). *Let B be a topological space and $p: \mathbb{X} \rightarrow B$, $p_1: \mathbb{X}_1 \subset \mathbb{X} \rightarrow B$, $q: \mathbb{Y} \rightarrow B$ three Banach bundles with fibers $p^{-1}(b) = X(b)$, $p_1^{-1}(b) = D_M(b)$ and $q^{-1}(b) = Y(b)$ for each $b \in B$ respectively. Assume that \mathbb{X}_1 is a subbundle of \mathbb{X} , and we have two continuous families $A(b) \in \mathcal{B}(D_M(b), Y(b))$ and $D(b) \in \mathcal{S}(D_M(b))$, $b \in B$ such that $A(b)|_{D(b)}: X(b) \supset D(b) \rightarrow Y(b)$ is a closed operator for each $b \in B$. Then the family of operators $\{A(b)|_{D(b)} \in \mathcal{C}(X(b), Y(b))\}_{b \in B}$ is continuous.*

PROOF. Set $W(b) := \text{graph}(A(b)|_{D_M(b)})$, $\widetilde{W}(b) := D_M(b) \times Y(b)$, $Z(b) := X(b) \times Y(b)$, $F := \bigcup_{b \in B} W(b)$, $\widetilde{F} := \bigcup_{b \in B} \widetilde{W}(b)$, and $E_2 := \bigcup_{b \in B} Z(b)$. Then we have a subbundle map $\widetilde{F} \rightarrow E_2$. Since the family $A(b) \in \mathcal{B}(D_M(b), Y(b))$, $b \in B$ is continuous and $W(b)$ is complemented in $\widetilde{W}(b)$, we have subbundles $F \rightarrow \widetilde{F}$. Then our result follows from Corollary A.6.3. \square

REMARK A.6.5. (a) By Lemma A.3.1, our condition means $P_n \rightarrow P$ in $H^\sigma(\Sigma) \rightarrow H^\sigma(\Sigma)$ (in [76] Nicolaescu uses the notation L_σ^2 for the Sobolev space $H^\sigma(\Sigma)$) for $\sigma = 1/2$. We do not require the condition for $\sigma = 0$.

(b) In [19, Theorem 3.9 (d)], it is assumed that $P_n \rightarrow P$ in $H^\sigma(\Sigma) \rightarrow H^\sigma(\Sigma)$ for $\sigma = 0$. The proof is incomplete there. For the correct proof, see [20, Theorem 7.16].

A.7. Compact perturbations of closed subspaces

Let X be a Banach space and M a closed subspace of X . In this subsection we study compact perturbations of M .

We recall the notion of relative index between projections.

DEFINITION A.7.1. Let $P, Q \in \mathcal{B}(X)$ be projections and $QP: \text{im } P \rightarrow \text{im } Q$ is Fredholm. The *relative index* $[P - Q]$ is defined by

$$(A.17) \quad [P - Q] := \text{index}(QP: \text{im } P \rightarrow \text{im } Q).$$

The relative index have the following properties.

LEMMA A.7.2. *Let X be a Banach space and $P, Q, R, P_1, Q_1 \in \mathcal{B}(X)$ projections.*

- (a) *we have $[P - Q] = \text{index}(\text{im } P, \ker Q) = [(I - Q) - (I - P)]$.*
- (b) *If $P - Q$ is compact, $QP: \text{im } P \rightarrow \text{im } Q$ is Fredholm.*
- (c) *If $P - Q$ or $Q - R$ is compact, we have $[P - Q] + [Q - R] = [P - R]$. In particular, we have $[P - Q] = -[Q - P]$ if $P - Q$ is compact.*
- (d) *If $PP_1 = P_1P = 0$, $QQ_1 = Q_1Q = 0$, we have $[(P + P_1) - (Q + Q_1)] = [P - Q] + [P_1 - Q_1]$.*
- (e) *If $T \in \mathcal{B}(X, Y)$ is invertible, we have $[TPT^{-1} - TQT^{-1}] = [Q - P]$.*

PROOF. (a) We have $\ker(QP: \operatorname{im} P \rightarrow \operatorname{im} Q) = \operatorname{im} P \cap \ker Q$. Note that $\operatorname{im} P + \ker Q = Q(\operatorname{im} P) + \ker Q = \operatorname{im}(QP) + \ker P$. Then we have

$$X/(\operatorname{im} P + \ker Q) = (\operatorname{im} Q + \ker Q)/(\operatorname{im} P + \ker Q) \simeq \operatorname{im} Q/\operatorname{im}(QP).$$

So we have

$$\begin{aligned} [P - Q] &= \operatorname{index}(\operatorname{im} P, \ker Q) = \operatorname{index}(\operatorname{im}(I - Q), \ker(I - P)) \\ &= [(I - Q) - (I - P)]. \end{aligned}$$

(b)-(e) See [105, Lemma 2.2, 2.3]. Note that (c) follows from the proof of [105, Lemma 2.3], and (d) follows from the definition. \square

DEFINITION A.7.3. Let X be a Banach space and M, N be closed subspaces of X .

(a) We define $M \sim^f N$ if $\dim M/(M \cap N), \dim N/(M \cap N) < +\infty$, and call N a *finite change* of M (see [72, p. 273]).

(b) We define $M \sim^c N$ if there exist closed subspaces $M_1 \subset M, N_1 \subset N$ and a compact operator $K \in \mathcal{B}(X)$ such that $I + K$ is invertible, $N_1 = (I + K)M_1$ and $\dim M/M_1, \dim N/N_1 < +\infty$, and call N a *compact perturbation* of M . In this case we define the *relative index* $[M - N] := \dim M/M_1 - \dim N/N_1$.

LEMMA A.7.4. Let X be a vector space and M, N, W three linear subspaces. If $N + W \subset M$, we have $\dim M/(N \cap W) \leq \dim M/N + \dim M/W$.

PROOF. We have

$$\begin{aligned} \dim M/(N \cap W) &= \dim M/N + \dim N/(N \cap W) \\ &= \dim M/N + \dim(N + W)/W \\ &\leq \dim M/N + \dim M/W. \end{aligned} \quad \square$$

LEMMA A.7.5. Let X be a Banach space and M, M_1, M_2 be closed linear subspaces. Assume that M_1, M_2 are subspaces of M with finite codimension in M , and there exists a compact operator $K \in \mathcal{B}(M_1, X)$ such that $(I_{M_1} + K)M_1 = M_2$ and $I + K \in \mathcal{B}(M_1, M_2)$ is invertible. Then there exists an invertible operator $L \in \mathcal{B}(M)$ such that $M - I_M$ is compact and $L_{M_1} = I_{M_1} + K$. In particular, we have $\dim M/M_1 = \dim M/M_2$.

PROOF. Let V_1, V_2 be finite-dimensional subspaces of M such that $M = M_1 \oplus V_1 = M_2 \oplus V_2$. Let $A \in \operatorname{gl}(V_1, V_2)$ be a linear map. Set $L := (I_{M_1} + K) \oplus A$. Then $L \in \mathcal{B}(M)$ is Fredholm and

$$\operatorname{index} L = \operatorname{index} A = \dim V_1 - \dim V_2.$$

For any bounded set B of M , the sets $\{x \in M_1; x + v \in B \text{ for some } v \in V_1\}$ and $\{v \in V_1; x + v \in B \text{ for some } x \in M_1\}$ are bounded. Since K is compact, the set $L(B) = \{Kx + (A - I_M)v; x \in M_1, v \in V_1, x + v \in B\}$ is a sequentially compact set. Thus $L - I_M$ is compact and $\operatorname{index} M = 0$. So we have $\dim V_1 - \dim V_2 = \dim M/M_1 - \dim M/M_2 = 0$. Then we can choose A such that A is invertible. In this case L is invertible. \square

Now we are ready for the third main result of this appendix:

- PROPOSITION A.7.6. (a) The relations \sim^f and \sim^c are equivalence relations.
 (b) If $M \sim^c N$ holds, the relative index $[M - N]$ is well-defined. In the case of $[M - N] = 0$, there exists a compact operator $K \in \mathcal{B}(M, X)$ such that $(I_M + K)M = N$ and $I_M + K \in \mathcal{B}(M, N)$ is invertible.
 (c) If $M \sim^c N$ and $\dim M_1, \dim N_1 < +\infty$ hold, we have $[M - N] = -[N - M]$ and $[M_1 - N_1] = \dim M_1 - \dim N_1$.
 (d) If $M \sim^c N \sim^c W$ holds, we have $[M - N] + [N - W] = [M - W]$.
 (e) If $M \cap M_1 = N \cap N_1 = \{0\}$, $\dim M_1, \dim N_1 < +\infty$, $M \sim^c N$ holds if and only if $M + M_1 \sim^c N + N_1$. In this case we have $[(M + M_1) - (N + N_1)] = [M - N] + [M_1 - N_1]$.
 (f) Assume that $M \in \mathcal{S}^c(X)$. Then $M \sim^c N$ holds if and only if $N \in \mathcal{S}^c(X)$, and there exist projections $P, Q \in \mathcal{B}(X)$ such that $P - Q$ is compact, $\text{im } P = M$ and $\text{im } Q = N$. In this case we have $[M - N] = [P - Q]$. In the case of $[P - Q] = 0$, there exists a compact operator $K \in \mathcal{B}(X)$ such that $I + K$ is invertible and $(I + K)M = N$.
 (g) If $M \sim^c N$ and $M \in \mathcal{S}^c(X)$, we have $\mathcal{F}_{k+[M-N],M}(X) = \mathcal{F}_{k,N}(X)$.

PROOF. (a) (i) If $M \sim^f N$, we have $N \sim^f M$. If $M \sim^f N \sim^f W$, we have

$$\begin{aligned} \dim(M \cap N)/(M \cap N \cap W) &= \dim(M \cap N + N \cap W)/(N \cap W) \\ &\leq \dim N/(N \cap W) < +\infty. \end{aligned}$$

Then we have $\dim M/(M \cap N \cap W), \dim N/(M \cap N \cap W) < +\infty$. Similarly, we have $\dim W/(M \cap N \cap W) < +\infty$ and $M \sim^f W$.

(ii) If $M \sim^c N$, there exist closed subspaces $M_1 \subset M$, $N_1 \subset N$ and a compact operator $K \in \mathcal{B}(X)$ such that $I + K$ is invertible, $N_1 = (I + K)M_1$ and $\dim M/M_1, \dim N/N_1 < +\infty$. Then $(I + K)^{-1} - I$ is compact and $M_1 = (I + K)^{-1}N_1$. So we have $M \sim^c N$.

(iii) If $M \sim^c N \sim^c W$, there exist closed subspaces M_1, N_1, N_2, W_2 and compact operators $K, L \in \mathcal{B}(X)$ such that $I + K, I + L$ is invertible, $N_1 = (I + K)M_1$, $W_2 = (I + L)N_2$ and

$$\dim M/M_1, \dim N/N_1, \dim N/N_2, \dim W/W_2 < +\infty.$$

By Lemma A.7.4 we have $\dim N/(N_1 \cap N_2) < +\infty$. Set

$$\begin{aligned} M_3 &:= (I + K)^{-1}(N_1 \cap N_2) \subset M_1 \subset M, \\ W_3 &:= (I + L)(N_1 \cap N_2) \subset W_2 \subset W. \end{aligned}$$

Then we have that $(I + L)(I + K) - I$ is compact, $W_3 = (I + L)(I + K)M_3$, $\dim M_1/M_3 = \dim N_1/(N_1 \cap N_2) < +\infty$, and $\dim W_1/W_3 = \dim N_2/(N_1 \cap N_2) < +\infty$. Thus we have $\dim M/M_3, \dim W/W_3 < +\infty$ and $M \sim^c W$.

(b) Assume that we are given closed subspaces $M_j \subset M$, $N_j \subset N$ and compact operators $K_j \in \mathcal{B}(X)$, $j = 1, 2$ such that $I + K_j$ is invertible, $N_j = (I + K_j)M_j$ and $\dim M/M_j, \dim N/N_j < +\infty$ for $j = 1, 2$. Since $(I + K_2)(I + K_1)^{-1} - I$ is compact and $(I + K_j)(M_1 \cap M_2) \subset N$, by Lemma A.7.5 we have

$$\dim N/((I + K_1)(M_1 \cap M_2)) = \dim N/((I + K_2)(M_1 \cap M_2)).$$

So we have

$$\begin{aligned}
\dim M/M_1 - \dim M/M_2 &= \dim M_2/(M_1 \cap M_2) - \dim M_1/(M_1 \cap M_2) \\
&= \dim((I + K_2)M_2)/((I + K_2)(M_1 \cap M_2)) \\
&\quad - \dim((I + K_1)M_1)/((I + K_1)(M_1 \cap M_2)) \\
&= \dim N/((I + K_2)M_1) - \dim N/((I + K_1)M_2) \\
&= \dim N/N_1 - \dim N/N_2,
\end{aligned}$$

and therefore $\dim M/M_1 - \dim N/N_1 = \dim M/M_2 - \dim N/N_2$. Then $[M - N]$ is well-defined when $M \sim^c N$. If $[M - N] = 0$, by the proof of Lemma A.7.5 we get the desired K .

(c) By definition.

(d) We use the notations of the proof of (a) (iii). Then we have

$$\begin{aligned}
[M - N] - [N - K] &= (\dim M/M_3 - \dim N/(N_1 \cap N_2)) \\
&\quad + (\dim N/(N_1 \cap N_2) - \dim W/W_3) \\
&= \dim M/M_3 - \dim W/W_3 = [M - W].
\end{aligned}$$

(e) Since $M \sim^c M + M_1$ and $N \sim^c N + N_1$, by (b), $M \sim^c N$ holds if and only if $M + M_1 \sim^c N + N_1$. By (d), in this case we have

$$\begin{aligned}
[(M + M_1) - (N + N_1)] &= [(M + M_1) - M] + [M - N] \\
&\quad + [N - (N + N_1)] \\
&= \dim M_1 + [M - N] - \dim N_1 \\
&= [M - N] + [M_1 - N_1].
\end{aligned}$$

(f) (i) If $M \sim^c N$, there exist closed subspaces $M_1 \subset M$, $N_1 \subset N$ and a compact operator $K \in \mathcal{B}(X)$ such that $I + K$ is invertible, $N_1 = (I + K)M_1$ and $\dim M/M_1, \dim N/N_1 < +\infty$. Then we have $\dim((I + K)M)/N_1, \dim N/N_1 < +\infty$. Since $M \in \mathcal{S}^c(X)$, $(I + K)M \in \mathcal{S}^c(X)$. By Proposition A.2.2, $N_1 \in \mathcal{S}^c(X)$ and $N \in \mathcal{S}^c(X)$.

Let V_1, V_2 be finite-dimensional subspaces such that $(I + K)M = N_1 \oplus V_1$ and $N = V_2 \oplus N_1$. Set $W_1 := (I + K) \ker P$, $W_2 := \ker Q$ and $Y := (N_1 + W_1) \cap (N_1 + W_2)$. By Lemma A.7.4 we have $\dim X/Y < +\infty$. Set $W_3 := W_1 \cap (N_1 + W_2)$ and $W_4 := W_2 \cap (N_1 + W_1)$. Then W_3 and W_4 are closed, and we have

$$Y = N_1 \oplus W_3 = N_1 \oplus W_4.$$

So $W_1/W_3 \simeq (N_1 + W_1)/Y$ and $W_2/W_4 \simeq (N_1 + W_2)/Y$ are finite-dimensional. Let $V_3 \subset W_1$, $V_4 \subset W_2$ be finite-dimensional subspaces such that

$$W_3 \oplus V_3 = W_1 \text{ and } W_4 \oplus V_4 = W_2.$$

So we have

$$\begin{aligned}
\text{(A.18)} \quad X &= ((I + K)M) \oplus W_1 = (N_1 \oplus V_1) \oplus (W_3 \oplus V_3) \\
&= N \oplus W_2 = (N_1 \oplus V_2) \oplus (W_4 \oplus V_4)
\end{aligned}$$

$$\text{(A.19)} \quad = (N_1 \oplus V_2) \oplus (W_3 \oplus V_4).$$

The projection of X on $(I + K)M$ defined by (A.18) is $\tilde{P} := (I + K)P(I + K)^{-1}$. Denote by \tilde{Q} the projection of X on N defined by (A.18). Then $\text{im}(\tilde{P} - \tilde{Q}) = \tilde{P}(V_2 \oplus V_4)$ is finite-dimensional. Since $P - \tilde{P}$ is compact, $P - \tilde{Q}$ is compact.

(ii) If there exist projections $P, Q \in \mathcal{B}(X)$ such that $\text{im } P = M$ and $\text{im } Q = N$, we set $R := QP + (I - Q)(I - P)$. Assume that R is Fredholm with index 0. Set

$$\begin{aligned} V_5 &:= \ker(QP: \text{im } P \rightarrow \text{im } Q), \\ W_6 &:= \text{im}(QP: \text{im } P \rightarrow \text{im } Q), \\ V_7 &:= \ker((I - Q)(I - P): \ker P \rightarrow \ker Q), \\ W_8 &:= \text{im}((I - Q)(I - P): \ker P \rightarrow \ker Q). \end{aligned}$$

Let W_5, W_7 be closed subspaces and V_6, V_8 finite-dimensional subspaces such that

$$\text{im } P = V_5 \oplus W_5, \text{im } Q = V_6 \oplus W_6, \ker P = V_7 \oplus W_7, \ker Q = V_8 \oplus W_8.$$

Then we have a bounded invertible linear map

$$\tilde{R} := R|_{W_5 + W_7}: W_5 + W_7 \rightarrow W_6 + W_8$$

with $\tilde{R}(W_5) = W_6, \tilde{R}(W_7) = W_8$. Since $\text{index } R = 0$, we have $\dim V_5 + \dim V_7 = \dim V_6 + \dim V_8$. Let $A \in \text{GL}(V_5 + V_7, V_6 + V_8)$ be invertible. Set $L := \tilde{R} \oplus A$. Then $L \in \mathcal{B}(X)$ is invertible, $L(W_5) = W_6, L(W_7) = W_8$. Note that $\dim M/W_5, N/W_6 < +\infty$.

In this case, we have

$$[M - N] = \dim M/W_5 - \dim N/W_6 = \dim V_5 - \dim V_6 = [P - Q].$$

If $[P - Q] = 0$, we can require $A(V_5) = V_6, A(V_7) = V_8$ and then $L(M) = N$.

Now assume that $P - Q$ is compact. Then $R - I = (Q - P)(2P - I)$ is compact and $\text{index } R = 0$. So we can apply the above argument. In this case $L - I$ is compact and our result is obtained.

(g) By Lemma A.2.6, we have $\mathcal{F}_{k+[M-N],M}(X), \mathcal{F}_{k,N}(X) \subset \mathcal{S}^c(X)$. Let $W \in \mathcal{S}^c(X)$. By (f), there exist projections $P, Q, R \in \mathcal{B}(X)$ such that $\text{im } P = M, \text{im } Q = N, \text{im } R = W$, and $P - Q$ is compact. By (f) and Lemma A.7.2.a,c we have

$$\begin{aligned} \text{index}(M, W) &= [P - (I - R)] = [P - Q] + [Q - (I - R)] \\ &= [M - N] + \text{index}(N, W), \end{aligned}$$

and one side of each equality is well-defined if and only if the other side is. Thus we have $\mathcal{F}_{k+[M-N],M}(X) = \mathcal{F}_{k,N}(X)$. \square

COROLLARY A.7.7. *Let X be a Banach space with a complemented closed linear subspace M . Let $P \in \mathcal{B}(X)$ be a projection onto M with $N := \ker P$. Set*

$$(A.20) \quad \mathcal{CP}_0(X, M) := \{(W \in \mathcal{S}^c(X); W \sim M, [W - M] = 0)\},$$

$$(A.21) \quad \mathcal{CP}_0(P) := \{W \in \mathcal{CP}_0(X, M); X = W \oplus N\}.$$

Then

- (a) *the set $\mathcal{CP}_0(P)$ is an affine space (hence contractible), and*
- (b) *the set $\mathcal{CP}_0(P)$ is dense in $\mathcal{CP}_0(X, M)$, and the set $\mathcal{CP}_0(X, M)$ is path connected.*

PROOF. (a) Let $W \in \mathcal{CP}_0(P)$. By Lemma A.4.5 we have $W = \text{graph}(A)$ for some $A \in \mathcal{B}(M, N)$. Denote by P_W the projection of X onto W along N , and we have $P_W(x + y) = x + Ax$ for $x \in M, y \in N$. By Proposition A.7.6.f, A

is compact. Conversely, for a given compact operator $A \in \mathcal{B}(M, N)$, the space $W := \text{graph}(A) \in \mathcal{CP}_0(P)$. So we have

$$(A.22) \quad \mathcal{CP}_0(P) = \{\text{graph}(A); A \in \mathcal{B}(M, N) \text{ is compact}\}$$

and the set $\mathcal{CP}_0(P)$ is an affine space (hence contractible).

Let $W \in \mathcal{CP}_0(X, M)$. By the proof of Corollary A.4.7, there exists a path $c: [0, 1] \rightarrow \mathcal{S}^c(M)$ such that $c(0) = W$, $c(s) \sim^c W$, $c(s) \in G(X, N)$ and $[W - c(s)] = 0$ for $s \in (0, 1]$. Since $W \sim^c M$ and $[W - M] = 0$, by Proposition A.7.6.a,d we have $c(s) \sim^c M$ and $[c(s) - M] = 0$. So we have $c(s) \in \mathcal{CP}_0(X, M) \cap G(X, N) = \mathcal{CP}_0(P)$ for $s \in (0, 1]$. Our results then follow from (a). \square

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List of Symbols

Except a few standard notations, all symbols are explained at their first occurrence. We recall a few very standard notations and then we provide an index to the used more peculiar symbols.

$\mathcal{B}(X, X'), \mathcal{C}(X, X'), \mathcal{F}(X, X')$	Bounded, closed, Fredholm operators
$C(\dots), C^\infty(\dots)$	Continuous resp. smooth functions
$C^\infty(M; E)$	Smooth sections of the vector bundle E over M
$C_0^\infty(M; E)$	Smooth sections with support in the interior $M \setminus \partial M$ of M
D^n	Unit ball in Euclidean n -space
$\mathrm{GL}(n, \mathcal{A})$	Invertible $n \times n$ matrices with entries in \mathcal{A}
$\mathrm{gl}(n, \mathcal{A})$	$n \times n$ matrix algebra over the algebra \mathcal{A}
\mathcal{H}	Generic name for a Hilbert space
I	Identity operator
$L^2(M; E)$	L^2 sections of the Hermitian vector bundle E
M, Σ	Generic names for Riemannian manifolds w/wo boundary
$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$	Natural, integer, real, complex numbers
\mathbb{R}_+	Nonnegative real numbers
S^n	Unit sphere in Euclidean $(n + 1)$ -space
$\mathrm{span}(x_1, \dots, x_k)$	Linear span
$\mathrm{supp}(f)$	Support of the section (or distribution) f
V^\perp	Orthogonal complement of subspace in Hilbert space
X	Generic name for a complex vector space or Banach space
X^*	Dual space
\mathbb{Z}_+	Nonnegative integers
$[\cdot]$	Integer part of a real number

Δ diagonal (i.e., the canonical Lagrangian) in product symplectic space, [41](#)

$\Delta(s; \sigma)$ diagonal in $S(s; \sigma) \times S(s; \sigma)$, [81](#)

$\Gamma(\lambda, \mu, t), \Gamma(\lambda, \mu, \lambda_0, V, t)$ crossing form, [44](#), [45](#)

$\Omega, \Omega(s)$ L^2 inner product, [77](#)

$\Omega : X \times Y \rightarrow \mathbb{C}$ generalized \mathbb{R} -linear form on pair of complex vector spaces, [12](#)

$\Omega : X \times Y \rightarrow \mathbb{C}$ non-degenerate sesquilinear map, [68](#)

$\bar{\Omega}$ induced non-degenerate reduced form for pairs of vector spaces, [13](#)

Ω^b annihilator map, [13](#)

$\Sigma, \Sigma(s)$ smooth closed boundary part, [76](#)

α symplectic action function, [62](#)

$\beta(A)$ space of abstract boundary values, [5](#), [73](#)

γ abstract trace map, [5](#), [72](#)

$\hat{\gamma}$ minimum gap (angular distance), [85](#)

κ Cayley transform, [69](#)

$\lambda^{\Omega, r}, \mu^{\Omega, l}$ right and left annihilators of subspaces in components of a pair of complex vector spaces with generalized bilinear form, [12](#)

λ^ω symplectic complement (annihilator) of subspace λ , [15](#)

λ^\perp dual complement of subspace λ , [18](#)

$\omega : X \times X \rightarrow \mathbb{C}$ symplectic form, [14](#)

$\omega : Z \times Z \rightarrow \mathbb{C}$ induced symplectic form on $Z := X \times Y$, [68](#)

- ω_{can} canonical strong symplectic form on $X \times X^*$, 19
 ω_l, ω_r decomposition symplectic structures, 28, 38, 39
 $\omega(s)_{\text{Green}}$ by Green's form induced symplectic form, 6, 72
 $\tilde{\omega}$ induced form on symplectic reduction, 23, 50
 π natural projection, 76
 $\pi_1(\mathcal{FL}(\lambda, \cdot))$ fundamental group of Fredholm Lagrangian Grassmannian, 7
 $\rho(A)$ resolvent set of closed linear relation, 67
 σ order of Sobolev space, 77
 $\sigma(A)$ spectrum of closed linear relation, 67
 $\sigma(\cdot, \cdot; \cdot, \cdot)$ Hörmander index, 57
 $\tau : Y \rightarrow X^*$ conjugate-linear injection, 68
 φ trivialization, 76
 $A(s, P(s))$ self-adjoint Fredholm extension, 79
 $A^t, A^t(s)$ formal adjoint of differential operator, 78
 $A^\sharp, A(s)^\sharp$ induced operator on cut manifold, 81
 A_m closed symmetric operator, 72
 $A_m(s; \sigma), A_M(s; \sigma)$ bounded two-parameter extensions in Sobolev chain of elliptic differential operator, 77
 A_m^ω adjoint operator (maximal extension), 72
 $\{A(s)\}$ curve of Dirac operators, 63
 \mathbb{A} (big) smooth linear differential operator, 77
 A_D extension (realization) of operator A with domain D , 5, 73
 A_M intermediate extension, 73
 $B, B_t, B_{s,t}$ tangential operators, 62
 $\mathcal{B}^\times(X)$ general linear group of bounded invertible operators on X , 37
 $\mathcal{B}^{\text{ad}}(X, Y)$ space of all continuous linear-complex-conjugate operators from X to Y , 18
 $\mathcal{B}^{\text{sa}}(\mathcal{H})$ space of bounded self-adjoint operators, 5
 $\text{CD}(A)$ abstract Cauchy data space, 5
 $D^\sharp(s)$ pasting domain for induced operator on cut partitioned manifold, 81
 $D_m(s; \sigma), D_M(s; \sigma)$ two-parameter domains in Sobolev chain, 77
 d Hausdorff metric, 84
 d order of differential operator, 77
 \hat{d} gap distance, 85
 D_M intermediate Banach space, 72, 73
 D_M/D_m reduced weak symplectic von-Neumann space, 73
 D_m minimal domain, 72
 D_{max} maximal domain, 72
 D_{max}/D_m von-Neumann space of abstract boundary values, 72
 $\text{dom } A$ domain of relation A , 67
 \mathbb{E} (big) Hermitian vector bundle over a (big) compact Hausdorff space \mathbb{M} , 76
 \mathcal{F} family of semi-norms on first factor of paired vector spaces, 14
 $\mathcal{F}(X)$ space of bounded Fredholm operators on X , 5
 $\mathcal{FL}(X, \omega)$ Fredholm Lagrangian Grassmannian, 5, 17
 $\mathcal{FL}_k(X), \mathcal{FL}(X, \mu), \mathcal{FL}_k(X, \mu), \mathcal{FL}_0^k(X, \mu)$ subspaces of the Fredholm Lagrangian Grassmannian, 17
 $G(X, N)$ set of closed subspaces of X complementary to N , 40
 $\mathbb{G}, \mathbb{G}_0, \mathbb{X}, \mathbb{Y}$ Banach bundles over $[0, 1]$, 74
 $\text{graph}(A)$ graph of operator A , 5, 67
 $H^{1/2}(\Sigma; E|_\Sigma)$ weak symplectic Sobolev space, 1, 6, 79
 $H^\sigma(M; E), H_0^\sigma(M; E)$ Sobolev spaces, 77
 $\text{Hom}(X, Y)$ space of linear maps from X to Y , 28
 $\text{im } A$ range of relation A , 67
 $\text{index}(\lambda, \mu)$ index of Fredholm pair, 17
 $\text{index } A$ index of Fredholm relation, 67
 J operator associated to symplectic form, 1, 2, 16
 $J, J_t, J_{s,t}$ skew-self-adjoint bundle isomorphisms, 62
 $j, j(s)$ embedding, 76
 $\ker A$ kernel of relation A , 67
 $\mathcal{L}(X, \omega)$ Lagrangian Grassmannian, 5, 15
 ℓ_\pm co-oriented gauge curve, 32
 $\text{Lin}(X)$ set of linear subspaces of the vector space X , 24
 $L(t, s)$ local frame, 37
 $M(s)^0$ interior of manifold with boundary, 76
 $M, M(s)$ smooth compact manifold closed and partitioned with hypersurface $\Sigma, \Sigma(s)$, 80
 $M, M(s)$ smooth compact manifold with boundary $\Sigma, \Sigma(s)$, 1, 62
 $M, M(s)$ smooth compact manifold with boundary $\Sigma, \Sigma(s)$, 76
 $M^\sharp, M^\sharp(s)$ cut partitioned manifold, 80
 $\text{Map}(S^1, M)$ loop space, 2
 \mathbb{M} (big) compact Hausdorff space, 76

- $\mathcal{M}(X, \omega, \lambda_0)$ Maslov cycle, 7
 $m^\pm(Q), m^0(Q)$ positive (negative) Morse index and nullity of quadratic form Q , 34, 46
 $\text{Mas}(\lambda(s), \mu(s))_{s \in [0,1]}$ Maslov index, 5, 38
 $\text{Mas}_\pm\{\lambda(s), \mu(s)\}$ oriented Maslov index, 32, 38
 $P_0, P_0(s), P_0(t, s)$ decomposition projections, 28, 38, 39
 $p_y(\cdot)$ family of semi-norms, 14
 $Q(A)$ quadratic form associated to a symmetric relation, 68
 $Q, Q(s), Q(\lambda, t)$ intersection form, 28, 34, 43, 44
 $Q, Q(s), Q(s; \sigma)$ Calderón projection, 78
 $R_W(\lambda)$ symplectic reduction of isotropic λ along co-isotropic W , 3, 23
 R_{W_1/W_2^ω} inner symplectic reduction, 24
 $S(s; \sigma)$ two-parameter boundary traces in Sobolev chain, 77
 $\mathcal{S}(X)$ set of closed linear subspaces in Banach space X , 51, 67, 83
 s variational parameter, 62, 76
 $\mathcal{S}^c(X)$ set of closed complemented subspaces in Banach space X , 26, 83
 $\text{sf}\{A(s)\}$ spectral flow of a curve of self-adjoint Fredholm relations, 70
 sf_ℓ spectral flow through gauge curve ℓ , 5, 32
 sign signature of quadratic form, 47
 S_M unit sphere of linear subspace in Banach space, 84
 t inward normal coordinate, 62
 T_l, T_r reduction isomorphisms, 28
 \mathcal{T}_Ω weak topology induced by pairing Ω (Ω -topology), 14
 U, V unitary generators, 32
 W/W^ω reduced symplectic space, 3, 23, 50
 $\mathbb{W}, \mathbb{W}_0, \tilde{\mathbb{W}}$ Banach subbundles for parametrization of symplectic reduction, 50
 (X, ω) symplectic vector or Banach space, 3
 X^{ad} space of all continuous complex-conjugate functionals on X , 17, 18
 \mathbb{X} Hilbert or Banach bundle, 31
 $\|x\|_{\mathfrak{g}}$ graph norm, 72
 $X^\mp(s)$ symplectic splitting, 31
 $[\cdot - \cdot]$ relative index, 56
 \sim^c, \sim^f compact, finite change, 56
 \rightarrow converging to, 4

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