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Research article

Which kind of mathematics was known and referred to by those who wanted to integrate mathematics in «Wisdom» –Neopythagoreans and others?

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Abstract: Plato, so the story goes, held mathematics in high esteem, and those philosopher-kings that ought to rule his republic should have a thorough foundation in mathematics. This may well be true – but an observation made by Aristotle suggests that the mathematics which Plato intends is not the one based on theorems and proofs which we normally identify with "Greek mathematics". Most other ancient writers who speak of mathematics as a road toward Wisdom also appear to be blissfully ignorant of the mathematics of Euclid, Archimedes, Apollonios, etc. – though not necessarily of their names. The aim of the paper is to identify the kinds of mathematics which were available as external sources for this current (on the whole leaving out of consideration Liberal-Arts mathematics as not properly external). A number of borrowings can be traced to various practitioners' traditions – but always as bits borrowed out of context.

Keywords: Ancient Greek mathematics; Neopythagoreans; Plato; Practitioners' mathematics; Recreational mathematics; Side-and-diagonal-numbers algorithm

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Sonja Brentjes gewidmet,
und
in Erinnerung an Gerlinde Wußing

Remarks about Plato

In Aristotle's *Metaphysics* N, $1090^b14–1091^a5$ there is a short polemical passage dealing with the "ideal numbers" and the supposedly Platonic "mathematical" numbers intermediate between ideal and sensible number. About the former it is said that "not even is any theorem true of them, unless we want to change mathematics and invent doctrines of our own" [trans. Barnes 1984: 1723], and about the latter that they are either "the same" as ideal number or an absurdity.

We must presume Aristotle to have known Plato's and other contemporary doctrines better than we do. This would not necessarily have prevented him from distorting such doctrines for polemical purposes; but we must also assume that his *audience* knew these doctrines, and this must have kept Aristotle from making too gross distortions if he wanted to convince.

Aristotle's formulation implies that the current mathematics about numbers which he refers to contains theorems; we should hence describe it as "theoretical arithmetic", as different from practical computation.

This might raise some doubts about that passage in *Republic* VII (525A) where Socrates/Plato distinguishes two branches of knowledge about number, logistics and arithmetic, normally taken to correspond to the two approaches to number of which he speaks in the following (525B-526C): the vulgar approach of retailers, and the noble approach which suits the guardians, that is, the approach serving war and contemplation – from which the particular approach dealing with the contemplation of merely intelligible number is singled out as particularly worthy.

Actually, there is nothing in the text which suggests that this contemplation should deal with theorems, demonstration or anything of the kind. It may be true that the *arithmetic* spoken of in 525A belongs to the same family as *Elements* VII–IX, but in that case there is nothing which identifies it specifically with the discipline about numbers accessible only to thought which should be taught to the future guardians. Alternatively, *arithmetic* might really be intended to designate this latter discipline (less likely, since the word is used before Glaucon understands the distinction), but then there is no reason to believe that Plato thinks of the theoretical discipline we know from the *Elements*.

Things become even more blurred if we look at 587D, where Socrates shows that the distance between the tyrant's imagined pleasure and real pleasure is the "plane number" $3{\cdot}3 = 9$ when regarded as "number of the length" (τοῦ μήκους ἀριθμός). He goes on to claim it to be "clear, in truth, how great a distance it is removed according to *dýnamis* and third increase" (κατὰ δύναμιν καὶ τρίτην αὔχην). Glaucon comments that it is "clear at least to the logistician" (δήλος τῷ γε λογιστικῷ). We may plausibly link this reference to a logistic art concerned with the second and third power of (seemingly pure) numbers to Diophantos's description of his own concern as "theoretical arithmetic", of which the *dýnamis* is an “element” (στοιχεῖον) [ed. Tannery 1893: I, 4] – which might imply that Plato's distinction between arithmetic and logistics was not the same as that between theory and non-theoretical practice, and that it did not coincide with the distinctions made in later time.[1] In any case it makes it even more obvious that nothing in Plato's text forces us to believe that the guardians should learn a theoretical arithmetic containing theorems.[2]

Plato himself had certainly encountered theoretical mathematics consisting of theorems. There are references to it in the dialogues, even though most of them do not prove intimate familiarity. Some, however, do prove direct knowledge at least of rather technical *results* – for instance, the references to mathematical harmonics and to the system of heavenly circles in *Timaeus* 35A–36D.[3] Moreover, Eudemos was so close in time and so close to Aristotle (that is, to somebody who was quite reserved as regards Plato's mathematics) that his narrative of mathematicians working together at Plato’s Academy must be considered reliable, at least *grosso modo*. But it might be time to revise the reading which has been current since the Renaissance, according to which this was the kind of mathematics that Plato saw as conducive to “wisdom”.

I am quite aware that I am not the first to propose such a revised reading. I shall only mention Review Netz' delicious simile of “the book according to the film” [1999: 290]:

> We all know the fate of a book which suddenly becomes a bestseller after being turned into a film – in the version “according to the film”. This process was originated in south Italy in the late fifth century bc, but it was Plato who turned “Mathematics: the Movie” into a compelling vision. This vision remained to haunt western culture …

and his summing up of the curriculum passage in *Republic* VII as “Do it, but only in a certain, limited way” (p. 303).

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[1] Summarized in [Heath 1921: I, 14–16]. Both Geminos and a scholiast to Plato's *Charmides* take logistics to deal with concrete, not abstract number. Plato's elder contemporary Archytas, taken by Heath to think along similar lines, instead states that logistics is “far ahead of other arts in relation to wisdom or philosophy”, and that it “seems to make the things of which it chooses to treat even clearer than geometry does; moreover, it often succeeds even where geometry fails”. If anything, this supports the assumption that the meaning of the word had changed between Plato and Geminos.

[2] Thus, however much some latter-day mathematicians would like the philosopher-kings to be mathematicians, they were *not* (in any sense in which the mathematicians would recognize themselves). Were they philosophers? My personal hunch, built on the strength of the description of *light* in the myth of the cave, and also in the Seventh Letter (independently of whether the latter text is really written by Plato or by a close disciple) is that their long preparation was meant to guide them to *mystical* experience and insight.

In an observation about Whitehead's dictum that European philosophy is a series of footnotes to Plato, Imre Toth once made the point [1998] (I do not have the book at hand for a precise reference) that the same holds for Plato himself: philosophy *is* footnotes, namely critique, commentary and second thoughts. Philosophy thus begins with Aristotle – Plato was a sage.

[3] Familiarity with mathematical *results* and *facts* is also abundant in Theon of Smyrna's *Expositio*. If chronology did not forbid it, Plato might probably have learned all his mathematics from Theon.

What was at hand?

I shall not go on with Plato but concentrate on less prestigious readers of the "book according to the film" – more precisely those "quasi-gnostic" writers[4] who claimed mathematics to be a road toward Wisdom. Which were the types of mathematics that were around for those who, for lack of competence or sympathy, would not read Eudoxos, Euclid, Archimedes, Apollonios, etc. – leaving aside that arithmology which the group itself and its tradition created.[5]

On a general level, an answer is offered by *Republic* VII, 525a–527c: the arithmetic of merchants, and the practical geometry used in warfare; inherent in the etymology of γεωμετρία is also the geometry of surveying, to which we might add that of city-planners and architects. But on that level of generality we find no information of relevance for our question.

We should therefore first ask what went *together* with the mathematics of merchants and accountants, and with the practical geometries. Indeed, the everyday routine of these groups was too pedestrian to be paraded as kindred to "wisdom".

Fortunately for the various quasi-gnostics, the same need for something beyond pedestrian everyday turns up in all professions which use their particular knowledge as a means to demarcate themselves. In the oral cultures of mathematical practitioners, this need was catered for by "neck riddles" – riddles which one had to be able to answer in order to show oneself an authentic member of the group, and which, in order to serve this purpose, should look as if they had something to do with the particular practice of the group. Among the various kinds of mathematical practitioners, this gave rise to a phrase which, with variations, is often found in writings situated at the interface between oral practitioners' culture and literate mathematics, "tell me, if you are a diligent calculator, ...", accompanying so-called "recreational"

4 Since some of these writers might be characterized as Neopythagoreans, others as Neoplatonists, others again as late Platonists, I introduce this ad-hoc neologism.

5 I shall also leave aside what we find in the handbooks serving or reflecting Liberal-Arts mathematics. Part of what they include derives from sources that somehow saw mathematics as a way toward *gnosis*, and many of those who belong to the quasi-gnostic tradition may have known the mathematical substance of their own tradition by way of its presence in this kind of teaching – which could imply that the very notion of "their own tradition" is problematic, this "tradition" possessing perhaps no inner continuity beyond the mere idea that "mathematics" or "number" were conducive to higher insight. It is true that the purpose of Liberal-Arts teaching was to impart culture rather than Wisdom; but the reason that the mathematical disciplines were at all accepted (at least by a minority – the actual curriculum seems not to have gone much beyond grammar and rhetoric) as a constituent of necessary culture was probably their supposed affinity with Wisdom. In consequence, Liberal-Arts mathematics cannot be distinguished sharply from quasi-gnostic mathematics as a separate and external entity, which excludes it as an independent source for quasi-gnostic mathematics *per se*; at the same time, however, its different pretensions forbids an identification of the two.

With the partial exception of Proclos, I shall also not consider those Neoplatonists who did understand their Euclid, as well as the young Platonizing Augustine, whose *De musica* [PL 32, 1184] sees "sensible number" as a step toward understanding "immutable number", but who was competent enough to have read Euclid on his own (and still could not forget it when writing *De civitate Dei*).

problems.[6] In the original context, “recreational” is thus a misnomer – this label corresponds to the role of the problems within the new, literate context.

These problems often go together in clusters, depending (so we must presume) on clusters of social groups in professional interaction. As a matter of course we have no direct evidence from the non-literate groups which were their original carriers, but the problems may turn up in written sources after having been adopted by widely scattered literate traditions – often solved by means of techniques developed by these traditions. The obvious parallel is Apuleius's taking a fable “as old women tell them”, inserting into it the names of Amor and Psyche and twisting it for his own (moral-religious) purposes.

The “Silk Road” cluster

The best known cluster – in the sense that it gathers very well-known recreational problems, not that it is normally thought of as a cluster[7] – may have been carried by the community of long-distance traders interacting along the Silk Road and/or the sea routes over the Chinese Sea and the Indian Ocean. Within this cluster we find:

– unity doubled 30 or 64 times (the “chessboard problem”);
– pursuit problems;[8]
– problems of the type “a hundred fowls”;[9]
– problems of the type “give and take”;[10]
– problems of the type “purchase of a horse”.[11]

The “purchase of a horse” appears as a pure-number problem in Diophantos's *Arithmetica* I.24 [ed. Tannery 1893: I, 56], from where I borrowed the numbers of the example. I. 22 and I.23 contain a variant where each (of three respectively four) receives only a given fraction ($\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ and, in the four-number case, $\frac{1}{6}$) of the neighbouring one. With different dress the same bizarre mathematical structure turns up with three respectively six participants in the Chinese *Nine Chapters* VIII.12–13 [ed., trans. Vogel 1968: 87; ed., trans. Chemla & GUO 2004: 641, 643] from the first century CE.[12]

[6] I have discussed this relationship in [1990a] and [1997a] (and elsewhere).

[7] Most of these problems are listed in [Tropfke/Vogel et al 1980] together with a wide range of occurrences from China to Western Europe.

[8] For instance, “one man starts 100 steps in front of another one; the first takes 60 steps while the second takes 100”. Variants with increasing or decreasing speeds are also widespread.

[9] For example, “I go to the market and buy 100 fowls for 100 dinars. A goose costs three dinars, a hen costs two, and chicks go three to a dinar”.

[10] For instance, “One man says to another, if you give me 30 dragmas of your money, I shall have twice what you have left. The other says, if you give me 50 of yours, I shall have thrice what you have left”.

[11] For instance, “three men go to the market in order to buy a horse; the first man asks for $\frac{1}{3}$ of what the others have in order to be able to pay it, the second needs $\frac{1}{4}$ of the possession of the others, and the third only needs $\frac{1}{5}$ of what the first and the second have”.

[12] My example of the pursuit was borrowed from problem VI.12 of the same Chinese treatise [ed., trans. Vogel 1968: 62; ed., trans. Chemla & GUO 2004: 519].

A strange passage in *Republic* I (333B–C) suggests that Plato knew about the problem, and supposed his readers to be in the same situation. At least he refers to the need to associate with an expert in horses when one is going to buy *in common* or sell a horse; since horses were not used for any purpose which made common possession meaningful, this is likely to be an oblique reference to the problem.[13]

A single case of even striking similarity proves nothing; but there are other similarities of the same kind.

Firstly, it is obvious that Zeno's paradox of Achilles and the tortoise has the structure of the pursuit problem. Even though Diogenes refuted it by walking (if we are to believe Diogenes Laërtios – VI.39, trans. [Jürß 1998: 267]), the original intention may well have been not to refute the common sense of everybody but that of calculators – or at least to make a point with reference to a familiar mathematical problem, where the inexperienced calculator is likely to fall into the trap needed for the paradox.

Secondly, there is a problem which the Latin and Italian Middle Ages borrowed from the second-century Roman jurisprudent Salvianus Julianus [Cantor 1875: 146–149] – as found in Jacopo da Firenze's *Tractatus algorismi* from 1307 [ed. trans. Høyrup 2007: 259] it runs:

> A man is ill and wants to make testament. And he has a wife, who is pregnant. And this one devises that if his wife makes a male child, he leaves to him $\frac{2}{3}$ of everything of his, and to the wife he leaves $\frac{1}{3}$. And if the wife makes a female child, he leaves to the girl $\frac{1}{3}$. And to the wife $\frac{2}{3}$ of all his possession. Now it happened that the good man departed from this life, and in due time the wife gave birth and made a male child, and a female child.

One may suspect the Roman jurisprudent to have taken over a mathematical recreational problem belonging to the same cluster and giving it a dress corresponding to his own field (juridically, the conditions would be impossible). Indeed, the earliest extant Chinese mathematical manuscript, the *Suàn shù shū* from no later than c. 186 BCE, contains a problem about a fox, a wild-cat and a dog going through a customs-post and sharing the tax according to the ratios between the value of their skins, which again are pairwise 1:2 [trans. Cullen 2004: 45]. A similar story about animals (now eating in the same proportions) is found in the *Nine Chapters* [ed., trans. Vogel 1968: 28; ed., trans. Chemla & Guo 2004: 285–287].[14]

Taken alone, neither Plato's reference to the collective purchase of a horse, nor Zeno's paradox or the twin inheritance is more than a suggestion. Taken together, and seen in the light of the shared structure of *Arithmetica* I.22–23 and *Nine Chapters* VIII.12–13, they make it plausible that the cluster of problems to which they belong and which reached from the Mediterranean to East Asia in the Middle Ages, was already known over most of the same area in Antiquity.[15] However, there is only one fairly certain set-off

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[13] Below I shall present more evidence for the familiarity with the problem type in Plato's times.

[14] There is no reason to conclude that all these problems *originated* in China; China just happens to be the only region outside the Mediterranean where documents of the kind from the epoch have survived.

[15] Further supportive evidence could be found in the arithmetical epigrams of the *Greek Anthology*. The doubling of 1 “until 30 times”, first found in a text from Mari in Iraq from the eighteenth century BCE, is also known from a Greco-Egyptian papyrus of CE-date, and again in the Carolingian *Propositiones ad acuendos iuvenes*, which in the main might consist of problems that had circulated in the Gallic region since Roman times – see [Høyrup 1990b: 23*f*].

in the "book according to the film", namely Iamblichos's account of "Thymaridas's bloom",[16] a technique that can be used to solve problems belonging to the family of the "purchase of a horse". If we can trust Iamblichos's ascription to Thymaridas (I have never seen any doubts raised), this shows that at least somebody in the Pythagorean environment of Plato's times was interested in number problems of a kind which was derived from the purchase-of-a-horse family, and which is also represented by Diophantos's *Arithmetic* I.22–25; since Iamblichos discusses the method in detail with examples, the interest must have remained alive in at least some Neopythagorean circles.

This, however, is the only trace I have found in wisdom-oriented writings about mathematics pointing to the "Silk Road" problem cluster. Zeno, if he really referred to the pursuit problem, rather proved that the insights gained from mathematics are deceptive. In general, the sometimes elegant, sometimes convoluted tricks used to solve problems from this category may give the same impression, which of course might make them unsuited for the purpose. (Indeed, even Plato's reference, if it is one, is to a situation where you should better not trust your own reason.)

Surveyors' riddles

Another cluster consists of geometric proto-algebraic riddles about squares and rectangles:[17]

– given, for a square, the sum of or the difference between the area and either one or all four sides, to find the side;
– or about a rectangle, given the area and either the sum of or the difference between the sides, to find the sides;
– still for a rectangle, given the diagonal and the area, to find the sides;
– to find a rectangle where the sum of length and width equals the area;
– for two squares, given the sum of or the difference between the areas together with the sum of or the difference between the sides, to find the sides;
– for a circle, to find the perimeter or diameter from the sum of perimeter, diameter and area;
– and a few more.

These riddles appear to have been invented in a Near Eastern lay surveyors' environment around the outgoing third millennium. Their first manifestation in written culture is in the "algebra" of the Old Babylonian scribe school, which expanded their scope immensely. With the collapse of the Old Babylonian culture around 1600 BCE, this sophisticated discipline disappeared, but the original riddles turn up twice in the late Babylonian period, first in the fifth century BCE. In written sources from the Seleucid period (the third and second century BCE) some further riddles are added, for instance to find the sides of a rectangle from the area and the sum of length, width and diagonal. In these texts, the solutions of rectangle problems go via the sum and the difference between the sides – until then, their half-sum and half-difference (in other words, the average side and the deviations from the average) had been used. At this stage and in this characteristic form, (some of) the riddles also turn up in Demotic papyri.[18] The geometrical section of Mahāvīra's ninth-century *Ganita-sāra-sangraha* shows that both the original and

[16] Ed. [Pistelli 1975: 62–67], cf. [Heath 1921: I, 94–96]. Heath (p. 69) describes Thymaridas as "an ancient Pythagorean, probably not later than Plato's time").

[17] See, for instance, [Høyrup 2001].

[18] Concerning the Seleucid-Demotic period, see for instance [Høyrup 2002].

the Seleucid-Demotic version of the riddles had reached India long ago [Høyrup 2004] – apparently in separate waves.

Both versions also had an impact in Mediterranean classical Antiquity. Most of Euclid's *Elements* II is in fact a “critique” of the traditional solutions (in the Kantian sense – *in which sense* and *to which extent* they are true); Diophantos' *Arithmetica* I.27–30 are pure-number versions of traditional rectangle riddles; and chapter 24 of the pseudo-Heronian *Geometrica* contains the four-sides-and-area riddle about a square [ed. Heiberg 1912: 418].[19] All of these build on the original riddles and methods; but certain problems in the Latin *Liber podismi* [ed. Bubnov 1899: 511*f*], which according to its title must be based on a Greek original (πόδισμός means “mensuration in feet”), and also those in the Papyrus graecus genevensis 259 [ed. Sesiano 1999] descend from the Seleucid-Demotic type [Høyrup 2002: 21*f*].

Geometrica 24 also contains pure-number problems (discussed in [Sesiano 1998]) which betray some kind of inspiration from these geometric riddles; probably they are witnesses of that kind of “theoretical arithmetic” which we know best from Diophantos, and therefore constitute evidence that this discipline had some of its ultimate roots in the geometric riddles, though rather in the riddles than in the methods used to solve them.

Once again, I know of *one* piece of mathematical knowledge in the “book according to the film” that points to these geometrical riddles. However, it occurs several times.

Firstly, Plutarch has the following in *Isis et Osiris*, chapter 42[20],

> The Egyptians have a legend that the end of Osiris's life came on the seventeenth of the month, on which day it is quite evident to the eye that the period of the full moon is over. Because of this the Pythagoreans call this day “the Barrier”, and utterly abominate this number. For the number seventeen, coming in between the square sixteen and the oblong rectangle eighteen, which, as it happens, are the only plane figures that have their perimeters equal to their areas, bars them off from each other and disjoins them, and breaks up the ratio of eight to eight and an eighth [8:9, the whole tone/JH] by its division into unequal intervals.

The text speaks of surface numbers, not surfaces, which might make us believe that it refers to a representation of numbers by means of ψῆφοι, calculi. If so, however, the counting of the total number of calculi and of those on the perimeter is certainly meaningful – but the statement is false, cf. Figure 1, bottom. It is only true if rectangular areas and their sides are thought of. There is thus no doubt that both Plutarch and those Pythagoreans whom he refers to thought of the upper configuration and its square counterpart.

Secondly, there are two references to the equality of square perimeter and area in the *Theologumena arithmeticae* (II.11 and IV.29)[21] – once under the dyad and once under the tetrad. Under the dyad, the fact that in smaller squares the perimeter is larger than the area and in larger squares it is smaller explains

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[19] Chapter 24 is actually an independent treatise (or rather a conglomerate of several independent problem collections), which happens to be in the same composite codex as one of the main components of what Heiberg put together as *Geometrica* but is well separated from it. See [Høyrup 1997b: 92*f*].

[20] Ed., trans. [Babbitt 1936: 101–103]. This passage must be the one to which Heath [1921: I, 96] tells to have found a reference in a letter from Sluse to Huygens without being able to locate it.

[21] Ed. [de Falco 1975: $11^{11–13}$, $29^{6–10}$], trans. [Waterfield 1988: 44, 63].

why 16 is "a sort of mean between larger and lesser"; the second, taken over from the mid–third-century bishop and computist Anatolios of Alexandria, explains that 4 "is called `justice', since the square which is based on it is equal to the perimeter"; both observations refer to themes that fit early as well as later Pythagorean currents – and, in general, fit the metaphorical use of mathematics in the service of (social) Wisdom. So does Plutarch's account of the matter. The observation might thus have been borrowed already in Plato's times or before (it does not ask for that level of mathematical competence which Thymaridas must have possessed, and which is evident in Archytas's discussion of the various means in Fragment 2[22]). But it may also have been borrowed much later.

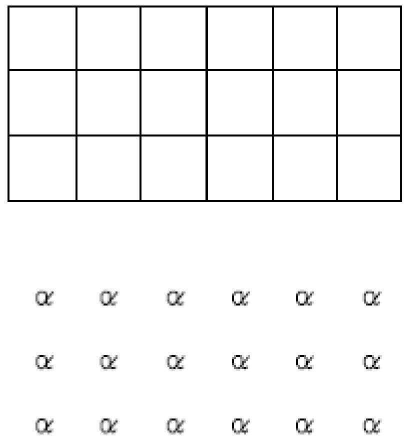

Figure 1. 6 × 3 represented as a surface and as a surface number.

Summations until 10

A third cluster, first known from Seleucid and Demotic sources, is constituted by summations of various series. In the tablet AO 6484[23] (a mixed anthology text from the early second century BCE), we find among other things two summations "from 1 to 10". Obv. 1–2 finds 1 + 2 +...+ 2^9, while obv. 3–4 determines 1 + 4 +...+ 10^2. The latter summation is solved as

$$Q_{10} = \sum_{i=1}^{10} i^2 = (1 \cdot \tfrac{1}{3} + 10 \cdot \tfrac{2}{3}) \cdot 55 \ ,$$

a special case of the formula

$$Q_\mathrm{n} = \sum_{i=1}^{n} i^2 = (1 \cdot \tfrac{1}{3} + n \cdot \tfrac{2}{3}) \cdot T_n \ , \quad \text{where} \quad T_n = \sum_{i=1}^{n} i \ .$$

The determination of the factor $(1 \cdot \tfrac{1}{3} + n \cdot \tfrac{2}{3})$ is described in detail; unless we assume gross stylistic inhomogeneity, the unexplained number 55 must therefore have been found as T_{10} in an earlier problem of the original thematic text from which the anthology borrowed the two summations.

This is mathematically impressive, but totally isolated within the cuneiform tradition. The idea of

[22] Ed. [Diels 1912: 333*f*].

[23] Ed. Neugebauer in [MKT I, 96–99].

taking precisely 10 members in both cases might therefore be a quirk of the author, or it might agree with a more general pattern.

The Demotic P. British Museum 10520[24] (probably of early Roman date) is helpful. In literal translation it says that "1 is filled up twice to 10"; as the numbers show, this refers to the sums

$$T_{10} = \sum_{i=1}^{10} i \quad \text{and} \quad P_{10} = \sum_{i=1}^{10} T_i \ .$$

The answers given correspond to the (correct) formulae

$$T_n = \frac{n^2+n}{2} \quad \text{and} \quad P_n = (\frac{n+2}{3}) \cdot (\frac{n^2+n}{2}) \ .$$

This does not overlap with the series dealt with in AO 6484, but the four summations are sufficiently close in style to be reckoned as members of a single cluster. Moreover, the formula for T_{10} is just what (as argued) must have been contained in the thematic text on which the Seleucid anthology text is based; further, the Seleucid formula for Q_n follows from the Demotic formula for P_n when combined with the observation that $i^2 = T_i + T_{i-1}$.

In the formulae for T_n, P_n and Q_n it is noteworthy that the latter two are expressed in terms of the former (represented by the number 55); also worth noticing is that T_n is *not* found as the product of mean value and number of terms, as normal in most mathematical cultures.

In modern symbolism, the formula is easily derived from the identity $n^2 = T_n+T_{n-1}$, from which follows

$$n^2 + n = T_n + T_{n\_1} + n = T_n + T_n \, ,$$

and thus

$$T_n = \frac{1}{2} \cdot (n^2 + n) \, .$$

This was evidently not the way things were expressed in Antiquity, but the structure corresponds to an observation made by Iamblichos in his commentary to Nicomachos's *Introduction*[25] – that 10 × 10 laid out as a square and counted "in horse-race" as shown in Figure 2 shows that 10 × 10 = (1 + 2 +...+ 9) + 10 + (9 +...+ 2 + 1), whence 10 × 10 + 10 = $2T_{10}$.

Exceptional as the formula is in the general historical record, it is fairly certain that the Neopythagorean observation and the Seleucid-Demotic formulae are linked. Since both the Seleucid and the Demotic text postdate Euclid, they *could* prima facie have borrowed a result obtained by early Greek arithmeticians (perhaps Pythagoreans, perhaps not). However, the same texts contain nothing else which might remind of Greek theoretical mathematics, which speaks against a borrowing of just these summation formulae – in particular because this very selective adoption should have happened both in Egypt and in Mesopotamia.

There is a further reason to doubt a Greek invention. The determination of

[24] Ed., trans. [Parker 1972].

[25] Ed. [Pistelli 1975: 75[25–27]], cf. [Heath 1921: 113*f*].

The diagram described by Iamblichos is used also by several modern historians – thus by J. Dupuis in his edition of Theon of Smyrna's *Expositio* [1892: 69 n. 14] and by Ivor Bulmer-Thomas in a commentary to an excerpt from Nicomachos [Thomas 1939: 96 n. a].

$$Q_{10} = 1^2 + 2^2 + ... + 10^2 \text{ as } (1 \cdot \tfrac{1}{3} + 10 \cdot \tfrac{2}{3}) \cdot \sum_{i=1}^{10} i$$

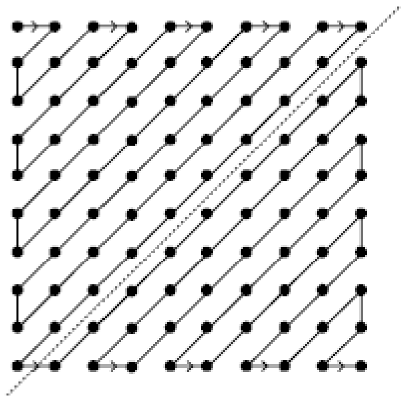

Figure 2. 10 × 10 arranged as a "race-course".

also turns up in the *Theologumena arithmeticae* (X.64, ed. [de Falco 1975: 86], trans. [Waterfield 1988: 115]), in another quotation from Anatolios (in a passage dealing with the many wonderful properties of the number 55). Anatolios, however, gives the sum in abbreviated form, as "sevenfold" T_{10}, that is, in a form from which the correct Seleucid formula cannot be derived; this in itself does not prove that earlier Greek arithmeticians did not know better; but at least it shows that the Seleucid-Demotic cluster cannot derive from the form in which the formula was known to Anatolios. In addition, the absence of the formula from any earlier Greek source derived from the theoretical or the Pythagorean tradition (including Theon of Smyrna and Nicomachos) suggests that the learned Anatolios has picked it up elsewhere.

The shape of the summation formulae in combination with the reference to the race-course arrangement points with high certainty toward a derivation or proof based on ψῆφοι. If we accept the axiom that only Greek and Greek-inspired mathematics can have been based on (even heuristic) proofs and that everything else has been "empirical", then we may still conclude that the formulae *must* be of Greek origin, in spite of contrary evidence. Without this prejudice, the evidence instead suggests that (heuristic) proofs based on pebbles were no Pythagorean or otherwise Greek invention. Instead the technique will have been part of the heritage which the Greeks adopted from the Near East. Most plausibly, the source was that practitioners' melting pot of which the various shared themes and formulae of Seleucid and Demotic mathematics bear witness. Since Epicharmos Fragment B 2[26] refers to the representation of an odd number ("or, for that matter, an even number") by a collection of ψῆφοι as something trivially familiar, the adoption must be placed no later than the early phase of Pythagoreanism – whence it may well even have been pre-Pythagorean, all reliable evidence for Pythagorean mathematics being later. However, there is no doubt that at some moment the representation of numbers by ψῆφοι (and, as Iamblichos shows, heuristic proofs) were taken over by Pythagoreans and other quasi-gnostics[27]

[26] Dated no later than c. 475 BCE. Ed. [Diels 1912: I, 118*f*].

[27] It may be an accident, but the ever-recurrent summation until precisely ten suggest that even the sacred

ψῆφος arithmetic is known to have been used for other purposes than the summation of series – the Epicharmos fragment refers to the "doctrine of odd and even", apart from which the figurate numbers (including the summations just discussed) constitute its most conspicuous application. The Seleucid-Demotic material suggest that even the Near Eastern predecessors of the Greeks had used it to argue about triangular and square numbers and the corresponding pyramid numbers P_n and Q_n; since these turn up together (and always together with the sum $\sum_{i=1}^{n} i^3 = T_n^2$) in Indian sources and in al-Karajī's *Fakhrī*,[28] it is a fair assumption that these were all dealt with before the borrowing took place; the absence of higher polygonal numbers from all these sources (of which the Indian sources, Āryabhaṭa as well as Bhāskara II and Brahmagupta, are more systematic than can be expected from the random surviving papyri and fragments of clay tablets) indicates that these represent further Greek explorations of the tool – explorations that did not spread eastward.

Westward they did spread, or rather an unintended repercussion. The higher polygonal numbers were taken over by the Roman agrimensors, who mistook these inhomogeneous expressions for area determinations of regular polygons.[29] No Near Eastern surveyor would have made this mistake, nor would Hero or even the less able compilers of the *Geometrica*. There is thus little doubt that they came from mistaken Greek theory (maybe a reminiscence of the teaching of Liberal-Arts arithmetic, where these numbers played a conspicuous role). On the other hand, the side of the regular polygons in the treatise mentioned in note 29 is invariably 10, which seems to be a heritage from the Near Eastern tradition – see [Høyrup 1997b: 91].

The higher figurate numbers play a role in the handbooks for Liberal-Arts arithmetic, but I have not noticed them in quasi-gnostic contexts. Here, only the Near Eastern heritage (perhaps in watered-down form, witness Anatolios) turns up.

Pythagorean ten could have been a borrowing; whether even its sacredness was a borrowing is a matter of guessing (my own guess being that it was not). If an accident, the coincidence must have pleased the Pythagoreans.

28 See [Clark 1930: 37] (Āryabhaṭa), [Colebrooke 1817: 290–294] (Brahmagupta), [Colebrooke 1817: 51–57] (Bhāskara II), and [Woepcke 1853: 61] (*Fakhrī*).

29 For instance Epaphroditus & Vitruvius Rufus, ed. [Bubnov 1899: 534–545]. But the nonsense survived into the late medieval abbacus tradition.

Fields to be measured would hardly ever by regular pentagons, hexagons etc. – and if they were, standard measurement would only reveal them to be equilateral, not equiangular, for which reason their area would anyhow have to be found by subdivision. We may therefore safely assume that the wrong formulae were never used in practice; though no riddles, these formulae, giving an impression of completeness, were supra-utilitarian adornments.

Side-and-diagonal numbers

The last topic I shall take up in some detail is a likely borrowing from architectural geometry, at some moment transferred into a number algorithm. I refer to the "side-and-diagonal-number algorithm", an algorithm for producing increasingly precise approximations to the ratio between the diagonal and the side of a square (in anachronistic terms, to √2). The basis for this algorithm is what I shall call the side-and-diagonal *rule:* if *s* and *d* are the side and diagonal of a square, then the same holds for $\Sigma = s + d$ and $\Delta = 2s + d$. Experience combined with common sense shows that iteration of the process from values s_1 and d_1 which do not fulfill the condition $d_1^2 = 2s_1^2$ leads to convergence of the ratio $d_n^2:s_n^2$ toward 2:1. In particular, if we start from $s_1 = d_1 = 1$, we get the successive pairs 1:1, 3:2, 7:5, 17:12, 41:29, 99:70, 239:169, ...; this is the side-and-diagonal *algorithm*.[30]

The algorithm is not described by any of the "great" or "genuine" mathematicians, but it was known by both Theon of Smyrna (*Expositio* I.XXXI, ed. [Dupuis 1892: 70–74] and Proclos;[31] a final reference is found in Iamblichos's commentary to Nicomachos [ed. Pistelli 1975: 91]. We may assume it to have circulated in quasi-gnostic circles, which was part of the shared background of these three authors (and Iamblichos's principal background).

In his edition of Proclos' commentary to the *Republic*, Kroll supposed that the rule was proved by means of *Elements* II.10,[32] which he further took to be of Pythagorean origin (actually, it is the justification of one of the old two-square riddles, which antedates Pythagoras by an abundant millennium). Another (quasi-algebraic) proof can be based on the diagram which Leonardo Fibonacci employs in the *Pratica geometrie* [ed. Boncompagni 1862: 62] when solving the problem $\Delta - \Sigma = 6$ (see Figure 3); by simple counting, the same diagram can also be used to prove *Elements* II.10. The proof is of a type that is familiar from the Seleucid rectangle riddles, and strong arguments can be given that the similarity is based on an actual historical link.

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30 Asymptotically, each added step reduces the error of the ratio *d*:*s* by a factor $\frac{1}{1+\sqrt{2}}$.

31 Proclos describes it in a commentary to a passage in *Republic* 546C ([ed. Kroll 1899: II, 24*f*]; cf. discussion in [Vitrac 1990: 351*f*]); there is also an oblique but unmistakeable reference in his commentary to *Elements* I ([ed. Friedlein 1873: 427^{21-23}], trans. [Morrow 1972: 339]), where it is spoken of as σύνεγγυζ, "proximate".

It has been assumed that Plato's reference to "a hundred numbers determined by the rational diameters of the pempad lacking one in each case" in *Republic* 546C, trans. [Shorey 1930: II, 247]) shows him to be familiar with the same algorithm. Actually, all it shows for certain is that he was familiar with the use of 7 as an (approximate) value for the diagonal in a square with side 5.

Heath [1926: I, 399] supposes that the "lacking one" refers to the fact that 7^2 is lacking 1 compared to the square on the true (irrational) diameter in the square with side 5, which is an essential feature of the sequence of approximations produced by the algorithm. Actually, as pointed out to me by Marinus Taisbak (personal communication), Plato's point is rather that the number 48 (the number which is required) is lacking one with regard to the "number on the rational diameter 7" (and 2 with regard to that on the irrational diameter *dynámei*, as Plato goes on). This is indeed also Proclos's explanation, cf. Hultsch in [Kroll 1899: II, 407].

32 Using the letters of Figure 4: If *CB* is bisected by *G*, and prolonged by *BE*, then $\square(CE)+\square(BE) = 2\cdot(\square(CG)+\square(GE))$.

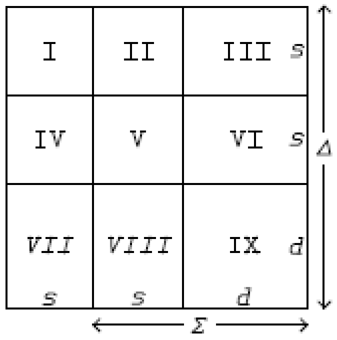

Figure 3. Fibonacci's implicit proof of the side-and-diagonal algorithm.

However, the rule can also be observed rather directly in the construction of a rectangular octagon by superposition of two identical squares (see Figure 4): if *CG* = *GB* = *DG* = *s*, then *CD* = *AC* = *AF* = *d*. Therefore Σ = *DJ* = *DG* + *GJ* = *s* + *d*, while the corresponding diagonal is Δ = *FD* = *FC* + *CD*. But *FC* = *CD* = 2*s*, whence Δ = 2*s* + *d*.

In the pseudo-Heronian *De mensuris* 52 [ed. Heiberg 1914: 206], a reduced version of Figure 4 is used for the octagon construction: the oblique square is omitted, but it is used that *AB* = *EC* = *AO* (etc.). This follows from exactly the same arguments as lead to the side-and-diagonal-rule. It is difficult to believe this construction to have been invented directly, without the passage over the superimposed squares.

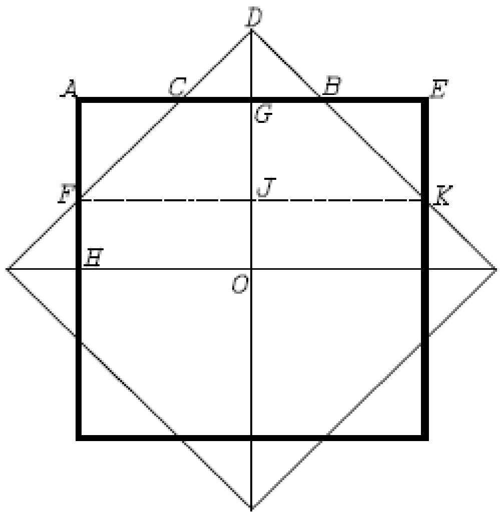

Figure 4. A regular octagon produced by superimposed squares.

The reduced construction turns again up in Abū'l-Wafā''s *Book on What is Necessary from Geometric Construction for the Artisan* VII.xxii [ed., Russian trans. Krasnova 1966: 93]; in the *Geometria incerti auctoris* no. 55 [ed. Bubnov 1899: 360*f*]; and in Mathes Roriczer's late fifteenth-century *Geometria deutsch* [ed. Shelby 1977: 119*f*]. Roriczer's *Wimpergbüchlein* [ed. Shelby 1977: 108*f*] makes use of the superimposed squares and shows (though this is not the topic) that Roriczer knew some of their relevant properties. The superimposed squares producing the regular octagon are found as an illustration to the determination of its area (via the octagonal number!) in Epaphroditus & Vitruvius Rufus [ed. Cantor 1875: 212, Fig. 40[33]]. As I have been told by Hermann Kienast (personal communication) they can also be seen to have been used in the ground plan of the Athenian "Tower of the Winds" from the first century BCE.[34] All in all there is thus no doubt that both constructions were known by practical geometers in the classical age; the places where we find references to the algorithm or material traces of the construction (all late and far removed from the theoretical tradition) make it unlikely that the idea *originated* among Greek theoretical mathematicians – including the Pythagorean *mathematikoi*.

The rule and even its transformation into an algorithm could be of much earlier date. Two Old Babylonian tablets (YBC 7289 and YBC 7243, in [MCT, 43, 136]) give the value 1;24,51,10 for the ratio between the square diagonal and side. In their commentary, Neugebauer and Sachs noticed that this value is the sixth step in an alternating iteration by arithmetic and harmonic means,[35] the fourth step of which is the value 1;25, which also turns up in cuneiform sources (as we now know, of Old Babylonian as well as Seleucid date). As shown by David Fowler and Eleanor Robson [1998], however, the calculations require repeated divisions by sexagesimally irregular numbers; approximation by regular divisors would lead to roundings which would either yield a result which was less or which was even more precise. This explanation can therefore be discarded; so can the iteration suggested in note 35, which suffers from the same defect.

This seems to leave us with the side-and-diagonal algorithm. Indeed, 1;24,51,10 is the rounded value of 239÷169; it can be found by a single division by an irregular number, which we know Mesopotamian scribes to have been trained to perform already before the mid-third millennium BCE. The approximation 1;25, also found in Babylonian sources as we remember, is nothing but 17÷12. This value *can* be found by the iteration of note 35 from above, starting from the value $1\,{}^{1}/_{2}$; but the plausible use of the side-and-diagonal for the better approximation speaks in favour of its use even here.[36] *If* this is so, a possible link between the Old Babylonian (plausible) use of the rule and its certain presence in the classical world is at hand. Indeed, the peculiar Old Babylonian way to express the perimeter of the circle in terms of the diameter (not multiplying by 3 but taking the triple) pops up again in Greek practical geometry, and finds its explanation in a construction described by Roriczer and in an Icelandic manuscript from the early fourteenth century – a construction which allows to measure the perimeter without calculation; it is likely to have been carried by the profession of master builders.[37] However, the reader

33 The text is also in [Bubnov 1899: 539], but the diagram is omitted.

34 Vitruvius's description of how the ground plan was made (*De architectura* I.vi.4) is thus an *a posteriori* reconstruction – "rational", but wrong.

35 More likely than this alternation would be the equivalent iteration of the approximation $\sqrt{n^2+d} \approx n + \frac{d}{2n}$, which can be argued geometrically. This eliminates half of the steps from the Neugebauer-Sachs procedure, but leaves the relevant ones.

36 One Old Babylonian text uses this iteration from below, but none from above (which in general is much less common in the sources until the outgoing Middle Ages).

37 See [Høyrup 2009: 368–370].

counting the occurrences of words like “plausible”, “seems” and “if” in the course of this argument will realize that it is far from compulsory – and definitely insufficient to decide with any certainty between a borrowing and independent (re)discovery, either of the construction or of the number algorithm.

We have no indication as to when the algorithm was adopted by the quasi-gnostic environment; it may have been in the age of Thymaridas and the Pythagorean *mathematikói*, or much closer to Theon's late first century CE. But we cannot avoid noticing that all sources we possess for the algorithm link mathematics with Wisdom, while the evidence we have for the diagram behind it is an architectural real-life construction; if mathematicians with no esoteric affinity had once worked on the topic, they seem to have lost all interest in epochs from which sources survive.[38] None of our explicit sources – that is, neither Theon nor Proclos – show convincingly to know the “principles and causes” behind the algorithm.

A final note about fractions and ratios, and a conclusion

I promised in note 5 to leave aside Liberal-Arts arithmetic together with its impact in the mathematics of Wisdom. I shall permit myself a slight breach of this promise, a mere reference to a publication which in my opinion by far has not received the attention it deserves: Kurt Vogel's habilitation thesis from [1936].

One of the points made by Vogel (p. 449) is that the Greek vocabulary for ratios is shaped after that for fractions. For reasons I shall not discuss here, the Euclidean (but not the Diophantine) brand of theoretical arithmetic as well as the arithmetic of Liberal-Arts handbooks avoided fractions.[39] Instead, as we know, theoretical Greek mathematics had recourse to ratios, and a large part of Liberal-Arts arithmetic is dedicated to the classification and naming of ratios – an interest which is also visible in some of the quasi-gnostic writings (first of all of course in Nicomachos's *Introduction*, if we count that, next in Iamblichos's commentary to this work). The whole apparatus built up around this classification was quite adequate for those who felt attracted to the easy “royal road” to mathematics – in particular when it was taught exclusively through numerical examples and without even paradigmatic proofs built on single cases. Ultimately, this is another case of mathematics coming from base practice and taken over as “wisdom”. With the difference, however, that only the transposition to ratios called for the creation of the classification system – for fractions most of it would have been obviously superfluous.

Apart from this, however, that “royal road” to mathematical Wisdom whose existence Euclid denied

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[38] There is one just possible impact on the theoretical tradition: the proof of *Elements* II.10, the diagram of which is nothing but the section of Figure 4 designated by the letters *KEBGCDD* (but in the general case, without the specific ratio between *GB* and *BE*). The proposition states that $\square CE + \square BE = 2(\square EG + \square BG)$, which is obviously fulfilled when $\square BE = 2\square BG$, $\square CE = 2EG$, as happens in the case of the superimposed squares. Whereas the proofs of *Elements* II.1–8 all correspond to the techniques by which the rectangle riddles were solved already in the Old Babylonian epoch, those of II.10 and the closely related II.9 are of a wholly different kind.

Isolated as that similarity is, the preceding observation can be nothing but a suggestion. Euclid and his predecessors were certainly able to devise their own diagrams as they needed.

[39] The avoidance may have to do, both with the fateful answer “a collection of units” once given to the question “what is a number”, and (in Plato's case, according to the curriculum passage of *Republic* VII) with the use of fractions by petty traders. A supplementary stimulus for interest in ratios (but *not* for avoiding fractions in general) is the creation of mathematical harmonics.

(as Proclos's story goes) was in part paved with material borrowed from those who designed common roads or moved their merchandise along them – but borrowed piecemeal, mostly as bits without coherence and out of context. The internal coherence of quasi-gnostic mathematics, to the extent it can be seen to have possessed one, was probably provided by arithmology and by the interests it shared with Liberal-Arts mathematics.

References

1. Babbitt, F. C. (ed., trans.), 1936. Plutarch's *Moralia*, vol. IV. London: Heinemann / New York: Putnam.
2. Barnes, J. (ed.), 1984. The Complete *Works* of Aristotle. The Revised Oxford Translation. 2 vols. Princeton: Princeton University Press.
3. Boncompagni, B. (ed.), 1862. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. II. *Practica geometriae* et *Opusculi*. Roma: Tipografia delle Scienze Matematiche e Fisiche.
4. Bubnov, N. (ed.), 1899. Gerberti postea Silvestri II papae *Opera mathematica* (972 – 1003). Berlin: Friedländer.
5. Cantor, M., 1875. *Die römischen Agrimensoren und ihre Stellung in der Geschichte der Feldmess-kunst. Eine historisch-mathematische Untersuchung*. Leipzig: Teubner.
6. Chemla, K., & GUO S. (ed., trans.), 2004 *Les neuf chapitres. Le Classique mathématique de la Chine ancienne et ses commentaires*. Paris: Dunod.
7. Clark, W. E. (ed., trans.), 1930. The *Āryabhaṭīya* of Āryabhaṭa. Chicago: University of Chicago Press.
8. Colebrooke, H. T. (ed., trans.), 1817. *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhascara*. London: John Murray.
9. Cullen, C. (ed., trans.), 2004 The *Suàn shù shū*, "Writings on Reckoning": A Translation of a Chinese Mathematical Collection of the Second Century BC, with Explanatory Commentary. (Needham Research Institute Working Papers, 1). Cambridge: Needham Research Institute. Web edition available from: http://www.nri.org.uk/suanshushu.html.
10. de Falco, V. (ed.), 1975. [Iamblichi] *Theologumena arithmeticae*. Stuttgart: ^{2}Teubner. ^{1}1922.
11. Diels, H., 1912. *Die Fragmente der Vorsokratiker, Griechisch und Deutsch*. Dritte Auflage, erster Band. Berlin: Weidmann.
12. Dupuis, J. (ed., trans.), 1892. Théon de Smyrne, philosophe platonicien, *Exposition des connaissances mathématiques utiles pour la lecture de Platon*. Traduite pour la première fois du grec en français. Paris: Hachette.
13. Fowler, D. H., & E. Robson, 1998. Square Root Approximations in Old Babylonian Mathematics: YBC 7289 in Context. *Historia Mathematica* **25**, 366–378.
14. Friedlein, G. (ed.), 1873. Procli Diadochi *In primum Euclidis Elementorum librum commentarii*. Leipzig: Teubner.
15. Heath, T. L., 1921. *A History of Greek Mathematics*. 2 vols. Oxford: The Clarendon Press.
16. Heath, T. L. (ed., trans.), 1926. *The Thirteen Books of Euclid's Elements*. 2nd revised edition. 3 vols. Cambridge: Cambridge University Press / New York: Macmillan.
17. Heiberg, J. L. (ed., trans.), 1912. Heronis *Definitiones* cum variis collectionibus. Heronis quae feruntur *Geometrica*. Leipzig: Teubner.
18. Heiberg, J. L. (ed., trans.), 1914. Heronis quae feruntur *Stereometrica* et *De mensuris*. Leipzig:

Teubner.
19. Høyrup, J., 1990a. Sub-Scientific Mathematics. Observations on a Pre-Modern Phenomenon. *History of Science* **28**, 63–86.
20. Høyrup, J., 1990b. Sub-scientific Mathematics: Undercurrents and Missing Links in the Mathematical Technology of the Hellenistic and Roman World. *Filosofi og videnskabsteori på Roskilde Universitetscenter*. 3. Række: *Preprints og Reprints* 1990 nr. 3. To appear in *Aufstieg und Niedergang der römischen Welt*, II vol. 37,3 (if, by miracle, that volume is ever going to appear). Manuscript available from http://ruc.dk/~jensh/Publications/1990%7bg%7d_Undercurrents.PDF.
21. Høyrup, J., 1997a. Mathematics, Practical and Recreational, pp. 660–663 *in* Helaine Selin (ed.), *Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures*. Dordrecht etc.: Kluwers.
22. Høyrup, J., 1997b. Hero, Ps.-Hero, and Near Eastern Practical Geometry. An Investigation of *Metrica*, *Geometrica*, and other Treatises, pp. 67–93 *in* Klaus Döring, Bernhard Herzhoff & Georg Wöhrle (eds), *Antike Naturwissenschaft und ihre Rezeption*, Band 7. Trier: Wissenschaftlicher Verlag Trier.
23. Høyrup, J., 2001. On a Collection of Geometrical Riddles and Their Role in the Shaping of Four to Six 'Algebras'. *Science in Context* **14**, 85–131.
24. Høyrup, J., 2002. Seleucid Innovations in the Babylonian 'Algebraic' Tradition and Their Kin Abroad, pp. 9–29 *in* Yvonne Dold-Samplonius et al (eds), *From China to Paris: 2000 Years Transmission of Mathematical Ideas*. Stuttgart: Steiner.
25. Høyrup, J., 2004. Mahāvīra's Geometrical Problems: Traces of Unknown Links between Jaina and Mediterranean Mathematics in the Classical Ages, pp. 83–95 *in* Ivor Grattan-Guinness & B. S. Yadav (eds), *History of the Mathematical Sciences*. New Delhi: Hindustan Book Agency.
26. Høyrup, J., 2007. *Jacopo da Firenze's* Tractatus Algorismi *and Early Italian Abbacus Culture*. Basel etc.: Birkhäuser.
27. Høyrup, J., 2009. The Rare Traces of Constructional Procedures in 'Practical Geometries', pp. 367–377 *in* Horst Nowacki & Wolfgang Lefèvre (eds), *Creating Shapes in Civil and Naval Architecture*. Leiden & Boston: Brill.
28. Jürß, F. (ed., trans.), 2001. Sextus Empiricus, *Gegen die Wissenschaftler*, Buch 1–6. Würzburg: Königshausen & Neumann.
29. Krasnova, S. A. (ed., trans.), 1966. Abu-l-Vafa al-Buzdžani, *Kniga o tom, čto neobxodimo remeslenniku iz geometričeskix postroenij*, pp. 42–140 *in* A. T. Grigor'jan & A. P. Juškevič, *Fiziko-matematičeskie nauki v stranax vostoka.* Sbornik statej i publikacij. Vypusk I (IV). Moskva: Izdatel'stvo »Nauka«.
30. Kroll, W. (ed.), 1899. Procli Diadochi *In Platonis Rem publicam commentarii*. 2 vols. Leipzig: Teubner, 1899, 1901.
31. MCT: O. Neugebauer & A. Sachs, *Mathematical Cuneiform Texts*. New Haven, Connecticut: American Oriental Society, 1945.
32. MKT: O. Neugebauer, *Mathematische Keilschrift-Texte*. I-III. Berlin: Julius Springer, 1935–1937.
33. Morrow, G. R. (ed., trans.), 1970. Proclus, *A Commentary on the First Book of Euclid's Elements*. Princeton, New Jersey: Princeton University Press.
34. Netz, R., 1999. *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*. Cambridge: Cambridge University Press.
35. Parker, R. A., 1972. *Demotic Mathematical Papyri*. Providence & London: Brown University Press.

36. Pistelli, H. (ed.), 1975 Iamblichos, *In Nicomachi Introductionem Arithmeticam*. [2]Stuttgart: Teubner. [1]1894.
37. PL : J. P. Migne (ed.), *Patrologiae cursus completus, series latina*. 221 vols. Paris, 1844–1864.
38. Sesiano, J., 1998. An Early Form of Greek Algebra. *Centaurus* **40**, 276–302.
39. Sesiano, J., 1999. Sur le Papyrus graecus genevensis 259. *Museum Helveticum* **56**, 26–32.
40. Shelby, L. R. (ed.), 1977. *Gothic Design Techniques. The Fifteenth-Century Design Booklets of Mathes Roriczer and Hanns Schmuttermayer*. Carbondale & Edwardsville: Southern Illinois University Press.
41. Shorey, P. (ed., trans.), 1930. Plato, *The Republic*. 2 vols. London: Heinemann / New York: Putnam, 1930, 1935.
42. Tannery, P. (ed., trans.), 1893. Diophanti Alexandrini *Opera omnia* cum graecis commentariis. 2 vols. Leipzig: Teubner, 1893–1895.
43. Thomas, I. (ed., trans.), 1939. *Selections Illustrating the History of Greek Mathematics*. 2 vols. London: Heinemann / New York: Putnam, 1939, 1941.
44. Toth, I., 1998. *Aristotele e i fondamenti assiomatici della geometria. Prolegomeni alla comprensione dei frammenti non-euclidei nel «Corpus Aristotelicum» nel loro contesto matematico e filosofico*. Milano: Vita e Pensiero.
45. Tropfke, J./Vogel, K., et al, 1980. *Geschichte der Elementarmathematik.* 4. Auflage. Band 1: *Arithmetik und Algebra*. Vollständig neu bearbeitet von Kurt Vogel, Karin Reich, Helmuth Gericke. Berlin & New York: W. de Gruyter.
46. Vitrac, B. (ed., trans.), 1990. Euclide d'Alexandrie, *Les Éléments*. Traduits du texte de Heiberg. 4 vols. Paris: Presses Universitaires de France, 1990-2001.
47. Vogel, K., 1936. Beiträge zur griechischen Logistik. Erster Theil. *Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayerischen Akademie der Wissenschaften zu München* 1936, 357–472.
48. Vogel, K. (ed., trans.), 1968. *Chiu chang suan shu. Neun Bücher arithmetischer Technik. Ein chinesisches Rechenbuch für den praktischen Gebrauch aus der frühen Hanzeit (202 v. Chr. bis 9 n. Chr.)*. Braunschweig: Friedrich Vieweg & Sohn.
49. Waterfield, R. (trans.), 1988. *The Theology of Arithmetic. On the Mystical, mathematical and Cosmological Symbolism of the First Ten Number* Attributed to Iamblichus. Grand Rapids, Michigan: Phanes.
50. Woepcke, F., 1853. *Extrait du Fakhrî, traité d'algèbre* par Aboû Bekr Mohammed ben Alhaçan Alkarkhî; *précédé d'un mémoire sur l'algèbre indéterminé chez les Arabes.* Paris: L'Imprimerie Impériale.