KANT'S SCHEMATISM
AND
THE FOUNDATIONS OF MATHEMATICS

KLAUS FROVIN JØRGENSEN

SECTION FOR PHILOSOPHY AND SCIENCE STUDIES
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by

Klaus Frovin Jørgensen, M.Sc.

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Abstract

The theory of schematism was initiated by I. Kant, who, however, was never precise with respect to what he understood under this theory. I give—based on the theoretical works of Kant—an interpretation of the most important aspects of Kant's theory of schematism. In doing this I show how schematism can form a point of departure for a reinterpretation of Kant's theory of knowledge. This can be done by letting the concept of schema be the central concept. I show how strange passages in, say, the first *Critique* are in fact understandable, when one takes schematism serious. Likewise, I show how we—on the background of schematism—get a characterization of Kant's concept of 'object'. This takes me to an analysis of the ontology and epistemology of mathematics. Kant understood himself as a philosopher in contact with science. It was science which he wanted to provide a foundation for. I show that, contrary to Kant's own intentions, he was not up-to-date on mathematics. And in fact, it was because of this that it was possible for him to formulate his rather rigid theory concerning the *unique* characterizations of intuition and understanding. I show how phenomena in the mathematics of the time of Kant should have had an effect on him. He should have remained more critical towards his formulation and demarcation of intuition, understanding and reason.

Finally I show how D. Hilbert in fact gives the necessary generalization of Kant's philosophy. This generalization provides us with a general frame work, which functions as a foundation for an understanding of the epistemology and ontology of mathematics.

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Resumé

Denne afhandling handler om skematisme, som er en teori, der påbegyndes af I. Kant. Kant var aldrig helt præcis med hensyn til, hvad han forstod under denne teori. Baseret på Kants teoretiske værker giver jeg en fortolkning af de vigtigste dele af Kants skematisme, og jeg viser, hvorledes skematisme giver anledning til en nyfortolkning af Kants erkendelsesteori ved at lade begrebet være det centrale begreb. Således viser jeg, hvorledes mærkværdige passager i f.eks. den første *Kritik* kan gøres forståelige, når Kants teori om skemaerne tages alvorligt. Jeg viser ligeledes, hvordan vi på baggrund af skemaerne får karakteriseret Kants begreb om 'objekt', hvilket leder frem til en analyse af matematikkens ontologi og erkendelsesteori. Kant så sig selv som en filosof i kontakt med videnskaberne. Det var videnskaberne han søgte at fundere med sin erkendelsesteori. Jeg viser, hvorledes han desværre var tidsligt et stykke bagefter med hensyn til matematikken. Dette førte ham til at formulere sin noget rigide teori angående den *entydige* og *endelige* karakterisering af anskuelsen og forstanden. Jeg viser, at fænomener i matematikken, som den så ud på Kants tid, burde have fået Kant til at forholde sig mere kritisk i forhold til netop sin formulering af anskuelse, forstand og fornuft.

Endelig viser jeg, hvorledes D. Hilbert faktisk giver en generalisation af Kants filosofi. Denne generalisation giver anledning til formuleringen af en generel ramme, der kan fungere som grundlag for en forståelse af matematikkens erkendelsesteori og ontologi.

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Prologue and Acknowledgments

In contemporary philosophy of science there is a renewed interest in Kant's theory of knowledge and his philosophy of science. One of the reasons for this is the comprehensive work done by Michael Friedman which has prompted a variety of responses. The present thesis can also been seen as a reaction on Friedman's work on Kant. I was introduced, through Friedman (1992), to Kant's notion of schematism in geometry. Now, by reading the *Critique of Pure Reason* I became acquainted with that fact that Kant claims to have a theory of schemata, not only for geometry, but for all the concepts of the understanding—pure as well as empirical. As it was clear to me, that the theory of schemata could provide an alternative understanding of Kant, it became a goal to get a deeper understanding of it. But also other interests motivated me.

Today, within the philosophy of mathematics, there is also a growing interest in the use of diagrammatic reasoning. Euclid is paradigmatic in this respect—some find the reasoning style used by Euclid highly problematic, others do not. Kant belongs to the latter category. Euclid uses diagrams throughout *Elements* and Kant wants to give an account of and a foundation for this kind of reasoning. With Kant's theory of schematism follows a notion of *schematic construction in pure intuition*—and this is precisely Kant's device when providing an epistemological frame work for Euclid's reasoning. Schematic reasoning is not exclusively about justification, nor exclusively about discovery in mathematics—it is generally about the whole reasoning process which the mathematician makes from discovery to justification. Mathematical schematic reasoning is not in particular about mathematical proofs, it is generally about mathematical thinking.

I have not written this thesis for the sake of writing history, rather it is for the sake of understanding what mathematics really is about. Understanding Kant's schematism is on my path. Nevertheless, for an understanding of schematism I need to use and produce elements from history of science and history of philosophy. On the other hand I will also use theories and concepts from contemporary philosophy and mathematics in order to get something meaningful out of Kant's schematism. We *need* to interpret Kant's theory of schemata, as Kant does *not* present a clear, nor a detailed theory; rather he outlines some remarkable and very fruitful ideas. But these ideas are much in need of elaboration. I am interested in understanding Kant's theory of knowledge, not for the sake of history, but for the sake of truth. Thus this thesis should be read as a thesis on philosophy of mathematics, not as a thesis on history of philosophy. In consequence of this I allow myself to use notions and concepts which were not know at the time of Kant.

Chapter 1 and 2 treat my interpretation of Kant's schematism. The former of the two chapters shows how the geometrical schemata and Euclid's postulates go hand in hand. The geometrical schemata are the part of Kant's schematism which is best explored in the literature. In this thesis, this chapter on the geometrical schemata functions mostly as an introduction to schematism and the central notions of schematism such as types, tokens and rules. Chapter 3 is on the schematism of the pure concepts of quantity. I show how Kant operates with both a concept of a particular number and a concept of a schema determining the properties of numbers. A particular number, to Kant, signifies unity in a collection of objects falling under a concept. The unity that the collection can posses is, that by counting the elements of the collection we reach a finite number just in case we can judge the collection to be a unit. I furthermore show that numbers are not determined extensionally. Rather they are determined by a schema—an intensional element.

Chapter 3 and 4 deal with the relations between space, schemata, geometry and the notion of object. It has always been difficult for me to understand the following words: "Space is represented as an infinite **given** magnitude" (B40). I also show in chapter 6 that also Hilbert and Bernays had difficulties here. As it turns out, my interpretation of the arithmetical and geometrical schemata as presented in chapter 1 and 2 actually provide a framework for an understanding of Kant's concept of infinity. Geometrical schemata need a space to exercise in; a space which is unbounded. How unbounded is the space? Well, it is in-finite in the sense that no magnitude can be ascribed to space. In chapter 4 I elaborate on an observation due to Carl Posy. It is well-known that according to Kant all objects are completely determined, in the sense, that given any object x and any predicate P, then either $P(x)$ is true or false. Posy (1995) observes that this is expressible by the following first order formula:

$$\exists y(y = x) \rightarrow P(x) \vee \neg P(x).$$

I note that it is not quite clear to which language this formula belongs. In elaborating on the formula I provide a class of modal models which (semi-)validates a modalized version of Posy's formula.

I chapter 5 I critically discuss Kant's philosophy of mathematics, and I generalize some of his notions in order to incorporate some of the new elements which have been discussed and introduced in mathematics since the time of Kant. This leads me to:

Chapter 6. In this chapter I outline a relation between Kant's general theory of knowledge and Hilbert's philosophy of mathematics. Through Kantian schematism we will reach a notion of *constructibility* in pure intuition. In fact this notion *characterizes* Kant's notion of 'mathematical object'. But as it turns out, such a notion is to narrow. This insight was realized by Hilbert who generalized the concept of 'mathematical object' by introducing ideal elements. I show how the the pair, schema and so-called quasi-schema, can fully describe Hilbert's finitary and ideal parts of mathematics, respectively. I furthermore hope the reader can see that my analysis of schematism founds an outline of a general philosophy of mathematics which I present by the end of chapter 6.

I will assume that the reader is familiar with the central notions of Kant's philosophy such as intuition, understanding, pure concepts, reason, ideas and regulative principles.

When making references to Kant I refer to the standard German edition of Kant's works, *Kant's Gesammelte Schriften*, edited by the Royal Prussian (later German) Academy of Sciences, Georg Reimer, later Walter de Ruyter & Co., 1900– , Berlin. For the first *Critique* I use the standard pagination referring to the first and the second edition; the translation of this work is due to P. Guyer and A. W. Wood (Kant; 1998). Unless stated otherwise, remaining translations are my own. When I refer to chapters or sections in Kant's work, like the chapter presenting schematism in the first *Critique* I use: Schematism; but when I refer to the element schematism in Kant's theory I use: schematism.

I wish to acknowledge my great debt to my supervisor Professor Stig Andur Pedersen. He introduced me several years ago to logic and philosophy and to a very fruitful, I think, view on mathematics. I have had enormous benefit in discussing these things with him.

I also want to thank Frederik Voetmann Christiansen and Claus Festersen, who read drafts of some of the chapters. Thanks also to Torben Bräuner who provided a reference, which turned out somewhat important. Thanks also to Thomas Bolander and Vincent F. Hendricks for discussing some of the logical things with me.

It is, however, my family that I am most grateful towards. Especially Anne Dorte supported me constantly, and gave me the space and time that I (apparently) needed—though it was tough by the end. By that end it probably sounded pretty odd when I kept saying "I am 95 percent done".

> Zwei Dinge erfüllen das Gemüth mit immer neuer und und zunehmender Bewunderung und Ehrfurcht, je öfter und anhaltender sich das Nachdenken damit beschäftigt: **der bestirnte Himmel über mir und das moralische Gesetz in mir.** (Ak. 5, 161)

Contents

CHAPTER 1

Kant and Geometrical Schematism

1.1 Introduction to Schematism

The first chapter of the second book in the *Transcendental analytic* is entitled "On the schematism of pure concepts of the understanding". This important chapter is not a long chapter—it makes up 11 pages of the whole *Critique* which constitutes 883 pages in the second edition. Although the chapter is short the content is very central in Kant's theory of knowledge: Schematism explains *how* concepts and sensibility are connected and the chapter is therefore at the heart of Kant's synthesis of rationalism (Leibniz–Wolff) and empiricism (Locke). Given these facts, it is not surprising how difficult and condensed the text is. When reading the chapter for the first time, one gets the impression that it is a careless writer, not knowing precisely what he wants to say.[1] The wonder does not diminish when realizing that nothing is changed in the second edition.[2] Kant made substantial notes in his own copy of the first edition, but none of them found their way to the second edition. In my view this is a peculiarity, but I am not the first to be puzzled:[3]

> His [Kant's] doctrine of the schemata can only have been an afterthought, an addition to his system after it was substantially complete. For if the schemata had been considered early enough, they would have overgrown his whole work. (Peirce; 1.35)

Whether an afterthought or not, the doctrine does not appear early in Kant's writings. According to P. Guyer and A. W. Wood the first mention of schematism is around 1779/80 (Ak. 18, 220).[4] The next is from the period immediately following

[1] Kant writes, for instance: "The schema of actuality is existence at a determinate time." (A144/B184). This is stated without any further indication of *how* this should be seen as a transcendental schema which generally is understood as a universal rule for transcendental time-determination.

[2] Except for two things: In the running header "the Categories" changes to "pure concepts of understanding", and then an "is" is missing in the first edition (A145).

[3] E. Shaper (1964-65, 270) puts it like this:

> The opaqueness and obscurity of the Schematism chapter—the chapter which Kant himself thought to be one of the most important pieces of the *Critique*, and to which Hegel paid tribute as being among the finest pages of the entire Kantian *oeuvre*—has often been—stressed with undertones ranging from wonder to irritation. From among the earliest statements we recall F. H. Jacobi's assessment of schematism as "the most wonderful and most mysterious of all unfathomable mysteries and wonders" [...], and Schopenhauer's characterization of schematism as a curiosity "which is famous for its profound darkness, because nobody has yet been able to make sense of".

[4] See their commentary to the *Critique* (Kant; 1998, 728).

the composition of the first edition of the *Critique* (Ak. 18, 267-68). And interestingly, there is a late note (Ak. 18, 685-87) from the end of 1797 where Kant summarizes: "The Schematism is one of the most difficult parts. – Even Mr. Beck could not understand it. I think this chapter is one of the most important chapters".[5]

I think Kant was aware of the incompleteness of the chapter when he wrote that the schematism is "a hidden art in the depths of the human soul, whose true operations we can divine from nature and lay unveiled before our eyes only with difficulty." (A141/B180–1) And further down the page: "Rather than pausing now for a dry and boring analysis of what is required for transcendental schemata of pure concepts of the understanding in general, we would rather present them".

The Schematism is deeply connected with the so-called figurative synthesis (*synthesis speciosa*) as introduced in § 24 of the Deduction in the second edition of the *Critique*. Now, what seems to be missing in the *Critique* is a general discussion of that ability of the mind which the figurative synthesis designates. This discussion is apparently left out due to its general dryness and boredom. The philosophical literature of today is *still* in need of a detailed, thorough and careful interpretation of the *source* of schemata in Kant's theory of knowledge, which is deeply connected with an analysis of the figurative synthesis and more generally the transcendental imagination. Such an interpretation requires that, especially a) the B-deduction is analyzed with respect to the synthesis, and that b) the general role of imagination in Kant's thinking is taken into account. Such a project is not within the scope of this text.[6] I will rather analyze and interpret what, especially, the geometrical and arithmetical schematism consist of. This will be of the highest importance for the complete interpretation of the Kantian notion of schematism and his concept of figurative synthesis. Fortunately, these two types of schemata form the most accessible parts of Kant's theory of schemata.

1.2 What Schematism is generally about

The Analytic has three very important parts: The Deduction, the Schematism and the Principles. All three of them concern the relation between (pure) concepts and sensibility. Roughly it can be said, that the Deduction shows (or is intended to show) that there is harmony between the pure concepts of the understanding and the way appearances are given. This harmony ensures that appearances *can* be cognized under categories. The Schematism shows (or is intended to show) *how* the appearances are subsumed under categories. The Principles show (or are intended to show) what the general *conclusions* to be drawn are.

[5] Jakob S. Beck was a disciple of Kant.

[6] But see (Longuenesse; 1998) concerning a) and (Gibbons; 1994) concerning b).

But in fact the Schematism concerns the connection between concepts and sensibility more generally; it is not only about the schematism of the categories but about schematism of:

1. Empirical concepts,

2. Pure sensible (i.e., geometrical) concepts,

3. Pure concepts of understanding.

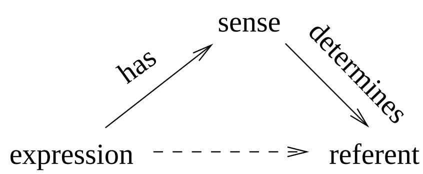
I treat the schematism of empirical concepts in section 1.4, the geometrical concepts in section 1.6 and the schematism of the categories of quantity are analyzed in chapter 2.

As a first definition of schema Kant writes: "Now this representation of a general procedure of the imagination for providing a concept with its image is what I call the schema of this concept" (A140/B180–1). Any concept has a schema associated and such a schema constructs and/or reflects on mental images in *accordance with rules*. The images are necessary for subsuming appearances under concepts as the schema is a "general condition under which alone the category can be applied to any object" (A140/B179). This is exemplified in the case of the geometrical schemata, which are put on a par with mathematical axioms:

> Now in mathematics a postulate is the practical proposition that contains nothing except the synthesis through which we first give ourselves an object and generate its concept, e.g., to describe a circle with a given line from a given point on a plane; and a proposition of this sort cannot be proved, since the procedure that it demands is precisely that through which we first generate the concept of such a figure. (A234/B287)

The schema which is associated with a geometrical concept makes the very concept *possible* in the first place: "We cannot think of a line without **drawing** it in thought, we cannot think of a circle without describing it" (B154). In this way, our capacity for producing images by means of schemata can be seen as a *transcendental* condition for knowledge and objective representation. As schemata and axioms are two sides of the same coin it does not make sense for Kant to talk about uninterpreted formulas of mathematics which can either be true or false depending on a particular interpretation. Kant writes: "On **axioms.** These are synthetic *a priori* principles, insofar as they are immediately certain" (A732/B760). In saying this, "certain" is certainly not to be understood as being true in some model-theoretic sense—rather it is to be understood as "correctness"; correct axioms describe human procedures which make concepts possible in the first place.

Schemata are a mediating "third thing" (A138/B177) preventing that situations where "[t]houghts without content are empty, intuitions without concepts are blind" (A51/B75). This mediating element is providing concepts with images which in turn provide concepts with meaning. Although a theory of language with a modern distinction between intensional and extensional semantics is not to be found in the *Critique*, Kant's theory of schemata is an anticipation of work found in for example Frege. "Thus the schemata of the concepts of pure understanding are the true and sole conditions for providing them with a relation to objects, thus with significance [*Bedeutung*]" (A145–6/B185). This can be compared with the popular sense-reference triangle:



1.3 What Schematism is to account for

As we know, Kant wants to formulate a theory of knowledge in which we find an account for the possibility of the *synthetic a priori*. In particular, Kant is of the opinion that "mathematical judgments are all synthetic" and "a priori" (B14-15). But more ambitiously, Kant also intends to give an account of human thinking in general. In answering both of these question Kant understands how important and central mental images are for cognition. The empiricists, such as Berkeley and Locke, were definitely aware of the epistemological significance of mental images.[7] But Kant is the first to view images *relatively* to schemata.

An image, according to Kant, is not to be confused with the schema. Rather than being singular, the schema is a *general* procedure. Schemata are rule-governed operations and they *provide* and/or *reflect* on images. In doing this the schemata make it possible for objects to be subsumed under concepts. We will see below, that this general idea—schemata being responsible for the construction and reflection on images—is seen as a foundation for geometrical concepts. Thus, the geometrical concepts are ultimately founded on two things: Firstly, they are grounded in space as a form of intuition, and secondly, they are made possible by the transcendental imagination through schemata. See chapter 3 for a detailed analysis on the epistemological interplay between geometrical schemata and space.

[7] The neo-Kantian Harald Høffding (1905, 200) formulates this position in a rather precise way: "We cannot think without images [. . .] Every thought, be it as abstract and sublime as it may supposes images".

The role of the transcendental imagination in connection with geometrical concepts is, however, more complicated than just described. As we saw above, we "cannot think of a line without **drawing** it in thought". Here Kant not only means that images, or the production of images, are necessary, he also means that by thinking a concept (line) of *outer* sense, we must also produce an *inner representation* (which happen to be an image) of the concept. It is the relation between inner and outer which is interesting here. Representations are part of our inner life, and as such they are given—and can only be given—in time; the inner sense. Thus it seems that geometrical concepts are made possible through the transcendental imagination by way of time. But the relation between inner and outer is in this respect symmetrical: When Kant in fact introduces the figurative synthesis in §24 in the B-deduction—in the section where talks about drawing lines—it is also postulated that "we cannot even represent time without [. . .] **drawing** a straight line (which is to be the external figurative representation of time)" (B154). Thus it seems that pure temporal concepts are made possible by spatial concepts. The figurative synthesis is the central ingredient for this and the schemata are—as we shall see—the explanation of *how* it can be possible. But note, how the transcendental imagination (here represented by the figurative synthesis) is the connection between inner and outer. This is one of the very important consequences of this synthesis that I will explore in this chapter.

1.3.1 Schemata in mathematics

In mathematics and in geometry, in particular, schemata play an indispensable role according to Kant.

> **Philosophical** cognition is **rational cognition** from **concepts,** mathematical cognition that from the **construction** of concepts. But to **construct** a concept means to exhibit *a priori* the intuition corresponding to it. For the construction of a concept, therefore, a **non-empirical** intuition is required, which consequently, as intuition, is an **individual** object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept. Thus I construct a triangle by exhibiting an object corresponding to this concept, either through mere imagination, in pure intuition, or on paper, in empirical intuition, but in both cases completely *a priori*, without having had to borrow the pattern for it from any experience. The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition we have taken account only of the action of constructing the

> concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent, and thus we have abstracted from these differences, which do not alter the concept of the triangle. (A713–4/B741–2)

It is precisely the schematism which puts Kant in a position where he can account for the mentioned characteristic properties connected with the the mathematical method: Constructibility of mathematical concepts—of which constructibility of mathematical objects is seen as a special case; inference to the general from the singular; and necessity of mathematical propositions. Kant is serious when he says that we can conclude universal statements, even though we have only examined a particular instance; and even more problematic: This instance can be given to us as an empirical figure. By the end of the 19th century a mathematician such as M. Pasch rejected the use of diagrams in geometry. Diagrams, it was claimed, could be very useful in the context of discovery, but one should never rely on them when proving theorems. In fact he—and also Hilbert—turned towards a linguistic conception of proofs as sequences of sentences, in which diagrams seemed to play no role. Moreover, they were both of the opinion that when investigating the mathematical science called geometry only the highest rigor was sufficient—and, perhaps therefore, the mathematician should avoid thinking of the meaning when proving theorems: "[I]f geometry really has to be deductive, then the deduction has to be completely free of any form of [reference to] *meaning* [*Bedeutung*] of the geometrical concepts, and likewise independent of figures. Thus, we will acknowledge only those proofs in which one can appeal step by step to preceding propositions and definitions" (Pasch; 1882/1926, 90). Hilbert (1902b, 602) was also skeptical to the use of diagrams: "One could also avoid using figures, but we will not do this. Rather, we will use figures often. However, we will *never rely on them*."[8] But Hilbert's standpoint on this is not as simple as one would think on the face of it. I will return to this in section 1.6.6.

One of the reasons why a certain skepticism towards the use of diagrams was expressed was probably due to a discovery by K. Weierstraß announced in 1872.[9] His result was the following. If a is odd, $b \in [0,1[$ and $ab > 1 + 3\pi/2$, then the function

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n x)\pi,$$

is continuous on the whole of $\boldsymbol{R}$ but nowhere differentiable. This is a counterexample to the idea which arises when one draws continuous functions by hand. When do-

[8]The translation of Hilbert is due to Paolo Mancosu (2005, 15).

[9]The result was published by du Bois Reymond (1875, 29).

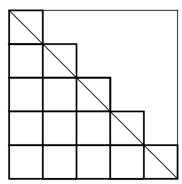
ing this it seems evident that any continuous functions is smooth, i.e., differentiable except at isolated points.

Despite the problems (which I will come back to in section 1.6.5) Pasch and Hilbert seem to find in the use of diagrams we, nevertheless, find Kant in a position, where he claims that in fact we can use diagrams (intuitions) as an essential and reliable part of mathematical arguments. We see that the problems Kant wants to solve with his theory of schemata are not minor problems.

I hope to show in this chapter that Kant is actually approaching an analysis of what can be seen as the type–token distinction. And in doing this I hope to be able to give an explanation of *why* I take, for instance, the following argument to be a proof of:

$$1+2+\ldots+n=\frac{n(n+1)}{2}.$$

We argue for the case where $n=5$:



The diagram shows $1+\cdots+5$ placed in a square with area $5\cdot 5$. To get $1+\cdots+5$ we divide the area of the square by 2 and add half of the elements on the diagonal. Therefore:

$$1+2+\ldots+5=\frac{5\cdot 5}{2}+\frac{5}{2}=\frac{5(5+1)}{2}.$$

From the diagram it is seen that nothing special about the number 5 is used, thus the proof will apply to any number.[10]

1.3.2 Schemata for pure concepts of the understanding

It might be the case that Kantian claims about geometrical schemata, such as drawing lines in thought, seem awkward. But the claims about transcendental schemata are worse. Here a "transcendental time-determination" figures prominently:

[10]The example is found, for instance, in (Brown; 2005, 64). Interestingly, Brown writes "The simple moral I want to draw from this example is just this: We can in special cases correctly infer theories from pictures, that is, from visualizable situations. An intuition is at work and from this intuition we can grasp the truth of the theorem." This sounds very much like Kant at A713/B742. Note, however, that Brown sees his own position as 'full-blooded Platonism'; a term which can hardly be said to characterize Kant's position.

> Now a transcendental time-determination is homogeneous with the **category** (which constitutes its unity) insofar as it is **universal** and rests on a rule *a priori*. (A138–9/B178–9)

The central aspects are (again) universality, rules and a prioriness, but now supplemented by time-determination. Transcendental schemata are of special interest to the philosophy of mathematics, as "number" is the schema for the category of quantity. Precisely this very fact, that "number" is a transcendental schema has as a consequence that for a mathematical theory of numbers "there are nevertheless no axioms" (A164/B204). Today this sounds odd, and is in any circumstance in contrast with the situation within geometry where there certainly are axioms according to Kant.

In the following I give an interpretation of Kant's theory of schemata. As already mentioned, Kant operates basically with three different kinds of schemata: These are labeled 1) empirical schemata, 2) geometrical schemata, and 3) transcendental schemata. What they have in common is the following: The schemata are a product of the transcendental imagination, and they are responsible for production of and reflection on images. The images are necessary when objects in our experience are subsumed under concepts. In all three kinds, the schemata can be seen as essential for the type-token relation. I will, however, emphasize the geometrical and arithmetical schemata.

1.4 Empirical schematism

Although Kant's chapter on schematism is entitled as a chapter on the schematism of *pure* concepts, there are several examples of the schematism of empirical concepts. Now, empirical concepts are a posteriori concepts and they are as such derived from the contingent experience we may have of, say, dogs. The concept of a dog is a collection of so-called marks (*Merkmale*),[11] and as the concept is not pure this collection may vary among humans. Typically we would, however, tend to think of a dog as an animal which has four feet (and is inclined towards barking). But we are not able to refer to anything, unless the concept is used *together* with its schema. When this is the case, then:

> The concept of a dog signifies a rule in accordance with which my imagination can specify the shape of a four-footed animal in general, without being restricted to any single particular shape that experience offers me of any possible image that I can exhibit *in concreto*. (A141/B180)

[11] See page 60 for an my analysis of the intension and extensions of Kantian concepts.

Here we are introduced to (what seems to be) the more simple aspects of the theory. The schema of dog is a procedure by which the imagination can produce a paradigmatic mental image, which functions as a prototype or a representative example giving me a *figurative representation* of a typical dog.[12] By using this I can think of a dog (as such) even though no dogs are present, and when there is a dog present I can subsume this animal under my concept dog, since I find the characteristics (the marks) to be present in the dog, because of essential similarities between the dog and my mental image of a dog in general. I can compare dogs, and I can count collection of dogs. Kant uses this example, I think, as an introduction to the more complicated parts of schematism. On the face of it, the empirical schemata may all seem unproblematic and straight-forward. But I think there is more to it. The problem is that, according to Kant, empirical concepts are dynamical, since "**empirical** concepts cannot be defined at all but only explicated", where "**to define** properly means just to exhibit originally [*ursprünglich*] the exhaustive concept of a thing with its boundaries" (A727/B755). Therefore, to give a definition of the concept "dog", would be to give a priori (*originally*) the necessary and sufficient characteristics for objects to be subsumed under dog. But empirical concepts are derived from experience and thus different persons may understand different things under the concept. Thus it seems problematic to require universality of empirical schemata. Another, though related, problem in this connection seems to be, in the words of Mary Tiles (2004, 127) that "any empirical object will have more specificity than is contained in the schematic image". As a consequence of this it follows that empirical objects are not—and cannot be—determined by the schematic rule. Thus the empirical schema does not characterize uniquely the reference.[13] As images referring to empirical objects are essential for transcendental schemata I will come back to this problem in section 2.1.

1.5 Euclidean reasoning: The challenge for geometrical schemata

There is no doubt that Kant has a special interest in geometry. And in connection with the theory of schemata this is particularly clear. I think there are three reasons for this.

The first reason: The challenge from mathematics. Kant understands mathematics and mathematical knowledge as based on constructions taking place in time.

[12]Thus Kant's theory of schemata are a clear predecessor of a contemporary theory of prototypes found in cognitive psychology as given by Rosch.

[13]Therefore there is a conflict between empirical schemata and Frege's "sense". In Frege's theory "sense" should determine uniquely the reference.

Therefore mathematics is synthetic. Moreover, the propositions of mathematics are necessary, therefore a priori. Simultaneously, Kant aims at giving a general account of human knowledge and here the sciences play a central role. So Kant's theory intends at being a framework for understanding the human activity of science. Mathematics is a special kind of science: The objects are in a certain sense non-empirical; nevertheless mathematical knowledge is synthetic. Thus, Kant wants to develop a theory in which we are 'guaranteed' that there really *are* meaningful mathematical objects, fulfilling the criteria concerning constructibility and necessity. The theory of schemata is a basic pillar in explaining this.

The second reason: The possibility of abstract concepts. Kant's more general problem is to explore the possibility of subsuming objects under concepts. If Kant can explain how geometrical concepts, such as triangle, 'function' then, perhaps, he can lift this to a *general* theory of the relation between concepts and objects? In fact, I think this is Kant's strategy

The third reason: The source of schematism. As a Kantian unification of the two foregoing I find it most likely that Kant got the idea about schematism from his studies of Euclid. Lisa Shabel sharpens this claims by saying that Euclidean reasoning through diagrams "provides an interpretive model for the function of a transcendental schema" (2003a, 109).

As evidence for the third reason let me mention that Kant frequently lectured on Euclidean geometry, and that he was well-trained in Euclidean reasoning. Having this in mind it seems reasonable that his insights from Euclidean geometry inspired him, at least, for chosen the term "schema". When looking up the Greek word "Schēma" in Liddell & Scott's *Greek-English Lexicon*, one the the possible meanings are "geometrical figure" with reference to—not surprisingly—Aristotle.

1.5.1 Reasoning style in Euclid's Elements

Euclid's *Elements* comprises 13 books with content ranging from basic plane geometry, over arithmetic and incommensurables to solid geometry. A list of definitions, postulates and common notions open Book I. Whereas the postulates and common notions remain the same throughout the 13 books, each of the subsequent books add new definitions to this list. It is beyond the scope of this thesis to give a full survey of the type of reasoning used in *Elements*. For the purpose at hand it suffices to give representative examples. I will give an analysis of the proof of proposition 32 from Book I.[14] I.32 claims that the three interior angles of any triangle equal two right

[14]Below I will refer to propositions and definitions by $X.Y$, where X is the number of the book and Y the number of the proposition/definition in that book. Proposition 13 from Book I, for instance, is referred to as "proposition I.13". When it is clear from the context whether it is a proposition or a

angles. In modern terms, that the sum of the angles of a triangle equal 180 degrees. Let me begin by stating the postulates (Euclid; 1956, 154–155).[15]

POSTULATES.

Let the following be postulated:

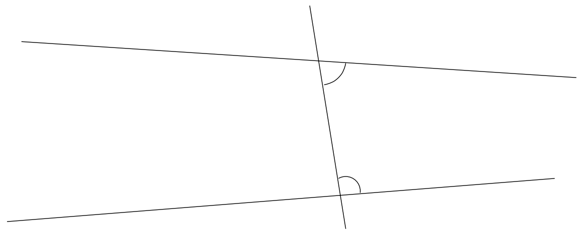
1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The first three postulates concern constructions of objects and it is commonly taken that those are the objects that can be constructed with a straightedge and compass. Thus these postulates can be seen as construction-axioms. An important feature of the objects which are constructible is that only *finite* objects are constructed: Given two points a line of finite length can be constructed, and given any centre and any (finite) distance a circle with a finite area can be constructed.

The fourth and the fifth postulate can be seen as structural axioms, making Euclidean geometry special. The fourth, for instance, guarantees invariance of right angles. And given we have notions of "smaller than" and "greater than" the fourth postulate puts us in a position where we can recognize the situation given in postulate 5, (where the two interior angles are less than two right angles), and thus postulates that if we extend the two lines far enough they will meet. Note again the finitary attitude expressed: If a certain state of affairs occurs, then if we extend the two lines continuously then the extensions will meet after construction in a finite amount of time. A figure illustrating the postulate is:

definition I will refrain from specifying this.

[15]The objects that the five postulates are about are defined in the 23 definitions presented just before the postulates. Definition I.1 says for instance: "A **point** is that which has no part" and I.2: "A **line** is a breadthless length". Furthermore, we learn in I.3 that: "The extremities of a line are points".



Let us now take a look at proposition I.32 and its proof as given in (Euclid; 1956):

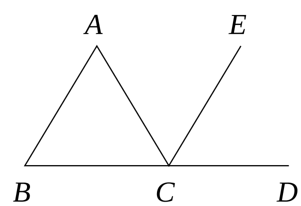
PROPOSITION 32.

In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

Let *ABC* be a triangle, and let one side of it *BC* be produced to D;

I say that the exterior angle *ACD* is equal to the two interior and opposite angles *CAB*, *ABC*, and the three interior angles of the triangle *ABC*, *BCA*, *CAB* are equal to two right angles.

For let *CE* be drawn through the point *C* parallel to the straight line *AB*. [I. 31]



Then, since *AB* is parallel to *CE*, and *AC* has fallen upon them, the alternate angles *BAC*, *ACE* are equal to one another. [I. 29]

Again, since *AB* is parallel to *CE*, and the straight line *BD* has fallen upon them, the exterior angle *ECD* is equal to the interior and opposite angle *ABC*. But the angle *ACE* was also proved equal to the angle *BAC*;

therefore the whole angle angle *ACD* is equal to the two interior and opposite angles *BAC*, *ABC*.

Let the angle *ACB* be added to each;

therefore the angles *ACD*, *ACB* are equal to the three angles *ABC*, *BCA*, *CAB*.

> But the angles *ACD*, *ACB* are equal to two right angles; [I. 13]
>
> therefore the angles *ABC*, *BCA*, *CAB* are also equal to two right angles.
>
> Therefore etc.

The use of diagram is already seen in the formulation of the proposition. Here the angles are classified as either interior or exterior. These terms cannot be understood unless one uses the diagram—"exterior" and "interior" are only defined implicitly through diagrams. Now, the diagram of I.32 depicts what we are given: a triangle *ABC*. But more is shown in the diagram. Out of the given triangle we construct line *BD* by postulate 2. The extremity of any line is a point, thus *D* is constructed and exists. Furthermore, due to I.31 a line *CE* can be drawn parallel to *AB*. I.31 is, in turn, proved by appealing to postulate 1 and 2 and propositions I.23 and I.27, and the construction taking place in this proof is also visualized by a diagram (as all the proofs in *Elements* generally are).

The diagram used in the proof of I.32 shows properties about line *CE* which can *only* be inferred by use of the diagram: From the text we do not know the *direction* in which the construction of *CE* goes. But from the diagram we see that it is drawn upwards and thus splits *ACD* in two angles *ACE* and *ECD*. Euclid's fifth common notion states that "[t]he whole is greater than the part"—but it is through the diagram we learn what the whole is (*ACD*) and what the parts are (*ACE* and *ECD*).[16]

Now that we have determined relations between the lines and we know what is interior and exterior we see—due to proposition I.29—that *BAC* and *ACE* are equal (in size). In fact this is the only non-obvious part of the proof. And I.29 is proved—through a diagram—by appealing essentially to postulate 5 and I.15 (which relies I.13 and postulate 4).[17]

The rest of the proof of I.32 is more elementary in character: We apply common notions and I.13 which says that "*If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles*" (Euclid; 1956, 275).

The reasoning style used by Euclid is very rigorous and in consequence of this it is natural to view Euclid as the founder of the axiomatic-deductive method. As a

[16]The other four common notions in (Euclid; 1956) are:

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.

[17]Of course common notions (1 and 2) are also used in the proof of I.29.

result of this it is perhaps also natural to require that he should conform to only the strictest contemporary standards. But the picture is, however, not that simple. On a first reading it looks as if (almost) every step is justified by appeal to postulates, or common notions or previously proved propositions. But as seen above from the proof of I.32 *diagrams* are used in an essential way. In my understanding of the reasoning style used by Euclid I tend to agree with Shabel when she writes:

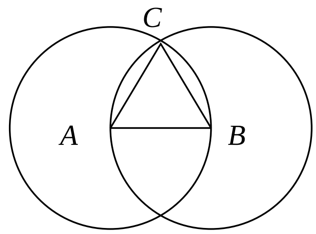
> [Euclid] offers a list of definitions and common notions that, properly understood, help us to read information off of diagrams constructed in accordance with the postulates [... T]he diagrams *enable* the reasoning of the demonstration by warranting deductive inference. Crucial steps in common Euclidean derivations are taken by virtue of observations made on the bases of a diagram. (Shabel; 2003a, 38)

This is of course not the same as to say that diagrams are *necessary* when proving geometrical theorems. In fact the schematic reasoning more generally concerns mathematical reasoning as such, not only in connection with justification, but also in the general mathematical reasoning-process. Mathematics and its working-process concerns both proofs and discoveries and diagrammatic reasoning is a cornerstone in the mathematical process.

Euclid's reasoning through diagrams has been—within the context of justification— criticized for its "many deficiencies" (Kline; 1972, 1007). One of the claimed problems is that Euclid is relying on unjustified continuity assumptions. According to the contemporary understanding of geometry which originates in Pasch (1882/1926) (see page 6) one should only use diagrams on the heuristic level. This was a part of the whole foundational 'project' by the end of the 19th century to eliminate essential use of diagrams in proofs.

An instance where continuity seems to be needed is in the very first proposition I.1: "*On a given finite straight line to construct an equilateral triangle*" (Euclid; 1956, 241). Euclid's proof goes like this: Given any line *AB* construct a circle with center *A* and radius *AB* (postulate 3). Likewise construct another circle with center *B* and radius *AB*. The circles cut in *C*, and *CA* and *CB* can be constructed (postulate 1). By construction[18] it follows that *ABC* is the required triangle.

[18] And by definition 15 and common notion 1.



It could happen that A and B have rational coordinates $(0,0)$ and $(0,1)$, respectively and C would therefore have $(1/2,\sqrt{3}/2)$ as coordinates. Now, we could view the Euclidean postulates as axioms not associated necessarily with any specific model—thus with no fixed reference. Then, it could (perhaps) be argued that a model having only $\mathbf{Q}^2$ as domain would make the Euclidean postulates true. As $\sqrt{3} \notin \mathbf{Q}$ it follows that the non-existence of C is consistent with the postulates. The existence of C does therefore not follow from the postulates, and Euclid is accordingly not warranted in claiming its existence. A criticism along these lines was stated by Pasch (1882/1926, 44–5). The first obvious objection to this argument is that it is rather doubtful that $\mathbf{Q}^2$ should exhaust the points constructible by the postulates. But I will raise two other and more principle objections to the Paschian criticism.

The first bear on the general project concerning the development of the axiomatic method and the rigorization of mathematics in the 19th century. This development led to new views on the methodology of mathematics, one of them being—in a rough and ideal form—the *formal* method of mathematics: Mathematics is written in an uninterpreted language which can have several different models. This methodology has two important consequences. The formal method allows—for instance in the case of Hilbert's *Grundlagen der Geometrie*—for a study of new structures, which can *only* be studied through the axiomatic method. I will elaborate on this in section 1.6.6. The other consequence is of course the rigor accompanying more formal methods. Thus Pasch writes, that "we will acknowledge only those proofs in which one, step by step, only refers to preceding theorems and definitions" (Pasch; 1882/1926, 45). This idea—which ultimately is the idea of mathematics done in an uninterpreted language, where theories allow for a variety interpretations in different models—is a thought very remote for Euclid and Kant. They do not want to separate mathematics from the models—as they have only one model in mind. Thus the method used by Euclid is not the same as the method used by Pasch and Hilbert—and is not intended to be either.[19] See section 1.6.6 for more details in this respect.

[19]This observation is in turn also an objection to the so-called "model-interpretation". According to this interpretation one understands non-Euclidean geometry as a *confirmation* of Kant's epistemology. It follows from Kant's considerations in regard to the modalities that consistency does not imply existence.

Shabel (2003a) has raised another objection. She claims that Euclid understands a circle as a disc. This follows from three of Euclid's definitions: Definition I.13 says that a "**boundary** is that which is an extremity of anything", and definition I.14 that a "**figure** is that which is contained by any boundary or boundaries"—note the use of "contained". And finally, definition I.15: "A **circle** is a plane figure contained by one line such that all straight lines falling upon it from one point among those lying within the figure are equal to one another" (1956, 153). In consequence of his understanding of a circle, Euclid constructs in the proof of I.1 a disc with center A from the given AB. Likewise he uses B and AB to construct the second disc. As seen in the diagram, this construction implies that the two discs have a non-empty intersection, which itself is a figure because it is contained in both circles. The boundary of the new figure is given by the boundary of the first circle which is contained in the second circle and the boundary of the second circle contained in the first. The boundary of the intersection is therefore constructed out of two partial circumferences created in accordance with postulate 3. These two curved lines have extremities. One of them is C. Thus, when Heath writes in his commentary that "Euclid has no right to assume [...] that the two circles *will* meet in a point C" (Euclid; 1956, 242), I think he puts it wrongly: Euclid does not *assume* that the point exists, rather he *constructs* the point.

There are other criticisms of Euclid such as the use of the so-called "principle of superposition" (that objects can be moved through space for purposes of comparison, e.g., proposition I.4), but it is beyond the scope of this text to analyze this.[20] Another criticism is that Euclid has no proper theory of order,[21] but as indicated above, the diagrams—at least partially—take care of this.

It is, however, interesting that in her interpretation Shabel (2003a, 21–28) goes through (equivalents of) the different groups of axioms for Euclidean geometry as provided by Hilbert (1902a)—axioms of connection, order, parallels, congruence, and continuity—and shows how the interplay in Euclid's geometry between definitions, common notions and construction through diagrams in accordance with the postulates can be seen as warranting all the axioms of Hilbert. Thus Euclid was not *necessarily* sloppy or inattentive to the finer details of geometry, but—to quote

"That in such a [sensible] concept no contradiction must be contained is, to be sure, a necessary logical condition; but it is far from sufficient for the objective reality of the concept" (A220/B267–8). Thus the existence of consistent non-Euclidean geometries does not present a problem, it just shows that Euclidean geometry is not logically necessary—although it is the only really possible geometry, being the only correct model of our spatial intuition. Also Friedman (1992, 100–104) argues against this interpretation—but for other reasons.

[20] See (Shabel; 2003a, 29–34) for a discussion.

[21] C.F. Gauss complained for instance about this (Kline; 1972, 1006); but also—according to Friedman (1992, 56)—"Russell, who habitually blamed all the traditional obscurities surrounding space and geometry—including Kant's views, of course—on ignorance of the modern theory of relations".

Shabel:

> That they were not expressed as axioms at the outset of Book I of the *Elements* should not lead us to conclude that Euclid failed to recognize necessary axioms, nor that he left them wholly implicit; instead we should come to realize that he did not see his project as foundational in the modern sense. The foundation of Euclid's project lies in the systematic use of the diagrams constructed and understood in accordance with his definitions, postulates and common notions. (Shabel; 2003a, 28)

1.6 Geometrical schematism

References to Euclidean geometry are found throughout the *Critique*. Let me give some examples.

In a letter to Schütz of 25 June 1787 Kant writes (Ak. 10, 489) that he was thinking of proposition I.5 when he—in the introduction to the second edition (Bxi-xii)—wrote about the mathematical method Thales used when he "demonstrated the isosceles triangle". And the triangle was one of Kant's favorite examples. In the Aesthetic (A24/B39) proposition I.20 is written out almost verbatim, and in the section Axioms of intuition (A164/B205) it is clear that Kant is thinking of proposition I.22. A final example; on A716/B744 Kant mentions proposition I.32—which also happens to be Aristotle's favorite illustration of geometry.[22] I will come back to the two latter examples. Moreover, Kant continuously mentions the first three postulates[23] or paraphrasings hereof.[24] Common notions are mentioned; as analytical propositions (B16–17; A64/B204–05). What I want to stress by mentioning this, is that it is precisely the Euclidean way of reasoning that Kant wants to give an account of and a foundation for. He does this by his *schematic construction in pure intuition*. Therefore, there is more than a grain of truth in Leibniz's comment on I.1 which according to Kline is: "Leibniz commented that Euclid relied upon intuition when [proving I.1]" (Kline; 1972, 1006).

Kant wants to give this account because, this is *how* he understands what he took to be the most important mathematical discipline. Moreover, the Euclidean model of construction provides a general model for Kant's notion of *constructibility*.[25]

[22] See Heath commentary on page 320 in (Euclid; 1956).
[23] But only the first three(!)
[24] See (A24/B39; B154; A163/B204; A234/B287; A239/B299; A261/B317; A300/B356; A511/B539; A716/B744).
[25] As was perhaps first observed by Hintikka (1967).

1.6.1 Constructibility and schemata

It is "schemata that ground our pure sensible concepts" (A140/B180). This grounding is deeply connected with the so-called "**construction** of concepts". What it means to "**construct** a concept" is expressed by Kant on pages A713–4/B741–2 (see page 5 for the full citation). Here we learn that to construct a concept means to construct in accordance with rules a non-empirical intuition, which should represent the concept universally. The production of the intuition can be done either purely by the imagination (the figurative synthesis) or it can be a figure drawn on paper. In the latter case the empirical intuition functions as a symbol which refers by analogy to the pure intuition.[26] The universality of the image results when the particular image is taken together with the procedure generating the image. Such a procedure is deeply connected with the Euclidean style of reasoning:

> Give a philosopher the concept of a triangle, and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on this concept as long as he wants, yet he will never produce anything new. He can analyze and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts. But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and obtains two adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that here there arises an external adjacent angle which is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and a the same time general solution of the question. (A716–7/B744–5)

Geometrical knowledge evolves in precisely the manner that we have seen the production of mathematical knowledge in *Elements*. Here Kant explicitly refers to the proof of Euclid's proposition I.32. The geometer *constructs* a triangle. But he already knows something in advance, namely (a generalization of) proposition I.13. *Therefore* he constructs in accordance with postulate 2 an extension of one of the

[26] In consequence of this I adopt the use of "symbolic reference" as used by Mary Tiles (1991).

lines. Then he divides—a construction validated by proposition I.31—the external angle, and *so on*. But it "is always guided by intuition" by the construction of an image (purely mental or empirical with symbolic reference to the pure image) in accordance with some basic postulates, definitions and common notions; after wards discoveries are realized through the constructed image.[27] Then a 'general solution' can be found. I will treat the problem concerning universality in section 1.6.3. Here I concentrate on what precisely the schemata are.

Kant seems to claim that any meaningful concept has a schema. For instance, "[t]he schema of the triangle [. . .] signifies a rule of the synthesis of the imagination with regard to pure shapes in space" (A141/B180). So the natural question is: What does this rule consist of? The schema for the concept "triangle" must be a rule by which we can construct any (image of a) triangle. Kant writes: "Thus we think of a triangle as an object by being conscious of the composition of three straight lines in accordance with a rule according to which such an intuition can always be exhibited" (A105). And later in the *Critique*;

> "three lines, two of which taken together are greater than the third, a triangle can be drawn," then I have here the mere function of the productive imagination, which draws the lines greater or smaller, thus allowing them to abut at any arbitrary angle. (A164/B205)

These two quotations show that the "function of the productive imagination" which Kant is referring to is the function defined in the proof of I.22:

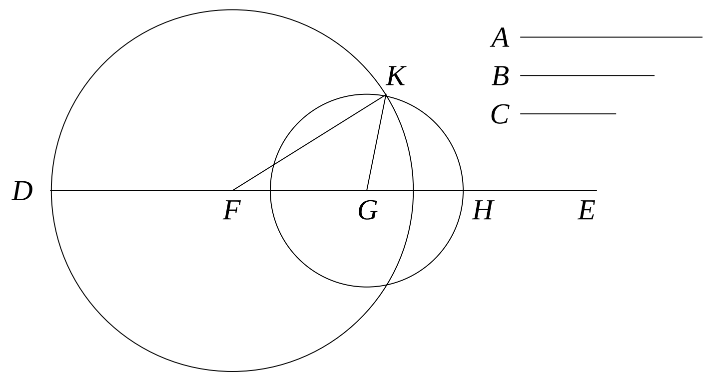
PROPOSITION 22.

> *Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one.*
> [I.20]

The proof is of course given by constructing a triangle out of the three given lines A, B and C, which fulfill the requirement that any two are greater than the remaining one. The requirement was proved in I.20 to be a property of the collection of the three sides of any triangle. On a line DE constructed to be long enough (postulate 2) we construct DF to be equal (in length) to A; we construct FG to be equal (in length) to B and construct GH equal (in length) to C. This can be done by the procedure

[27] Thus my interpretation differs from Friedman's (1992, 90), where it is rejected that intuition "enable us to "read off" the side-sum property [in the case of proposition I.32]". Of course the 'reading of'-procedure is not just a simple visual inspection, as Kant also rejects in the case of I.5 referred to on page Bxi-xii. See section 1.6.3 for an analysis of how the 'reading of' can yield universality.

given in the proof of proposition I.3. Now describe two circles with center F and diameter FD and center G and diameter GH, respectively (postulate 3). The circles intersect (as seen from the diagram) and thus they meet in K. Connect K with F and G (postulate 1) and the triangle is constructed.



The proof of the proposition depends on proposition I.3 which depends on postulate 3 and proposition I.2, which in turn depends on proposition I.1. The latter propositions depend on the first three postulates. Of course common notions and definitions are also used in the proofs, and additionally properties about order and continuity must be inferred from the diagrams.

Now, the schema for the concept "triangle" is, according to Kant, a rule-governed operation which "draws the lines greater or smaller, thus allowing them to abut at any arbitrary angle" (A164/B205) (postulate 2). Out of these three lines a triangle is constructed. This construction is with respect to postulates ultimately founded on the first three postulates. But the schema operation is of course also based on the common notions, definitions and the more implicit diagrammatic rules as indicated above. The rule-governed operation producing all triangles must, however, also conform to the insight expressed in the *assumption* given in the formulation of I.22: That any two of the lines are greater than the third. This assumption is necessary according to I.20.[28]

Let us understand the concept "triangle" as similar to a definition: x is a *triangle*, if and only if, x has three lines and the three lines are. . . etc. The concept is like a (Peircean) type, whereas the individually constructed images are the tokens falling under the type. The schema corresponding to the concept is, following this line of thought, a complete method which exhaust the relation between a type and the tokens

[28]Note, that I.20 is proved by referring, among other things, to the parallel postulates. So, the schema for the concept "triangle" also contains a notion about parallelism, although I.20 could perhaps be proved without reference to these postulates.

falling under that type. By the schema we can construct all triangles, but we can also decide whether a given figure is a triangle or not, by examining whether it can it be constructed by the schema. Thus the concept itself is something passive, whereas the schema amounts to certain active construction procedures.

In accordance with the analysis of the proof of I.22 I think it is reasonable to interpret Kant in the following way. There are certain geometrical schemata which are more basic than others. The schema for triangle can, on the one hand, be seen as founded in basically the first three postulates together with the relevant common notions and definitions. This must, however, be supplemented with the more implicit rules used when reasoning through Euclidean diagrams.

Some handwritten notes of Kant from 1790 support my interpretation. They belong to the Kant-Eberhard dispute and are notes which were used by Kant's disciple Johann Schulze in Schulze's review of Eberhard's *Philosophisches Magazin*. In the notes Kant writes:[29]

> [I]t is very correctly said [by Kästner] that "Euclid assumes the possibility of drawing a straight line and describing a circle without proving it"—which means without proving this possibility *through inferences*. For *description*, which takes place a priori through the imagination in accordance with a rule and is called construction, is itself the proof of the possibility of the object. [... T]hat the possibility of a straight line and a circle can be proved, not *mediately* through inferences, but only immediately through the construction of these concepts (which is in no way empirical), is due to the circumstance that among all constructions (presentations determined in accordance with a rule in a priori intuition) some must still be *the first*—namely the *drawing* [*Ziehen*] or describing (in thought) of a straight line and the rotating of such a line around a fixed point [...] (Ak. 20, 410–11)

Thus some schemata are more basic than others: The drawing of a line and the describing of a circle. These very concepts are made possible only through the figurative synthesis by construction in pure intuition. And so, constructions by rule and compass are the primary and paradigmatic schematic geometrical constructions. Here Kant confirms what he writes in §24 of the B-deduction where he introduces the figurative synthesis (*synthesis speciosa*),[30] and later in the *Critique*:

[29] The following translation is Friedman's (2000, 189).

[30] "We cannot think of a line without **drawing** it in thought, we cannot think of a circle without describing it" (B154).

> Now in mathematics a postulate is the practical proposition that contains nothing except the synthesis through which we first give ourselves an object and generate its concept, e.g., to describe a circle with a given line from a given point on a plane; and a proposition of this sort cannot be proved since the procedure that it demands is precisely that through which we first generate the concept of such a figure. (A234/B287)

Central to the notion of schematic construction is that the construction is *pure*. The construction of, say the geometrical configuration, used for the proof of I.32, is an ideal construction. The diagram enables and supports the reasoning, but we are not measuring on the concrete diagram drawn on our paper. Thus the construction is (our) pure construction and we are not proving an "empirical proposition (through measurement of its angles), which would contain no universality, let alone necessity, and propositions of this sort are not under discussion here" (A718/B746), but *if* we use an empirical diagram then

> in the case of this empirical intuition we have taken account only of the action of constructing the concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent, and thus we have abstracted from these differences (A714/B742).

Therefore, when we *construct a geometrical concept a priori* we generate an empirical or pure image representing the concept. This image makes it possible for us to think the concept. But in order to get a concept construction a priori, the image must be understood *together* with the rules generating any image of that concept.

1.6.2 Geometry and syntheticity

Geometrical objects are constructible, *therefore* mathematics is synthetic. The type of constructibility and constructivism we find in propositions like I.22 and I.32—and ultimately the constructivism given by the first three postulates when seen together with diagrams—is paradigmatic to Kant. The geometrical objects are not as such temporal objects. They belong to outer sense and are therefore spatial and not temporal. A triangle is a pure form, which *can* be found in experience. Therefore, although the geometrical concepts are constructible, they are not subjective and contingent ideas. To *grasp* geometrical concepts, however, we need intuitions constructed by the figurative synthesis—the transcendental imagination. That the methodology based on concept construction is reliable is due to the fact that the type "triangle" is connected in a complete and reliable way to its tokens by the schema. Viewed in this way, mathematics, or at least geometry is a spatio-temporal *process* made possible by the transcendental imagination which mediates between inner and outer. Thus the inner and outer senses are connected in an essential way, namely by the imagination.[31] Thus our geometrical knowledge is synthetic in the sense that time functions as a pre-condition—without time we would not have any access to the mathematical properties.

If the knowledge expressed by, say, proposition I.32 had been analytic then the philosopher could have analyzed the three concepts (or rather the “angle”, “line” and the number “three” and he could have inferred that the sum of the three angles equal two right angles. But this is not possible, rather we have to “go beyond” the concepts, go to intuition and use our general geometrical capacities—the schemata—by which we in time construct new lines, new angles etc. all which together constitute an intuition. This is a singular object which taken *together* with the generating rules represent the concept of triangle. Then by analyzing this intuition and the rules generating it we discover I.32. One of the points is that “[t]houghts without content are empty”—we *need* intuitions in order to reason about mathematical concepts; we have access to the abstract concepts *only* through particular and concretely given instances falling under the concepts.

> I put together in a pure intuition, just as in an empirical one, the manifold that belongs to the schema of a triangle in general and thus to its concept, through which general synthetic propositions must be constructed (A719/B746)

But if we think through concretely drawn diagrams how can we obtain necessary and universal results? How do we gain universal knowledge from particulars?

1.6.3 Schemata and universality

Central to the schematism is that the schematic rules are universal: “[T]his representation of a general procedure of the imagination for providing a concept with its image is what I call the schema for this concept” (A140/B180–1). In the case of geometrical schemata the problem is specifically the problem of drawing a universal conclusion based on a singular instance. Let me quote again:

[31] Especially important evidence for this is found in §24 of the B-deduction, but is also at A162–3/B203–4: “I cannot represent to myself any line, no matter how small it may be, without drawing it in thought, i.e., successively generating all its parts from one point, and thereby first sketching this intuition. It is exactly the same with even the smallest time. I think therein only the successive progress from one moment to another, where through all parts of time and their addition a determinate magnitude of time is finally generated.” See also Ak. 20, 410-11 and A234/B287; both cited above.

> The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition we have taken account only of the action of constructing the concept (A714/B742).

My interpretation is that when we argue for a universal statement from a singular instance we use schemata in the following way. We want to prove that some property belongs generally to a mathematical concept:

1. We recognize a particular constructed figure—presented as a mental image or a figure drawn on paper—as a token of a type. We use schema or schemata for recognizing this.

2. Then we prove that the property we are after holds for this particular token.

3. We recognize that in this proof we have not used anything about that particular token which would not hold for any other token of that type. We use the schema in recognizing this.

4. Therefore, the type has the property, i.e., any token of that type has the property.

This is precisely what happens in the proof of I.32; "we have taken account only of the action of constructing the concept", and therefore we have not used properties such as "the magnitude of the sides and the angles" and since "we have abstracted from these differences" the figure represents the concept universally. Now this is of course nothing but a very typical way of proving universal statements in mathematics. We prove that some property A holds for a, then we recognize that none of our assumptions concerns a, except that a is of a certain type, say, the natural numbers; therefore we can conclude that every x of that type must have property A. Precisely this way of reasoning is codified by the natural deduction rule:[32]

$$\frac{\begin{array}{c}\vdots\\ A(a)\end{array}}{\forall x A(x)}$$

[32]Note, that all this is in fact not far away from what Aristotle says in the Metaphysics:

> [I]f we regard objects independently of their <accidents> and investigate any aspect of them as so regarded, we shall not be guilty of any error on this account, any more than when we draw a diagram on the ground and say that a line is a foot long when it is not; because the error is not in the premises. (Aristotle; 1933, 192)

1.6.4 Schemata and abstract concepts

One of the reasons why geometrical schemata can be successful in the way just described is that geometrical concepts are well-defined. As discussed on page 9 this is in contrast with empirical concepts. According to Kant (A727/B755) we can give necessary and sufficient criteria for, say, triangles.[33] Below I will argue that the objects of pure intuition are precisely those objects which can be constructed using the schematic rules. Therefore, we can *decide,* in a finite amount of time, whether an intuition is an instance or not of some concepts.

The distinction between image and schema is a way of making abstract concepts possible, and thus the distinction can be seen as a refutation of a crude empiricist claim that mathematical concepts, as simple generalizations, are illusionary. It is in this way that schemata ground or found mathematical concepts, as claimed for instance on A141/B180, and described on A234/B287. See page 21 for an analysis of the latter.[34]

Generally Kant views schemata as the mediating element "which must stand in homogeneity with the category on the one hand and the appearance on the other" (A138/B177). This applies as well to empirical concepts. The schema of dog is mediating between the empirical dog and the concept of dog in that it produces a mental image which as an representation mediates between the empirical and the intellectual.

The situation is different in the case of geometrical and mathematical concepts: The objects are not empirical objects, but pure intuitions. This leads Shabel (n.d., 24) to conclude that "[i]n the case of mathematical concepts then, schemata are strictly redundant: no "third thing" is needed to mediate between a mathematical concept and the objects that instantiate it since mathematical concepts come equipped with determinate conditions on and procedures for their construction".[35] I think this is a somewhat peculiar view. Though certainly, as Shabel notes, mathematical concepts are well-defined. Thus we can define precisely what it means to be an triangle. But a definition does not necessarily include a description of the schema. As Kant writes on A716/B744 "[g]ive a philosopher the concept of a triangle"—from the concept

[33]When using one of Locke's favorite examples Kant says on A728/B756 that this is not the case of empirical concepts: "[I]n the concept of **gold** one person might think, besides its weight, color, and ductility, its property of not rusting, while another might know nothing about this".

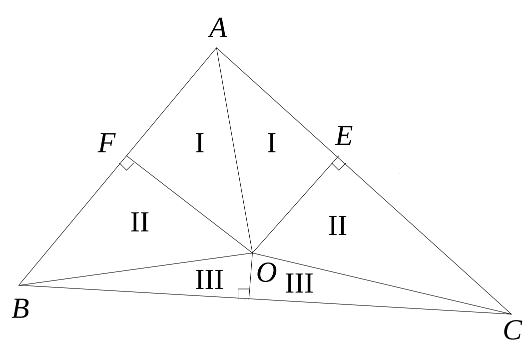
[34]That Leibniz had a huge in influence on Kant is beyond no doubt as Carl Posy (1995, 1998, 2000) has documented. But it is interesting that even this distinction between image and schema and the importance was already seen by Leibniz. In his discussion on Locke's *New Essays on Human Understanding* Leibniz notes "how essential it is to distinguish *images* from *exact ideas* which are composed of definitions" (Leibniz; 1996, 137).

[35]In this respect she follows Guyer (1987, 159).

alone, the philosopher can learn nothing about the sum of the interior angles. And, although the certain relation between a right angle and the sum of the interior angles is discoverable only by the schema, the relation itself belongs to concept, not to the schema. The schema of triangle is essentially a part of a larger enterprise being reducible, perhaps, to some fundamental capacities such as equivalents of Euclidean postulates, basic axioms resulting in general capacities for reasoning using intuitions. Of course I do not understand the concept triangle unless I posses its schema—but this is not the same as saying that the concept and its schema are identical. It is more reasonable to view a geometrical concept as a (passive) type, images as tokens, and the schema of the type as the constructive (i.e., active) relation between the type and its tokens. Such a constructive relation is to be understood, on the one hand, as our capacity for recognizing in finite time something as a token of a type; on the other hand it gives rise to a rule for constructing a paradigmatic and pure instantiation of that concept. The schema is therefore a *decidable and constructive* procedure in the sense that we can decide in finite time the type to which a token belongs, but also that we can 'go from type to token' through construction.

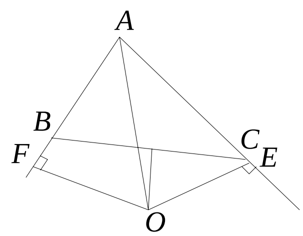
1.6.5 A contemporary critique of Euclid

As already mentioned it is not uncommon to find sharp criticism of Euclid's method. Kline (1972, 1005) mentions the "[d]efects in Euclid" and presents—what is taken to be—a counterexample to the Euclidean use of diagrams. The example is the following "proof" that every triangle is isosceles. Let a triangle *ABC* be given. Construct an angle bisector at *A* and construct a line bisecting *BC*. If these two lines are parallel the triangle is isosceles. Suppose they are not parallel and that they meet in *O*.



Construct *OF* perpendicular to *AB*; and likewise *OE* perpendicular to *AC*. It is easy to see that the triangles marked I in the diagram are congruent; therefore *AF* and *AE* are equal. Construct also *OB* and *OC*. By construction triangles III are congruent. From this it follows that triangles II are congruent, so FB and EC are equal. Consequently, AB and AC are equal.

Now, one could question the position of O and in fact it necessarily falls outside the triangle. So we are apparently dealing with this configuration:



But we can again "prove" ABC to be isosceles. Now, every triangle is of course not isosceles, so what is wrong? The positions of E and F are wrong. In fact, it *can* be proved that either E is between A and C and F is outside the triangle, or F is between A and B and E is outside. Given the correct determination of the positions of E and F the "proof" does not go through.

> But this means that we must be able to determine the correct position of F with respect to A and B and E with respect to A and C before starting the proof. Of course one should not rely upon drawing a correct figure to determine the locations of E and F, but this was precisely what Euclid and the mathematicians up to 1800 did. Euclidean geometry was supposed to have offered accurate proofs of theorems suggested intuitively by figures, but actually it offered intuitive proofs of accurately drawn figures" (Kline; 1972, 1007).

I do not find this criticism reasonable. Firstly, the above "proof" is (of course) not to be found in *Elements*. Secondly, I find it highly implausible, that Euclid would ever produce a proof of this character. Euclid, as we have seen, is always very careful and refers to propositions proved earlier. Thus, to produce a correct proof (which does not exists) one would have to know the positions of E and F. Euclid would have established these in a proposition used as a lemma.

But more importantly: It is reasonable to characterize the "proof" as empirical. Attention is *only* paid towards the constructed empirical figure (drawn incorrectly), which *should* symbolically refer to the pure intuition. *If* this reference is taken into account then the particular properties of the empirical figure are of no importance. But in this case, the reference is not taken into account. About such a methodology Kant writes: "[This] would yield only an empirical proposition (through measurement of its angles), which would contain no universality" (A718/B746). On the other hand, Euclid's methodology consists of taking the diagram *together* with the rules producing the diagram. Thus I find it difficult to understand how it should be a counterexample to Euclid. In principle there is nothing wrong with Euclid's use of diagrams. Of course Euclid is sometimes mistaken, sometimes his diagrammatic reasoning is flawed. As is well-known, diagrams *can* be misleading.[36]

1.6.6 Hilbert and the axiomatic method

Also Hilbert has reservations in regard to the use of diagrams, as seen on page 6, but he represents a rather subtle position.

In his famous lecture delivered before the International Congress of Mathematicians at Paris in 1900 Hilbert he argues (1902c, 442–3) that there are many important styles of reasoning within mathematics, and argues strongly against a kind of reductionism which advocates—in the strive for rigorous proofs—an elimination of all concepts which do not belong to analysis and, ultimately, to number theory. Geometrical rigor certainly exists:

> The use of geometrical signs [*Zeichen*] as a means of strict proof presupposes the exact knowledge and complete mastery of axioms which underlie those figures; and in order that these geometrical figures may be incorporated in the general treasure of mathematical signs, there is necessary a rigorous axiomatic investigation of their conceptual content [...T]he use of geometrical signs is determined by the axioms of geometrical concepts and their combination. (Hilbert; 1902c, 443)

Note how close Hilbert seems to Kant here. Hilbert argues, that we *can* use diagrams when we have exact knowledge about the procedures which ground the figures—i.e., when we posses and understand the schemata, we can use figures in a fully rigorous way; the use of symbols, such as diagrams, is determined by axioms. But how does this relate with the views expressed by Pasch and Hilbert earlier.

It seems that Pasch is interested in the formalistic conception of proof for the sake of rigor. This, however, is not the main motivation for Hilbert when applying formal methods in geometry. For Hilbert the motivation is new discoveries, made *possible* by the axiomatic method.

Hilbert's presentation in the seminal *Grundlagen* (1902a) deals with different kinds of geometries. The axioms are not treated as self-evident axioms which are true in some absolute sense. Hilbert is interested in the groups of axioms, their relations and consequences. There are five groups:

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[36]As mentioned on page 6, diagrams could lead one to think that any continuous function is differential except in isolated points—which is false.

I. Axioms of connection.

II. Axioms of order.

III. Axioms of parallels.

IV. Axioms of congruence.

V. Axioms of continuity.

Hilbert shows that the groups are pairwise independent. This is done by providing different models, validating different groups of axioms. Thus the groups give rise to different geometries such as Euclidean, non-Euclidean, Archimedian and non-Archimedian geometries. Therefore, when Hilbert talks about lines they can be lines as Euclid understands them, or they can be great circles on a surface of a sphere, or they can be something quite different. Hilbert's treatment is much more abstract and the objects that the theory refers are not fixed.

The core of Hilbert's method is that he detaches the geometrical concepts from their semantics. Hilbert is not studying one, and only one, particular model. He is open minded towards a whole variety of models. *This is not for the sake of rigor—it is for the sake of discovery*. By his axiomatic method Hilbert opened a gate-way for new discoveries such as the connections between projective geometry and algebra as was found in the beginning of the 20th century.[37]

Thus, by the axiomatic formal method discoveries are made possible; discoveries which are not possible in the Euclidean-Kantian way of doing geometry. The separation of language and interpretation; the whole idea of an uninterpreted language is as remote as it can be for Kant. When Hilbert is making the detachment he is not anymore studying what Kant takes to be the object of study for mathematics: The concrete schematic images. Rather, Hilbert studies the axioms—the schemata. But *the schemata are taken as formal rules with no fixed semantics*. Thus postulate 1, does not necessarily produce a Euclidean line given two Euclidean points. For Kant, on the other hand, *schemata are semantics* and therefore it is a very drastic step to treat a schema not as a rule determining some specific semantics.

For Kant schemata produces the mathematical *objects* such as lines, circles, triangles and the like; and these are the only mathematical objects. When doing mathematics schemata are applied whereby new properties are discovered in synthetic

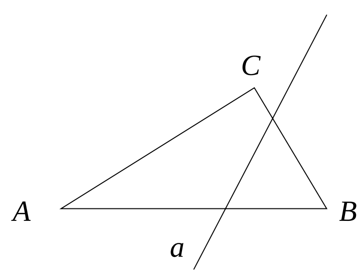
[37] Take for example, relations between geometrical theorems such as the theorems of Desargues and Pappus on the one hand, and properties of rings on the other. Desargues' theorem, taken as an axiom, corresponds to additional algebraic conditions on the addition and multiplication to form a skew field. (i.e. a field minus commutativity of multiplication). Also, that the multiplication operation be commutative corresponds to a further geometric axiom, namely the Pappus theorem.

ways. But, where schemata produce objects for mathematics *they are not themselves objects*. As we shall see below, on Kant's view there is only one source for mathematical objects: Spatial objects. Thus to Kant, geometry concerns the properties of objects constructed in pure intuition; the sum of the angles in a triangle, for example. Hilbert's geometry is on a level higher—his main concern is not to prove new theorems about spatial objects. His investigation concerns the axioms. Thus Hilbert also rejects Kant's thesis that the axioms are transcendental procedures which cannot themselves be objects. To Hilbert axioms are true objects of another type and they are studied by the axiomatic method

Nevertheless, geometry as such is *not* seen as some formal and meaningless theory with no content; Hilbert's geometry is an investigation of how we conceptualize space, thus "[t]his problem is tantamount to the logical analysis of our intuition of space" (Hilbert; 1902a, 1)

Hilbert rejects Kant's thesis, that schemata are not the objects of mathematics. As he also raises questions about the existence of one and only one geometry, he is positive towards the possibility that our conception of space is not necessarily Euclidean. But as seen above, Hilbert does not, in general reject, the Euclidean use of diagrams in reasoning—but we must be completely explicit about the "exact knowledge and complete mastery of axioms which lie at the foundation of those figures" that we may want to use in our proofs. Thus he includes, for instance, Pasch's axiom (Hilbert; 1902a, 7). It is certainly not trivial that it is necessary to include this axiom:

> Pasch's axiom. Let A, B, C be three points not lying in the same straight line and let a line a be a straight line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the straight line a passes through a point in the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.



1.6.7 Conclusion and perspectives on geometrical schemata

Kant wants to show *that* and *how* Euclidean geometry is possible. By the theory of schemata we now know how. Kant argues that geometry is our conceptualization of space. We have a concept of a triangle, for example. We can think and reason about our concepts of space through, and only through, singular pure intuitions. These intuitions are produced by geometrical schemata, and are therefore not simple generalizations on perceptions, rather they are constructed *a priori* to any (empirical) experience. The intuitions gain, nevertheless, their objectivity from the fact that they *can* in principle be met in experience.[38]

A concept and its schema are two different things, but we only understand a concept when we are able to use its schema. The schema is, moreover, a condition for the concept in the sense that it makes it possible, and that we through schemata can learn properties of these abstract concepts. As such the interplay, primarily, between basic schemata (equivalents to Euclid's postulates), their concepts and intuitions explains how we can reason about our conceptualization of space and obtain knowledge about it. In principle I find Kant's analysis completely sound and satisfactory as a theory about Euclidean reasoning: Proving theorems about mathematical concepts through the use of basic postulates and other axioms by using particular diagrams (intuitions).

In fact one can understand Kant's theory of schemata as the beginning of a theory of diagrammatic reasoning. Take for instance, the diagram-proof of

$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

given on page 7. We want to prove this property with respect to all numbers. Using the schema for numbers we realize that any sequence $1,2,\ldots,n$ can be represented in a square with sides of length n. In realizing this we are also using our geometrical schema of square. Take the number 5, and represent it by a square. By some geometrical reasoning we prove the wanted property for this particular token of the number 5. Using the schema for the concept of square we see that the argument applies to any square. And we know that the sequence $1,2,\ldots,n$ can be represented with respect to all its properties by a square. Therefore the theorem is proved for all numbers.

[38]This is a crucial point in Kant's philosophy of mathematics. I will take this up later in this text, see for instance the long quotation on page 44. The relation between pure and empirical intuition is discussed in subsection 3.6 found in chapter 3. Moreover, in chapter 5 I discuss in subsection 5.2 this thesis of objectivity that Kant endorses. I criticize it to the effect that numbers, for instance, can be objects.

CHAPTER 2

Kant and Arithmetical Schematism

In the forgoing chapter the Kant's concept of geometrical schema was analyzed. We saw that Euclidean geometry and the general reasoning style in *Elements* are essentially what Kant wants to give an epistemological account of. This leads Kant to his schemata of pure sensible concepts. In the coming chapter we will see how this theory functions as the paradigmatic example, which Kant in certain ways builds his more general theory of transcendental schemata upon. We will, however, only work with one type of categories: The categories of quantity.

> The pure **schema of magnitude** [***Größe***] (*quantitatis*), however, as a concept of the understanding, is **number,** which is a representation that summarizes [*zusammenfaßt*] the successive addition of one (homogeneous) unit to another. (A142/B182)

Now, it must be admitted that even with regard to some of the most fundamental aspects of his epistemology Kant is not completely clear. Here, however, it seems clear that "magnitude [*Größe*]" is a category—but not which one. In the "Table of Categories" (A80/B106) there are four main divisions of the categories; the first one being "Of Quantity [*Der Quantität*]". This consists of the three categories *unity, plurality* and *totality*. Now, "magnitude" could either refer to the collection of the three categories, or it could be one of them. Let us elaborate a little on this.

2.1 Number as schema

Quite generally Kant claims that it is equivalent to cognize and to perform judgments.[1] Therefore "the Clue to the Discovery of all Pure Concepts of the Understanding" (A70/B95) goes via the different ways we form judgments. In respect to quantity, Kant is completely Aristotelian: There are *singular, particular* and *universal* judgments (for instance, 'this body has mass', 'some bodies have mass' and 'all bodies have mass'). These different types of judgments lead Kant to the three categories unity, plurality and totality. The singular judgment corresponds to the

[1] See for instance the important A69/B94:

> We can, however, trace all actions of the understanding back to judgments, so that the **understanding** in general can be represented as a **faculty for judging.** For according to what has been said above it is a faculty for thinking. Thinking is cognition through concepts. Concepts, however, as predicates of possible judgments, are related to some representation of a still undetermined object.

category of unity; the particular judgment to plurality and the universal to totality.[2] Totally there are 12 categories, and just after stating what "the schema of magnitude" is Kant continues the Schematism by listing the nine remaining schemata—one after the other (A143–7/B182–7). This can be taken as evidence indicating that Kant took "the schema of magnitude" to be a schema common to the three categories unity, plurality and totality. And as Longuenesse notes (1998, 254) the three different categories seem to be involved in the definition of the schema: unity ("units"); plurality ("successive addition of one (homogeneous) unit to another"); and totality ("a representation that summarizes the successive addition of one (homogeneous) unit to another"). I think, however, that it is more plausible, to understand "the schema of magnitude" to be the schema of *totality*.[3] The third category in all the four divisions is always understood as a "combination of the first and second in order to bring forth the third" (B111). "Thus **allness** (totality) is nothing other than plurality considered as a unity" (B111). In *Prolegomena*, when listing the categories, Kant terms the first category of Quantity as "Unity (Measure [*das Maß*])" (Ak. 4, 303). Therefore, *Unity* is the concept we use when we judge something to be a unit; 'this body—taken as a unit—has mass'; but also 'this unit can be taken as a unit, when we want to count—it can be seen as (giving rise to) a measure'. A determination of a unit 'body' together with a judgment 'here is a plurality of divisible bodies' can form a new judgment, when we think another unit: 'All bodies in *this collection* are divisible'. In this way it is seen that totality is something more and something different than the two first categories, because:

> to bring forth the third [pure] concept requires a special act of the understanding, which is not identical with that act performed in the first and second. Thus the concept of a **number** (which belongs to the category of allness) is not always possible wherever the concepts of multitude and of unity are (e.g., in the representation of the infinite) (B111).

Thus, the category of totality is not reducible to unity and plurality and, moreover, it is "number" which belongs to totality. If we re-read the definition of "the schema of magnitude" in this light we see that it corresponds very well to the act performed,

[2]It is remarkable that it is not even completely clear from Kant's texts precisely what the correspondence between the Quantity of Judgments and the Table of Categories is in this case. Just like Paton (1936, II, 44) and Longuenesse (1998, 249), I understand Kant in such a way that the (Aristotelian) judgments are given in the traditional order: Universal–particular–singular, whereas the order of the categories are given as unity–plurality–universal. Therefore, the order of one of the tables is the reverse of the other. This interpretation is in contrast with Hartnack (1968, 39) and (Tiles; 2004).

[3]Here I am close to Longuenesse (1998, 253–255), although she seems to *identify* plurality and totality.

when totality—as a combination of unity and plurality—is performed: A certain unit (measure) has been determined; and we have encountered a plurality of these units. When reflecting *on* this plurality, we form a set out of the homogeneous elements and *enumerate* (summarize) the elements. This enumeration ends with a number, which is the number of elements in the set.[4] Therefore, when we apply "the schema for magnitude" we think unity in plurality. Generally "the schema of magnitude" is the ability humans have for determining finite extensions of empirical concepts. For instance 'there are five fingers on my left hand': The unit is 'finger'; the context is 'my left hand'; and there are totally five of them. The rule determining this act of enumerating and counting is the schema called "number".

But how precisely do we operate with this schema? It is a transcendental schema and there are certain important differences between the transcendental schemata and the geometrical schemata. "The schema of a pure concept of the understanding, on the contrary [to empirical and geometrical schemata], is something that can never be brought to an image at all" (A142/B181).[5] Rather than producing images, transcendental schemata provide a "transcendental time-determination" (A138–9/B178–9) of the objects given in experience. As empirical objects are given to us in outer sense (space) and time-determination is a determination of moments according to inner

[4]But as Kant notes, we cannot reduce the third category to the two forgoing, because sometimes we encounter an infinite plurality of units which do not form a set. Thus, the set-class distinction from set theory is reflected in Kant's epistemology. We can for instance, form a sequence of growing finite (in the sense that the volume is finite) empirical spaces. The space containing my office; the space containing my city; the space containing my country, the earth and so on. All these spaces are possible objects of intuition. But the *collection* of all finite space—the absolute empirical space—which is needed for Newtonian mechanics in the definition of absolute motion is not a possible object of intuition and "absolute space is *in itself* therefore nothing and indeed no object at all" as Kant writes in *Metaphysical Foundation* (Ak. 4, 481). I can measure all the finite spaces, but their collection is not measurable. Nevertheless, the concept of "*motion in absolute* (immovable) *space*" "in general natural science is unavoidable" (Ak. 4, 558), so "[a]bsolute space is therefore necessary, not as a concept of an actual object, but rather as an idea", i.e., as a concept of *reason* (Ak. 4, 560). In set theory any *set* of ordinals is itself measurable by an ordinal, but the class of all ordinals, is not a set, because it is not measurable by any ordinal.

[5]In fact, this fundamental difference between the arithmetical schema number and the geometrical schemata leads Kant to reject the possibility axioms for arithmetic. This well-known claim of Kant is expressed on pages A163–4/B205. This view seems rather awkward in a contemporary understanding and I will elaborate on it below.

sense,[6] this determination proceeds *mediately* by way of inner images.[7] Thus the schematism of pure concepts—in contrast to geometrical and empirical concepts—is more about *reflection* on images, rather than *construction*. This observation somehow runs counter to Shabel's (2003a, 109) claim (which I also mentioned on page 10 as the "sharpened claim") that the diagrammatic reasoning in Euclid—which Kant supplies an epistemological analysis of in terms of geometrical schemata—"provides an interpretive model for the function of a transcendental schema". Nevertheless, it will become clear that Kant's theory of geometrical schemata and his theory of transcendental schemata have many properties in common. The most important one being that schemata provide a foundation and explanation of the use of types and tokens.

In the following I will give my interpretation of the transcendental "**schema of magnitude**". As will be clear this really is an interpretation. Kant does not write much about the transcendental schemata and many of the details are 'left to the reader as an easy exercise'. But this exercise is not an easy one, as Kant certainly is not very clear on the whole issue. This is in contrast with the geometrical schemata. Kant's theory of these is much more clear than when he is discussing the transcendental schemata. In consequence of this I will in the first place not stay close to the text. This is for the sake of giving a coherent (as coherent as it can be at least) interpretation of the schema of magnitude. After having provided my interpretation I will go back to Kant's text and see whether or not I am able to supply difficult passages with meaning. If I succeed in doing this my interpretation will be justified.

Let me return for a moment to the empirical schemata which I will give the following interpretation. Let us view the collection of all the different images representing empirical objects as a constructive and non-monotone open-ended universe. It is not monotonic as our empirical concepts may vary over time and it is constructive in the sense that we produce images in time in accordance with rules. Furthermore, it is open-ended as our collection of empirical concepts is certainly not fixed once and for all. However, given a point i in time we can take the universe of images which are *in principle constructible* according to the collection of empirical concepts we may possess at i. Let us call this snapshot U_i. Any empirical concept partitions U_i in two sets. One of the sets consisting of the images representing the concept, the other set

[6] "Time is nothing other than the form of inner sense, i.e., of the intuition of our self and our inner state. For time cannot be a determination of outer appearances; it belongs neither to a shape or a position, etc., but on the contrary determines the relation of representations in our inner state." (A33/B49-50)

[7] "[T]ime is an *a priori* condition of all appearance in general, and indeed the immediate condition of the inner intuition (of our souls), and thereby also the mediate condition of outer appearances." (A34/B51)

consisting of those which do not represent the concept.[8]

"[M]agnitude (*quantitatis*)" concerns the question: "How big is something?" (A163/B204). An example could be: 'How many fingers are there on my left hand?'. Let $\prec$ denote the order of time.[9] I have experienced a plurality of fingers and let $i \prec j \prec k \prec l \prec m$ be the different moments in time corresponding to these experiences. Although the sequence of image-universes is generally not monotone, let me assume that *locally* there is monotonicity such that the partitions of $U_i, \ldots, U_m$ consisting of finger-images is the same from i to m. Alternatively we could assume that the schema belonging to concept 'finger-on-my-left-hand' is constant. Such a pragmatic assumption seems reasonable.[10] Therefore *any* image from the partition can be used to represent any finger from time i to time m. Let x be such an image. x represents any of the fingers that I experience on my left hand. The minimal requirement making the images different is their location in time.[11] Time "determines the relation of representations in our inner state" (A33/B49-50). Therefore we have *temporized* images: x_i, x_j, x_k, x_l and x_m. This is my interpretation of Kant's "transcendental time-determination", and note that the temporal indexes are *necessary* for my judgment 'there is a plurality of fingers'. Now, by judging unity in plurality I form the set M consisting of the temporized images which I simultaneously count ("summarize"). This is done by enumerating M. Mathematically speaking this is a determination of a set of natural numbers which is equinumerous with M. In other words, we determine a set of natural numbers N and establish an injection f from N to M. The canonical domain for f is, of course, $\{1,2,3,4,5\}$. Figure 2.1 represents this mental process.

On pages A142–3/B182 Kant defines the "schema of magnitude" in one paragraph. I gave most of the first half of that paragraph on page 32. The second part goes:

> Thus number is nothing other than the unity of the synthesis of the mani-

[8] Of course this is an idealization which is perhaps not fully justified. Empirical concepts are vague concepts, and therefore it is perhaps not possible, given a concept P, to form two *disjoint* sets U_i^P and $U_i^{\neg P}$ such that $U_i = U_i^P \cup U_i^{\neg P}$. This problem is, however, not a problem which threatens the interpretation of "the schema of magnitude", and therefore I will make this idealization.

[9] Kant's precise understanding of the order of time is not important for this example. For our example it only matters that the collection of past moments is linearly ordered.

[10] The assumption of monotonicity locally around a concept seems reasonable, given that I in the time from i to m do not discover new essential properties of the concept, and given that I do not forget any of the essential properties either.

[11] We could also imagine another way to make the images different, namely that they are cognized as fingers with coordinates in space. This would also separate the images. But this would only be an additional property making the images different, because we cannot help that the images are *constructed* in time. Thus the images with spatial coordinates would be different both with respect to time and coordinates.

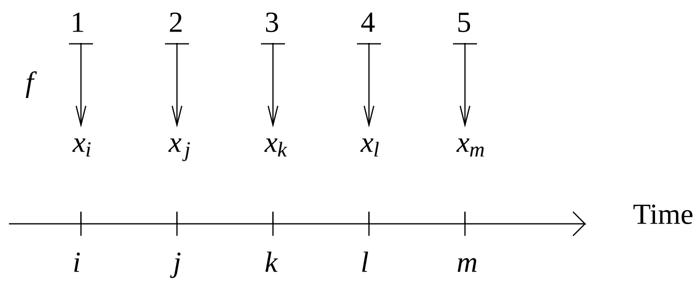


FIGURE 2.1

> fold of a homogeneous intuition in general, because I generate time itself in the apprehension of the intuition. (A142–3/B182)

In terms of the finger-example my interpretation is the following.

Firstly, I can produce the injection f "because I generate time itself in the apprehension of the intuition"—it is I who locate the images according to inner sense. This temporization is necessary. Secondly, that "number is nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general" means that the number 5 describes the unity of my set of fingers in the sense that a bijection (f is of course also surjective) can be established between the set of natural numbers less than or equal to 5 and M. But what precisely does the correspondence between 5 and M consist of?

The order of time $\prec$ induces a natural order on the temporized images; let $\prec$ denote also the induced order. The function f specified in Figure 2.1 is actually showing that $(\{1,2,3,4,5\},<)$ and $(M,\prec)$ are isomorphic. Could it be a notion along these lines that Kant has in mind when saying that number 5 *is* "the unity of the synthesis"? The bijection established would then show that the two sets are *equal* up to isomorphism. I think this is not Kant's idea. We are only interested in the *size (größe)* of M, not its order. Thus it is 5 as a cardinal number and not as an ordinal number Kant is interested in. The unity Kant is after is the unity which is expressed through *cardinality*. A contemporary mathematical understanding of size is the following. A and B are *equinumerous* or *equal in cardinality*, if a bijection between the two exists:

$$A =_c B, \text{ if and only if, there exists a bijection } f \text{ such that } f : A \to B.$$

Of course, given a set N of temporized images with n elements any bijection g between N and $\{1,\ldots,n\}$ gives rise to an isomorphism between $(\{1,\ldots,n\},<)$ and $(N,<_g)$, where $<_g$ is the order induced by g—but this additional information is not what Kant has in mind. It is information which lies *in* the process determining the cardinality of any finite set, but one has to pay attention to this. I understand Kant as saying that this type of information is not what we are after when applying the category of totality, although it can be unwinded from the process. Thus, in the finger-example 5 reflects the unity of the synthesis with respect to *cardinality*. The equality-relation is not isomorphism, rather it is $=_c$. Thus, the function f depicted in Figure 2.1 could be any bijection between M and $\{1,2,3,4,5\}$.[12]

I can now give my general interpretation of the "schema of magnitude". The task of the schema is to give unity to a succession of objects thought under the same concept; objects that we represent as temporized images. In our interaction with the world we temporize images. Before temporization the images belong to the universe of images constructible in principle. For the sake of presentational simplicity let us assume that this universe is constant, i.e., that my empirical schemata generating images are the same over time. Later I will dismiss this restriction. From this universe we form, by "transcendental time-determination" a temporized derivative, namely, the universe U of temporized images.

The general problem concerns the possible unity of a plurality of experiences of objects falling under a concept. Let us in accordance with this purpose form a second order universe $\mathscr{U}$ consisting of sets of temporized images representing the same concept. Thus the definition of $\mathscr{U}$:

$$M \in \mathscr{U} \text{ iff } M \text{ is a subset of } U \text{ and the members of } M \text{ represent the same concept.}$$

Clearly the notion of *equal in cardinality*, $=_c$ is an equivalence relation on $mathscrU$. The "schema of magnitude" which Kant calls "number" is the general rule which generates this equivalence relation. When we judge unity in a plurality of homogeneous units we determine the equivalence class to which the unit belongs. And if no such class can be found we cannot judge unity: A plurality of experienced units can be judged to be a unit itself *only if*, we can determine a cardinality of this plurality. Therefore, the generation of the members of $\mathscr{U}$ proceeds by production of the equivalence classes. An M becomes a member of $\mathscr{U}$, when it becomes a member of an equivalence class of $\mathscr{U}$. Thus it is a genuine *constructive* notion of existence.

[12] Of which there are many: 5!.

Let us now see, what the content of the rule “number” is. First of all, it consists of a “transcendental time-determination”, as otherwise we cannot distinguish between images representing the same concept. But what is the content of the act which partitions $\mathscr{U}$ in accordance with $=_c$? It is our ability to enumerate finite sets, and the mathematical description of this, is the ability to produce bijections between finite sets.

In this way, we see that the “schema of magnitude” is reflective because of reflection on already constructed first-order level images. But it is constructive on a second-order level; the sets of $\mathscr{U}$ are determined (second-order constructed) by the schema, and they live in $\mathscr{U}$ only when this determination has taken place. Moreover, the schema is an *act* of the understanding which itself takes place in time, temporal succession is a transcendental condition:

> No one can define the concept of magnitude in general except by something like this: That it is the determination of a thing through which it can be thought how many units are posited in it. Only this how–many–times is grounded on successive repetition, thus on time and the synthesis (of the homogeneous) in it. (A242/B300)

Let us now dismiss the restriction posed on the universe of images. Thus a certain invariance of meaning of images can happen, as an image can represent a concept at a certain time and not represent that concept at another time. Thus the universe $\mathscr{U}$ is a dynamical floating universe where the equivalence classes defined by $=_c$ are not the same over time. Does this pose a problem for the status of the *pure* concept of totality? Certainly not, given any variant of $\mathscr{U}$ the rules determining equivalence classes according to $=_c$ are precisely the same. The underlying universe may vary, as our empirical concepts vary, but the structure imposed on it, is the same. Of course there are extensions of empirical concepts which we are unable to determine, of which Sorites paradox provides excellent examples. But this is not a problem for our theory of magnitude, it is (perhaps) a problem for our theory of empirical concepts.

2.2 Number as concept

It must be admitted that my interpretation is somewhat involved. I am, for instance, using notions like equivalence-class and second order object—notions which were not present in the mathematics of Kant’s time. On the other hand, it is also true that the bijections involved for example are constructively fully meaningful: We are only working with functions from finite sets to finite sets—the essence of that is simply to pair objects from two different finite sets. Moreover, the universes mentioned should certainly *not* be understood in some kind of Platonic sense—rather they are universes constructed by a cognizing human.

Some interpretation *is* needed as Kant's own text is not crystal clear neither sufficiently detailed. Kant designates the constructive procedure used when counting as "**number**".[13] But from his text (or texts) alone it is not clear precisely what he means. For a full justification of my interpretation I should be able, however, to explain central themes in the Kantian theory of transcendental schemata. One of the most distinctive ones is, that arithmetic has no axioms. I will take that up in chapter **??**. Another important distinction in Kant's theory is the distinction between 'number as concept' as opposed to 'number as schema'.

Kant notes on A142/B181 that a transcendental schema "is something that can never be brought to an image at all". His point is that there are no generic images of the pure concepts. In contrast to empirical schemata, pure schemata do not provide images representing the pure concept. How would a paradigmatic image of causality look like, for instance? Somehow in contrast to this Kant writes when discussing the difference between image and schema, that

> "if I place five points in a row, ..., this is an image of the number five. On the contrary, if I only think a number in general, which could be five or a hundred, this thinking is more the representation of a method for representing a multitude (e.g., a thousand) in accordance with a certain concept than the image itself (A149/B179)

Thus there is a concept for the number five, but there is also a schema called "number" which is an act of the understanding or a "representation of a method for representing a multitude". Therefore we have to distinguish between the concept number, and the schema number. The number five is *not* the rule synthesizing my finger-images x_i, x_j, x_k, x_l, x_m, but rather a concept of the specific size—the cardinality—of the corresponding set. This cardinality is realized through the enumeration which simultaneously determines set-hood (unity) and cardinality. Thus number is a concept under which a multiplicity is thought. This concept is thought through the schema "number". The schema is the *procedure*, i.e., a rule-governed activity, that we use to determine whether a given collection of sensible things exhibits unity in plurality.[14]

[13]Note also that in German to count is *zählen*—a derivative of *Zahl*.

[14]In the A-deduction Kant writes: " If, in counting, I forget that the units that how hover before my senses were successively added to each other by me, then I would not cognize the generation of the multitude through this successive addition of one to the other, and consequently I would not cognize the number; for this concept consist solely in the consciousness of this unity of the synthesis" (A103).

Let $\overline{n}$ denote the class of elements in $\mathscr{U}$ which are equivalent $\{1,\ldots,n\}$ with respect to cardinality, in other words:

$$\overline{n} = \{\, \{x_1,\ldots,x_n\} \in \mathscr{U} \mid \{x_1,\ldots,x_n\} =_c \{1,\ldots,n\} \,\}.$$

Now, I propose to understand the *concept* of a particular number n as a type, in fact more specifically as the corresponding equivalence class $\overline{n}$. We know, however, that the elements of this equivalence class is not constant: An element of $\mathscr{U}$ is constructed as a second order object at a certain moment in time and trough this construction the element becomes a member of an equivalence class. Therefore, if we understand the concept five as an equivalence class in the set theoretic sense described above, then the equivalence class is *not* determined by its extension, it is rather determined intensionally, namely the "schema of magnitude" which amounts to the capacity of producing bijections. Only this *intensional* aspect can guarantee that number concepts remain the same over time. Therefore, if the $\overline{x}$ and $\overline{y}$ are number concepts, possibly 'found' at different moments in time, then the equality of $\overline{x}$ and $\overline{y}$ is determined not by their extensions but by the bijections—understood as rules—on which they are generated. In other words $\overline{x} = \overline{y}$, if and only if, the two canonical bijections are the same.

Thus we understand a concept of a certain number as a cardinal number being a *type* whose tokens are members of the corresponding equivalence class. The "schema of magnitude" decides the relation between the type $\overline{n}$ and the tokens falling under this type. It is due to the intensional aspect that the properties a particular number concept are independent of time. Moreover, each and all of the (finite) cardinal numbers are founded by the same schema. See Figure 2.2 for a diagram representing my interpretation.[15]

So the schema and the concept of number certainly are different.

2.3 The arithmetical schema and universality

In the case of geometrical schemata we saw that they make reasoning about say the general triangle possible through reasoning about one particular. My claim is that Kant holds the view that we are in a similar position in the case of arithmetic. The form of universality which emerges in the case of arithmetical schemata can be gained by using "the schema of magnitude". On the face of it, this may seem problematic as the extension of an arithmetical equivalence class is not constant. On the other

[15] Although generally my interpretation is different from the one given by Longuenesse (1998) I think my understanding of the difference between the concept number and the schema is close to her's being "that the concepts of number is the concept of a *determinate* quantity, and that number as a schema is the schema of *determinate* quantity" (1998, 256).

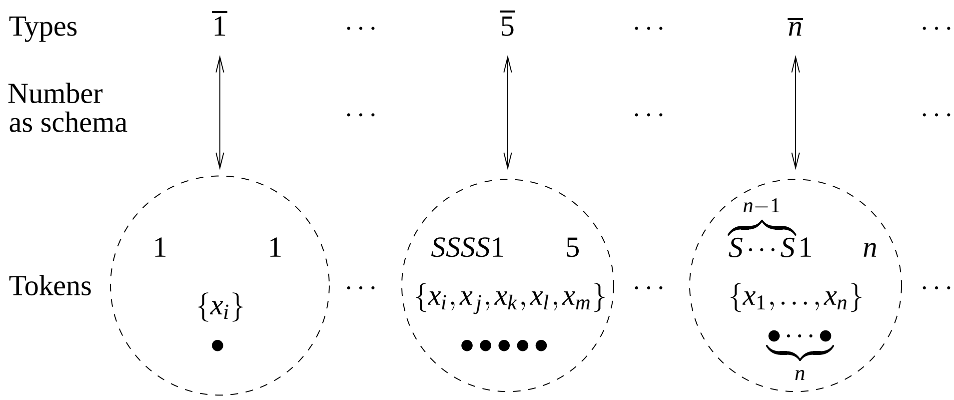


FIGURE 2.2

hand, the members of the geometrical equivalence classes are pure intuitions and a pure intuition belongs to the type triangle, if and only if, it can be constructed in pure intuition by the schema of triangle. This property is invariant over time—therefore the geometrical types are extensionally determined. Nevertheless, when explaining the synthetic nature of propositions in arithmetic Kant writes:

> The concept of twelve is by no means already thought merely by my thinking of that unification of seven and five, and no matter how long I analyze my concept of such a possible sum I will still not find twelve in it. One must go beyond these concepts, seeking assistance in the intuition that corresponds to one of the two, one's five fingers, say, or (as in Segner's arithmetic) five points, and one after another add the units of the five given in the intuition to the concept of seven. For I take first the number 7, and, as I take the fingers of my hand as an intuition for assistance with the concept of 5, to that image of mine I now add the units that I have previously taken together in order to constitute the number 5 one after another to the number 7, and thus see the number 12 arise. (B15–6)

This quote is central for an understanding of Kant's conception of numbers and arithmetic. Kant is claiming two important properties of arithmetic:

1. The simple propositions of arithmetic like $7+5=12$ are not analytic; "[o]ne must go beyond these concepts, seeking assistance in the intuition" in order to realize that it actually is the case that 5 added to 7 yields 12.

2. For the verification of the correctness of $7+5=12$ I can use contingent *empirical* representations of 5—fingers on my hand.

I will treat the syntheticity of numbers in the next section. Here I focus on the second point. How can I obtain a necessary proposition about all possible sets with cardinality 5 by using one particular set with cardinality 5? It could happen that I by accident could loose a finger, or that my concept of "finger-on-my-hand" would change over time such that a thumb would no longer be a finger. The answer is of course that due to the "schema of magnitude" we realize the following: When we use the set of (temporized images of the) five fingers we use it *only* with respect to its magnitude: We understand that we could have used any other set with five members, i.e., we could have used any other member of the equivalence class $\overline{5}$ for the verification. The situation is strikingly close to geometrical schemata only there, however, it is about construction and not reflection:

> The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition we have taken account only of the action of constructing the concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent, and thus we have abstracted from these differences, which do not alter the concept of the triangle. (A713–4/B741–2)

In the case of the set of five finger-images, we realize they represent "the concept without damage to its universality, for in the case of this empirical intuition we have taken account only" of its size, i.e., its cardinality, as determined by the "schema of magnitude". The construction which takes place is on a higher level, namely that *we think* of the five fingers as constituting a set. We realize, however, that we could have used any other set with this magnitude, and the similarity to the geometrical reasoning is striking: When we prove a proposition about a concept, say the number five or a triangle, we take an intuition representing the concept *together* with the rules determining any intuition falling under that concept. The "schema of magnitude" thus secures that we can operate with specific images of numbers, taking them as representatives of their types, prove properties about the images and be sure that these properties apply, not only of the specific images (i.e., the tokens) but to any image representing the type.

Precisely this aspect of Kant's schematism was not understood by Frege (1980). In §, 13 Frege ascribes to Kant's position that "each number has its own peculiarities. To what extent a given particular number can represent all the others, and at what point is own special character comes into play, cannot be laid down generally in advance." Well, the schema will take care of this, according to Kant.

It should be noted that in contrast to geometry it is not by first-order construction rather it is by reflection the determination of cardinality takes place. By this is meant the following. The construction which takes place in geometry is construction of pure intuitions which are first-order objects belonging to U. On the other hand arithmetical reflection is construction of second-order objects, subsets of images, belonging to $\mathscr{U}$.

Let me end this section by noting that now we are in a position to fully appreciate the diagrammatic proof given on 7 of

$$1+2+\ldots+n=\frac{n(n+1)}{2}$$

There are at least two types of universality arising. First we realize by the "schema of magnitude" that an empirical intuition, a square, really represents the concept universally. Secondly, the property we prove about the particular token, is really—it is seen by geometrical schemata—about any token square. Finally, again by the "schema of magnitude" we realize that the property about any square can be transfered to the number any such square represents universally.

2.4 Syntheticity of arithmetical propositions: Space and time

Both space and time ground our notion of number. Kant's point when saying "[t]houghts without content are empty" is precisely that if, say the equivalence class corresponding to some number is empty, then thinking about that particular number is not possible. The elements of the equivalence classes are mental objects, of which there are two kinds: pure intuitions and images referring to empirical objects. But in either case they are referring mediately or immediately to spatial objects. This relation between concepts and intuitions is exemplified in the relation between numbers as types and equivalence classes, by saying "[o]ne must go beyond these concepts, seeking assistance in the intuition" (B15). Pure intuitions, however, gain objectivity through and only through the empirical:

> Now the object cannot be given to a concept otherwise than in intuition, and, even if a pure intuition is possible *a priori* prior to the object, then even this can acquire its object, thus its objective validity only through empirical intuition, of which it is the mere form. Thus all concepts and with them all principles, however *a priori* they may be, are nevertheless related to empirical intuitions, i.e., to *data* for possible experience. Without this they have no objective validity at all, but are rather a mere play, whether it be with representations of the imagination or of the understanding. One need only take as an example the concepts of mathematics, and first, indeed, in their pure intuitions. Space has three dimensions,

> between two points there can be only one straight line, etc. Although all these principles, and the representation of the object with which this science occupies itself, are generated in the mind completely *a priori*, they would still not signify anything at all if we could not always exhibit their significance in appearances (empirical objects). [...] In the same science [mathematics] the concept of magnitude seeks its standing and sense in number, but seeks this in turn in the fingers, in the beads of an abacus, or in strokes and points that are placed before the eyes. The concept is always generated *a priori*, together with the synthetic principles or formulas from such concepts; but their use and relation to supposed objects can in the end be sought nowhere but in experience, the possibility of which (as far as its form is concerned) is contained in them *a priori*. (A239–40/B298–9)

If the proposition $7+5=12$ had been analytic then merely thinking about the concepts of $5,7$ and $+$ would yield the result 12. But this is not possible. Our concept of 5 is not a collection of marks, rather it is an abstract type, whose semantics is determined by the "schema of magnitude" when the subject is interacting with the world, and therefore numbers only have meaning in connection with intuitions. We have to operate with particular tokens which are found "in the fingers, in the beads of an abacus, or in strokes and points that are placed before the eyes". A consequence of this is that the notion of number is meaningless, only if all equivalence classes in $\mathscr{U}$ are empty. In other words, our notion of number is meaningless, only if no objects are representable. Thus it becomes an issue to guarantee that these classes are not empty, and this seems to lead to a problem about large numbers. As Frege puts it:

> I must protest against the generality of Kant's dictum: without sensibility no object would be given to us. [...] Even those who hold that the smaller numbers are intuitable, must at least concede that they cannot be given in intuition any of the numbers greater that $1000^{1000^{1000}}$, about which nevertheless we have plenty of information. (Frege; 1980, 101)

Tait formulates in the paper *Finitism* a similar criticism, which is extended to a critique of Kant's notion of number through a critique of Kant's dictum (in the words of Tait) that "existence is restricted to what can be represented in intuition" (2005, 7). Tait writes:

> It is clear—and was so to Kant and Hilbert—that there are numbers, say 10^{10} or 30, which are not in any reasonable sense representable in intuition. Kant seems to have responded to this by saying that at least their

> parts are representable in intuition [...] The real difficulty, however, is that the essence of the idea of Number is iteration. However and in whatever sense one can represent the operation of successor, to understand Number one must understand the idea of iterating this operation. But to have this idea, itself not found in intuition, is to have the idea of number *independent of any sort of representation*. (Tait; 2005, 35)

In should be clear from my exposition of Kant's notion of number that Kant is not in any way affected by the latter part of Tait's criticism. Kant does not hold an empirical understanding of number—rather the concept of a general number is founded on the "schema of magnitude" which flows from the intellect, certainly not from the empirical.

In the former part of the criticism Tait seems to have the same premise as does Frege. Intuition in their understanding does not include pure intuition. Granting that we have *pure intuition* Kant would respond by saying that by *the very writing* of $1000^{1000^{1000}}$ and 10^{10} or 30 we in fact *have* intuitions. The very inscriptions provided by Frege and Tait are intuitions. They are understood in terms of the exponentiation function which is basically a primitive recursive function. Thus, principally we can determine the equivalence-class-membership of these inscriptions—they are numbers, as we have an intuition and a rule determining how to operate with this intuition.

Let me give a general solution concerning meaningfulness of large numbers. As it turns out it is our concept of space which ultimately provide arithmetic with its objects. Our primary geometrical schemata (some equivalents of Euclid's postulates) lead to a production of a sequence of finite spaces,

$$S_1, S_2, \ldots, S_n, \ldots$$

where S_i is smaller than S_j, if $i < j$. This sequence of pure spaces is a constructive but potential infinite sequence. Thus given any natural number the equivalence class corresponding to that number is inhabited, at least due to this sequence of increasing finite pure spaces. But the justification of this argument, which refutes Frege's criticism, rests on Kant's notion of space which I will take up in the next chapter. The chapter following includes a general account of Kant's philosophy of mathematics and incorporated in this is an elaboration of the status of the natural numbers and their axiomatization.

Time, however, is also a necessary condition for the concept of magnitude. The concept of iteration is a necessary element in the "schema of magnitude" ("the successive addition of one (homogeneous) unit to another"). Without iteration it would be impossible to determine the magnitude for any given thing. Kant assumes nothing particular about the objects for numbers—they can be anything—but adding unit to unit always takes place in time: “this how–many–times is grounded on successive repetition, thus on time” (A242). Therefore, the natural numbers as a sequence of numbers can only be *represented* as a progression in time. Furthermore, also the most simple operations of arithmetic, say addition, takes place in time according to Kant. In a letter to Schultz Kant writes: “If I view $3+4$ as the expression of a *problem*” then the results found “through the successive addition that brings forth the number 4, only set into operation as a continuation of the enumeration of the number 3” (Ak. 10, 556). So I can take first three fingers together with four fingers and enumerate all of them. This enumeration ends by judging the fingers to constitute a set with cardinality 7.[16]

So, number as *schema* generates under the condition of inner sense (time) the synthesis of representations of objects subsumed under a concept. Thus cognition, number, pure concept, representations of objects falling under empirical concept, transcendental time–determination and the transcendental imagination are closely related *in inner sense*, as also Figure 2.1 illustrates: It all takes place under the conditions of inner sense.

On the other hand Kant claims in the same letter to Schultz as quoted above that:

> Time, as you correctly notice, has no influence on the properties of numbers (as pure determinations of magnitude) [...] The science of numbers, notwithstanding the succession that every construct of magnitude requires, is a purely intellectual synthesis, which we represent to ourselves in thought. But insofar as specific magnitudes (*quanta*) are to be determined in accordance with this science; and this grasping must be subjected to the condition of time. (Ak. 10, 556-57)

The natural numbers are pure concepts (types) of the understanding, in the sense that they are not derived from experience, but from the structure of our representation.[17] Time has no influence on these types, as they are not dependent on time. The rules determining the tokens (members of the equivalence classes) are pure intellectual rules, which remain the same over time. This is the intensional aspect of number. The objects in the equivalence classes, on the other hand, are ultimately empirical objects, therefore spatial. But reasoning about numbers proceeds necessarily by way

[16] See also A164/B205.

[17] It is of course an interpretation to say that the numbers are pure concepts of the understanding, as Kant claims there are only 12 of these categories. It would, however, not make much sense, I think, to regard the numbers as nothing but pure. In this respect I completely agree with M. Young (1992, 174), and I am generally sympathetic towards his short interpretation of Kant’s schematism.

of mental images over time. When we do mathematics and examine the properties of numbers, time is a necessary condition for the representations of numbers.[18] The interplay between pure concepts, tokens, time and space is summarized by Kant in Dissertation by saying that

> there is a certain concept which itself, indeed belongs to the understanding but of which the actualization in the concrete (*actuatio in concreto*) requires the auxiliary notions of time and space (by successively adding a number of things and setting them simultaneously side by side). This is the concept of *number,* which is the concept treated in ARITHMETIC. (Ak. 2, 397)

Thus both space and time conditions our access to and the constitution of the natural numbers: They are constituted by non-temporal schemata but any use will be temporal, and the meaning is ultimately provided by intuitions.

2.5 Conclusions on schemata

Numbers are not characterized extensionally—this would not be meaningful in Kant's framework. Number is rather given an *intentional* characterization in terms a collection of rules. In order to fully appreciate this and to give a coherent account we need Kant's full theory of schemata—all the way through empirical, geometrical and transcendental schemata. On the other hand we really get a coherent interpretation when the unwinded theory of schemata is taken into consideration. I think that Charles Parsons failed to realize this in his article "Arithmetic and the Categories" when concluding that "Kant did not reach a stable position on the place of the concept of number in relation to the categories and the forms of intuition" (1992, 152). In contrast to this, I hope my interpretation of Kant's theory of schemata has shown, precisely how Kant's notion of number relates to the categories, and to the two forms of intuition: Space and time.

I do agree with Shabel (2003a, 109) claiming that Euclidean reasoning provides a handle for an understanding of the general theory of schemata. I think it is clear, that Kant anticipates the type–token distinction by the triple: Concept–schema–image. This is the heart of his theory: A schema gives an account for the relation between a concept understood as a type and the objects to be subsumed under the concept. The objects to be subsumed can be either pure or empirical. In the first case the objects are given to us immediately in pure intuition by construction in a finite number of steps. In the latter case the objects are subsumed mediately by temporized images. This

[18] Therefore, the situation is not as in mechanics, where time is *studied* together with the alteration of placement in space.

difference find expression in the aspects concerning construction and reflection and in the not very well-explained notion of transcendental time determination which only relates to the pure schemata. But common to both the geometrical and arithmetical schemata is, that they are rule-governed procedures carrying universality, in such a way that schemata found concepts and provide them with meaning through the particular.

CHAPTER 3

Kant on Space, Schemata and Geometry

Infinity is perhaps the most important but philosophically the most difficult notion in mathematics. When it comes to Kant it is certainly not trivial to give an account of the role of the infinite in his theory of knowledge. The forgoing chapters dealing with Kant's (pre-)theory of schemata helps, nevertheless, immensely when trying to understand the role of the infinite in Kant's theory.

When arguing for space as being, not a concept, but an a priori intuition he writes in the Transcendental Aesthetic that "[s]pace is represented as an infinite **given** magnitude [*Größe*]" (B40).[1] It has always been difficult for me to understand Kant's claim here as anything but a positive claim saying that space is *actual infinite*. This impression is confirmed by the letter exchange between Kant and Schulze from 1790 (belonging to the Kant-Eberhard dispute). Here Kant discusses the ideality of space and claims that "*actu infinitum* (the metaphysical given) is *non datur a parte rei, sed a parte cogitantis* [not given on the side of the object, but on the side of the thinker]" (Ak. 20, 421). On the face of it we could understand *actu infinitum* simply as an *idea*, but this is not in agreement with Kant's continuation: "This latter mode of representation [*actu infinitum*], however, is not for this reason invented and false. On the contrary, it absolutely underlies the infinitely progressing construction of geometrical concepts" (Ak. 20, 421).

The nature of space and the the understanding of infinity are also central to Kant's investigations in *Metaphysical Foundation* where he provides (or tries to provide) an epistemological foundation for Newtonian mechanics. Here Kant writes that "*motion in absolute* (immovable) *space*" "in general natural science is unavoidable" (Ak. 4, 558–9), and that "[a]bsolute space is therefore necessary, not as a concept of an actual object, but rather as an idea", i.e., as a concept of *reason* (Ak. 4, 560).[2] The confusion, however, does not diminish when the First Antinomy is read as Kant here attempts to prove physical space to be—given our conceptual framework—neither infinite nor finite.

It thus comes as no surprise that Kant's understanding of infinity has been criticized from various points of views. In *The Principles of Mathematics* Bertrand (Russell; 1903, §§ 249, 435–6) simply finds Kant to be inconsistent in this respect. And throughout the 1920s and 1930s Hilbert and Bernays continuously attacked Kant's claim that space is "an infinite **given** magnitude". In the programmatic article "On

[1]In comparison with the first edition "**given**" is emphasized in the second. This only stresses the difficulties when interpreting the sentence.

[2]The translation given here is Friedman's; (Kant; 2004).

the infinite" Hilbert writes:[3]

> In the attempt to prove the infinitude [*Unendlichkeit*] of space in a speculative way, moreover, obvious errors were committed. From the fact that outside of a region of space there always is still more space it follows only that space is unbounded but by no means that is is infinite. (Hilbert; 1926, 374)

In this chapter I will present my analysis of the Kantian conception of infinity. In doing this I will resolve the apparent disagreement between Kant and Hilbert/Bernays by showing that Hilbert and Bernays misunderstood Kant. What they failed to understand is that Kant uses magnitude (*Größe*) with two different meanings. Magnitude can mean both *quantum* and *quantitas*. Given a quantum the "schema of magnitude" can try to determine the quantitas of that quantum. The schema can succeed, if and only if the size of the quantum is finite. Thus, in case the schema does not succeed the quantum is not finite, therefore *in-finite*. My analysis will in fact show that Kant and Hilbert are (in a certain sense) in complete agreement with respect to the objectivity of the infinite.[4]

My analysis will in turn take us to Kant's understanding of what it means to be an *object*. This will be important for understanding why—on Kant's view—there are no postulates for number theory. Central here is that the role of postulates in the Kantian epistemology is to produce objects—but numbers are rules of the understanding and cannot be objects. The only real objects of mathematics thus turns out to be the geometrical objects.

These questions concerning the Kantian notions of infinity, space and objects have of course been dealt with quite extensively in the literature. I think, however, that my analysis of the role of geometrical and arithmetical schematisation allows us to make a reinterpretation of most of the central notions in Kant's theory. Among these are the notion of infinity and the notion of object.

But let us first of all see how space as a pure intuition both warrants and constraints the geometrical schemata.

The Transcendental Aesthetic is divided in two parts according to the two different forms of intuition. The former part about space serves as a model for the latter part about time; both concerning structure and argumentation. I find the section on space more original and thorough and I concentrate my exposition on that.

[3] In (Hilbert; 1933) precisely the same critique is formulated with explicit reference to Kant. Another place where Hilbert's criticism is expressed is in (Bernays; 1930).

[4] This is important for the understanding of the development in the philosophy of mathematics in the 20th century as it was precisely understanding of infinity which led Hilbert and Brouwer in two incompatible directions. I will return to these matters later in this thesis, but see also (Posy; 1998).

It is impossible not to compare the Aesthetic with the Schematism. Whereas the Schematism is difficult and certainly in need of an interpretation the Aesthetic is quite clear (at least according to Kantian standards). As noted in the previous chapter the Schematism is (surprisingly) revised *only* with respect to a running header, whereas the Aesthetic is given a thorough revision. The B-version of the Aesthetic is a laudable presentation which will be my point of departure here.

The part on space has three sections: The Metaphysical exposition, the Transcendental exposition, and the Conclusions. In the Metaphysical exposition § 2 Kant gives what in contemporary mathematics would be termed an intuitive or perhaps metaphysical[5] discussion of space. In a certain sense it is a pre-mathematical account of some general and—according to Kant—a priori properties of space and spatial capacities that we as humans have. After having established what these properties are Kant elaborates in the Transcendental exposition § 3 on the connection between geometry as a science and these general properties of space. Stated briefly we can say that the essence of these sections is that space is infinite and a pure (a priori) intuition and that it is the structure (or pure form) of sensibility. In the Conclusions it is further established that space is *nothing more* than pure intuition and structure of sensibility leading to Kant's empirical reality and transcendental ideality of space.

Although the whole section is generally clear there are of course difficulties in the Metaphysical exposition. The full title of the section is "On space. § 2 Metaphysical exposition of this concept". The surprising element is that space seems to be a concept. This presents a curiosity as Kant is about to analyze sensibility—why does he begin with a concept? Moreover, later in the very same section Kant concludes that "space is an ***a priori*** **intuition,** not **a concept**" (B40). That space is introduced as a concept could be a slip of the tongue, but when outlining the purpose of the section Kant writes:

> [W]e will expound the concept of space first. I understand by **exposition** [***Erörterung***] (*expositio*) the distinct (even if not complete) representation [*Vorstellung*] of that which belongs to a concept; but the exposition is **metaphysical** when it contains that which exhibits the concept **as given** *a priori*. (B38)[6]

We know that Kant divides the elements of human cognition exhaustively into intuitions and concepts. Concepts belong to reason and to the understanding, but "[t]he

[5]These terms of course bear on Kant's own designations.

[6]In the first edition the first sentence of this quote runs "we will first consider space." The rest of the quotation is new in the second edition, which suggest Kant's persistence on the *conceptual* element of space.

understanding is not capable of intuiting anything, and the senses are not capable of thinking anything. Only from their unification can cognition arise. But on this account one must not mix up their roles" (A51-2/B75-6). We have empirical concepts, say of a dog, but we also posses pure concepts (categories) such as the concept of magnitude (totality). In addition to these concepts we have "pure sensible concepts" (A140/B180). Such a concept is *constructible* in the sense that an image can be constructed in a finite number of steps which together with the schema allow for *universal* reasoning about the concept. A category, on the other hand, is not constructible, this "is something that can never be brought to an image at all" (A142/B181).[7] From what Kant writes later in the *Critique* it is clear that space is neither an empirical nor a pure concept of the understanding. It could be that space is a pure sensible concept. But this does not harmonize, for instance, with the fact that the sensible concepts are constructible. We can construct finite spaces, but how should we construct a paradigmatic image of space as such?

3.1 The notion of *Vorstellung*

Related to the problem outlined in the foregoing paragraphs is the problem of understanding what is meant by "the representation of space"—a term which is used throughout the Aesthetic. In a large part of the important secondary literature on the *Critique* focus is put on "the representation of space" and not just on "space".[8] What could be meant by the expression "the representation of space"? And is there a difference between "space" and "the representation of space"? On pages A319-20/B375-7 Kant outlines his rather technical terminology concerning *representation*:

> The genus is **representation** [***Vorstellung***] in general (*repraensentatio*). Under it stands the representation with consciousness (*perceptio*). A **perception** that refers to the subject as a modification of its state is a **sensation** (*sensatio*); an objective perception is a **cognition** (*cognitio*).

[7]See also A57/B81, where Kant introduces *aesthetic* concepts in addition to empirical and pure concepts.

[8]See for instance (Allison; 2004) and (Warren; 1998). Warren opens his important article like this: "The first major section of the *Critique of Pure Reason*, the Transcendental Aesthetic, is concerned with the nature of space and time, and with the nature of our representation of them." This more than suggests that there is an important difference between say, "space" and "the representation of space" (1998, 179). Warren also has a brief, but not really satisfying, discussion of what is meant by the expression "the representation of —" (1998, 182n). Allison (2004, 117) writes, for instance in his comments to B40: "It should be noted that the question concerns our *representation* of space, not space itself." Allison's use of representation is not completely unambiguous it seems. Sometimes he uses representation with a meaning close to Kant's *Vorstellung*, sometimes he uses it with a modern meaning, as "the representation of objects" (Allison; 2004, 173).

> The latter is either an **intuition** or a **concept** (*intuitus vel conceptus*). The former is immediately related to the object and is singular; the latter is mediate, by means of a mark, which can be common to several things. A concept is either an **empirical** or a **pure concept,** and the pure concept, insofar as it has its origin solely in the understanding (not in a pure image of sensibility), is called *notio*. A concept made up of notions, which goes beyond the possibility of experience, is an **idea** or a concept or reason. (A320/B376-B377)

From these pages it is clear that a representation—*eine Vorstellung*—can be various elements within human cognition.[9] It can be a sensation, an intuition, an object falling under a concept a so on. *Vorstellung* is the very general word Kant uses in very much the same way as Locke uses *idea*.[10] Space is a very crucial element within human cognition and in the course of determining its function Kant denotes it sometimes as a concept, sometimes as a *Vorstellung*; both understood as a notion of human cognition, whose role we are about to determine. Thus space as a *concept* should be understood, not in some precise sense,[11] but broadly as a notion, and *representation* should be understood as an element of human cognition. The latter, by the way, completely in parallel to how the "pure **schema of magnitude** [...] is a representation" (A142/B182).

In the two arguments—which I will treat below—for the a prioriness of space given on A23-4/B38-9 Kant uses both "the representation of space" (*die Vorstellung des Raumes*) and "[s]pace is a [...] representation". I propose that both should be understood as 'space *as* a representation'. It is certainly not the case that there is a space which we try to represent in some way or another and that it is the latter which is the object of our analysis. Although, It is true that Kant uses this understanding of representation when e.g., he talks about representing time spatially as a continuous line,[12]. A variant of this usage of *representation* is our modern understanding of representation, which is used when we say that a mental image of dog is representing

[9] But note that *Vorstellung* is also used in a more non-technical way as—what we today would term—a *linguistic presentation*. See for instance the quote from B38 as given above on page 52; or see the subtitle on A158/B197. Both of these instances are standardly translated as representation (Kant; 1998), although I find "presentation" to be the semantically correct translation.

[10] With implicit reference to Locke, Kant writes that he wants to "preserve the expression **idea** in its original meaning, so that it will not henceforth fall among the other expression by which all sorts of representations are denoted in careless disorder" (A319/B376).

[11] Such as being either a category, an empirical concept or a pure sensible concept.

[12] "[W]e cannot even represent time without [...] **drawing** a straight line (which is to be the external figurative representation of time)" (B154).

an empirical object dog.[13] But this is not what is at stake here. Here Kant discusses space as a *Vorstellung*.[14]

Thus space is a notion, a representation, a principle which this section is an exposition of. Now, an exposition (*expositio*) in the Kantian terminology is a discussion of a notion, but the discussion is not necessarily a complete and final explication or presentation of everything which is contained in the notion.[15] In § 3 the exposition is furthermore metaphysical, which means a priori. Therefore, metaphysics is understood as a priori to geometry which it epistemologically precedes. Consequently the Metaphysical exposition is a (possibly partial) outline of an epistemological foundation of space in the first place, geometry in the second. That is, an outline of an underlying capacity which partially warrants but also constraints our specific abilities to construct and understand spatial objects.[16] In chapter 1 we have seen that Kant's analysis of these abilities sums up to the fundamental geometrical schemata, which are conceptualized first and foremost in the Euclidean postulates. In consequence of this the Metaphysical exposition should be seen as a kind of pre-mathematics. A pre-mathematics where Kant examines certain primitive spatial procedures and their presuppositions.

Kant seems to have two options: Either he can analyze our concept of space, completely independent of geometry, or he can grant that we have some geometrical capacities, first and foremost the geometrical schemata, which we seek some epistemological foundation for. I argue that Kant primarily goes for the second option. The main target of the exposition is to discuss space as a principle which warrants and constraints capacities, whose codification are the fundamental geometrical schemata. The Metaphysical exposition is therefore about epistemic presuppositions and its results are epistemologically prior but presupposed by geometry as a science. Let us take a look at this non-mathematical and perhaps only partial outline of what the properties of space are.

[13] Kant is close to this usage of *representation* on A69/B94.

[14] Thus, my understanding is different from Warren's (1998) and Allison's (2004)—see footnote 8. Their interpretation implies that there is a space which we try to represent in some way or another. Space as such would then be a kind of Nomenon, of which we have no access. Nevertheless we are able to represent it. And this, certainly, seems not to be in accordance with Kant's general understanding of Nomenon.

[15] An exposition is understood in contrast to a definition. Kant uses the term "**exposition,** which is always cautious and which the critic can accept as valid to a certain degree while yet retaining reservations about its exhaustiveness" (A729/B757). See also page 9 for a discussion of Kant's use of *definition*.

[16] Although my approach is different from the approach found in (Shabel; 2003b) I end up very much with the same general view on space, namely the view just formulated.

3.2 The a prioriness of space

The section contains four numbered arguments. When I go through the different arguments I will take Kant's idea of schematism as serious as possible and relate the arguments to geometrical schemata as much as possible. The first argument is:

> I) Space is not an empirical concept that has been drawn from outer experiences. For in order for certain sensations to be related to something outside me (i.e., to something in another place in space from that in which I find myself), thus in order for me to represent them as outside <and next to> one another, thus not merely as different but as in different places, the representation of space must already be their ground. Thus the representation of space cannot be obtained from the relations of outer appearance through experience, but this outer experience is itself first possible only through this representation. (A23/B38)[17]

I see myself and other objects as occupying different places. I have a capacity to spatially relate such objects to myself and I can, likewise, relate one object to another one. This capacity, I suggest, that Kant has in mind is the capacity which mathematically is codified by Euclid's first postulate: "To draw a straight line from any point to any point" (see page 11). From me to an outer object I can pre-mathematically think a relation; mathematically speaking this relation is an (imagined) line drawn *from* me *to* the object. Kant's claim here is that there *must* be a space in which this takes place. Space precedes and makes possible the relations we can make between ourselves and outer objects. This is necessarily so, and therefore space cannot be empirical.[18] Note that Kant is modest in this argument: He claims only that some kind of space precedes (and is therefore a priori to) the capacity to spatially relate objects. Nothing except a prioriness is claimed about the nature of the space. Such a space has to be presupposed in order to partially explain the possibility of a very basic non-mathematical feature of our experience, namely that we can spatially relate objects.

[17] The three words between $<$ and $>$ are only found in the second edition.

[18] A great number of commentators have found this understanding of the first a priori argument both to be "tautologous" *and* that it "proves too much"; see for instance (Allison; 2004, 100–4), but see also (Guyer; 1987, 346) and (Strawson; 1966, 58). It is difficult for me to see that because we can use relations (which we happen to call spatial relations) such as 'outside', 'next to', . . . , then by a tautology space exists. Given that a tautology is something like a logically true formula in propositional logic it is difficult for me to see how the argument could be a tautology and *at the same time* prove too much. Warren (1998) gives good arguments why the tautologous argument does not prove to much. He shows how the argument does not work for an empirical concept like brightness: From the existence of an a posteriori relation like 'is brighter than' it does not follow that a one-dimensional so-called brightness-space is presupposed.

The connection between the pre-mathematical discussion given here in the Metaphysical exposition and the fundamental geometrical schemata was already seen, I suppose, by Johann Schultz. Schultz was a colleague of Kant at the University of Königsberg and a true defender of Kant's critical philosophy, who wrote the very first commentary to the *First Critique*. Schultz wrote: "If I should draw a line from one point to another, I must already have a space in which I can draw it" (Allison; 2004, 113). The argument here (or perhaps one should say claim) is the same as Kant's, except that the context here is geometry and not space as such. I will return to this thought after presenting the last three arguments in § 3.

Kant's second a priori argument has the same target, although it is even more simple than the first.

> 2) [. . .] One can never represent that there is no space, though one can very well think that there are no objects to be encountered in it. It is therefore [. . .] an *a priori* representation. (A24/B38–9)[19]

The argument is that we cannot even *imagine* an experience of an (outer) object without the object is given in some space. We can imagine an empty space, but not an object not presented in a space. Therefore space is not a posteriori to the objects, and space cannot be derived from the experience of objects.

'Imagine', 'imagination' and 'power of imagination' are very important concepts in the discussion of space. Imagination and pure space are deeply connected according to Kant. This is an important premise for him when arguing for the infinity of space, as we will see below. The scientific study of space as done in geometry is when it comes right down to it a study of our spatial power of imagination as codified by the geometrical schemata. The schemata are a product of the transcendental imagination. Following this line of thought makes it clear that the geometrical counterpart of Kant's argument here is, that we can construct geometrical spaces which are empty regarding figures; but we can not construct figures independent without some space in which these figures are constructed. Our spatial procedures for construction of spatial relations between objects need some kind of space to exercise and live in. In this sense space precedes and warrants geometrical schemata.

[19]Again we see subtleties in the translation of *Vorstellung*. The German text is: "Mann kann sich niemals eine Vorstellung davon machen, daß kein Raum sei". I think one should understanding *Vorstellung* in this context, not as having the technical meaning, such as being a sensation or an object or something like that, but rather as one cannot *imagine* that there is no space. The hypothetical subjunctive in "sei" supports this understanding. The second use of *Vorstellung*, however, in: "Er [. . .] is eine Vorstellung a priori" is of course to be understood in the technical sense. (The italic of "a priori" is only in the English translation [sic!]).

3.3 The arguments for space as intuition

Until now Kant has only argued that there must be a space, but we know nothing about it, except that it is a priori. The last two arguments numbered as 3) and 4) of the Metaphysical exposition purports to show that space cannot be a concept. As cognition is either conceptual or intuitive it follows that space must be intuitive.[20] More specifically, by analyzing the pre-mathematical notion of space Kant wants to show in the first intuition argument that space cannot be a general concept. A general concept to Kant is—not surprisingly—a concept which generally applies to a variety of things and thus gives rise to a collection of objects, such as the collection of all men or the collection of all triangles.

The first of the intuition arguments runs as follows (note that, as usual Kant begins with the conclusion):

> $<$3)$>$ Space is not a discursive or, as is said, general concept of relations of things in general, but a pure intuition. For, first, one can only represent a single space, and if one speaks of many spaces, one understands by that only parts of one and the same unique [*alleinigen*] space. And these parts cannot as it were precede the single all-encompassing space as its components (from which its composition would be possible), but rather are only thought **in it.** It is essentially single; the manifold in it, thus also the general concept of spaces in general, rests merely on limitations. (A25/B39)[21]

On the face of it we could understand the argument in the following way. We can only *imagine* one space. We can construct and experience many spaces, but we can only imagine them as being part of one space—being *in* an all including singular space. Therefore the all-including singular space is the foundation for all kinds of spaces, because we cannot imagine the situation in any other way. But the argument is, of course, more subtle than this.[22]

What is at stake here are the epistemological relations between a general concept of space, particular (finite) spaces and the all-including singular space, which Kant

[20] "[B]esides intuition there is no other kind of cognition than through concepts" (A68/B93).

[21] It is the sentence "erstlich kann man sich nur einen einigen Raum vorstellen" which is translated into "first, one can only represent a single space".

[22] In the literature there is a general discussion on the so-called 'argument from geometry', which is a transcendental argument with the structure, that as we have a geometry with certain characteristics it follows that space must be intuitive. Allison (2004, 116–8) claims that this argument is found in the Transcendental exposition whereas Friedman (1992, 70) and Shabel (2004) claim to find the argument in the Metaphysical exposition. To my best knowledge we find it in its clearest form here in the first intuition argument.

wants to demonstrate exists as something different from the general concept. Moreover, he wants to show that the all-including space has priority over and precedes both the 'smaller' particular spaces and the general concept of space (of which the particular spaces are instances).

A central claim of my chapter 1 is that throughout the *Critique* Kant argues that humans are able to perform general spatial procedures which can be reduced to a codification given by some fundamental geometrical schemata. Such procedures, however, need to exercise in some kind of space—this we partially know from the a priori arguments. By using our schemata we can construct spatial objects like triangles and finite spaces. In chapter 1 I also gave an analysis of how the concept of triangle is founded on the schema of triangle, which in turn rests on the fundamental schemata. A similar analysis could be given with respect to the general concept of space and its schema.

I claim that this relation between general spatial concepts and the fundamental schemata is what Kant is thinking of in the sentence "these parts [the finite spaces] cannot as it were precede the single all-encompassing space as its components". This view is reflected in the conclusion of the Schematism, namely that our pure sensible concepts are possible, only because of schemata. From this it follows that the singular space is epistemologically prior to the general concept of space. Let S denote the all-including singular space, and let S_i be the smaller parts of S which are constructed on the basis of schematic limitations of S. Then, the argument says, S cannot be constructed out of its parts, in mathematical notation:

$$S \neq \bigcup S_i,$$

because S precedes epistemologically any S_i.

Understood in this way Kant's argument makes sense.[23] The argument is also found in the Kant-Eberhard discussion:

> [T]he representation of space (together with that of time) has a *peculiarity* found in no other concept, viz., that all spaces are only possible and thinkable as parts of one single space, so that the representation of parts already presupposes that of the whole. (Ak. 20, 419)[24]

[23]As already noted, Allison does not understand the first intuition argument in this way. He finds the argument to be based on the notion of (Kantian) intension (see below for explanation of this term). Thus space cannot be a general concept because the intensions of concepts "are logically prior" to the concept itself (Allison; 2004, 110). It is firstly unclear what is meant by "logically"; secondly, certain general concepts can precede intensions not yet determined.

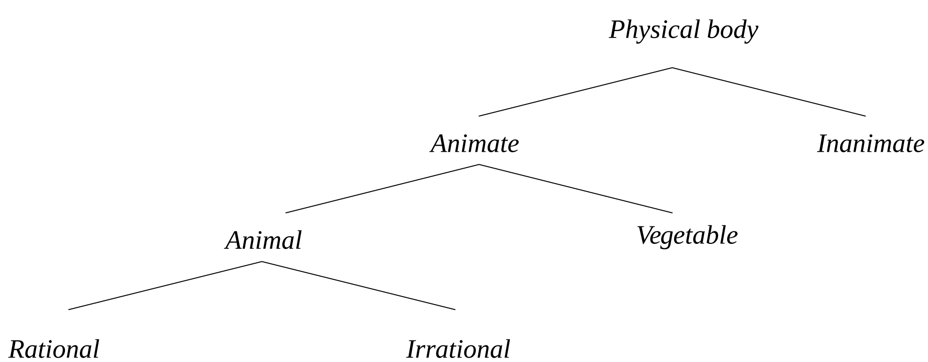
[24]The translation is Allison's (1973, 176).

Or at least the argument makes sense inasmuch as it shows that space cannot be a general sensible concept. It is, however, much more problematic to see that the argument should demonstrate the all-including singular space to be *unique*. This is a problem I will elaborate on in ??

Turning to the infinity of space Kant argues:

> 4) Space is represented as an infinite **given** magnitude. Now one must, to be sure, think of every concept as a representation that is contained in an infinite set of different possible representations (as their common mark [*Merkmal*]), which thus contains these **under itself;** but no concept, as such, can be thought as if it contained an infinite set of representations **within itself.** Nevertheless space is so thought (for all the parts of space, even to infinity, are simultaneous). Therefore the original representation of space is an ***a priori*** **intuition,** not **a concept**. (B39–40)

The standard way to understand this deals with the Kantian notions of intension and extension of a concept. The extension of a concept A is the collection of concepts which have A as their common marks (*Merkmale*). In the graph below all the concepts standing under 'Physical body' are contained in the extension of it. The intension, on the other hand, is made up by the sub-concepts (*Merkmale*) which constitute the concept; 'human', for example, is constituted by 'rational' and 'animal'. In this sense the intension of a concept is an unlimited but finite conjunction of concepts, whereas the extension is potentially infinite.



The usual understanding of the second intuition argument is thus that no concept, according to Kant, can have "an infinite set of representations **within itself**". Of course, Kant claims this in opposition to Leibniz. According to Leibniz certain complete concepts exist. A complete concept is a concept completely describing a unique object, or monad as Leibniz prefers to call the objects. Two objects x and y

are different, only if there is a least one property P which differ with respect to x and y:

$$\forall x,y\big(x \neq y \rightarrow \exists P(P(x) \leftrightarrow \neg P(y))\big).$$

A complete concept is thus a possibly infinite *conjunction* of primitive concepts. Aggregates, on the other hand, are made up of more objects, and disjunctions can occur in their descriptions. According to Kant Leibnizean complete concepts are meaningless, because we as human finite beings do not have access to them.[25]

Now, another premise for Kant's argument is that "[s]pace is represented as an infinite **given** magnitude [*Größe*]". Before presenting the implicit argument for this premise let me introduce Kant's distinction between *quantity* and *quantum*. In his *Lectures on Metaphysics* Kant says:

> Quantity: determination of a being, how many times it is posited [*Quantitas: determinatio entis, quoties sit positum*].
>
> Quantum: it is one thing, in which there is quantity [*Quantum: est unum, in quo est quantitas*]. (Ak. 28, 21)

The following argument showing space to be infinite is again an instance of the 'argument from geometry'. Let us suppose that we can construct some initial finite space S_0. Given the category of unity[26] we can use S_0 as a measure for determining the magnitude of other finite spaces. Using our geometrical schemata (basically Euclid's second postulate) we can construct a sequence of finite spaces

$$S_0, S_1, \ldots, S_n, \ldots$$

such that S_{i+1} is precisely 1 unit bigger than S_i. If we assume the all-including space S to be finite, say of size m, then S_m would be 1 unit bigger than m. But S_m is part of S, thus S cannot be finite. It is infinite. Thus S is a quantum but the quantitas in it is indeterminable. Kant uses *Größe* for both quantum and quantitas. Thus S is an infinite given *Größe* (quantum) in the sense that it is a real, non-fictional epistemological pre-condition, in which there is quantitas. In this sense also the *singularity* of space can be understood under "infinite **given** magnitude". Any attempt, however, to determine its magnitude, its quantitas, is doomed to fail. The schema of magnitude determines infinity in the sense that a quantum Q is infinite just in case the schema is not able to determine the size of Q. Therefore S is in-finite.[27] Note that for Kant

[25] See (Posy; 1995) for an elaboration of the connection between Kant's and Leibniz' paradigms.

[26] Which in *Prolegomena* is termed "Einheit (das Maß)".

[27] "[O]ne can only view as *infinite* a magnitude in comparison to which each specified homogeneous magnitude is equal to only a part" (Ak. 20, 419).

there is only one type of real (or objective) size, namely the finite. The contemporary mathematical understanding of the size of infinities such as $\aleph_0$ is as remote to Kant as can possibly be. S is infinite in precisely the same sense as the real numbers are uncountable to an intuitionist. In fact it is intuitionist understanding of negation; it is a negative claim: We can prove that any attempt to ascribe a finite number to S, like any any attempt to enumerate the real numbers will necessarily fail. If $\perp \equiv (0 = 1)$ then we can express it like this:[28]

$$\text{“}S\text{ is infinite”} \quad \equiv \quad \text{“(}S\text{ is finite)} \rightarrow \perp\text{”}.$$

S is infinite *in actu* in the sense that S is a space which cannot be finite although it necessarily exists. It is not potential like an acorn is an oak tree in potency, or like a cold body is hot in potency, or like the sequence $S_0, S_1, \ldots, S_n, \ldots$ is potentially infinite. It is actual as an epistemological warrant of our geometrical schemata and henceforth of our geometrical concepts.

We are now in a better position to understand Kant's second intuition argument. S contains an infinite collection of spaces. Thus S has—in the words of Kant—"an infinite set of representations **within itself**". But no concepts has infinite intensions (no complete concepts) in the Kantian epistemology, therefore S is not a concept, but an intuition.

To finish the section let me give Schultz' transcendental arguments (as found in (Allison; 2004, 113)):

> If I should draw a line from one point to another, I must already have a space in which I can draw it. And if I am able to continue drawing it as long as I wish without end, then this space must already be given to me as an unlimited one, that is, as an infinite one.

Schultz is crystal clear here, but Kant has the same argument in § 12 of Prolegomena, where postulate 2 presupposes space as intuition and they also belong to the Kant-Eberhard controversy.[29]

[28] See footnote 9 for an elaboration on the intuitionistic meaning of the logical symbols such as implication and negation. This meaning amount to the so-called Brouwer-Heyting-Kolmogorov (BHK) interpretation.

[29] Kant writes, in Allison's (1973, 176) translation: "[T]he geometer expressly grounds the possibility of his task of infinitely increasing a space (of which there are many) on the original representation of a single infinite space, as a singular representation, in which alone the possibility of all spaces, proceeding to infinity, is given" (Ak. 20, 420).

3.4 Evaluation of the arguments given in the Metaphysical exposition

I hope to have provided evidence for my general view on the Metaphysical exposition. Basically it is an argument in favor of epistemic presupposition. More precisely, it is here we find Kant's 'argument from geometry': Humans posses certain fundamental spatial procedures. In the Metaphysical exposition Kant discusses what can warrant but also constrain such procedures. He argues that there must be some a priori space in which these can operate and that this space cannot be conceptual as concepts, themselves, are founded on corresponding schemata. So far so good.

There are, however, certain problems with some of the details of the arguments.

Firstly, the claim that the all-including space should be unique is stated without giving any evidence. Viewed historically, it is not surprising that Kant find it difficult to argue for this. Perhaps it is not settled by now, what the structure of space is. General relativity theory tells us that locally space is Euclidean but globally it is hyperbolic. Thus the constitutive principles of space depends on the perspective. What Kant shows is, that there is necessarily some space—but he does not tell us much about that space. In fact he only examines that space in the light of basically the first two Euclidean postulates. For sure there is nothing about orthogonal, nor parallel lines (Euclid's postulate four and five). Therefore, after the Metaphysical exposition we only know that space as a governing principle for basic schemata must be a priori, infinite and in some sense non-conceptual.

Secondly, the arguments for space as intuition also have problems. Of course, the premise that cognition can only arise on the basis of concepts or intuition is disputable. Moreover, the analogy in the last argument between *Vorstellungen* in concepts and space, respectively, is perhaps a *poor* analogy. In the following I will use a modern linguistic approach to analyze this problem. Say an expression A has B and C as its intension. Using this we can syntactically identify A as $B \wedge C$.[30] Now, according to Kant, if a concept is expressible by an expression with an intension, then this intension consists of a finite conjunction of expressions, say $B_1 \wedge \ldots \wedge B_n$. On the other hand, suppose we have a sequence of spaces (S_i) which are definable by expressions C_i, such that $S_i = \{x \mid C_i(x)\}$, then if we want to have the full space S which is the union of all S_i, then

[30] For the following it does not matter whether we use the syntactic approach or the semantic approach which Kant used. If we let $\overline{A}$ denote the Fregean extension of A, that is the collection $\{x \mid A(x)\}$, then to say $A = B \wedge C$ is the same as $\overline{A} = \overline{B} \cap \overline{C}$.

$$\begin{aligned} S &= \bigcup S_i \\ &= \{x \mid \text{there exists } C_i \text{ such that } C_i(x)\}. \\ &= \{x \mid C_1(x) \vee C_2(x) \vee \cdots \vee C_n(x) \vee \cdots\} \end{aligned}$$

In this way we see, that intension are conjunctions, where as spatial inclusions are disjunctions. Thus the analogy is perhaps only a weak analogy, and consequently the argument does not carry the power Kant thought it had. And then I have not even mentioned another problem. Suppose that the analogy between the intension of a concept and the concept itself works. Then at least for empirical concepts it is (perhaps) true that the *Merkmale* constituting the intension of a concept precedes epistemologically the concept itself. But what about pure concepts, like causality? Here Kant would probably claim that the concept, causality, and its intension, cause and effect, are given simultaneously. If this is true, then the second intuition argument attempts to show that space cannot be an empirical concept. But this seems strange given the fact that the two a priori arguments show space not to be a posteriori.

What however strikes me when reading the two intuition arguments is that Kant does not use the following argument. Given the unbounded infinite increasing sequence $S_0, \ldots, S_n, \ldots$ it should be quite forward to argue for Kant that the following equation is meaningless:

$$S = \bigcup S_i,$$

simply because it is impossible to *imagine* any operation realizing the union of the spaces, here denoted by $\cup$. No human can ever posses a schema uniting an infinite union of increasing unbounded spaces. A union which in modern non-constructive terms is possible through transfinite recursion. Thus the equation is simply meaningless within Kant's framework. But in fact, later in the *Critique* Kant uses this argument together with an infinite variant of the first intuition argument:

> [I]t is by no means permitted to say of such a whole, which is divisible to infinity, that **it consists of infinitely many parts.** For though all the parts are contained in the intuition of the whole, the **whole division** is **not** contained in it; this division consists only in the progressive decomposition, or in the regress itself, which first makes the series actual. Now since this regress is infinite, all its members (parts) to which it has attained are of course contained in the whole as an **aggregate,** but the whole **series** of the **division** is not, since it is infinite successively and never is **as a whole;** consequently, the regress cannot exhibit any infinite

> multiplicity or the taking together of this multiplicity into one whole. (A524/B552)

Space is not made up of its parts.

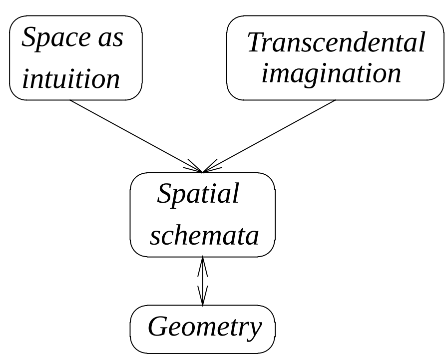
3.5 Pure space, geometry and the empirical space

Kant begins the Transcendental exposition by:

> I understand by a **transcendental exposition** the explanation of a concept as a principle from which insight into the possibility of other synthetic *a priori* cognitions can be gained. For this aim it is required 1) that such cognitions actually flow from the given concept, and 2) that these cognitions are only possible under the presupposition of a given way of explaining this concept. (B40)

The "concept" Kant is talking about here is, of course, space as intuition, which can also be understood as a principle governing human space. The difference between the metaphysical exposition and the transcendental exposition is that, in the former we consider presuppositions which precede geometry as science and in the latter we explain the relation between the metaphysical space and geometry.

Human spatial procedures are not as such part of the mathematical science geometry. They are basic epistemological capacities. They are, however, the subject of geometry and as such they are, according to Kant, codified by the postulates of Euclidean geometry. Therefore geometry, as a *scientific* study of space and the objects which we can construct in it on the basis of schemata, is possible only because these representations (*Vorstellungen*)—space as intuition and schemata—exist in the first place. Precisely in this sense do geometry and geometrical concepts "flow from" space as intuition, which therefore ground them in the first place. As geometry studies these a priori and constitutive principles it is itself a priori and apodictic.



As the account in chapter 1 showed the primary method in geometry is based on schematic construction in pure intuition. Now, how is this precisely to be understood? Kant thinks that Euclidean geometry as a science is a complete and correct codification/description of human spatial capacities. Therefore, constructions based on Euclidean postulates stands in a one-to-one correspondence with schematic construction in pure intuition. Thus geometrical knowledge is *synthetic a priori*.

3.6 The relation between pure and empirical space

In this section I will give my interpretation of the relation between pure and empirical space. I basically build on the Conclusions given just after § 3 in The Transcendental Aesthetic—the part on space. These conclusions need to be seen together with the Axioms of Intuition. But we will again see that Kant's theory of schematism provide us with important tools in our understanding.

The Metaphysical exposition took for granted that we posses some basic capacities which puts us in a position where we can use concepts like 'inside', 'outside', 'outside at different places', 'several spaces', 'is related to', and so on. Such concepts presuppose a space which has certain features. Space is a pure form of outer intuition. As opposed to the pure intuition we have empirical intuition and the decisive point here is how these two are related. Now, Kant actually claims that pure intuition provides us with a structure of the outer empirical sense, which is a consequence of the fact that "the empirical intuition is only possible through the pure intuition" (A165/B206). As with the Transcendental Aesthetic this is put forward under the premise that matter and form are separable. And as it turns out, the form is our perceptual grid.

The important lesson to draw from the Axioms of Intuition is that all intuitions, including physical objects, are extensive magnitudes, i.e., are objects which are capable of some mathematical description. How come it is like this? This fact of sensibility is explained by the fundamental role categories of quantity play in human cognition:

> All appearances [...] cannot be apprehended, [...] except through the synthesis of the manifold through which the representations of a determinate space or time are generated, i.e., through the composition of the homogeneous manifold in intuition in general, insofar as through it the representation of an object first becomes possible, is the concept of magnitude (*Quanti*). Thus even the perception of an object, as appearance, is possible only through the same synthetic unity of the manifold of given sensible intuition through which the unity of the composition of the ho-

> mogeneous manifold is thought in the concept of a **magnitude,** i.e., the appearances are all magnitudes. (B202–3)

In order to have an experience of an object, we need to experience a collection of units, which are homogeneous with respect to some property. Thus we experience a plurality of homogeneous units in which we judge unity. This is due to the "schema of magnitude".[31] This very fact of the constitution of the objects of the empirical world ensures that mathematics—at least numbers—are applicable. But of course, this does not tell us that the empirical space conforms according to the pure geometrical space.

"Space is nothing other than merely the form of all appearances of outer sense" Kant says (A26/B42). This he can claim on the basis that space is a priori, it is a necessary condition which human acquiring knowledge has to obey. As we know from the a priori arguments, space precedes all objects and therefore all objects are arranged according to space. Space is empirical real in this sense. But now, "if one abstracts from these objects, it is a pure intuition, which bears the name of space" (A27/B43). What does Kant mean by 'abstracting from objects'. Again the Schematism provide us with help.

The opening of the chapter on Schematism begins with the fundamental question of how do we subsume appearances under concepts. This is very question which the Schematism needs to answer. The example Kant gives deals with the relation between pure sensible forms and empirical forms met in objects. As we know, the central aspect which is necessary is homogeneity between different elements within our cognition. "Thus the empirical concept of a **plate** has homogeneity with the pure geometrical concept of a **circle,** for the roundness that is though in the former can be intuited in the latter" (A137/B176). This example is nothing but a reminiscence of the discussion on how an empirical intuition can be pure. Remember[32] a triangle drawn on paper can function as a pure intuition for universal reasoning, when the image (the token on the paper) is taken *together* with the constructing method of that token, i.e., when the image of the triangle is understood only together with the schema of the triangle. This applies of course to all kinds of empirical objects. This turns an empirical intuition into a pure intuition is precisely what is lies in the act to "abstract from these objects". It means simply, *to give a geometrical description of the spatial form by geometrical schemata*. In such a geometrical description all breadth of lines, extension of points, inaccuracies in angles and the like is ignored. We could say that one geometricalizes the object.

Geometrical schemata do not exist without a space, these two elements of human cognition belong together. They form an a priori structure which is the structure we

[31] See the forgoing chapter on arithmetical schematism.
[32] See pages 5 and 22.

apply—and have to apply, according to Kant—to the sensible world. Objects have to be given in some space, and the science of space is geometry. Therefore, Kant says, the objects must conform to what we learn in geometry.

One must not think however, that space exists in any other sense than relatively to humans. Space is nothing in itself[33]

> "All things are next to one another in space," is valid under the limitation that these things be taken as objects of our sensible intuition. If here I add the condition to the concept and say "All things, as outer intuitions, are next to one another in space," then this rule is valid universally and without limitation. (A27/B43)

This is the transcendental ideality of space.

[33] Precisely this property of Kantian space makes it really puzzling, why there seems to be a difference between "the representation of space" and "space in itself" in the treatments by, say, (Warren; 1998) and (Allison; 2004).

CHAPTER 4

Truth, Knowledge and the Determination of Kantian Objects

In the foregoing chapters a question has been lying just underneath the surface which now need to be answered. The question is: ‘what does it mean to be an object?’. For our purpose this is interesting for a variety of reasons. First and foremost we need to know what Kant counts as a mathematical object. In the previous chapters we have encountered the problem concerning the natural numbers. Apparently natural numbers cannot be objects to Kant—they are rules of the understanding, which give rise to abstract concepts in sense of intensional types. Are there good reasons to maintain this position? Secondly, it is of course interesting in itself to clarify this notion of object within the Kantian epistemology. Thirdly, for a re-interpretation of the Kantian theory of knowledge taking schematism as a central concept it is an important test whether it can provide a useful characterization of the notion of object.

There is in the literature a surprising consensus with respect to the Kantian understanding of object.[1] Now, x is an object in case at least two conditions obtain:

1. x can be found in space or time.

2. x is understood as a unit, there is something that unites it.

Nothing surprises about 1. This is the at the core of the Transcendental Aesthetic, where we learn that all thought must “ultimately be related to intuitions, thus, in our case to sensibility, since there is no other way in which objects can be given to us” (A19/B33). Item 2 is also not so surprising. An object is a certain unity found in an appearance. But to cognize an object as an object we need more, we need concepts.

> [T]here are two conditions under which alone the cognition of an object is possible: first, **intuition,** through which it is given, but only as appearance; second, **concept,** through which an object is thought that corresponds to this intuition. (A92/B125)

Apparently there is a distinction here between an object given as an appearance and the object that “corresponds to this intuition”. An intuition is an object, or perhaps, rather a kind of pre-object, because it does not make sense to talk about objects which are objects independently of any cognition. Thus to be an object means that there is (at least) the possibility for the object to be judged as being an object. This seems to lead towards schematisation as an ingredient in the characterization. So to be an object means to be an intuition which *can* be subsumed under a concept. This thought

[1]See for instance (Posy; 2000), (Allison; 2004) and (Longuenesse; 1998).

is expressed later in the Transcendental deduction: "*object* is that in the concept of which the manifold of a given intuition is united".[2]

That we are able to experience objects as substances[3] is an important theme in the Deduction. It will, however, take me to far away to treat this here. Nevertheless, I want to note the following. According to Kant we *can* experience objective spatial groupings which are *not* merely accidental combinations.[4] It is due to the unity of the understanding that there is an objective element in our experience.[5] But our perceptions are qua intuition embedded in a *network* of imagined possible (past) and future events and this is a subjective element. The objective element, however, enters the picture when we experience a substance, as a substance, that is, something which is not changed over (the objective) time. This element of invariance over time which is inextricably bound up with the notion of object takes me to an aspect which Posy (1998, 316) terms Kant's "principle of complete determination for intuited (and thus existent) objects" or "Predicative Completeness" (Posy; n.d., 7).

4.1 The complete determination of objects

> Every **thing** [***Ding***], however, as to it possibility, further stands under the principle of **thoroughgoing determination;** according to which among **all possible** predicates of **things,** insofar as they are compared with their opposites, one must apply to it. (A571/B579)

The principle is rather interesting and has striking similarities with the Leibnizean–Wolffian principle that each individual thing is completely determined through its complete and individuating concept.[6] Now, the principle has at least two different interpretations: 1. It is only an ontological principle; every thing has fixed truth values once it becomes an object of experience, or 2. It a principle in the sense of 1 but also

[2] See also: "Wenn [. . .] durch den Verstandesbegriff die Verknuepfung der Vorstellungen [. . .] als allgemeingueltig bestimmt wird, so wird der Gegenstand durch dieses Verhaeltsnis bestimmt, und das Urteil ist objektiv." (Prolegomena, § 19)

[3] 'Substance' is here used, not as the category, but rather as "substantia phaenomenon"; see for instance A265/B321 and A277/B333.

[4] "Since, however, if representations reproduced one another without distinction, just as they fell together, there would in turn be no determinate connection but merely unruly heaps of them, and no cognition at all would arise; their reproduction must thus have a rule in accordance with which a representation enters into combination in the imagination with one representation rather than with others" (A121).

[5] See instance B142.

[6] See for instance the note made by Guyer and Wood (Kant; 1998, 746n). They point out that this Leibnizean principle is, in fact, called the "principle of thoroughgoing determination" by Baumgarten, for instance. To use another term by Posy, here we again see how Kant *humanizes* central elements of the Leibnizean paradigm.

an epistemological principle; truth values are only given insofar as they can be known to us—there are no transcendent truth values.

The argument for the first interpretation of the principle is intrinsically bound of with the deduction of the categories. Nevertheless, I claim that Kant has the second interpretation in mind. A brief argument for this is the following. Below I argue for an alternative characterization of object, namely that objects are appearances which are capable of being subsumed under a concept, through the *schema* of the concept. Therefore, due to the schema we have principal access to all aspects of a given object. This understanding of object leads to both an ontological and epistemological aspect of the principle of complete determination. Kant departs, given this understanding of the principle, from Leibniz/Wolff in a very essential ways. Given any primitive predicate P and an object x we will sooner or later be able to decide whether $P(x)$ is true or not. On the other hand, Kant does certainly not claim that there is a complete concept of a given object. This can only be an idea.[7] This is partially due to the fact that the collection of all possible predicates is a potential infinite collection, thus the whole collection can only be an idea,[8] partially because an infinite conjunction of concepts is incomprehensible to finite human beings. Therefore, the complete concept of an object, which is the real concept to Leibniz, can only be a limit-concept to Kant. Kant therefore only works with local decidability rather than global: Given an object, we have decidability for that object. But we do not have decidability for all kinds of questions, such as “is the empirical world finite?”.

According to Posy (1995, 1998) the principle of complete determination is expressible by the formula:

$$\exists y(y = x) \rightarrow \big(P(x) \vee \neg P(x)\big). \tag{4.1}$$

where P is a primitive predicate-symbol and free variables need not denote. The formula expresses that if an object is given then tertium non datur applies to the object. This is in contrast with a *global* version of tertium non datur:

$$P(x) \vee \neg P(x). \tag{4.2}$$

[7]The denial of a complete concept is stated by Kant on A580/B608).

[8]“The proposition **Everything existing is thoroughly determined** signifies not only that of every **given** pair of opposed predicates, but also of every pair of **possible** predicates, on of them must always apply to it [. . .] the thing itself is compared transcendentally with the sum total of all possible predicates. What it means is that in order to cognize a thing completely one has to cognize everything possible and determine the thing through it, whether affirmatively or negatively. Thoroughgoing determination is consequently a concept that we can never exhibit *in concreto* in its totality, and thus it is grounded on an idea which has its seat solely in reason, which prescribes to th e understanding the rule of its complete use.”(A573/B601)

Although I certainly am sympathetic towards Posy's idea, it is not completely clear and it needs some elaboration. It is not clear, for instance, which interpretation—the ontological or the combined ontological/epistemological—the formula models. This question is connected with the question of logic; it is not clear over which logic the formulae should be read. If we take classical logic as basic logic then the interpretation must be the ontological, but over this logic formulae (4.1) and (4.2) are clearly *equivalent*. Thus we need another logic. Posy is probably thinking of intuitionistic logic. Let us interpret (4.1) with the Brouwer-Heyting-Kolmogorov interpretation.[9] If I claim (4.1) to be valid then I should posses a method Φ such that given evidence of truth of the antecedent then Φ converts this into evidence of truth of the consequent. To make the antecedent true means to provide an object witnessing the existential quantifier. On the other hand, to make a disjunction true means to posses a method which shows that at least one of the disjuncts is true *and* show which one it is. Thus Φ should be a function which takes any intuited object x and any predicate P and decides whether $P(x)$ is true or false. As I argued above it is reasonable to interpret Kant as saying that in fact humans have such a method. On this view, Φ applied to P gives us *the schema corresponding to P*. Thus Φ is a generally a method which goes from concepts to schemata. Here we see again that a concept and the corresponding schema are two different *Vorstellungen*. The predicate P corresponds to the concept, whereas $\Phi(\cdot, P)$ is the schema; a rule governed method.

But when one investigates the details of this understanding of (4.1) problems show up.

Firstly, the Brouwer-Heyting-Kolmogorov interpretation really is not an

[9] The BHK-interpretation is a heuristic interpretation based on the notions of informal proof and construction. A proof should be understood as a *construction* which verifies a statement. The interpretation takes the meaning of prime formulae for granted. For complex A, BHK explains "p proves A", which we abbreviate by $p : A$, in terms of $p' : B$ for 'immediate' sub-formulae B of A. The defining clauses relevant for our problem are:

($\bot$) $\bot$ denotes contradiction and there is no proof of contradiction.

($\vee$) $p : A \vee B$ iff p is a pair (p_0, p_1), $p_0 \in \{0,1\}$ and $p_1 : A$ if $p_0 = 0$ and $p_1 : B$ if $p_0 = 1$.

($\rightarrow$) $p : A \rightarrow B$ iff p is a construction taking any q such that $q : A$ into $p(q)$ such that $p(q) : B$.

($\neg$) $p : \neg A$ iff p is a construction taking any q where $q : A$ into $p(q)$ such that $p(q) : \bot$.

($\forall$) $p : \forall x A(x)$ iff p is a construction taking any t from the intended domain into $p(t)$ such that $p(t) : A(t)$.

($\exists$) $p : \exists x A(x)$ iff p is a pair (p_0, p_1), where p_0 is an object and $p_1 : A(p_0)$.

The interpretation tells us under which circumstances we can (constructively) claim or assert a certain formula A: We can claim A, when we can constructively prove it. And BHK then analyzes this notion. For instance, a proof of an existential statement is constructive if it provides a witness together with a constructive proof of the desired property of that witness.

interpretation—it is only a heuristic reading of the intuitionistic symbols. It does not provide a real semantics.[10]

Secondly, intuitionistic logic—as a formal[11] language—does not really represent that we can talk about 'things' which cannot be intuited. An example of such a thing is 'the empirical world'. The problem is that if we have a term t being a name for that, then in connection with an interpretation in a model, t will denote. All terms in intuitionistic logic denote.

Thirdly, time does not enter the picture (in any correct way) when using only intuitionistic logic. It is correct though that somehow, in the Kripke semantics for intuitionistic time *does* show up: The possible worlds can be seen as states in time, and when a proposition is proved, then it becomes true. But before that moment, it was false. This, however, is not the notion of truth we want. We want a notion of truth with striking similarities to the concept of truth found in constructive mathematics. If we at time t_1 can formulate and understand a certain proposition then perhaps we can prove the proposition, perhaps we cannot. In the latter case we do not know at t_1 whether or not the proposition has a truth value. But if we at a later time t_2 are to discover that in fact the proposition is true (or false) then the proposition *had* a truth value also a t_1 and it was, of course, the same as that at t_2. Thus, in constructive mathematics the notion of truth is the following: P is true, if and only if P is understood and P is provable (perhaps at some later time). Take for instance Fermat's conjecture: The Diophantine equation

$$x^n + y^n = z^n,$$

has no nontrivial solutions for $n > 2$. The conjecture has now been shown true. Thus, it was also true at Fermat's time.

Due to these considerations I therefore propose an elaboration on (4.1) within a tensed version of modal logic.

4.2 A modal logic for Kantian epistemology

In the following I develop a mathematical model of some very central aspects of the Kantian theory of knowledge. The most important aspects being modelled are objects of experience, ideas of reason, truth, time and knowledge. As with all kinds of mathematical modelling idealization plays a significant role. We can therefore not expect a complete representation of all aspects of Kant's theory. I hope, on the other hand, that the reader will agree that central properties of the Kantian notions of objects of experience, ideas, truth, time and knowledge are in fact faithfully represented

[10] Semantics meant as a semantics in the sense of a model for a logic.
[11] In the sense that it needs a mathematical semantics.

in the model. Modelling using mathematics is a very important tool in science, as reflected in physics, computer science, biology, economics, etc. In order to predict and understand the interplay between elements we model these elements. It is for the same reasons that I now propose the model. I pretend in no way that there is a one-to-one correspondence between my model and Kant's theory, but what I hope for is the discovery of a structure which can represent central notions. This representation is for the benefit of thinking things through. In following R. Montague (1975) this is called *Formal Philosophy*.[12] The following model will be a true generalized mixture of models and elements found in (Kleene; 1952; Fagin et al.; 1995; Fitting and Mendelsohn; 1998; Hendricks; 2001).

Suppose we have the a first-order two-dimensional modal language $L(\Box,\circ)$ with symbols for variables $x, y, z, \ldots$; the propositional connectives $\wedge$ and $\neg$; an existential quantifier $\exists$; equality $=$; predicates (i.e., unary relation symbols) P, Q; and two modal operators $\Box$ and $\circ$.[13] The intended meaning of $\Box$ is 'always in the future' and $\circ$ means that 'the agent knows'.[14] Other symbols such as $\vee, \rightarrow$ and $\forall$ are defined classically in terms of the primitive symbols. Likewise we have a symbol $\Diamond$ which should be interpreted as 'possible in the future'. Let me approach a modal understanding of (4.1) semantically using Kripke semantics.

A possible world is to be understood as a situation. I will be consist of a moment in time and a flow of states up to that moment. Thus we will need a non-empty set S consisting of states. Time-moments will be represented by a set T together with a linear order $<$ on T. A full flow of states is then a function f from T to S. Let G denote some non-empty set of functions from T to S. A *possible world* is a flow of states up to a certain moment in time; in other words, a function restricted to a time-moment. Let $f|t$ denote the restriction of f to t.[15] With this notation, if $f \in G$ and $t \in T$, we have that $f|t$ is a possible world (or a situation). One can think of a possible world as a snap-shot together with a history. We want individual objects to be elements of a domain belonging to a possible world. An element of such a domain is to be understood as an object (in the Kantian sense) which the agent—whose epistemic access to the world we are now modelling—has experienced. Thus a domain of a world $f|t$ simply consists of the cognized objects at time t in the flow of states represented

[12] "*Formal methods are a hope for certainty in an uncertain profession.*" (Fitting; 2005, 13).

[13] In order to keep the simplicity of the logic at a minimum I refrain from adding constant symbols. Adding constant symbols can be done, but this should take into account the difference between rigid and non-rigid designators. Nevertheless, due to monotonicity in the time argument of the interpretation function i constants will locally be rigid in the models considered here. See also (Fitting and Mendelsohn; 1998, Chap. 9).

[14] The correct intuition about 'the agent' is, that it corresponds to a research community. Not as common knowledge, but as a community searching everywhere for truth.

[15] More precisely, $f|t(x) = f(x)$ for all $x \leq t$ and undefined otherwise.

by f restricted to t. Our description of a model will therefore include a function D which ascribes a domain to any possible world $f|t$. The models we are working with are therefore varying domain models. This means that the domains ascribed to the possible worlds are not the same. The reason for modelling knowledge-acquisition in this way is obvious: As time goes by we cognize more and more object. Therefore we will require that if $g|t'$ is temporally accessible from $f|t$ then the domain of $f|t$ should form a subset of $g|t'$.

The domain of a possible world $f|t$ is (thought of as) a set of cognized objects. The future, however, is open in the sense that it is not a priori determined which objects are going to be cognized. This is subjectively determined by the acts of the agent. On this understanding the future is open and is definable by a partial order.[16] Now, $g|t'$ is accessible from $f|t$ if and only if, $t < t'$ and f restricted to t and g restricted to t represent the same flow of states; in symbols:

$$f|t \prec g|t', \quad \text{if and only if,} \quad f|t \subseteq g|t'.$$

Therefore, if $f|t \prec g|t'$ then $t < t'$ and $f|t = g|t$. Our models will have, as mentioned above, varying domains; they are, more precisely, monotone in the sense that the domain function D is monotone, meaning that $f|t \prec g|t'$ implies $D(f|t) \subseteq D(g|t')$. The relation $\prec$ will be the accessibility relation providing a semantics for $\Box A$. But we also need a relation R defining the truth of $\circ A$. This is more complicated and can be done in various different ways, which I will discuss below. Relatively to a definition of R we can now define what a model is.

A *monotone domain model* $\mathfrak{M}$ (hereafter just model) consists of a (monotone domain) frame M and an interpretation i. Let T be the set of time-moments and S the set of states, and let G be a non-empty set of functions from T to S. We define from these two sets another set W which consist of the worlds, i.e.,

$$W = \{f|t \mid t \in T \text{ and } f \in G\}.$$

Then a *frame F* is a quadruple

$$(W, \prec, R, D)$$

[16] There is a choice regarding the temporal accessibility relation—a least two different notions of time/future are available. We could at the beginning have taken possible worlds to be pairs (t, f) with $t \in T$ and $f \in G$. Then a possibility is to take (t', g) to be temporally accessible from (t, f) if and only if, $t < t'$ and $f = g$. This concept of future will, however, imply a realist understanding of the future. We can understand a possible world as a situation within a *full flow of states*. Thus, once a a possible world is given, then the future of that situation is fully determined in the model. Here I have chosen the open understanding of the future

where $\prec$ and R are binary relations on W, telling us how the worlds are temporally and epistemically related, respectively. D is a monotone domain function ascribing a domain to each world $f|t \in W$ and the domain of the whole frame is the union of all the different world-domains. An *interpretation* of $L(\Box,\circ)$ in a frame F is a function i which assigns to each predicate symbol P and to each world $f|t$ a subset of $D(f|t)$. Thus

$$(P, f|t) \mapsto i(P, f|t) \subseteq D(f|t),$$

with the important restriction that i is monotone in the time-argument in the sense that:

$$t < t' \quad \text{implies} \quad i(P, f|t) = D(f|t) \cap i(P, f|t').$$

This monotonicity requirement ensures that determinate truth values of atomic formulae at $f|t$ are embedded into $g|t'$ whenever $f|t \prec g|t'$.

This leads to the definition of model. $\mathfrak{M}$ is a model if $\mathfrak{M}$ is a frame together with an interpretation in the frame, i.e.

$$\mathfrak{M} = (W, \prec, R, D, i).$$

As we also want to talk about truth values of formulae with variables, we need a valuation of the variables of the language. A valuation with respect to the model $\mathfrak{M}$ will be a function assigning elements to variables. A valuation function will take any variable from $L(\Box,\circ)$ in to the domain of the whole model together with countable set I. The elements of I are thought of as quasi-objects, thus $D(\mathfrak{M}) \cap I = \emptyset$. Consequently, if $\mathfrak{M}$ is the model and V the set of variables, then a valuation v is a function, such that $v : V \rightarrow \text{dom}(\mathfrak{M}) \cup I$. The valuation of variables ensures that we can speak of things which do not exist: When valuating the free variables in a model, a variable can 'refer' to either something which does not exist at all, or which only exist at certain times.[17] From this it follows, for instance, that $\forall x P(x) \rightarrow P(y)$ is not generally valid, because the valuation of y may be non-existent at a particular world $f|t$, although P is actually true for every object in the domain of $f|t$. This flexibility implies that we can express in the language what it means to exist: $E(x)$ is an abbreviation of $\exists y(y = x)$.[18]

In consequence, the interpretation of quantifiers is the *actualist* interpretation, which means that quantifiers are ranging over what actually exist at a given time.

[17] Through a valuation function v free variables act like constants; i.e., if $v(x) = a$ then x as free variable functions as name for a everywhere in the model. The denotation of a free variable will therefore vary from valuation to valuation. A solution to this problem is to work with constant symbols. But as mentioned above, I have chosen not to do this here in order to keep the complexity at a reasonable level. The generalization is, however, trivial.

[18] In consequence of this $\forall x P(x) \wedge E(y) \rightarrow P(y)$ is valid (where $\forall x$ binds stronger than $\wedge$ which again binds stronger than $\rightarrow$). See generally (Fitting and Mendelsohn; 1998) for details on this kind of logic.

This way of modelling truth of first order formulae ensures that universal formulae can change truth value over time. Suppose $A(x)$ is true of black cats. *If* all the cats which are experienced in the situation $f|t$ are black, then $\forall x A(x)$ is true at $f|t$. But if the agent at a later situation $g|t'$ experiences a cat with some (other) color, then $\forall x A(x)$ is false a $g|t'$. Thus a formula has a truth value relatively to a context; a possible world (a situation) and the objects which are experienced there. This also has consequences concerning the truth value of future events. First of all, future events are not possible objects of experience at the present. Thus a property, such as the Germans will win the sea-battle the first coming Saturday, will not have a truth value. Moreover, once we arrive at Saturday, when the battle becomes a possible object of experience, then it is possible both to have a situation where the Germans win and to have a situation where they loose. This example, however, also shows that we need to work with a three truth values.

We have to kinds of atomic formulae. If P is a predicate symbol, and x a variable then $P(x)$ is an atomic formula. Moreover, if x and y are variables, then $x = y$ is also an atomic formula. Literals are atomic formulae and negations of atomic formulae. Now, suppose $P(x)$ is an atomic formula of the first kind, then $P(x)$ is false in a bivalent logic at all worlds where x does not denote. This, however, will conflict with another feature that we want to represent. A formula like $P(x)$ with x not denoting should correspond to assertions about 'things' which are not yet, possibly never, objects of experience. Therefore, if x is not denoting at the moment then the truth-value of $P(x)$ should be neither true nor false, but undetermined. On the other hand, once x denotes an object in the domain of a world then $P(x)$ should be either true or false; *and this truth-value should remain the same in the future*. We will therefore follow Kleene (1952) and have a logic with three truth-values $\{0, {}^1\!/_2, 1\}$, with 0 being false, 1 being true and ${}^1\!/_2$ is undetermined. 0 and 1 are called *determined* truth-values and determined truth values of literals, will not change over time. Once it is true or false that the Germans won the battle, then this should remain as a fact for the future. In fact we want this property to hold for the class of purely existential formulae.[19] Below I will prove that this property is contained in every model. The behavior of the undetermined truth value should, however, be compatible with an increase of experience/information; thus a literal can go from undetermined to a determined truth value. Kleene used the term *regularity* for this property—today monotonicity is used (the undetermined truth value being below both of the determined truth values).

We can now give a definition of truth at a world. We let $[\![A]\!]^{\mathfrak{M},v}_{f|t}$ denote the

[19] A purely existential formula A is either a propositional formula or a propositional formula prefixed with only existential quantifiers. Thus the class of purely existential formulae is a pendant to the Σ_1 formulae of recursion theory.

truth value of A at $f|t$ in the model $\mathfrak{M}$ under the valuation v. In taking notational complexity into account we only write $[\![A]\!]_{f|t}$ when it is clear which model and which valuation we are talking about. The truth value function is defined recursively on the complexity of formulae. Let $\mathfrak{M}$ and v be given.[20]

$$[\![P(x)]\!]_{f|t} = \begin{cases} 0, & \text{if } v(x) \notin i(P, f|t) \text{ and } v(x) \in D(f|t), \\ 1/2, & \text{if } v(x) \notin D(f|t), \\ 1, & \text{if } v(x) \in i(P, f|t), \end{cases}$$

$$[\![x = y]\!]_{f|t} = \begin{cases} 0, & \text{if } v(x) \neq v(y), \\ 1, & \text{otherwise.} \end{cases}$$

The connectives are simple (and give rise to a version of Kleene's strong three-valued logic): The truth value of $\neg A$ is 1 minus the truth value of A and the truth value of $A \wedge B$ is the inf of the truth values of A and B.[21] The truth value of $\exists x A(x)$ is 1 at $f|t$ in case there is an object $a \in D(f|t)$ such that $A[a]$ has value 1; otherwise it is 0.[22] The semantics of the modal operators are defined as follows:

$$\begin{aligned} [\![\Box A]\!]_{f|t} &= \inf\{n \mid [\![A]\!]_{g|t'} = n, \text{ and } f|t \prec g|t'\}, \\ [\![\circ A]\!]_{f|t} &= \inf\{n \mid [\![A]\!]_{g|t'} = n, \text{ and } f|t\, R\, g|t'\}. \end{aligned}$$

The set $\{g|t' \mid f|t\, R\, g|t'\}$ should contain the situations the agent believes possible relatively to $f|t$. As for a definition of R there are many choices. One requirement could be that $f|t\, R\, g|t'$ obtains only if $t = t'$. Such an R would imply that the agent knows the time. But we need a stronger R, as with this definition R is not even reflexive. Why should R be reflexive? If we want knowledge of A to imply truth of A, then R need to be reflexive, i.e. reflexivity of R implies that, if $\circ A$ is true, then also A is true. An attempt to define R more specifically could be the following: Let R be the union of some subset of $\prec$ and the identity relation on worlds, then Kant's principle—in a proper modalized version—would be validated (in a certain sense, to be explained below). Such a definition would, however, be a very strong notion of

[20]That equality is always interpreted by the mathematical identity relation means that we are only working with normal models.

[21]A consequence of this definition is that validity is not a very interesting concept anymore. At least not, if it is understood in the classical meaning: A formula is valid, iff it has truth value 1 everywhere in every model. In our three-valued logic we loose for instance tautologies, as if A is indeterminate, then so is $A \to A$. Generally, the truth of $\to$ is given from the truth of $\neg$ and $\wedge$. Thus $[\![A \to B]\!]_{f|t} = 1 - \inf\{[\![A]\!]_{f|t}, 1 - [\![B]\!]_{f|t}\}$. But as we will see below, we can refine the classical meaning of validity in order to take this problem into account.

[22]More precisely, by saying '$A[a]$ has value 1 at $f|t$' means that there is a valuation w which agrees with v on every variable, except (possibly) x with $w(x) = a$ and $[\![A(x)]\!]^{\mathfrak{M},w}_{f|t} = 1$.

knowledge, as the agent in a situation would know everything which can be known, principally, in that situation.

But in fact we do not need a precise definition of R at the moment. We can view the definition of R as a parameter giving rise to different classes of models. As we want the most general definition of R it suffices to note that there is a non-empty class in which the Kantian principle is 'validated' (this will become completely clear when we below modally define Kant's principle and our concept of 'validation). Let us therefore attack the analysis from another point of view, assuming only that R at a minimum is reflexive. Thus we turn to an analysis of the formulae we want to be 'validated'. What can 'validation' mean in a three-valued logic? Note first of all, for a literal A, if $[\![A(x)]\!]_{f|t} = 1$ then also $[\![E(x)]\!]_{f|t} = 1$. It is this type of entailment we are interested in. But note also that although we have this entailment,

$$A(x) \rightarrow E(x),$$

will not be valid (meaning having truth value 1 everywhere), because we can always provide a model containing a situation such that $A(x)$ is undetermined, thus also $E(x)$ is undetermined and in consequence of this also $A(x) \rightarrow E(x)$ is undetermined.[23] We will therefore work with a notion of 'universally true' which, in the context of three values, is weaker. It is a generalization in the sense that over classical logic the two notions are the same.

For the rest of this chapter, when I talk about models I mean monotone domain models of the type defined above where $\prec$ is serial (reflecting the infinity of time) and R is at a minimum reflexive.

Definition ($\mathcal{K}$-semi-validity). Let $\mathcal{K}$ be a class of models. We say that a formula

$$A \rightarrow B,$$

is $\mathcal{K}$*-semi-valid*, if and only if, for any model $\mathfrak{M}$ in $\mathcal{K}$, any world $f|t$ in $\mathfrak{M}$ and any valuation v, if A is true at $f|t$, then B is true at $f|t$, i.e.,

$$[\![A]\!]_{f|t} = 1 \text{ implies } [\![B]\!]_{f|t} = 1.$$

The principle which we want to be semi-valid is Kant's principle of complete determination. Based on Posy's formula (4.1) a candidate for a formulation within our framework is

$$E(x) \rightarrow \Box\Diamond\circ P(x) \vee \Box\Diamond\circ\neg P(x),$$

[23] See footnote 21 for the definition of truth value of an implication.

where P is any predicate symbol. Thus the notion '$P(x)$ is knowable' is formalized by $\Box\Diamond\circ P(x)$.[24] But in fact, as for a formalization of Kant's principle, we can do with $A(x) \rightarrow \Box\Diamond\circ A(x)$, where A is any literal, as the two formulae are semi-valid for the same classes of models.:

Observation 1. Suppose $\mathcal{K}$ is a class of models and P any predicate symbol, then

$$E(x) \;\rightarrow\; \Box\Diamond\circ P(x) \vee \Box\Diamond\circ\neg P(x), \tag{4.3}$$

is $\mathcal{K}$-semi-valid, if and only if,

$$A(x) \rightarrow \Box\Diamond\circ A(x), \tag{4.4}$$

is $\mathcal{K}$-semi-valid for any literal A.

Proof. We need to show that under the assumption of $\mathcal{K}$-semi-validity of (4.3), then given any $\mathcal{K}$-model $\mathfrak{M} = (W, \prec, R, D, i)$, any world $f|t$ from W and any valuation v, the truth of the premise of (4.4) at $f|t$ will imply the truth of the conclusion of (4.4) at $f|t$; and vice versa.

Only if. Let a literal A be given and assume $[\![A(x)]\!]_{f|t} = 1$. From this it follows via the definition of truth that $[\![E(x)]\!]_{f|t} = 1$. As (4.3) is $\mathcal{K}$-semi-valid we have $[\![\Box\Diamond\circ P(x) \vee \Box\Diamond\circ\neg P(x)]\!]_{f|t} = 1$, for any P; therefore $[\![\Box\Diamond\circ P(x)]\!]_{f|t} = 1$ or $[\![\Box\Diamond\circ\neg P(x)]\!]_{f|t} = 1$. Assume without loss of generality that A is Q for some predicate symbol Q and, furthermore, for the sake of obtaining a contradiction, $[\![\Box\Diamond\circ Q(x)]\!]_{f|t} \neq 1$; but then $[\![\Box\Diamond\circ\neg Q(x)]\!]_{f|t} = 1$ would be the case. Now, as $\prec$ is serial this implies existence of $g|t'$ being accessible from $f|t$ such that $[\![\circ\neg Q(x)]\!]_{g|t'} = 1$. By reflexivity of R and the definition of truth of negation we get $[\![Q(x)]\!]_{g|t'} = 0$. This, however, contradicts $[\![Q(x)]\!]_{g|t'} = 1$ which follows from the assumption that $[\![Q(x)]\!]_{f|t} = 1$, and the fact that the interpretation i embeds determined truth values of literals in future worlds.

If. Let P be given and assume $[\![E(x)]\!]_{f|t} = 1$. Now, either $[\![P(x)]\!]_{f|t} = 1$ or $[\![\neg P(x)]\!]_{f|t} = 1$. In both cases we use $\mathcal{K}$-semi-validity of (4.4); here we only treat the former case. If $[\![P(x)]\!]_{f|t} = 1$ then $[\![\Box\Diamond\circ P(x)]\!]_{f|t} = 1$; therefore $[\![\Box\Diamond\circ P(x) \vee \Box\Diamond\circ\neg P(x)]\!]_{f|t} = 1$. $\dashv$

Due to Observation 1 we take

$$A(x) \rightarrow \Box\Diamond\circ A(x), \tag{4.5}$$

[24] As Baire Space can be used to give a concrete model of the type treated in this chapter, our notion of knowability generalizes (perhaps not surprisingly) Kelly's notion of *identification in the limit*. See (Kelly; 1996, 219). Another variant of this is found in (Hendricks and Pedersen; 2003, 353) under the name "Limiting convergence".

for any literal, to be the formalization in $L(\Box,\circ)$ of Kant's principle of complete determination, and we can use this formula to define the class M consisting precisely of all those models which semi-validate Kant's principle:

$$M = \{\mathfrak{M} \mid \mathfrak{M} \text{ semi-validates } A(x) \to \Box\Diamond\circ A(x),\ A \text{ being literal}\}.$$

This class is well-defined, non-trivial and closed under intersection and union of sub-classes.

We have in fact, for any given class K, (4.5) is K-semi-valid, if and only if,

$$A(x) \leftrightarrow E(x) \wedge \Box\Diamond\circ A(x)$$

is K-semi-valid. This leads to:

Observation 2. In M, the class of models semi-validating Kant's principle, we have that

$$A(x) \leftrightarrow E(x) \wedge \Box\Diamond\circ A(x),$$

is semi-valid.

The philosophical consequence of this equivalence is that Kant is operating with a strong notion of local knowability of primitive facts. Once a 'thing' becomes an object of experience then the truths about that object *are characterized* by what we in principle can know about this object. Kant therefore operates with precisely the same notion of truth as is the common understanding of truth in constructive mathematics: A proposition is true, iff it is understood and provable. See also my discussion on page 73.

Suppose now that x is cognized in a situation $f|t$. Let t_0 be the least time-moment such that x is cognized at $f|t_0$. Now form the set U_x such that

$$U_x = \{g|t' \mid f|t_0 \prec g|t'\},$$

Then we have that, if A is a literal

$$\begin{aligned}[\![\Box\Diamond\circ A(x)]\!]_{f|t} = 1, \quad &\text{iff} \quad \text{for all } g|t' \in U_x : [\![A(x)]\!]_{g|t'} = 1\\ &\text{iff,} \quad \text{there exists } g|t' \in U_x : [\![A(x)]\!]_{g|t'} = 1\end{aligned}$$

This can probably be strengthened to purely existential formulae.

Furthermore, this property of the operator $\Box\Diamond\circ$ has properties being similar with an S5 operator. In fact the S5 axiom

$$\neg\Box\Diamond\circ A(x) \to \Box\Diamond\circ(\neg\Box\Diamond\circ A(x)), \tag{4.6}$$

is semi-validated locally in every context U_x, by putting extra conditions on the accessibility relation R; if for instance R is also symmetric and transitive, then (4.6) is semi-validated locally. These conditions are, however, for obvious reasons to strong.

Now, what kind of logic is this? The general logic is a first order modal version of Kleene's strong three valued logic. This corresponds generally to some kind of intuitionistic logic. But underneath this there is a stronger logic. Once objects are intuited the truth-values are transcendentally determined. This is the principle of complete determination. This gives us an ontological relation of truth living unknown to (or as a transcendental principle for) the agent's understanding. Namely, in case x is intuited then the truth value for any predicate symbol is determined at the situation and in all later situations. Moreover, $\Box\Diamond\circ P(x)$ follows the ontological determination (if it is $P(x)$ which is true, otherwise its negation). The relation determining where $\Box\Diamond\circ P$ is to be found is living its own local life, so to speak. Thus, once an object x comes into being, transcendentally, truth values are decided. The locality is well-defined by U_x.

In our modelling of Kant's theory we thus have two different logics around. We have the global logic which concerns generally all terms, whether they denote objects or not. In this logic we can formulate propositions which we cannot be sure have determined truth values. This corresponds to the problem we generally face in the natural sciences:

> [I]n natural science [*Naturkunde*] there are an infinity of conjectures in regard to which certainty can never be expected, because natural appearances are objects that are given to us independently of our concepts, to which, therefore, the key lies not in us and in our pure thinking, but outside us, and for this reason in many cases it is not found; hence no certain account of these matters can be expected. (A480–1/B508–9)

On the other hand, underneath the global logic, which cannot be classical, a local logic which applies to objects we have experienced. This is stronger logic, where truth and knowability go hand in hand. This also shows how in Kant's theory, epistemology and ontology are two sides of the same coin. Note, that it is the fact that a term may denote or not denote which enables us with structure having the consequence that we can separate the two layers.

4.3 Distinguish-ability of objects

A consequence of the principle of complete determination for intuited, and thus existent, objects is the property that an object can be distinguished from other objects:

- Any object x can ultimately be distinguished from any other object y.

Actually, this property is a *characterizing* property of an object. x is a legitimate object only if it can be distinguished from any other object:

> [S]ince the agreement of cognition with the object is truth, only the formal conditions of empirical truth can be inquired after here, and appearance, in contradistinction to the representations of apprehension, can thereby only be represented as the object that is distinct from them if it stands under a rule that distinguishes it from every other apprehension, and makes one way of combining the manifold necessary. That in the appearance which contains the condition of this necessary rule of apprehension is the object. (A191/B236)

The principle of complete determination is indeed a useful principle. The principle also gives us a rule which works the other way around: in the case we want to show that something is *not* an object. If for a certain 'thing' we can show that there is a (well-typed) property, which the 'thing' is neither true nor false for, then the thing cannot be an object. Precisely this way of formulating the principle is what Kant uses when he demonstrates that the whole world is not a possible object of our experience. We are not, and will never be, in a position where we can show the world to be finite, nor infinite—understood as not being finite. Thus the principle is a powerful tool when we are to identify ideas of reason.

The principle has furthermore an important consequence for equality relations. Given any class of objects equality on that class will be decidable as different objects are ultimately distinguishable.[25]

4.4 The notion of mathematical objects

We know what it means to be a physical object. Now, Kant's answer to the question, 'what gives mathematics its objectivity?' is, however, somewhat provocative. Let us therefore be careful in the examination of it.

> [A]ll concepts and with them all principles, however *a priori* they may be, are nevertheless related to empirical intuitions, i.e., to *data* for possible experience. Without this they have no objective validity at all, but are rather a mere play, whether it be with representations of the imagination or of the understanding. One need only take as an example the concepts of mathematics, and first, indeed, in their pure intuitions. Space

[25] See also (Posy; 1998, 316).

> has three dimensions, between two points there can be only one straight line, etc. Although all these principles, and the representation of the object with which this science occupies itself, are generated in the mind completely *a priori*, they would still not signify anything at all if we could not always exhibit their significance in appearances (empirical objects). (A239–40/B298–9)

Apparently the only real objects are the empirical objects. Mathematics is without meaning had it not been for the empirical. Now, this indeed sound drastic and sounds like Kant is having some kind of empiricist foundation in mind. The claims, however, has to be seen in connection the transcendental ideality of space and the resulting Axioms of intuition. What Kant says in the quote does not downgrade the importance of geometry, nor its apodictic nature. On the contrary. Geometry is constitutive for empirical objects as geometry describes the a priori structure of human perception. Therefore, as geometry provides pure results synthetic a priori, these are valid for empirical objects. Geometry in this way provide us with knowledge about both the conditions of our sensible representations and—in consequence of this— also about the objects that appear to us under those conditions. Given this, it can come as no surprise that the objective validity of geometry has a necessary connection with the empirical.

And moreover, as with almost all central aspects of Kant's theory of knowledge, this standpoint is connected with Kant's doctrine of schematism. Above, Kant says "rather a mere play" and thus he refers to "[t]houghts without content are empty, intuitions without concepts are blind" (A51/B75). The concept of a triangle, for instance, means nothing without images. Now we can imagine an image of a triangle, but this again, means nothing without the possibility of an experience of a physical triangle.

But there is another problem with the quote given above. The claim, that mathematical concepts are meaningless without the possibility of empirical exemplification. The problem is that the claim is made on behalf of all of mathematics. Again, this is not a slip of the tongue:[26] "The mathematician can not make the least claim in regard to any object whatsoever without exhibiting it in intuition" (Ak. 11, 44).[27] As there can be no such thing as a number-intuition, number theory receives its objects,

[26]Another place in the *Critique* with the same message is from the B-deduction: "[A]ll mathematical concepts are not by themselves cognitions, except insofar as one presuppose that there are things that can be presented to us only in accordance with the form of that pure sensible intuitions. **Things in space** and **time,** however, are only given insofar as they are perceptions (representations accompanied with sensation), hence through empirical representation. (B147)

[27]The translation is Allison's (1973, 167).

ultimately, from the empirical intuition. The problem here is that there is only room for geometrical objects. Now, this only reflected the standard view on mathematics from the Antique to the 17–18th century: Geometry was in good shape with a solid, Euclidean foundation. Number theory, on the other hand, had no foundation, had no axioms. Although—as we have seen in chapter 2—Kant gives in fact a foundation for number theory. But in this foundation numbers are rules rather than objects. Or more precisely, number theory is founded on the "schema of magnitude" and numbers are not really perceivable, not like geometric objects. Tokens of numbers are found as sets of first order empirical objects, and therefore numbers are not really objects, as their tokens are second order objects.

Mathematics is about constructibility in pure intuition, but because of the axiom of intuition, this means that the objects for number theory come from geometry, and thus ultimately from empirical intuition.[28]

4.4.1 Schematisation and mathematical objects

Having concluded that the real mathematical objects are the geometrical objects let me finish with a different approach to the characterization of mathematical objects.

I propose that to understand geometrical, and thereby mathematical, objects like this:

- x is a geometrical object, if and only if, x can be schematised.

Space is an a priori structure. Together with the fundamental geometrical schemata it gives us our perceptual grid. This perceptual grid is the form of all outer intuition. Therefore, the form of any sensible object is ultimately describable by schemata.

Such a definition of object also explains why space, the all-including intuitive space, cannot be an object: It is not schematisable. No operation can construct the full space out of its parts. No schema can complete the operation

$$\bigcup S_i.$$

And this also explains why the geometrical spaces, as treated in the science of geometry, have to be finite. Otherwise, they would not be objects.[29]

> The question, however, as to how this single infinite space is given, or how we have it, does not occur to the geometrician, but concerns merely the metaphysician. Moreover, it is just here that the *Critique* proves that

[28]"Mathematics does not merely construct magnitudes *quanta*, as in geometry, but also mere magnitude (*quantitatem*)" (A717/B745).

[29]See again (Ak. 20, 419–20).

> space is not at all something objective, existing apart form us (*außer uns*), but rather consists merely in the *pure form of the mode of sensible representation of the subject as an a priori intuition*. (Ak. 20, 421)[30]

Thus we have a schematic understanding of a methodological difference between the metaphysician and the geometer. The metaphysician analyzes *Vorstellungen* which cannot be treated in any schematisable and therefore scientific way. This is just another way of saying that the Metaphysical exposition is a kind of pre-science.

As we know, quantum is something in which there is quantity. Sometimes the quantitas of a quantum is determinable by the “schema of magnitude”. Sometimes it is not. The situation is similar from modern class–set theory. Some classes have size determinable by an ordinal, some do not. This distinction between reason and understanding has many of the consequences that the distinction between set and class in set theory has. Some concepts are objective, can be experienced, like some classes are sets. But not all concepts, like not all classes can be intuited. This idea is also found in (Tiles; 2004), where it is even claimed that Kant foreshadowed some of Gödel’s insights.

Now, a speculative claim of mine is the following. Just like we can characterize the notion of geometrical object by the geometrical schemata, it should be possible to characterize the notion of empirical object by the ‘empirical’ schemata. This, however, is something which needs a full theory of Kant’s schemata; something which—to my knowledge—still has to be done.[31]

[30]Translation due to Allison (1973, 176).

[31]Evidence showing that this indeed was an idea of Kant, is found in two important places: A200/B245 and A280/B236.

CHAPTER 5

A Critical Discussion of Kant's Philosophy of Mathematics

Kant certainly is one of the founders of the constructive philosophy of mathematics, as he provides an epistemological foundation for the constructive mathematical reasoning he finds, primarily, in *Elements*. The corner stone in this foundation is schematism.

5.1 Schematisation of mathematical concepts

What Kant claims in Schematism is that any concept is founded on a schema. Without a schema any concept is void of meaning as we cannot bring the concept together with the objects which should be subsumed under the concept. This is in particular true of mathematical concepts.

In case of geometry Kant says that geometrical concepts are *definable*. A concept is given by an intension and in geometry this can be done a priori in an exhaustive way.[1] Take Kant's paradigmatic example, triangle:

$$x \text{ is a triangle,} \quad \text{iff,} \quad \begin{array}{l} x \text{ is a polygon with three vertices} \\ \text{and three sides which are straight lines.} \end{array}$$

Thus, the concept of triangle is something passive. It does not tell us how to produce or manipulate triangles. It gives us necessary and sufficient conditions for judging a given object to be a triangle. The concept, however, is useless without a schema.[2] The schema of triangle is basically a rule which puts us in a position where we can produce *any* triangle. This means that any object x which can be subsumed under the concept triangle is in principle constructible in a finite amount of time by the schema. Using modern terminology we say that the extension of triangle is exhausted by the schema, thus the schema determines the meaning of the concept. In this sense, Wittgenstein's dictum "meaning is use" is a derivative of Kant's theory of schemata.

As I discussed in section 4.4, schematism provides an alternative definition of geometrical objects. The geometrical objects are precisely those objects that are constructible through a geometrical schema. As it turns out, the only true mathematical

[1] "**[T]o define** properly means just to exhibit originally [*ursprünglich*] the exhaustive concept of a thing with its boundaries" (A727/B755).

[2] In the words of Kant: "Give a philosopher the concept of a triangle, and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on this concept as long as he wants, yet he will never produce anything new. He can analyze and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts." (A716/B745)

objects are the spatial—therefore geometrical—objects. This was expressed, for instance, in the long quotation found on page 44. Thus, on Kant's view all mathematical reasoning is ultimately based on what can be constructed in space. The notion of geometric constructibility is therefore at the end of the day responsible for the so-called mathematical optimism that Kant advocates. What precisely does this optimism consist of?

I discussed in chapter 4 that generally in science, the global logic is different from and weaker than the local logic. In the Doctrine of Methods Kant discusses this theme which connects ideas of reason, objects of experience and the logic of reasoning in mathematics and the natural sciences, respectively:

> Apagogic proof, however, can be allowed only in those sciences where it is impossible **to substitute** that which is subjective in our representations (*Vorstellungen*) for that which is objective, namely the cognition of what is in the object. Where the latter is the dominant concern, however, then it must frequently transpire that the opposite of a certain proposition either simply contradicts the subjective conditions of thought but not the object, or else that both propositions contradict each other only under a subjective condition that is falsely held to be objective, and that since the condition is false, both of them can be false, without it being possible to infer the truth of one from the falsehood of the other. (A791/B819)

Kant gives on A793/B821 the obvious example of the latter: The magnitude of the sensible world. The underlying subjective representation (*Vorstellung*) is the whole sensible world, which has been substituted for an object. This gives rise to the false assumption that the whole sensible is either finite or infinite. This would, according to Kant, be a misuse of the principle of complete determination; let x refer to the "whole sensible world" and let F represent the concept finiteness, then[3]

$$E(x) \rightarrow F(x) \vee \neg F(x).$$

We will, however, never find a situation where x is an object of our experience, therefore $F(x) \vee \neg F(x)$ will never have a determinate truth value.

The sentence following the long quote above is: "In mathematics this subreption is impossible; hence apagogic proof has its proper place there" (A792/B820). Given his general epistemology, *Kant can only have one good argument in support of this point of view*:

1. All mathematical propositions have determinate truth values.

[3]Here the ontological version of the principle suffices.

This, I claim, is in fact equivalent to each of the following statements:

2 All mathematical objects are schematisable,

3 All mathematical objects are constructible in a finite amount of time,

4 Ideas, whether regulative or not, play no role in mathematics,[4]

5 Every mathematical 'thing' is complete determined,

6 The mathematical universe is epistemically complete.

Take the model I provided in the foregoing chapter, and suppose we want it to be a model of the mathematical epistemology and ontology as Kant sees it. The set I containing the quasi-objects is empty, as all terms in mathematics denote. An object of experience is now, not an object that we actually have an experience of, but an object which we can principally have an experience of:

$$E(x) \text{ is true, iff, } x \text{ is constructible by a schema.}$$

As all mathematical objects in the Kantian epistemology are finitely constructible, we are not in need indeterminate truth values for prime formulae. Thus, the global tertium non datur

$$P(x) \vee \neg P(x), \tag{5.1}$$

is true *in any situation*, for any mathematical object x and any predicate P. Thus for any mathematical object x, we will always be in the context U_x. In consequence of this, the ontological tertium non datur (5.1) is equivalent to an unconditioned global epistemological version of the principle, .i.e.,

$$\Box\Diamond\circ P(x) \vee \Box\Diamond\circ\neg P(x).$$

In fact, the mathematical universe is determinate in the sense, that there is no distinction between the local and the global, i.e., for any mathematical object x, $U_x = W$.

Using a term from recursion theory, we would say that the mathematical universe is decidable—according to Kant!

[4]In the new edition of *The Cambridge Companion to Kant*, Shabel (n.d.) has a paper with the title "Kant's Philosophy of Mathematics". It is surprising that she does not mention this essential property of Kant's mathematics.

5.1.1 Friedman and Kant

Michael Friedman (1992) purports the view that Kant compensated ingeniously for the *logical* resources of his time and used 'construction in intuition' instead of logic. Thus Friedman suggests, it seems, that *if* Kant had had access to modern logic he would not have needed construction in intuition. In other words *if* Kant had had quantifiers, then he would not have formulated construction in pure intuition as something governed by rules, rather he would have used quantifier expressions like $\forall x \exists y \cdots$. As evidence for this, Friedman points out that in fact Kant only had polyadic logic which is not capable of expressing infinity. Thus Kant *had* to develop a theory of intuition. In formulating this point of view Friedman is partially repeating Russell's critique from the beginning of the 20th century.

I think this view on Kant is quite wrong. Kant viewed Euclidean geometry as paradigmatic for mathematics. The reasoning style Kant found in *Elements* was—as I described in Chapter 1—based on diagrammatic reasoning. This is a *premise* for Kant. *Then* he wants to formulate a philosophy which can account for this and arrives at the notion of intuitions being constructible in accordance with schemata. His notion of constructibility is at the heart of his philosophy of mathematics, and thus it is very remote from a Kantian point of view to say that Kant developed his notion of intuition *because* he lacked a timeless notion of logic which could generate infinity. On my view, if Kant had lived today and should he follow Friedman's and Russell's view he should give up his whole philosophy. Schematisation in finite time is at the heart of Kant's epistemology; and, moreover, it is a crucial feature of Kant's theory of knowledge that he needs only one kind of epistemology which works for both the empirical and the mathematical.

If quantifiers should play any role then I would say, following Posy, that Kant would read quantifiers constructively (this seems obvious due to the fact that Kant always talks about constructivity)! But then quanfifiers are only abbriviations for constructive procedures, and the play no principle role.

5.1.2 Consistency and existence

As mathematical objects are schematisable it follows that the mathematical objects have to be in accordance with intuition. Thus consistency does *not* entail existence. This feature is in fact true for any proposed object:[5]

> That in [. . .] a concept no contradiction must be contained is, to be sure, a necessary logical condition; but it is far from sufficient for the objective

[5] See also B14 for the mathematical case and A218–21/B266–8 for the general case of the categories of modality.

> reality of the concept. (A220/B267–8)

5.2 What are numbers?

According to Kant numbers cannot be objects, as arithmetic has no axioms.

> The self-evident propositions of numerical relation [...] are to be sure, synthetic, but not general, like those of geometry, and for that reason also cannot be called axioms, but could rather be named numerical formulas. (A164/B205)

The numerical formulae Kant is thinking of are propositions like "$7+5=12$". But "[s]uch propositions must [...] not be called axioms (for otherwise there would be infinitely many of them)" (A165/B205). In a sense Kant is very right: *There was no axiomatization of number theory at the time of Kant*. In fact Kant reflects, once again, the Euclidean paradigm. Euclid has no axioms for numbers. Number theory is treated in Book VII, and already in proposition 2 (the gcd of two numbers) he uses well-ordering of the natural numbers, but with out reference to some first principle.

But if Kant had had axioms for number theory, would he have regarded them as genuine axioms? Axioms should be synthetic a priori in analogy to the way that Euclid's postulates reflect schematic spatial procedures.

According to Kant geometrical schemata produce true objects in pure intuition. These are genuine objects in the sense that they are possible objects of experience. Let us use the terminology of a first-order universe U and a second-order universe $\mathscr{U}$, which I introduced on page 38. Thus the objects which the geometrical schemata produce are elements of U. Representations of numbers, however, are elements in the second-order universe $\mathscr{U}$. Thus, they are not really objects to Kant as the only true objects are first-order objects. This also explains why the pure "schema of magnitude" deals more with reflection than construction. The elements of $\mathscr{U}$ are according to Kant not really constructed, they rather reflect a certain relation between objects living in U. I think this is the only good reason Kant has when claiming that the numbers are not objects.

In the course of history we have learned—due to relativity theory—that space is only *approximately* Euclidean. Thus it seems that the objects we produce in Euclidean geometry are only approximations of possible objects of experience. On the other hand, the number five is still represented by the set of fingers on a normal left hand. If we allow second-order objects to be genuine objects, then in fact they are more objective (in the Kantian sense) than the objects of Euclidean geometry.

Today we can give an axiomatization of the natural numbers, which relatively to the slight generalization of the Kantian notion of object, is a set of axioms in the

Kantian sense.

We can define an arithmetic Q called Robinson arithmetic in which there is a constant 0 called zero, a unary operation S, called successor and two binary operations $+$ and $\cdot$ called plus and times, which satisfy the following axioms:

1. $\forall x(0 \neq Sx)$.
2. $\forall x,y\big((Sx = Sy) \rightarrow (x = y)\big)$.
3. $\forall x\big(0 \neq x \rightarrow \exists y(x = Sy)\big)$.
4. $\forall x(x + 0 = x)$.
5. $\forall x,y\big(x + Sy = S(x + y)\big)$.
6. $\forall x(x \cdot 0 = 0)$.
7. $\forall x,y\big(x \cdot Sy = (x \cdot y) + x\big)$.

The first three axioms are realized by the "schema of magnitude". They claim existence of a concept of iteration S which taken together with a symbol 0 gives rise to a paradigmatic representation of the natural numbers:

$$0,\ S0,\ SS0,\ SSS0, \ldots$$

Due to our ability of producing bijections, these paradigmatic numbers as intuitions put us in a situation where we can use and reason universally about *any* representation of the numbers. Recall, the schema together with any representation, whether empirical or pure, allow for universal reasoning. The sequence is furthermore potentially infinite, thus for any type $\overline{n}$, the number $\overline{n}$ is meaningful. The realization of the latter axioms is also due to our ability to produce and operate with bijections. Plus corresponds to composition of functions, which we found on page 43 was validated by the "schema of magnitude" and times iterates this concept.

If we thus allow second-order objects to be objects and accept the above axiom system, then there are numbers, just as much as there are triangles—or perhaps even more. We meet them as tokens, and it should be clear that the principle of complete determinability holds for numbers. Any number $\overline{n}$ is schematisable in the sense that we can produce a representation $S \cdots S0$ such that the representation together with the schema allows for universal reasoning. This also makes certain elements found the first *Critique* more coherent, as for instance when Kant writes about number images (A149/B179; A240/B299).

5.3 On the infinite

Kant only allows the potential infinite in mathematics. But there are, at least, two different notions of potential infinity: One in geometry and one in number theory.

5.3.1 Infinity in geometry

What does Kant understand under the term, ‘the mathematical space is infinite’? We saw in Chapter 3 that space as intuition is claimed by Kant to be actual infinite in the sense that for which ever magnitude we might want to ascribe to space, that magnitude will not suffice. It is “an infinite **given** magnitude” in the sense that it is actual as a necessary presupposition of our geometrical schemata and henceforth of our geometrical concepts. Itself, being a presupposition for schemata, can however never be schematised. The only infinity we have in geometry is the infinity we find when we construct a non-convergent increasing sequence of spaces

$$S_0, S_1, \ldots, S_n, \ldots \tag{5.2}$$

such that for any number n, the space S_n is constructible and we can construct another space S_{n+1} which is larger than S_n. This gives us a mathematical notion of infinity. This is the potential infinite. This concept is a real mathematical concept with a schema. The schema realizing it is the procedure which produces any paradigmatic sequence in the style of (5.2), each element in the sequence being constructible. This is a genuine concept of infinity, and we have a corresponding schema, which *realizes* the concept. This is the core of the theory of schematism.

Note, that the metaphysical infinities of time and space as the two forms of intuition are *necessary* conditions for a notion of potential infinity in mathematics. Time is a necessary condition for the concept of magnitude, as the concept of magnitude is realized by iteration—the schema of magnitude. Without iteration it would be impossible to think of the magnitude of any given thing.

As the concept of the potential infinite is fully schematisable it poses no problem in mathematics. But the concept of the actual infinite is. According to Kant we cannot form the set—it cannot be an object, it cannot be schematised. In the case of space I argued on page 64 that although the sequence (5.2) is fully schematisable, the complete union

$$\bigcup S_i$$

cannot form an object, as there can be no corresponding schema.

5.3.2 Infinity in arithmetic

The number sequence is also infinite. Of course this can also only be a potential infinite. But in comparison with the infinity found in geometry, the arithmetical infinity is to a certain extent more general. Recall that numbers are intensional, they are not extensionally determined.[6] Therefore (tokens) of numbers can be anything, they are not necessarily spatial objects, they can also be mental states. But as to the class of all numbers Kant writes in § 26 of *Critique of Judgment* that this can only be an idea.[7] In that paragraph Kant writes that the potentiality of the sequence of numbers is an *objective* fact. The potential infinity here is essentially of same kind as the potential infinity of the Euclidean space. We have a procedure realizing the sequence, namely the "pure **schema of magnitude**". We do, however, force ourselves—Kant says—and go to the limit of this sequence of numbers and form the class of all numbers. This is necessarily so, but only as an idea; therefore an incomplete object. There cannot be anything in intuition corresponding to the class of all numbers. We know that realization (schematisability) of mathematical concepts are what provides objective reality to the concepts, thus the class of all numbers can only be a regulative idea. It is a true class in the sense that it has no size; it cannot be schematised; it cannot be a mathematical object. And here we see a conflict in Kant's philosophy: It cannot even a regulative idea in mathematics, as there are no ideas or other subjective notions in mathematics—according to Kant.

5.3.3 Does Kant's notion of infinity suffice?

There are, however, problems with Kant's understanding of infinity in mathematics. Some notion of actual infinity is needed in mathematics. *Also* at the time of Kant. There are many examples of this. It is clear, that the sequence of increasing spaces as given above,

$$S_1, S_2, \ldots, S_n, \ldots,$$

is schematisable, because we can provide an algorithm realizing it. *But when do such algorithmic processes have limits?* Using contemporary language we would say precisely when they converge. Suppose we have a metric for our spaces. Then, for

[6] See page 41 for my discussion of this aspect of numbers.

[7] "Nun aber hört das Gemüt in sich auf die Stimme der Vernunft, welche zu allen gegebenen Größen, selbst denen, die zwar niemals ganz aufgefaßt werden können, gleichwohl aber (in der sinnlichen Vorstellung) als ganz gegeben beurteilt werden, Totalität fordert, mithin Zusammenfassung in eine Anschauung, und für alle jene Glieder einer fortschreitend-wachsenden Zahlreihe Darstellung verlangt, und selbst das Unendliche (Raum und verflossene Zeit) von dieser Forderung nicht ausnimmt, vielmehr es unvermeidlich macht, sich dasselbe (in dem Urteile der gemeinen Vernunft) als ganz (seiner Totalität nach) gegeben zu denken."

example, if S_{n+1} is only a little bigger than S_n, e.g., $1/2^{n+1}$ then

$$\bigcup S_n$$

should be a Kantian mathematical object, whereas if S_{n+1} is $1/n$ bigger than S_n, then it is not. The notion of convergence was not at all clarified at the time of Kant, and it is somewhat peculiar that Kant had not considered this to be a problem. Take for instance the method of exhaustion which is used for measuring the circumference of the circle. The circumference is found by inscribing polygons with more and more vertices in the circle. As the edges of the polygons get smaller the difference between the circumference and the total length of the sides of polygons gets smaller and smaller. *In the limit* we have the circumference. We know that Kant worked with this method,[8] but apparently he did not feel the foundation questionable.

Euler worked intensively in the so-called infinitesimal analysis. He wrote textbooks which contained mathematical concepts like "function", "sum of infinite series", "integral", "solution to a differential equation"—concepts which in no way were clearly schematisable in the strict Kantian sense.

A series which caused endless dispute was

$$1-1+1-1+1-1+\dots$$

Now, we could treat the series in the following way:

$$0=(1-1)+(1-1)+(1-1)+\cdots=0+0+0+\dots$$

but then again,

$$1=1+(-1+1)+(-1+1)+(-1+\dots$$

is also a possibility. Still another sum might be reasonable:

$$T=1-(1-1+1-1+1\cdots)=1-T,$$

thus the whole sum should be $1/2$. Guido Grandi argued in 1703 (68 years before the first *Critique*) that this was indeed the result. He used the geometric series

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots,$$

with $x=1$.[9] According to Kline (1983, 308) this argument was accepted by Daniel Bernoulli and Leibniz, though Leibniz found the argument more metaphysical than mathematical.

[8] Although with great difficulty (Addickes; 1924, 19).

[9] According to Kline: "He also argued that since the sum was both 0 and $1/2$, he had proved that the world could be created out of nothing" (Kline; 1983, 307).

Euler also worked with this example. He took the series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \tag{5.3}$$

and substituted -1 for x and obtained Grandi's result. But Euler worked with other examples as well. For instance, by substituting -1 for x in

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

he obtained

$$\infty = 1 + 2 + 3 + 4 + \ldots \tag{5.4}$$

Simultaneously he substituted 2 for x in (5.3) and with the following result:

$$-1 = 1 + 2 + 4 + 8 + \cdots. \tag{5.5}$$

Euler treated infinity algebraically simply as another number and, moreover, concluded that -1 is larger than infinity, since the terms of (5.5) exceed the corresponding terms of (5.4).[10]

We see that Euler treated divergent series algebraically and really did not care much about whether the series were finite or infinite. Sometimes he was not so lucky (as above) but sometimes he was, as with:

$$\begin{array}{ccccccccccccc} 1 & - & 1/3 & + & 1/5 & - & 1/7 & + & \cdots & = & \pi/4 \\ 1 & - & 1/3^2 & + & 1/5^2 & - & 1/7^2 & + & \cdots & = & \pi^2/8 \\ 1 & - & 1/3^3 & + & 1/5^3 & - & 1/7^3 & + & \cdots & = & \pi^3/32 \end{array}$$

The experience in mathematics of such phenomena led the Berlin Academy in 1784 to offer a prize for a "clear and precise theory of what is called the infinite in mathematics." Although the Academy was not entirely satisfied with any of the entrants the prize was awarded to the Swiss mathematician Simon Antoine Jean L'Huillier (1750–1840). And all in all, these strange results were finally understood due to work of mathematicians like Bolzano and Cauchy who introduced the notion of *convergence* of infinite series in the beginning of the 19th century.

Kant could perhaps dismiss some of the infinite series by saying that they are just thought-experiments which do not really have anything to do with mathematics. This would certainly have been very non-typical for him, as his agenda generally was that we have science, and it is this science he wanted to give a theory of knowledge for. π

[10] See Kline (1983).

is objective, Kant says, because we can describe it by geometrical schemata. Thus π is a quantum. But obviously it is then a finite object in the sense that the figure has a finite area or a finite circumference. As both of these depend parametrically on π we need π if want to measure such finite magnitudes. How do we know that our method for determining π converges? Another example is the constant e. This number can be given as an infinite series or as a converging sequence of rational numbers:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

These were problems known at the time of Kant, but it is apparently not truly legitimate methods.

Another challenge to Kant's philosophy of mathematics is the ‘vibrating string controversy’. I will not go into this here, but let me just note that Euler carved the way for a whole new theory of partial differential calculus by introducing a new concept of function. When doing this Euler allowed very irregular functions in analysis, as long as these were used for describing *phenomena* in space, such as a vibrating string. Many of these functions are only describable by heavy infinitary methods.

In conclusion let me say that it is not in any way clear how Kant wants to treat the use of infinitary methods in mathematics. They are necessary, when for instance measuring the area of a circle, or when describing a vibrating string. Infinitary methods were used throughout the mathematical landscape in the 18th century. But none of this is taken into account in Kant's philosophy of mathematics. Moreover, to my best knowledge, he does not even comment on it![11]

All this points to a dissatisfying fact about Kant's philosophy of mathematics. First of all, it did not fit to the mathematics, even, of his time. Second of all, and this is somehow disappointing, Kant had opportunities to be updated on the newest results in mathematics. He was in contact with Lambert, and most probably Kant *knew* of consistent non-Euclidean geometries from Lambert. Kant wrote admiring about Euler. But he does not mention the obvious problems anywhere, seemingly.

5.4 Partial versus total schemata

For Kant, a natural number $\overline{n}$ is the equivalence class consisting of sets with n elements. This equivalence class is not determined extensionally by its elements, rather it is determined by the “schema of magnitude”. This schema is the human ability

[11] This somehow runs counter to some of Friedman's understanding of Kant's philosophy. Friedman is of the opinion that Kant's epistemology is “a fruitful philosophical engagement with the sciences” (Friedman; 1992, xii). This was perhaps the case in physics, but *not* the case in mathematics.

to decide (in the sense of computability theory) whether or not a collection of objects belongs to this equivalence class. As I also mentioned in the forgoing chapter, given any collection of mathematical objects, the equality relation on the collection is decidable.

Consequently it is a total notion of computability that Kant ascribes to schemata, they are functions defined everywhere—not partial functions. *But* as soon as we generalize the notion of object, such that we will also accept the class of natural numbers as an object, then we also loose this nice property of the mathematical schemata. Say, we have infinite collections as objects and suppose we are asked to decide whether an object is finite. This can connot be done in any case. Thus, if we allow ideas to be objects, then we can only expect schemata to be partial functions.

5.5 Kant's two dogmas

5.5.1 Complete determination of mathematical objects

Given decidability of the methods for constructing objects, it is natural for Kant to formulate his Dogma: Any mathematical question can be decided. Therefore Kant has complete optimism with respect to mathematical problems as for instance formulated on A480/B508 and also in the Doctrine of Method, as discussed above: We are allowed to use full classical logic in mathematics. But this really is a disputable dogma of Kant. We need infinitary methods in mathematics—and these methods were needed, even at the time of Kant. We need to work with some notion of infinity when we want to describe space. In consequence of this it seems natural to introduce some subjective elements in mathematics, such as the *set* of all natural numbers, or an absolute space.

5.5.2 Sharp and unique borderlines between intuition, understanding and reason?

Why can space not be conceptual to Kant? Because then he would loose the constructive aspect. He needs space to be intuitive because he wants construction. Mathematics is not based on concept analysis, it is based on constructions in time. I accept the arguments Kant gives in the Transcendental Aesthetics for the a prioriness of space. But the arguments for the intuitiveness and uniqueness of space are not conclusive, see my discussion on page 63. Thus, the sharp borderlines between intuition and understanding are questionable.

But there are other reasons why the sharp borderlines cannot be maintained. We see that Kant is against the view that in arithmetic and in geometry there are subjective elements. This is because he wants arithmetic to be the unique codification of the objective interplay between the quantitative categories and the transcendental imagination. Geometry is the unique codification of the objective interplay between space as form of intuition and the transcendental imagination. There should be a one-to-one correspondance between the human abilities and the mathematical conceptualizations. Therefore, as soon as one allows concepts of reason to be involved in arithmetic and in geometry the absoluteness of the understanding and the absoluteness of space as form of intuition are destroyed. Such subjective elements *are*, nevertheless, involved. Thus, 'interests' of reason have a decisive factor when the structure and the properties of space are decided. The same applies to more advanced arithmetic including infinite series and the like.

Kant also claims a sharp borderline between inner and outer. Nevertheless it is also claimed that we cannot think spatial concepts such as lines and circles without drawing them in thought, i.e., picturing them in the inner sense. On the other hand, we cannot think temporal concepts without representing them in time (B154); see page 5 for a discussion of these elements. Thus it seems that sharp borderlines between the two forms of intuition—time and space—cannot be drawn as strict as Kant wants it. In fact this is compatible with some of the results from relativity theory, namely that time and space are not completely separable.

CHAPTER 6

A generalization of Hilbert's Philosophy of Mathematics

David Hilbert was perhaps the most important mathematician in the period from 1880 to 1940. He contributed in an impressing amount of areas within mathematics and mathematical physics. Moreover, together with L.E.J. Brouwer he was the main figure in the so-called Grundlagenstreit in the 1920s, and he became well-know for his program. I will not treat the program here. Rather, I want to show that his general philosophical view on the mathematics can be seen as a generalization of Kant's philosophy of mathematics.

6.1 Introduction

> The infinite is realized nowhere; it does not exist in nature, nor is it admissible as a foundation of our rational thought. And yet we cannot dispense with the unconditional application of the *tertium non datur* and of negation, since otherwise the gap-less and unified construction of our science would be impossible. (Hilbert; 1931, 488)[1]

Hilbert developed his philosophy of mathematics in order to incorporate a modern theory of the infinite. When doing this he founded his views on Kant's epistemology—on his philosophy of mathematics, his philosophy of science in general, and his general theory of knowledge.

Hilbert was not a philosopher, he was a mathematician. But he was very interested in philosophy of science and epistemology. It is clear that Hilbert read Kant in the 1880s and -90s. This was standard among intellectuals in Germany at that time. Perhaps Hilbert's interest in Kant stems from his reading of Hertz' *The Principles of Mechanics*, which influenced him strongly by the end of the 19th century.[2] Later on, in 1918 Paul Bernays was appointed as Hilbert's assistant, and Bernays took a very active part in the preparation of a series of lectures *Natur und mathematisches Erkennen* (Hilbert; 1919). Bernays—being both mathematician and philosopher—became a life-long assistant of Hilbert and had a great influence on Hilbert's conception of mathematics. In fact, I think it is reasonable to claim that with respect to philosophy of mathematics in the 1920s and 30s Hilbert's and Bernays' views are inseparable.[3] Therefore, when I write Hilbert in the following I often mean Hilbert and Bernays.[4]

[1]Translation due to William Ewald, as found in (Mancosu; 1998, 269).

[2]See (Corry; 2000) and (Rowe; 2000).

[3]This is something which I have also discussed with Paolo Mancosu who agrees with this point of view.

[4]If it is not clear from the context what is meant, I will state it explicitly.

I will not give a contextual nor a historical accurate account of Hilbert's position. Others have made rather precise historical analyzes of Hilbert's and Bernays' development.[5]

On the other hand, a thorough philosophical discussion of Hilbert's position is still missing, I think. Although M. Tiles (1991) has made initial steps in this direction, and Michael Hallet is working on a monograph on the topic, I still think that we are in need of a broader philosophical discussion of the relation between Hilbert and Kant. In such a discussion, *schematism should be a cornerstone*.

There is, however, a problem which connects with an interpretation of Hilbert. Hilbert and Bernays were from time to time unclear and confusing.[6] Hilbert's view certainly underwent changes over time. But instead of a accurate historical account of this, I will selectively collect from what they said when the position was developed in its most mature form. This was primarily around 1930. From this I will re-construct a coherent picture of the position. Now, as it turns out their position is, nevertheless, partly non-coherent or at least incomplete. Let me mention the following:

1. With respect to one of the most important aspect, namely the status of the infinite Hilbert (and Bernays) *misunderstood* the Kantian account of infinity.

2. Kant's philosophy of mathematics includes—as we have seen— the dogma that every mathematical problem is solvable, basically because all objects in Kant's mathematics are schematisable—this is equivalent to saying that there are no ideas in mathematics. Hilbert on the other hand generalized the concept of object in mathematics and allowed ideas (ideal elements are his words) as objects also in mathematics. It is a question whether he maintains the Kantian dogma—which has now become known under the name Hilbert's Dogma. If he does that his position becomes highly problematic.

3. Hilbert's characterization of the finitary has turned out to be, incomplete and not satisfying. At best. Completely wrong at the worst. This problem is ultimately connected with the fact, that Hilbert had only an implicit theory of schematism.

In consequence of these problems I will make a re-interpretation of what Hilbert's and Bernays' position perhaps could or should have have been.

[5] See for instance (Sieg; 2000), (Mancosu; 1998, 149–188) and (Zach; 2001).

[6] An example of this is that according to (Mancosu; 1998, 169) Hilbert and Bernays went from so-called intuitive perception to pure intuition as the intuitive foundation of mathematics.

6.2 Hilbert's philosophy of mathematics

Hilbert understands mathematics as consisting of a finitary part and an ideal part. Below I will treat more precisely what these two parts include; but very shortly described we can say that the finitary is viewed as completely safe and epistemologically unproblematic, whereas the ideal part is not unproblematic. This is due to the use of highly abstract, impredicative and non-constructive principles and styles of reasoning coming from the ideal. Criticism against ideal methods in mathematics was formulated by Jordan, Kronecker, Brouwer, Borel and Weyl—just to mention a few. Hilbert was terrified by the alternatives presented by the critics and in consequence of this he formulated his *program*: Basically an attempt to give a (transcendental) deduction within the secured context, namely the finitary; a deduction showing consistency of the most important ideal elements. Now, as it turned out, the program failed; his more general philosophy of mathematics, however, is still of great interest.

6.2.1 Hilbert's general description of the finitary

Hilbert described the domain of finitary reasoning in a well-known paragraph which appears with the same content in almost all of Hilbert's later philosophical papers (Hilbert; 1922, 1926, 1927, 1930, 1931).

> Even if today we can no longer agree with Kant in detail, nevertheless the most general and fundamental idea of the Kantian epistemology retains its significance: to determine the intuitive [anschauliche] a priori mode [einstellung] of thought, and thereby to investigate the condition of the possibility of all knowledge. In my opinion, this is essentially what happens in my investigations of the principles of mathematics. The a priori is here nothing more and nothing less than a fundamental mode of thought, which I also call the finite mode of thought: something is already given to us in advance in our faculty of representation [*Vorstellung*]: certain extra-logical concrete objects that exist intuitively [anschaulich] as an immediate experience before thought. If logical inference is to be certain, then these objects must be completely surveyable in all their parts, and their presentation, their differences their succeeding on another or their being arrayed next to one another is immediately and intuitively given to us, along with the objects, as something that neither can be further reduced to anything else, nor needs such a reduction. This is the fundamental mode of thought which is a necessary precondition for mathematics in particular and for science, thought and communication in general. (Hilbert; 1931, 485–486)

Hilbert is not exactly precise here—it is unclear in which sense "the Kantian epistemology retains its significance"; but we know that he quite early subscribed to a clear anti-logicist conception of mathematics:

> Mathematics is not without presuppositions. But they are not self-evident. (Hilbert; 1919, 15)

Let me elaborate on the longer quote above by using the concept of number as an example.

> In number theory we have the numerals
>
> $$1,\ 11,\ 111,\ 11111,\ \text{[sic!]}$$
>
> each numeral being perceptually recognizable by the fact that in it 1 is always again following by 1 [if it is followed by anything]. These numerals, which are the object of our consideration, have no meaning at all in themselves. In elementary number theory, however, we already require, besides these signs, others that mean something and serve to convey information [...] (Hilbert; 1926, 377)

When Hilbert talks about numerals he cannot possibly mean anything but what we today would call tokens. Tokens do not mean anything in themselves. They are meaningful, only when their relations with axioms are taken into consideration:

> [W]e would regard $\mathfrak{a}+\mathfrak{b}=\mathfrak{b}+\mathfrak{a}$ merely as the communication of the fact that the numeral $\mathfrak{a}+\mathfrak{b}$ is the same as $\mathfrak{b}+\mathfrak{a}$. Here too, the contentual correctness of this communication can be proved by contentual inference, and we can go very far with this intuitive, contentual kind of treatment. (Hilbert; 1926, 377)

My interpretation of these statements are the following. Certain objects are given to us a priori. These include proto-types $1\dots1$ of the natural numbers. These numbers, however, mean nothing unless they are take together with schemata. As Kant says, schemata are rules expressed by axioms, and one of the axioms determining the meaning of numbers is: $\mathfrak{a}+\mathfrak{b}=\mathfrak{b}+\mathfrak{a}$. This axiom expresses one of the properties of numbers, namely that if a set is enumerated by the number $\mathfrak{a}+\mathfrak{b}$ then also the number $\mathfrak{b}+\mathfrak{a}$ can be used. This provides meaning in the sense, that the axiom give rise to operative knowledge which shows us how we can operate with and manipulate the representations of numbers. The numbers are "completely surveyable in all their parts", i.e., schematisable and in consequence of this they satisfy the Kantian principle of complete determination; "their differences, their succeeding on another [are] given to us, along with the objects". 'Along-nes' here refers the intensional aspect of numbers, namely that two numbers are different, iff, the corresponding schema separates the two numbers.

If we generalize this interpretation to all the finitary concepts, then a finitary concept is a concept which can be fully schematised and therefore constructed in intuition.

> [I]t must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. (Hilbert; 1926, 376).

The intensional rules determining equality on the finitary objects cannot be reduced to anything else. They are basic rules necessary "for all scientific thinking, understanding, and communication."

In one Bernays's most mature presentations of Hilbert's philosophy of mathematics (Bernays; 1930), Bernays clearly has the distinction between type and token. Now the objects of finitism are characterized as *formal objects*—Bernays' term for types. These are recursively generated by a process of repetition; the stroke symbols are concrete representations of these formal objects:

> If we want to have the ordinal numbers [. . .] as unique objects free from all inessential features, then we must take as object in each case the bare schema of the respective figures obtained by repetition; this requires a very high degree of abstraction. However, we are free to represent these purely formal objects by concrete objects ("number signs" or "numerals"); these then possess inessential arbitrarily added characteristics, which, however, can be immediately recognized as such. (Bernays; 1930, 31–32)[7]

Bernays ads an interesting note:

[7] Translation is due to Paolo Mancosu and Ian Mueller, as a part of the so-called Bernays Project.

> [T]he representing thing contains in its constitution the essential properties of the object represented, so that the relationships to be investigated among the represented objects can also be found among the representatives and can be determined by consideration of the latter. (Bernays; 1930, 32)[8]

This is precisely Kant's point when discussing the universality of schematic reasoning. We can use concrete intuitions, i.e., tokens, when reasoning universally about all tokens of the same type, *only if* the reasoning about the token is taken together with the schema.[9]

Hilbert were never precise with respect to the content of the finitary. In the programmatic article (Hilbert; 1926) he mentions finitary reasoning about the natural numbers—what we today would understand as reasoning using only bounded quantifiers. This is generalized by Hilbert and Bernays to:

> A *universal* judgment about numerals can be interpreted finitistically only in a hypothetical sense, i.e., as a proposition about any given numeral. Such a judgment pronounces a law which must verify itself in each particular case. (Hilbert and Bernays; 1934, 32)

This *is* the Brouwer-Heyting-Kolmogorov interpretation:[10] $\forall x A(x)$ is finitarily meaningful if we have a procedure Φ such that for any a, if $A(a)$ is well-formed, then $\Phi(a)$ verifies finitarily $A(a)$.

> An *existential sentence* about numerals, i.e., a sentence of the form "there is a numeral $\mathfrak{n}$ with the property $\mathfrak{A}(\mathfrak{n})$," is to be understood finitistically as a "partial judgment," i.e., as an incomplete communication of a more specific proposition consisting in either a direct exhibition of a numeral with the property $\mathfrak{A}(\mathfrak{n})$, or the exhibition of a process to obtain such a numeral,—where part of the exhibition of such a process is a determinate bound for the sequence of actions to be performed. (Hilbert and Bernays; 1934, 32)

This interpretation is, perhaps, more restricted than the BHK-interpretation as a specific bound needs to be given, in case the witness can only be devised principally.

The quotations given here are the most precise statements they gave. Generally, Hilbert was vague about what constitutes finitary reasoning. But let me give other examples could:

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[8] Translation is due to Paolo Mancosu and Ian Mueller, as a part of the so-called Bernays Project.

[9] See my discussion of this aspect in the case of natural numbers on page 43.

[10] See footnote 9 on page 72 for a definition of the BHK-interpretation.

1. Simple reasoning about geometrical figures and finite graphs, such as triangles, circles, constant functions, simple real functions and the like.

2. The computation of simple functions on the rational numbers (and some expansions hereof).

3. Integration and differentiation of (some of the) functions given in 2. For example constant functions and simple real functions.

One, who has tried to make precise what Hilbert understood under 'finitary reasoning' is (Tait; 1981, n.d., 2005).

6.3 Tait's thesis

Tait formulates the thesis that the functions of mathematics which are finitary meaningful are precisely the primitive recursive functions (Tait; 2005, 29); in other words:

A function f is finitary, if and only if, f is primitive recursive.

The class of primitive recursive functions is the smallest class of functions:[11]

1. containing initial functions for zero, successor and projection.

2. closed under composition and under primitive recursion: Given primitive recursive g, h we have

$$\begin{aligned} f(x,0) &= g(x) \\ f(x,y+1) &= h(x,y,f(x,y)) \end{aligned}$$

Let us evaluate critically Tait's thesis.

The first problem we face is how to understand the expression 'the function f is finitary'. There are at least two possibilities:

1. 'The function f is finitary' means that given any x the operation f applied to x is epistemically unproblematic.

2. 'The function f is finitary' means that we have a *concept* f of a function. This concept has a corresponding schema which allows us to decide whether a given representation is an instance of f and to use this function, i.e., to compute it in the sense of 1.

[11] See (Odifreddi; 1989, 22) for details.

I understand Tait as promoting the former understanding.[12] In fact, the former can be derived from the latter in most cases. Suppose we have a definition of a primitive recursive function f. This is given as a *finite* piece of text which is generated according to the rules given in the definition above. We can understand this as an intensional definition of f which works simultaneously as a proto type for our concept of f; any other (definition of a) primitive recursive function g is the same if it is defined in precisely the same way. Thus intentional equality between primitive recursive functions is reduced to equality between literal definitions of primitive recursive functions. As the latter equality is completely unproblematic, 2 reduces in the intensional case to 1. The extensional case is more complex and I will treat it below in the case of general recursive functions.

It seems, however, that Hilbert and Bernays perhaps had the intensional understanding of a finitary function in mind:

> A [finitary] function, for us, is an intuitive [*anschauliche*] instruction [*Anweisung*] which on the basis of a numeral, or a pair of numerals, or a triple of numerals, ..., assigns another numeral. (Hilbert and Bernays; 1934, 26)

If f is primitive recursive, then computing $f(x)$ is unproblematic with respect to 'complete surveyability in all parts', 'immediacy' and 'intuitivity':

6.3.1 Tait's "if"

Given a primitive recursive function f and any number x computing $f(x)$ is completely unproblematic. The only 'complicated' element in a definition of a primitive recursive function can be the two operations: substitution and iteration. Say that g is given and that we define f by $f(0) = a_0$ and $f(x+1) = g(x, f(x))$. Suppose we want to compute $f(x)$. If x is 0 then a_0 is given. If x is not 0, then $x = 1$ or $x > 1$. In the first case we have $f(1) = g(1, a_0)$ which by assumption is finitarily computable—say the result is a_1. If $x > 1$ then $x = 2$ or $x > 2$. In the first case finitary reasoning gives us a_2. This process goes on until we reach x. The process is guaranteed to terminate as g is finitary and x is a (finitary) natural number.[13]

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[12] "So how can the finitist understand $f : A \to B$ [...] he can understand it as recording the fact that he has given a specific procedure for *defining a B from an arbitrary A* or, we shall say, of *constructing* a B from an arbitrary A. (Tait; 2005, 24).

[13] The logician could make a counter argument here by asking: "How do we know that the given number x is not a non-standard number being infinitely large". This is actually a skeptic counter-

6.3.2 Tait's "only if"

Let us define a function φ in the following way. Suppose $\varphi_1(a,b) = a+b$ and $\varphi_2(a,b) = a \cdot b$ and that $\varphi_3(a,b) = a^b$. Furthermore, let $\varphi_4(a,b)$ be the b-th element of the sequence

$$a, \quad a^a, \quad a^{(a^a)}, \quad a^{(a^{(a^a)})}, \ldots$$

Continue an unfolding definition of φ_n in this way. That is, $\varphi_{n+1}(a,b)$ is an iteration of $\varphi_n(a,a)$ b times.

The Ackermann function $\varphi : \mathbf{N} \to \mathbf{N}$ is defined as $\varphi_n(n,n)$.[14] It grows *really* fast:

$$\varphi(1) = 2, \qquad \varphi(2) = 4, \qquad \varphi(3) = 9, \qquad \varphi(4) = 4^{4294967296}.$$

The function is not a primitive recursive function as it majorizes any function which is primitive recursive. This was observed by Ackermann in 1928. It is a relevant question, however, whether Hilbert considered (or should have considered) it to be finitary?

The Ackermann function is defined as a recursive function using nested recursion. This gives rise to a rule which allows that we can compute the values of φ from below. In principle we can compute any value of in the sequence $\varphi(1), \varphi(2), \ldots, \varphi(n), \ldots$. Arguments for finitariry are:

- It fulfills the 'surveyability', 'immediacy' and 'intuitivity' criteria.
- It is mentioned by Hilbert (1926) in the programmatic article.
- In volume II of *Grundlagen der Mathematik* Hilbert and Bernays write:

  > Certain methods of finitistic mathematics which go beyond recursive number theory (in the original sense [i.e., primitive recursive]) have been discussed in §7 [of volume I of *Grundlagen*], namely the introduction of functions by nested recursion [e.g., the Ackermann function] and the more general induction schemata. (Hilbert and Bernays; 1939, 354)

- Ackermann (1924) usestransfinite induction up to $\omega^{\omega^{\omega}}$ for showing consistency of a second order version of PRA. In this system φ is definable . Moreover, this consistency proof was considered in the Hilbert school to be finitistic.

argument. Well, we are not working with non-standard numbers. Our objects are not formal objects—the are contentual or semantic objects. Thus our numbers are not interpreted in some kind of model. They are numbers and not reducible to or interpretable in anything. But the problem becomes a real problem once we allow for ideal elements.

[14]See page 126 for a definition of the function in Gödel's system T.

I find these reasons sufficient as a refutation of Tait's "only if".[15] But let me give another argument also.

Any primitive recursive function is defined by a finite piece of text. We can therefore provide an algorithm which enumerates all primitive recursive functions; $f_1, f_2, \ldots, f_n, \ldots$. Based on this algorithm we can construct a finitary function U which is the universal function taking two arguments n and x and then picks the n-th primitive recursive function and applies it to x.[16]. In other words $U(n,x) = f_n(x)$. It can be argued that U is a finitary function. Thus also $U(n,n)+1$ is finitary. It is, however, not primitive recursive. Assume it is. Then there would exists an m such that

$$f_m(n) = U(n,n) + 1. \tag{6.1}$$

On the other hand, because U is the universal function we have $f_m(m) = U(m,m)$, which, however, contradicts (6.1) if we substitute m for n. Therefore, there are more finitary functions than primitive recursive functions.

Let me mention one crucial point. In the logic community there is a tendency to equate Hilbert's finitism with primitive recursive arithmetic, perhaps even with PRA as a formal theory. Hilbert talked about the *contentual content* of the objects. He looked at it semantically; what are the objects. Perhaps, in PRA we can *code* many of the mathematical objects, functions and operations which Hilbert found finitistically meaningful. But it can only be a codification, in which we loose the mathematical intuition which is connected in a semantic way with the finitary objects.

An example of this is expressed by Caldon and Ignjatović. They are "accepting Tait's analysis and delineation of finitism [...]. Thus, we take Finitism to be the part of mathematics that corresponds to the theory of natural numbers formalized as Primitive Recursive Arithmetic" (2005, 779).

Today there is no agreement on what should be considered as *the* finitary. Later in this chapter I will present arguments showing that there is no *unique* characterization of the finitary. Just as with Kant's faculty of understanding the specification of what can be regarded as finitary depends on certain interests of reason. This is in complete line, actually, with what the later Bernays formulated: "[T]he sharp distinction between the intuitive and the non-intuitive, which was employed in the treatment of the problem of the infinite, can apparently not be drawn so strictly" (Bernays; 1976, 61)

[15] A more detailed analysis is given by Richard Zach in his thesis 2001.

[16] For simplicity I have assumed that the primitive recursive functions are 1-ary.

6.3.3 The ideal

The ideal elements were supposed by Hilbert to be abstract elements introduced in the development of mathematics in order to simplify, generalize and complete already existing mathematics. In such a process new mathematics would also arise and this was how Hilbert saw the expansion and progression of mathematics.

Thus, in sharp contrast to Kant, Hilbert introduces regulative ideas in the philosophy of mathematics:[17]

> The role that remains to the infinite is, rather, merely that of an idea—if, in accordance with Kant's words, we understand by an idea a concept of reason that transcends all experience and through which the concrete is completed so as to form a totality–an idea [. . .] ((Hilbert; 1926, 392)

Hilbert (1926) gives instructive examples:

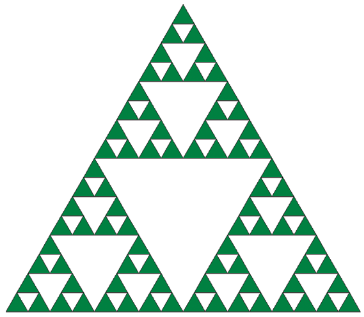
- The set of natural numbers being a number ω. Based on ω we can develop a theory of Cantor's transfinite numbers. (Hilbert; 1926, 374-5)
- The complex number $\sqrt{-1}$, giving rise to the fundamental theorem of algebra. (Hilbert; 1926, 379)
- Introduction of ideal factors "so that the simple laws of divisibility could be maintained even for algebraic integers". (Hilbert; 1926, 379)
- The completion of Euclidean geometry by supplying with elements at infinity. This becomes projective geometry which has particular simple and beautiful properties such as the duality principle. (Hilbert; 1926, 372–371) Geometry combined with the axiomatic method opens for a whole new discipline, namely algebraic geometry as also discussed on page 29.
- Full classical logic in all of mathematics and the axiom of choice. The latter is introduced through Hilbert's elegant ε operator. (Hilbert; 1926, 382)

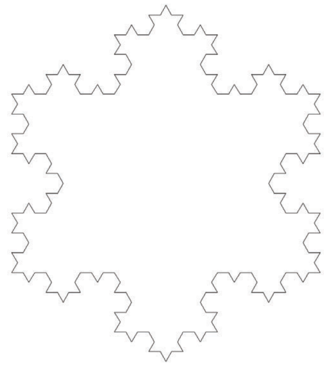
According to Hilbert ideal elements are *indispensable* to mathematics: "we cannot dispense with the unconditional application of the *tertium non datur* and of negation, since otherwise the gap-less and unified construction of our science would be impossible" (Hilbert; 1931, 488).

The mathematical universe is filled up with other examples, such as:

---

[17] "Die Rolle, die dem Unendlichen bleibt, ist vielmehr lediglich die einer Idee – wenn man, nach den Worten Kants, unter einer Idee einen Vernunftbegriff versteht, der alle Erfahrung übersteit und durhc den das Konkrete im Sinne der Totalität ergänzt wird [. . .]".





- The completion of the ring of rational numbers (i.e., the reals).
- Transfinite recursion and limit constructions in general.
- The embedding of 'simple' problems in 'complex' contexts.
- Extensionality.
- Zorn's lemma.
- Generating functions.
- Large cardinals.

They give new mathematics; generalize existing mathematics; complete, systematize existing mathematics. And they are *essential* for discoveries. Let us have a closer look at some examples:

Fractals are ideal elements. Here we take the von Koch curve and the Serpinski triangle.

Both of the fractals are obtained by letting a generating algorithm 'run' to infinity. In the case of Serpinski we will denote the n-th approximation by S_n. Now, S being the Serpinski triangle is constructed as the limit of this sequence of approximations. If we let $C(S_n)$ be the circumference of S_n and $A(S_n)$ be the area of S_n, then it is clear that

$$A(S_n) \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty$$

$$C(S_n) \longrightarrow \infty, \quad \text{as} \quad n \longrightarrow \infty$$

In this way we see that no one-dimensional dimension seems to fit the Serpinski triangle. We are thus led to a *generalized* concept of dimension—the Hausdorff dimension $\dim_H$—in which

$$\dim_H(S) = \frac{\log 3}{\log 2} \simeq 1,585$$

and the von Koch curve:

$$\dim_H(K) = \frac{\log 4}{\log 3} \simeq 1,263$$

Apart from broading our concepts, the fractals—being characterized by their broken dimensions—are useful and fruitful objects. Sometimes (or perhaps even in most cases) they give rise to a deepening of more 'basic' mathematical concepts. Thus the von Koch curve was introduced, just as the continuous nowhere differentiable Weierstraß function (as discussed on page 6) in order to scrutinize the concept of continuity. From the curve K it was realized that it was possible for a continuous curve to be nowhere differentiable. This came as a big surprise in the late 19th century and led into investigations of our concept of continuity.

6.3.4 Generating functions

Let me discuss a really characteristic part of the mathematical method: The embedding of 'simple' problems in 'complex' contexts. An example of this is the prime number theorem, which says that there is a certain order in the distribution of prime numbers. Let $\pi(n)$ denote the number of primes that do not exceed n, then:

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\log n} = 1$$

This theorem has basically two different proofs. There is an 'analytic' proof, which was the first proof. This proof goes via complex analysis and uses the Riemann zeta function. There is also a more simple proof, the so-called 'elementary' proof due to Erdös and Selberg (discovered independently). Their proof uses only real analysis and is therefore considered to be more simple than the original proof, although the later presupposes much more analytic machinery. Moreover, it is generally granted that the complex proof is far less intricate and more easily understood, and much shorter than the 'elementary' proof.

The Fibonacci numbers, probably introduced by Leonardo de Pisa around 1200, will serve as another example. Here another simple problem is easily solved when translated into complex analysis. And moreover, the discovery of the solution is very difficult to find, perhaps, unless the complex method using formal power series is used.

The sequence of the Fibonacci numbers is inductively defined as

$$f(0) = 0,\ f(1) = 1 \text{ and for } n \geq 2, f(n) = f(n-1) + f(n-2).$$

If we let χ be the characteristic function of the predicate "being equal to 1", i.e.,

$$\chi(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise} \end{cases}$$

and let ψ be the characteristic function of the predicate "being strictly greater than 0" then we can define the (slightly generalized) Fibonacci numbers in one formula:

$$\forall n \in \mathbf{Z} :\ f(n) = \psi(n)(f(n-1) + f(n-2) + \chi(n))) \tag{6.2}$$

The problem which now interests us is how to find a direct expression for the n-th Fibonacci number. In order to compute the n-th number we will, given the definition, have to compute the whole sequence of numbers between 0 and n in order to compute $f(n)$. This is of course very inefficient and we would thus like to have a closed expression telling us how to compute directly. This is seems to be a purely number theoretic question but the trick which makes the solution particularly easy and elegant is to embed the problem in complex analysis. Specifically we will look at the so-called generating function for the Fibonacci sequence.

In order to introduce and use generating functions we need some complex analysis. For any sequence $(a_k)_{k\geq 0}$ of complex numbers we can form the *formal power series in z with coefficients* a_k. This is defined as

$$P(z) := \sum_{k=0}^{\infty} a_k z^k := \big(\sum_{k=0}^{n} a_k z^k\big)_{n\geq 0},$$

$z \in \mathbf{C}$. Thus $P(z)$, which is also called the generating function for (a_k), is a sequence of finite sums. If the sequence $(\sum_{k=0}^{\infty} a_k z^k)$ converges then the formal power series converges. Irrelevant whether formal power series converges, we have certain operations we can perform on them. If $P(z)$ and $Q(z)$ are formal power series with coefficients $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$, respectively then we can add Q to P and obtain $(P+Q)(z)$ with coefficients $(a_k + b_k)_{k\geq 0}$. Likewise we can multiply with a scalar, form a certain Cauchy product and so on.

When we return to the Fibonacci numbers the formal power series with these numbers as coefficients is:

$$F(z) = \sum_{n=0}^{\infty} f(n) z^n, \qquad z \in \mathbf{C}.$$

When the generalized definition of the Fibonacci numbers (6.2) is used we get

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} \big(\psi(n)(f(n-1) + f(n-2) + \chi(n))\big) z^n \\ &= \sum_{n=0}^{\infty} f(n-1) z^n \; + \; \sum_{n=0}^{\infty} f(n-2) z^n \; + \; \sum_{n=0}^{\infty} \chi(n) z^n \\ &= \sum_{n=0}^{\infty} f(n) z^{n+1} \; + \sum_{n=0}^{\infty} f(n) z^{n+2} \; + z \\ &= z \sum_{n=0}^{\infty} f(n) z^n \; + \; z^2 \sum_{n=0}^{\infty} f(n) z^n \; + \; z \\ &= F(z)(z + z^2) + z. \end{aligned}$$

We isolate $F(z)$ in this equation we find that

$$F(z) = \frac{z}{1 - z - z^2}$$

This is a well-defined function whenever $1 - z - z^2$ is different from 0. If $|z|$ is close to 0 this function is analytic and can therefore be developed as a power series using for instance Taylor polynomials. There is, however, something which is more elegant in our case. For $\gamma = \frac{1+\sqrt{5}}{2}$ this can (using fairly simple methods) be re-written as

$$F(z) = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \gamma z} - \frac{1}{1 + \frac{z}{\gamma}} \right).$$

The body of the parenthesis can be developed as series:

$$\frac{1}{1 - \gamma z} = \sum_{k=0}^{\infty} (\gamma z)^k, \qquad \frac{1}{1 + \frac{z}{\gamma}} = \sum_{k=0}^{\infty} \left(\frac{-z}{\gamma} \right)^k$$

Putting things together we get

$$F(z) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{5}} \big(\gamma^k + (-1)^{k+1} \gamma^{-k}\big) z^k.$$

Therefore, by leaving the complex terms aside we get a closed expression for the n-th Fibonacci number

$$f(n) = \frac{1}{\sqrt{5}}\left(\gamma^n + (-1)^{n+1}\gamma^{-n}\right).$$

Interestingly, this also makes us able to say something about the asymptotic behavior of $f(n)$, namely when n is large then $f(n)$ is approximately $\frac{1}{\sqrt{5}}\gamma^n$, thus the theorem reveals information about how the sequence of the Fibonacci numbers, being a sequence of natural numbers, are connected with the real numbers. If there are explanations in mathematics, then they are to be found in examples like this. The direct expression of the n-th Fibonacci number can, of course, also be proved (easily) by induction. This proof, however, explains nothing about *why* the expression looks the way it does. Moreover, one need to know the theorem in advance when proving it. It is difficult to see how an induction proof could lead to some kind of discovery—this is more easily seen to be possible in the case the method using generating functions is used.

This Fibonacci-example, however, is nothing but a special case of a much more general well-known theorem:[18]

Theorem. Suppose $d \in \mathbf{N}$, and let $q_1,\dots,q_d \in \mathbf{C}, qd \neq 0$. Put $q_0 := 1$ and let $\alpha_1,\dots\alpha_e$ be the roots of the polynomial $q(z) = \sum_{k=0}^{d} q_k z^{d-k}$. Then for any sequence of numbers $(a_k)_{k\in\mathbf{N}}$ in the following are equivalent:

1. The sequence is recursively given, that is for every $k \in, k > d$
$$a_k + q_1 a_{k-1} + q_2 a_{k-2} + \cdots + q_d a_{k-d} = 0$$
2. There are polynomials $g_1(z),\dots,g_e(z)$ where $grad(g_i) < c$, for a certain $c \in \mathbf{N}$, $i = 1,\dots,e$ such that for all $k \in \mathbf{N}$
$$a_k = \sum_{i=1}^{e} p_i(k)\alpha_i^k$$

The proof of theorem reveals a method for finding the explicit expression and this method uses generating functions as in the example of the Fibonacci numbers. Here we use generating functions as a very powerful tool for solving questions of a much simpler character. But it is the very translation of a problem in a simple domain in to a complex domain which makes the solution particular easy, elegant and perhaps discoverable.

[18] I have to admit that I do not know who discovered the following theorem. I learned about it at a course I had in combinatorics.

6.3.5 Consistency, existence and ideal elements

It is somewhat ironic that Hilbert normally is seen as a leading figure within formalism. At least *if* Hilbert is a formalist to any extent it certainly is not the version being outlined by Corry (1951), where mathematics is *identified* with formal theories. The irony enters the picture as Hilbert is a neo-Kantian. And if there is something which is remote to the Kantian philosophy of mathematics then it is un-interpreted formal systems.[19]

Posy (1998, 301) mentions that the fact that intuitionistic logic as a formal theory "is sometimes taken as the core of intuitionism was Hilbert's ultimate victory". I very much agree with this. On the other hand, Brouwer also had a rather good victory: Quite often Hilbert's position *is* taken to be the core of formalism—this is certainly wrong—but it is partly due to Brouwer, who *introduced* the term 'formalism' in a lecture "Intuitionism and Formalism" which he gave in Amsterdam 1912 when he became appointed as full professor.

Looking back at Kant with a contemporary understanding of logic, he must be seen as an anti-formalist because of his sharp distinction between intuition and concepts. Concepts can never capture fully a continuum such as space. Space is not a concept, as I discussed intensively in Chapter 3.

According to Kant it is important to draw the distinction between logical and real possibility. The former is only obliged to conduct in accordance with constraints given by the intellect alone. The latter adds to these, the constraints of being a possible object of experience. That is, that there is a possibility for an object to be given in intuition, which corresponds to the concept: "Whatever agrees with the formal conditions of experience (in accordance with intuition and concepts) is **possible**." (A218/B265) Although consistency is a necessary condition it does not suffice:

> That in such a concept no contradiction must be contained is, to be sure, a necessary logical condition; but it is far from sufficient for the objective reality of the concept, i.e., for the possibility of such an object as is thought through the concept. (A220/B268–9)

But as we have seen earlier in this thesis, Kant only acknowledge objects in mathematics which satisfy the principle of complete determination, i.e., objects which are fully schematisable. Hilbert, on the other hand, operates with two different kinds of objects: real objects being schematisable in the sense of Kant and ideal objects being only pseudo-schematisable. But existence in connection with ideas are of a different

[19] See also my discussion of this in the section entitled "Hilbert and the axiomatic method" found on page 28.

type and here it is perhaps enough with consistency. Take for instance the concept of a completed infinity. Bernays and Hilbert (1934, § 1) puts it precisely in this way:

> The problem of the satisfiability of an axiom system (or a logical formula) can be positively solved in the case of a finite domain of individuals by exhibition; but in the case where the satisfaction of the axioms requires an infinite domain of individuals this method is no longer applicable because it is not determined whether an infinite domain of individuals cannot be considered as settled; rather, the introduction of such infinite domains is only justified by a proof of the consistency of an axiom system characterizing the infinite.

6.3.6 Hilbert's justification—the program

As seen by Hilbert the ideal elements of the mathematical method and universe are of indispensable value. However, Hilbert was at the same time aware of the fact that this progressiveness of the mathematical method was transcending the secured finitary parts of mathematics and it was therefore in need of some kind of justification. For this he developed his program, as mentioned on page 102

The program failed basically because of Gödel's incompleteness results. But how much of his philosophy can be saved? According to Bernays the problem is the following:

> [T]he epistemological perspective which motivated [the program's] formulation has become problematic [...] the sharp distinction between the intuitive [Anschaulichen] and the non-intuitive [Nicht-Anschaulichen], employed in the treatment of the problem of the infinite, can apparently not be drawn so strictly, and the reflection on the formation of mathematical ideas still needs a more detailed elaboration in this respect. (Bernays; 1976, 61)

This rest of this chapter mainly discusses the first of these problems: The distinction between the finitary and the ideal; i.e., the distinction between what can be fully schematised and what cannot.

6.4 Is Hilbert's position consistent?

There are some questions which we need to solve before turning to schematisation. There are, at least, two conflicts with a Kantian philosophy which need to be solved: Firstly, Kant and Bernays misunderstood the actual infinity of space as intuition; secondly we need to analyze the so-called Hilbert dogma: Is it to be taken as a constitutive or regulative principle?

6.4.1 The objectivity of the infinite

Hilbert formulated a critique of some of Kant's notions. In particular "the role and content of the a priori was extremely exaggerated" (Hilbert; 1930). In the lecture *Naturerkennen und Logik* it is clear that the problems Hilbert seem to find in the a priori is the postulated Euclidean structure of space and the sharp distinction between time and space.[20] But in other places Hilbert was critical towards the infinite in Kant.[21] In *On the infinite* Hilbert implicitly refers to Kant:

> In the attempt to prove the infinitude of space in a speculative way, moreover, obvious errors were committed. From the fact that outside a region of space there always is still more space it follows only that space is unbounded but by no means that it is infinite. (Hilbert; 1926, 372)

And in 1933 in a handwritten manuscript *Über das Unendliche* Hilbert writes "Until and with Kant, but also later in time, it was never doubted that space was infinite. This was, however, a conceptual misunderstanding".[22]

It it is, however, Bernays, who in the clearest way formulates the postulated tension:

> Furthermore, one cannot point to infinitely extended things like infinite lines, infinite planes, or infinite space as objects of intuition. In particular, space as a whole is not given to us in intuition. We do indeed represent every spatial figure as situated in space. But this relationship of individual spatial figures to the whole of space is given as an object of intuition only to the extent that a spatial neighborhood is represented along with every spatial object. Beyond this representation, the position in the whole of space is conceivable *only in thought*. (In opposition to Kant, we must maintain this view.) (Bernays; 1976, 36)

It is clear that Hilbert and Bernays did not really understand Kant in every detail. They thought that Kant had actual infinity "as an object of intuition". This, indeed, is really a misconception of Kant's notion of space. What Hilbert and Bernays most probably thought about Kant, is that he postulated actual infinity of space as objectively given, with the size of e.g., $\aleph_0$. Quite contrary to this—as I showed in

[20]Another general critique of Kant is "Even if today we can no longer agree with Kant in the details, nevertheless the most general and fundamental idea of the Kantian epistemology retains its significance" (Hilbert; 1931, 485–486).

[21]"The infinite is realized nowhere; it does not exist in nature, nor is it admissible as a foundation of our rational thought. (Hilbert; 1931, 488)

[22]As quoted in (Majer; 1993, 75).

Chapter 3—Kant maintained that space as intuition is infinite in the sense that it is in-measurable. The only measure Kant had was the finite numbers, therefore, Kant's infinite should be understood as in-finite. On the other hand, the potential infinite was objectively unproblematic. On the bottom line there is therefore no conflict between Hilbert and Kant.[23]

6.4.2 Hilbert's Dogma

In the following I want to discuss whether Hilbert's dogma: "Wir müssen wissen, wir werden wissen" is to be understood as a constitutive principle or regulative principle (to state it in Kantian terms). The Dogma followed Hilbert for all his life.

Hilbert's dogma (Hilbert; 1902c, 444). Any definite mathematical problem has a solution. There are two kinds:

1. Provide "an actual answer to the question asked", or
2. Give a "proof of the impossibility of its solution".

In mentioning 2 Hilbert is explicit in saying that he is thinking of the problems like the parallel-postulate or the squaring of the circle.

"We hear within us the perpetual call: There is the problem. See its solution. You can find it by pure reason, for in mathematics there is no *ignorabimus*." (Hilbert; 1902c, 402) Hilbert claims that every mathematician share this belief, but no one has yet been able to prove it.

There are three interpretations of Hilbert's dogma:

1. Either Hilbert completely follows Kant when he says that in mathematics every question is definite and solvable.
2. The principle is to be taken *only* as a regulative principle of the pure reason. This harmonies with "We hear within us the perpetual call" (Hilbert) with "we hear the voice of pure reason" (Kant).
3. Hilbert could have meant that it is a constitutive principle in the sense that solving a problem means providing a yes or a no, or showing that relatively to a frame work, the problem is not solvable.

[23]Note that Majer thinks (1993, 76) that Hilbert was against both the actual and the potential infinite. Thus, in Majer's reading, for Hilbert the potential infinite is a an ideal element. "Für Hilbert hingegen – der Kants "Schluss" als *Fehlschluss* durchschaut hatte – stellt sich das Problem des Unendlichen von Grund auf neu: Fuer ihn war nicht mehr die Frage nach dem *Modus* des Unendlichen, ob *aktual* oder *potentiell*, die entscheidende Frage, sondern, ob das Unendliche – als die *Verneinung* des Endlichen *ueberhaupt* – gerechtfertigt wernden kann".

The first interpretation is completely in line with Kant's philosophy of mathematics as I discussed it in Chapter 5. The second interpretation is more in line with the more general epistemology of Kant. And the third could be interpreted as a version of Kant's *Vernunftkritik* in the context of mathematics. I think that the status of the principle changes over time. I the earlier years he considered it to be constitutive, like something which can be proved, but in later years when he was more philosophically reflected it became regulative:

> As an example of the way in which fundamental questions can be treated I would like to choose the thesis that every mathematical problem can be solved. We are all convinced of that. After all, one of the things that attract us most when we apply ourselves to a mathematical problem is precisely that with us we always hear the call: here is the problem, search for the solution; you can find it by pure thought, for in mathematics there is no *ignorabimus*. Now, to be sure, my proof theory cannot specify a general method for solving every mathematical problem; that does not exist. But the demonstration that the assumption of the solvable of every mathematical problem is consistent falls entirely within the scope of our theory. (Hilbert; 1926, 384)

The principle is an ideal element. It is not fully schematisable: the proof "does not exist". But the principle is a reasonable ideal element, in the sense that we can consistently add it, but only as a regulative principle. Thus Kant does not claim Kant's principle of complete determination to be unconditioned true in mathematics. Hilbert realizes that by introducing ideal elements in mathematics he cannot expect[24]

$$E(x) \rightarrow \Box\Diamond\circ P(x) \vee \Box\Diamond\circ\neg P(x).$$

6.5 A neo-Kantian re-interpretation of Hilbert

6.5.1 The a priori

Hilbert was not completely clear on what precisely the a priori in mathematics consisted of. The main reason for this, I claim, is that Hilbert only had an implicit schema theory. If he had had a theory of schematism he could have been more precise here. Let me attempt a re-interpretation using elements of mathematical logic which have become known since the time of Hilbert.

It will be a result of the forthcoming pages that there is no *unique* a priori part of mathematics. But let us start with Kant's "schema of magnitude" which probably is

[24]Note the similarity between Hilbert's conception as expressed in: Give "the proof of the impossibility of its solution" (Hilbert; 1902c, 444) and Kant's "no answer is an answer" (A478n/B507n).

needed for most of our everyday life and certainly for all of the simple mathematics of size which exist. But mathematics as science needs other basic objects and categorizes. Here the natural numbers as a whole can be an object. Qua this generalization of the number concept, the mathematical schema of number cannot be expected to be total.[25]

6.5.2 A modern theory of schemata for basic and ideal mathematics

In general, concepts and schemata are intimately connected. Given any concept, say the definition of a function, the schema is the human ability to use and operate with this function. It is the ability to recognize and produce representations of the function.

For geometry we could simply follow the line from Kant. The objects are geometrical objects which can be created in the mind as pure forms of things that can be realized in the empirical world. To my best knowledge, Kant in fact never provided arguments for schemata which exceeded Euclid's first three postulates. As it therefore turns out, Kant's restricted notion of space is to a certain sense compatible with a non-Euclidean conception of space.

With arithmetic we leave Kant. As I argued on page 92 numbers are very naturally also objects, though of second order. The schema of number can be taken to be a primitive recursive functions realizing (in the sense of the BHK-interpretation) the axioms of Robinson arithmetic.[26] But as argued earlier in this chapter the primitive recursive functions are not enough, and we cannot generally expect totality. Thus we try the partial recursive functions: *When* they provide us with objects they lead us in a finite amount of time to knowledge. If not, they will say, forever "I don't know". This is completely satisfying given the fact, that we cannot know, if we are dealing with ideal elements or not.

Partial recursive functions are functions from numbers to numbers. They can be computed mechanically, step by step. It is a very robust class of functions and they can be characterized in many different ways using Turing machines, Register machines, lambda calculus, domain theory and Kleene schemata. All these different formulations of the effective functions have turned out to be equivalent and this have led to the so-called Church-Turing thesis: The partial recursive functions characterize effective computability.

Definition. The class of partial recursive functions is the smallest class of functions:[27]

[25] See my discussion on the previous chapter.

[26] See page 92 for the axioms of Robinson arithmetic.

[27] Here I use Kleene's symbol $\varphi \simeq \psi$ which is to be understood as "either φ and ψ are both undefined, or they are both defined with the same value".

1. Containing initial functions for zero, successor and projection.

2. Closed under composition and also under primitive recursion which is: Given ψ, γ we have

$$\begin{array}{rcl} \varphi(\boldsymbol{x},0) & \simeq & \psi(\boldsymbol{x}) \\ \varphi(\boldsymbol{x},y+1) & \simeq & \gamma(\boldsymbol{x},y,\varphi(\boldsymbol{x},y)) \end{array}$$

3. Closed under unrestricted μ-recursion, i.e., given ψ we have

$$\varphi(\boldsymbol{x}) \simeq \mu y\big(\forall z \leq y(\psi(\boldsymbol{x},z)\downarrow) \wedge \psi(\boldsymbol{x},y) \simeq 0\big).$$

The partial recursive functions are well-behaved. They can be given on a certain normal form, and as any partial recursive function is defined by a finite piece of text, they can be enumerated. This is the Enumeration theorem: There is a sequence, $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$ of partial recursive functions, such that any partial recursive function is within that enumeration. Let $\langle \boldsymbol{x} \rangle$ be the primitive recursive encoding of the sequence $\boldsymbol{x}$. Then there exists a universal partial recursive function $\varphi(e,x)$ such that for any partial recursive function ψ of n variables there is e such that

$$\psi(\boldsymbol{x}) \simeq \varphi(e, \langle \boldsymbol{x} \rangle).$$

As discussed above we could understand these finitary functions either intensionally of extensionally. The intensional understanding is simple and unproblematic, but what about the extensional notion.

Say that φ_n and φ_m are extensionally equal, in symbols,

$$\varphi_n \approx \varphi_m, \text{ if and only if, for every } \boldsymbol{x},\ \varphi_n(\boldsymbol{x}) \simeq \varphi_m(\boldsymbol{x}).$$

Then any definition of a partial recursive function φ_n generates an equivalence class $[\varphi_n]$. The question now is, is such a concept schematisable? It is not trivially so. If we consider $[\varphi_n]$ as a type, then membership of that type is not recursively decidable.

The question of extensionality of functions is an interesting question, which I will come back to below.

We could also provide some basic schematic rules for *set theory*. We have a determinate concept of a set A in case we have a schema, say a partial recursive function, which constructs a representation of A and the schema also recognizes representations of the concept. Given that two sets A and B are determinate objects, then also $A \cup B$ and $A \cap B$ and (A, B) are determinate objects.

But there are also quasi-schemata. These are 'rules for constructing' more abstract objects, such as the first infinite number ω or a converging sequence of rational numbers taken as a completed object. Characteristic for these quasi-schemata is that we allow constructions to run in an infinite amount of time. This is in contrast with the real schemata which are rules running only in a finite amount of time, if they give output. Examples of quasi-schemata are:

1. Given a set A and a equivalence relation $\sim$ on A form the quotient set $A/\sim$.
2. Zorn's Lemma or any of its equivalents: The axiom of choice, the maximal chain principle or the well-ordering principle.
3. Adjunction of ideal elements.[28]
4. Extensionality of functions and functionals.
5. Taking limits.
6. Power set construction.

Sometimes these principles are ideal, sometimes they are not. The axiom of choice is in an intensional context with intuitionistic logic fully schematisable, whereas in a set-theoretic context we have that extensionality and the axiom of choice imply full classical logic, as shown by Carlström (2004). Power set constructions can be unlimited or limited, as they can for instance be restricted to finite subsets or only the first-order definable subsets.

A simple example of an ideal concept with schema could be the symbol ω which can be seen as denoting the object

$$\{n \mid n \text{ is a natural number.}\}$$

The schema for constructing this set is the quasi-schema: "Take the limit".

The notion of schema and quasi-schema give rise to a relativised notion of construction in mathematics. Here there are at least two different notions of constructions.

Definition. Suppose X is an already constructed object, whether ideal or not. Then we say that *Y is (quasi)-schematisable relatively to X* in case one of the following situations obtains, either

- Y can be constructed using a (quasi) schema on X, or

[28] For example $\mathbf{Q} \longrightarrow \mathbf{Q}(\sqrt{-2})$.

- Y is obtained by adding some already constructed (ideal) elements to X.

Thus there are basically two different types of ideal elements. Some simple examples are

1. **Z** is schematisable relatively to **N**,
2. **Q** is schematisable relatively to **Z**,
3. **R** is quasi-schematisable relatively to **Q**,
4. **C** is schematisable relatively to **R**,
5. Projective geometry is schematisable relatively to Euclidean geometry.

This notion of ideal elements is open towards some dynamical aspects of ideal elements. Our constructions and construction-procedures are not fixed once and for all. Thus it took historically a considerable amount of time to provide an interpretation of the (imaginary) complex numbers, which reduces a complex number to a point in the real plane. Also projective geometry arose by supplying Euclidean geometry with points at infinity. Later on it was discover that we can in fact give a a model of projective geometry within Euclidean geometry by moving up in dimension. There seems to be, however, a very fundamental border between that which can be finitely schematised and that which cannot. I find it very hard to believe that we can eliminate the ideal element when we construct **R** out of **Q**.

In mathematics there are also more abstract ideal methods which do not directly correspond to some kind of quasi-schema. Examples of such are a) in mathematics there is no ignorabimus; b) seek generalizations; c) embed simple problems into to complex areas; d) Axiomatization, i.e., making things abstract—lifting away from intuition; and so on.

6.6 The relativised a priori

Let us come back to the question whether there exists a unique *a priori*. Recall Bernays' word:

> [T]he sharp distinction between the intuitive [Anschaulichen] and the non-intuitive [Nicht-Anschaulichen], employed in the treatment of the problem of the infinite, can apparently not be drawn so strictly, and the reflection on the formation of mathematical ideas still needs a more detailed elaboration in this respect. (Bernays; 1976, 61)

The following example is taken from the mathematical discipline proof theory and will serve to prove, both that the sharp distinction between the intuitive and the non-intuitive does not exist and that the formation of fully schematisable concepts depends on interests of reason—on which ideal elements we include in our theories. I will show that different ways of providing semantics to formal theories give different collections of concepts being fully schematisable. For that we will need a generalization of primitive recursive arithmetic.

6.6.1 Finite Type Theory

T is a type theory and thus it has a certain type structure. The ground type of T is type o which represents the natural numbers. Now, the types of T are generated inductively: if σ and τ are types then $\sigma \to \tau$ is also a type. Each type is thought of as representing a class of objects, and for o this is the natural numbers. Going to higher types, $\sigma \to \tau$ is the type containing operations/functionals from σ to τ. If F is a functional of type $\sigma \to \tau$, then we write this as is standard in mathematics $F : \sigma \to \tau$. Thus $F : (o \to o) \to o$ is a functional taking an arithmetical function as an argument and gives a natural number. The question now is of course: Which objects inhabit T?

The language of T is multi-sorted, meaning that each symbol is assigned a certain type. It includes a symbol for the number zero, $0 : o$ and a symbol for the successor function $S : o \to o$.[29] T has axioms concerning these objects stating for instance that $0 : o$ is the first number and that $S : o \to o$ is injective. T also has the standard axioms from propositional logic, but in order to do, among other things, primitive recursive arithmetic it has combinators k, s and recursor R. The axioms concerning these are[30]

$$\mathsf{k}xy = x \qquad \mathsf{s}xyz = xz(yz),$$

and

$$\begin{array}{rcl} \mathrm{R}xy0 & = & x \\ \mathrm{R}xy(Sz) & = & y(\mathrm{R}xyz)z \end{array}$$

One can think of the equations as reduction rules: The terms on the left 'reduce' to the terms on the right, thus defining rules for calculations. As such it is seen that the prescribed operations are step-by-step calculable.

By using projector k and combinator s one can introduce the λ-operator as a defined notion. The operator behaves in the following way: if $\lambda x.t[x]$ is a term of type $\sigma \to \tau$ and s is a term of type σ then $(\lambda x.t[x])s = t[s]$ where $t[s]$ is of type τ. With the

[29] The intended meaning of $S : o \to o$ is of course the mathematical function $f : \mathbf{N} \to \mathbf{N}$ defined by $n \mapsto n+1$.

[30] With respect to terms, parentheses are associated to the left; thus $t_1 t_2 \ldots t_n$ is short for $(\ldots((t_1 t_2)t_3)\ldots t_n)$.

lambda-notation it is simple to define basic arithmetical operations as for instance $+$ which is of type $o \to (o \to o)$. A definition of $+xy$, or in more usual notation $x+y$, can be given by $\mathrm{R}x(\lambda w, u.Sw)y$.

As I have already discussed a version of the Ackermann function is found in Hilbert's classical article Hilbert (1926). The Ackermann function, as introduced on page 108, is the number theoretic function φ defined as $n \mapsto \varphi_n(n,n)$. The function is not primitive recursive, but as argued above it is computable. It is definable in system T by recursion with function parameters. Let the iteration function be

$$I :\equiv \lambda f, x, y.\mathrm{R}y(\lambda u, v.fv)x.$$

Then the Ackermann function is definable by

$$\varphi :\equiv \lambda x.\mathrm{R}(Sx)(\lambda x, \psi, y.I\psi yy)x.$$

Let me give some reasons why functionals of T should be considered as schematisable. For sure, one can define fairly complicated functionals in T. But the more complicated functionals are always defined *inductively* by a chain of definitions where each step defines a new functional in terms of previously defined ones. Now, the single steps describe simple calculations. However, given a concrete well-formed closed term we cannot directly read of how many calculations we have to perform before the overall calculation is done.[31]. However, we are given inductive rules in order to do the computation—but at the beginning, however, we just do not know how many computations we will have to perform.

Therefore, when we define new procedures (or functionals) out of previously defined ones by the schemes we get a calculable functional, since the equations defining k, s and R prescribe constructive operations. Epistemologically, I consider it as a general human ability to carry out such inductive operations and consequently, I find it justified to consider Gödel's functionals as schematisable—at least in the intensional sense.

6.6.2 Reductions

Given we have accepted T as constructively unproblematic the next task is to consider elements which are ideal elements, at least on the face of it. These elements can perhaps be given an interpretation relative to T. Let us analyze the following principles:

1. *Extensionality* of functionals.

[31] By (non-constructive) mathematical reasoning we can, of course, give bounds for any such term

2. *Markov's principle*: If $A(x)$ is decidable for any x and if $\neg\neg\exists x A(x)$ holds then $\exists x A(x)$, i.e.

$$\forall x\big(A(x) \vee \neg A(x)\big) \wedge \neg\neg\exists x A(x) \to \exists x A(x).$$

3. *Independence-of-premise* (for certain classes of formulae). If $A \to \exists y B(y)$ holds and y is not a free variable of A then $\exists y\big(A \to B(y)\big)$ holds, i.e.

$$\big(A \to \exists y B(y)\big) \to \exists y\big(A \to B(y)\big).$$

I have already discussed some of the problems which seem to be connected with extensional notions of recursive functions. To see why, say, 3 is not immediately schematisable let the following be noted. The constructive reading (which is the BHK-interpretation) of $A \to \exists y B(y)$ is that given a proof of A we can construct an object t and a proof of the fact that t has property B, i.e. $B(t)$. The principle says that in case $A \to \exists y B(y)$ holds then in fact $\exists y\big(A \to B(y)\big)$. But the constructive reading of the conclusion is that we can construct an object t such that given any proof of A we can prove t to have property B. Thus the principle says that t can be constructed independently of the proof of A, and this is not constructively evident given $A \to \exists y B(y)$.

One of our main purposes here is to sketch an analysis of 1–3 in order to provide coherent and consistent systems that partly contain these *prima facie* non-constructive/non-finitary principles. Systems that nevertheless can be seen as constructively meaningful, in the sense that some of the principles are completely schematisable/realizable. A system is constructively meaningful when it has several important properties, say the existence property, in a crucial way. In the case at hand this means, for instance, that when an existential statement is proved in one of the theories then there is a number or a function from Gödel's system T actually realizing the existential quantifier. And this realizer can be found given any proof of the statement. Thus existence in the disputed theories is shown to be based on objects coming from T which we take to be constructive.

The general idea is that given the conceptual acceptance of a certain system, what can (or maybe should) we then accept on the basis of this; in other words, Given the acceptance of some system S_0 (which in our case is T), if S_1 reduces to (is interpretable in) S_0, then we also accept S_1.

The concept of *reduction* or *interpretation* that we work with here is the following:

S_1 reduces to S_0, in symbols $S_1 \preceq S_0$ if (i) S_1 has the existence property and the disjunction property.[32]

(ii) The terms realizing (i) live in S_0.

If we accept S_0 and have $S_1 \preceq S_0$ then we consequently accept S_1 on the basis of S_0, since S_1 has various nice properties and the realizers which make this possible come from S_0, found to be constructively unproblematic.

6.6.3 Specific Reductions

We will now take a look at some specific ideal elements. First extensionality. In any of the following formal theories we have equality only between objects of the lowest type—that is, equality between numbers. Functions and functionals are equal if they are equal on equal arguments. Equality in higher types is thus *defined* in terms of lower types and is therefore extensional. As the types can be fairly complicated it is not obvious that functionals respect this defined notion of equality. However, we can introduce axioms which claim all functionals to respect it. Let σ be type $\sigma_1 \to (\cdots \to (\sigma_n \to o)\dots)$. Then z of type σ is extensional, i.e. respects equality iff

$$\forall x_1^{\sigma_1}, y_1^{\sigma_1}, \dots, x_n^{\sigma_n}, y_n^{\sigma_n} \Big(\bigwedge_{i=1}^{n} x_i = y_i \to z\boldsymbol{x} = z\boldsymbol{y} \Big)$$

where $\boldsymbol{x}$ denotes $x_1, \dots, x_n$. Claiming full extensionality is to claim that all functionals of the theory are extensional. However, it turns out that we cannot always interpret full extensionality. In certain situations we can interpret only *weak* extensionality. This is not introduced by axioms but rather as a rule which within natural deduction has the following form:

$$\frac{\begin{array}{c}\Delta \\ \vdots \\ s = t\end{array}}{r[s] = r[t]}$$

where Δ consists only of quantifier free formulae, and s,t and $r[x]$ are terms.[33]

In the following there will be two different theories on top of which we will put ideal principles. They are two different versions of of intuitionistic number theory

[32] A theory S is said to have the existence property in case the following rule holds: $S \vdash \exists x A(x) \Rightarrow S \vdash A(t)$ for some term t. Likewise, S has the disjunction property if we have that for any provable closed formula $A \vee B$ there is a term deciding whether A or B holds.

[33] Full extensionality could also be formulated as a rule instead of using axioms. Then it has the same form as weak extensionality, just without any restrictions on the assumptions.

generalized to finite types; one will have full extensionality, E-HA$^\omega$, and the other one will only have weak extensionality, WE-HA$^\omega$.[34] Different elements which are generally non-constructive will then be added to these theories. First of all, two different versions of the principle of independence-of-premise. The principle in its general form was defined on page 127. Here we will work with the case where the formula A does not contain any existential quantifier, nor any disjunction—this will be denoted IP. As it turns out, this principle is constructively in conflict with Markov's principle. Thus, we will have to consider a more restricted version, namely the case where the formula A is purely universal. This principle will be denoted IP$_\forall$. Markov's principle is found on page 127. However, in our case the formulae which are decidable are the quantifier free formulae. Consequently, the form of Markov's principle (MP) which we will look at is

$$\neg\neg\exists x A_{\mathrm{qf}}(x) \to \exists x A_{\mathrm{qf}}(x),$$

where A_{qf} is quantifier free.

Finally, the axiom of choice, denoted AC, is also added.

Let Γ be any arbitrary but fixed set of true existence and disjunction free sentences and likewise, let $\Gamma_\forall$ be any arbitrary but fixed set of true purely universal sentences. Then we define the following theories:

$$\begin{array}{rcl} \mathrm{T}_1 & := & \text{E-HA}^\omega + \mathrm{IP} + \mathrm{AC} + \Gamma \\ \mathrm{T}_2 & := & \text{WE-HA}^\omega + \mathrm{IP}_\forall + \mathrm{MP} + \mathrm{AC} + \Gamma_\forall \end{array}$$

By modified realizability as defined by G. Kreisel (1962) T_1 is interpretable in Gödel's system T, that is $\mathrm{T}_1 \preceq \mathrm{T}$. On the other hand we have that Gödel's functional interpretation Gödel (1958) interprets T_2 in T and so we have both $\mathrm{T}_1 \preceq \mathrm{T}$ and $\mathrm{T}_2 \preceq \mathrm{T}$. Therefore we accept both T_1 and T_2 as truly constructive theories.[35] A crucial point here, which I will return to, is that these two theories are constructively incompatible.

An objection against this way of getting confidence in complex theories is found by asking: How do we know that the realizers actually do what is required? If we look at the details of the reductions then we see that this fact is actually proved using the exact principles in question. Using only basic mathematical reasoning we know for instance that if $\mathrm{T}_1 \vdash \exists x C(x)$, for any formula C then there exists a term in T which we can find such that $\mathrm{T}_1 \vdash C(t)$. The thing is that we really need the full theory T_1 in order to prove that t has property C. However, I do not find this problematic. The whole thing is, first of all, computationally meaningful, and the properties that the theories possess are certainly not enjoyed by theories based on classical

[34] See (Jørgensen; 2001) for full definitions of these theories.

[35] Note, however, that T_1 is not closed under Markov's *rule* as shown in (Jørgensen; 2001).

logic. But one could still argue that the theories incorporate principles which are non-intuitionistic and one may *therefore* be skeptical towards the theories. Now, such an attitude is dogmatic: Why should precisely intuitionistic logic monopolize constructivity? The interpretations referred to here show that the respective principles are locally schematisable—not that, say, Markov's principle generally is schematisable. But it is schematisable relative to typed arithmetic with weak extensionality together with a restricted form of the independence-of-premise and the axiom of choice, in the sense that there is always a primitive recursive functional making the principle true. Note that Markov's principle is not only validated for numbers (type o) but is also validated in higher types. It is easy to attach some understanding to the principle for numbers, but this is surely not straight forward in higher types.

The really interesting thing is that many of these principles are in several *combinations* constructively problematic. As I have shown in (Jørgensen; 2001) the combination of IP and MP is demonstratively non-constructive (relative to arithmetic) and also full extensionality together with MP is likewise problematic; also when MP is restricted to the much weaker rule:

$$(\mathrm{MR}^{\sigma}) \qquad\qquad \mathrm{T} \vdash \neg\neg\exists x^{\sigma} A(x) \Rightarrow \mathrm{T} \vdash \neg\neg\exists x^{\sigma} A(x).$$

In (Jørgensen; 2001, 76) the following theorem was proved:

Theorem. There is a quantifier free formula $A_{\mathrm{qf}}(x^0)$ of $L(\text{E-HA}^{\omega})$ such that

$$\text{E-HA}^{\omega} \vdash \neg\neg\exists x^0 A_{\mathrm{qf}}(x), \ \text{ but } \ \text{E-HA}^{\omega} + \mathrm{IP}^{\omega}_{\mathrm{ef}} + \mathrm{AC} \nvdash \exists x^0 A_{\mathrm{qf}}(x).$$

It follows from this theorem, using mainly Spector's result (1962), that:

- $\text{E-HA}^{\omega} \pm \mathrm{IP}^{\omega}_{\mathrm{ef}} \pm \mathrm{AC}$ are not closed under MR^0.
- $\text{E-HA}^{\omega} + \mathrm{MR}^0 \pm \mathrm{IP}^{\omega}_{\mathrm{ef}} \pm \mathrm{AC}$ have not the existence property. The realizers are not even bar-recursive.
- E-PA^{ω} proves totality of a type 3 functional, which cannot be proved total in $\text{WE-PA}^{\omega} + \text{QF-AC}$.[36]

But in case one wants to work with Markov's principle and an extensional notion of equality, then one will have to work with weak extensionality only.

[36] PA^{ω} is HA^{ω} with classical logic.

6.7 Conclusions on the relativised a priori

There are several conclusions to draw from these investigations. It should be clear that there is no unique global characterization of the a priori. We see that in certain contexts Markov's principle is schematisable, but in others—where we have full extensionality—it is not. Likewise versions of the principle of independence of premise is realized by primitive recursive functionals. But also here one has to be careful with respect to the context. It is indeed a *very* subtle issue to combine the different principles which treated separately can be given an schematic interpretation. One can obtain constructive methodologies in various different ways which are not in harmony with one and each other, since the constructivity of mathematical methods depends (also) on the context in which the methods are applied. Thus the formation of concepts which are schematisable depends crucially on the collection of ideal elements. Interests of reason (also) determines the more constitutive parts. Markov's principle is, for instance, very useful—but maybe it is not such a good idea always to have it in your tool box, as their can be problems at hand which are solved (more easily) in contexts without the principle—contexts which perhaps endorse full extensionality of functionals.

CHAPTER 7

Closing Remarks

Based on Kant's notion of schematism I have presented in this thesis first of all a reinterpretation of Kant's theory of knowledge. This interpretation is centered around a developed and detailed theory of schematism. As it turned out, notions like concept, schema and object found their modern formulations in type, computable procedure and token; respectively. The theory provides a foundation for an understanding of the use of constructive categories and ideas and provides, furthermore, a characterization of the notion of object. Essential for the whole theory is that schematism partially explains *how* concepts and objects are related. The upshot of the theory is, that it is basically the same type of theory which applies both to the empirical and to the mathematical. This can be done due to a substantial generalization and relativization of Kant's theory. I provided arguments showing that Kant's deductions of the uniqueness of both intuition and categories of the understanding cannot be expected to carry definite results.

I the last chapter, my interpretation of Hilbert's and Bernays' philosophy of mathematics is fused together with the revised Kantian epistemology. I then outlined a philosophy of mathematics where constructive categories are partly determined by subjective elements of reason. The constructive categories are constitutive in the sense that they define a part of mathematics where we have—due to schemata—privileged access to the mathematical objects. This is in contrast to the part of mathematics where the concepts and principles are only quasi-schematisable.

My future work will consist of two projects. First of all, to provide a deeper and more detailed analysis of how e.g., equivalence classes are understood as types which are or can be (quasi-)schematisable within different areas of mathematics. Second of all, to work out a variety of examples in the hope of getting a closer look at notions like discovery, explanation and unification within mathematics. My hope is that a general theory of schematism can provide the foundation for this work.

REFERENCES

Ackermann, W. (1924). Begründing des “tertium non datur” mittels der Hilbertschen Theorie der Wiederspruchsfreiheit, *Mathematische Annalen* **93**.

Ackermann, W. (1928). Zum Hilbertschen Aufbau der reellen Zahlen, *Mathematische Annalen* **99**: 118–133.

Addickes, E. (1924). *Kant als Naturforscher*, Vol. 1, De Gruyter, Berlin.

Allison, H. E. (1973). *The Kant-Eberhard Controversy*, The Johns Hopkins University Press, Baltimore and London.

Allison, H. E. (2004). *Kant’s Transcendental Idealism: An Interpretation and Defence*, 2. edn, Yale University Press, New Haven and London.

Aristotle (1933). *The Metaphysics*, Vol. 2 of *Loeb Classical Library*, Harvard University Press, Cambridge, Mass. Edited and translated by H. Tredennick.

Bernays, P. (1930). Die Philosophie der Mathematik und die Hilbertsche Beweistheorie, *Blätter für deutsche Philosophie* **4**: 326–67. Reprinted in (Bernays; 1976, 17–61). My page-references are to this reprint.

Bernays, P. (1976). *Abhandlungen zur Philosophie der Mathematik*, Wissenschaftliche Buchgesellschaft, Berlin.

Brown, J. R. (2005). Naturalism, Pictures, and Platonic Intuitions, *in* P. Mancosu, K. F. Jørgensen and S. A. Pedersen (eds), *Visualization, Explanation, and Reasoning Styles in Mathematics*, Synthese Library, Springer Verlag, pp. 57–73.

Caldon, P. and Ignjatović, A. (2005). On mathematical instrumentalism, *Journal of Symbolic Logic* **70**(3): 778–794.

Carlström, J. (2004). $\mathrm{EM} + \mathrm{Ext}\_ + \mathrm{AC}_{\mathrm{int}}$ is equivalent to $\mathrm{AC}_{\mathrm{ext}}$, *Math. Log. Quart.* **50**(3): 236–240.

Corry, H. B. (1951). *Outlines of a Formatist Philosophy of Mathematics*, North-Holland Publishing Company, Amsterdam.

Corry, L. (2000). The Empiricist Roots of Hilbert’s Axiomatic Approach, *in* V. F. Hendricks, S. A. Pedersen and K. F. Jørgensen (eds), *Proof Theory: History and Philosophical Significance*, Vol. 292 of *Synthese Library*, Kluwer Academic Publishers, Dordrecht.

du Bois Reymond, P. (1875). Versuch einer Classification der willkürlichen Functionen reeller Argumente nach ihren Aenderungen in den kleinsten Intervallen, *Journal für die reine und angewandte Mathematik,* pp. 21–37.

Euclid (1956). *The Thirteen Books of Euclid's Elements Translated from the Text of Heiberg*, Dover Publications, New York. With introduction and commentary by T.L. Heath.

Fagin, R., Halpern, J. Y., Moses, Y. and Vardi, M. Y. (1995). *Reasoning about Knowledge*, MIT Press, Cambridge, Mass.

Fitting, M. (2005). Interview on Formal Philosophy, *in* V. F. Hendricks and J. Symons (eds), *Formal Philosophy*, Automatic Press, pp. 13–19.

Fitting, M. C. and Mendelsohn, R. (1998). *First-Order Modal Logic*, Synthese Library, Kluwer, Dordrecht.

Frege, G. (1980). *The Foundations of Arithmetic*, Northwestern University Press, Evanston.

Friedman, M. (1992). *Kant and the Exact Sciences*, Harvard University Press, Cambridge, Massachusetts.

Friedman, M. (2000). Geometry, Construction, and Intuition in Kant and his Successors, *in* G. Sher and R. Tieszen (eds), *Betweeen Logic and Intuition: Essays in Honor of Charles Parsons*, Cambridge University Press, Cambridge, pp. 186–216.

Gibbons, S. L. (1994). *Kant's Theory of Imagination*, Oxford University Press, New York, Oxford.

Gödel, K. (1958). Über eine bisher noch nicht benüzte Erweiterung des finiten Standpunktes, *Dialectica* **12**: 280–287. Reprinted with English translation and introduction in (Gödel; 1990, 241–251).

Gödel, K. (1990). *Collected Works*, Vol. II, Oxford University Press, New York. Edited by S. Feferman et al.

Guyer, P. (1987). *Kant and the Claims of Knowledge*, Cambridge University Press, Cambridge.

Hartnack, J. (1968). *Kant's Theory of Knowledge*, MacMillan, London. Translated by M. Holmes Hartshorne.

Hendricks, V. F. (2001). Active Agents, *Journal of Logic, Language and Information* **12**: 469–495.

Hendricks, V. F. and Pedersen, S. A. (2003). Assessment and Discovery in the Limit of Scientific Inquiry, *in* A. Rojszczak, J. Cachro and G. Kurczewski (eds), *Philosophical Dimensions of Logic and Science*, Kluwer Academic Publishers, pp. 345–371.

Hilbert, D. (1902a). *The Foundations of Geometry*, Open Court, Illinois.

Hilbert, D. (1902b). **Grundlagen der Geometrie**. Unpublished lectures, Mathematisches Institut, Göttingen.

Hilbert, D. (1902c). Mathematical problems, *Bulletin of the American Mathematical Society* **8**: 437–479. Paris-lecture 1900 translated by Mary Winston.

Hilbert, D. (1919). *Natur und mathematisches Erkennen*, Birkhäuser Verlag, Basel, Boston, Berlin. Edited with introduction by David Rowe.

Hilbert, D. (1922). Neubegründung der Mathematik, *Abhandlungen aus dem mathematischen Seminar der Hambugerischen Universität* **1**: 157–177.

Hilbert, D. (1926). Über das Unendliche, *Matematische Annalen* **95**: 161–190. Page references are given to the English translation found *in* van Heijenoort, J., *From Frege to Gödel*, Harvard University Press, pp. 369–392.

Hilbert, D. (1927). Die Grundlagen der Mathematik, *Abhandlungen aus dem mathematischen Seminar der Hamburgerischen Universität* **6**: 65–85.

Hilbert, D. (1930). Naturerkennen und Logik. A lecture given by Hilbert at the congress of natural scientist in Königsberg, 1930. Published in *Naturwissenschaften 1930, pp. 959–963,*.

Hilbert, D. (1931). Die Grundlegung der elementahren Zahlenlehre, *Mathmatische Annalen* **104**: 487–494.

Hilbert, D. (1933). Über das Unendliche. Handwritten notes, Mathematisches Institut, Göttingen.

Hilbert, D. and Bernays, P. (1934). *Grundlagen der Mathematik*, Vol. 1, 2. edn, Springer-Verlag, Berlin.

Hilbert, D. and Bernays, P. (1939). *Grundlagen der Mathematik*, Vol. 2, 2. edn, Springer-Verlag, Berlin.

Hintikka, J. (1967). Kant on the mathematical method, *Monist* **51**: 352–375.

Høffding, H. (1905). On Analogy and its Philosophical Importance, *Mind* **14**: 199–209.

Jørgensen, K. F. (2001). *Finite Type Arithmetic: Computable Existence Analysed by Modified Realisability and Functional Interpretation*, Master's thesis, Roskilde.

Kant, I. (1998). *Critique of Pure Reason*, The Cambridge Edition of the Works of Immanuel Kant, Cambridge University Press, Cambridge. Translated and edited with introduction by P. Guyer and A.W. Wood.

Kant, I. (2004). *Metaphysical Foundations of Natural Science*, Cambridge Texts in the History of Philosophy, Cambridge University Press, Cambridge. Translated and edited with introduction by M. Friedman.

Kelly, K. (1996). *The Logic of Reliable Inquiry*, Oxford University Press, New York, Oxford.

Kleene, S. C. (1952). *Introduction to Metamathematics*, Wolters-Noordhoffs Publishing And North-Holland Publishing Company, Amsterdam.

Kline, M. (1972). *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York.

Kline, M. (1983). Euler and Ifinite Series, **56**(5): 307–14.

Kreisel, G. (1962). On weak completeness of intuitionistic predicate logic, *Journal of Symbolic Logic* **27**: 139–158.

Leibniz, G. W. (1996). *New Essays on Human Understanding*, Cambridge University Press, Cabridge. Translated and edited by P. Remnant and J. Bennett.

Longuenesse, B. (1998). *Kant and the Capacity to Judge*, Princeton Univeristy Press, Princeton, Oxford.

Majer, U. (1993). Hilberts Methode der Idealen Elemente und Kants regulativer Gebrauch der Ideen, *Kant Studien* **84**: 51–77.

Mancosu, P. (1998). *From Brouwer To Hilbert*, Oxford University Press, New York, Oxford.

Mancosu, P. (2005). Visualization in Logic and Mathematics, *in* P. Mancosu, K. F. Jørgensen and S. A. Pedersen (eds), *Visualization, Explanation, and Reasoning Styles in Mathematics*, Synthese Library, Springer Verlag, pp. 13–30.

Montague, R. (1975). *Formal Philosophy*, Yale University Press, New Haven, Conn.

Odifreddi, P. (1989). *Classical recursion theory*, Vol. 125 of *Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Co., Amsterdam.

Parsons, C. (1992). Arithmetic and the Categoreis, *in* C. J. Posy (ed.), *Kant's Philosophy of Mathematics*, Kluwer, pp. 135–158.

Pasch, M. (1882/1926). *Vorlesungen über neuere Geometrie*. Reprint of the 1926 edition with M. Dehn by Springer, 1976.

Paton, H. J. (1936). *Kant's Metaphysitcs of Experience*, Vol. two, Himanities Press, New York.

Peirce, C. S. (n.d.). *Collected Papers of Charles Sanders Peirce*, Vol. 1-8, Harvard University Press, Cambridge, Mass. Edited by C. Hartshorne, P. Weiss and A. Burks.

Posy, C. J. (1995). Unity, Identity, Infinity: Leibnizian Themes in Kant's Philosophy of Mahtematics, *Proceedings of the Eight International Kant Congress*, Vol. I, Marquette University Press, Milwaukee, pp. 621–642.

Posy, C. J. (1998). Brouwer versus Hilbert: 1907–1928, *Science in Context* **11**(2): 291–325.

Posy, C. J. (2000). Immediacy and the Birth of Reference in Kant: The Case for Space, *in* G. Sher and R. Tieszen (eds), *Betweeen Logic and Intuition: Essays in Honor of Charles Parsons*, Cambridge University Press, Cambridge, pp. 155–185.

Posy, C. J. (n.d.). Kant and the Continuum, *in* S. A. Pedersen and F. Stjernfelt (eds), *The Continuum in Philosophy and Mathematics*. Forthcoming, page references are relative to a preprint.

Rowe, D. (2000). The Calm Before the Storm: Hilbert's Early Views on Foundations, *in* V. F. Hendricks, S. A. Pedersen and K. F. Jørgensen (eds), *Proof Theory: History and Philosophical Significance*, Vol. 292 of *Synthese Library*, Kluwer Academic Publishers, Dordrecht.

Russell, B. (1903). *The Principles of Mathematics*, Cambridge.

Shabel, L. (2003a). *Mathematics in Kant's Critical Philosophy*, Routledge.

Shabel, L. (2003b). Reflections on Kant's concept (and intuition) of space, *Stud. Hist. Phil. Sci.* **34**: 45–57.

Shabel, L. (2004). Kant's "Argument from Geometry", *Journal of the History of Philosophy* **42**(2): 195–215.

Shabel, L. (n.d.). Kant's Philosophy of Mathematics, *in* P. Guyer (ed.), *The Cambridge Companion to Kant*, 2. edn, Cambridge University Press. The volume has not been published yet – page-references are to the final pre-print version.

Shaper, E. (1964-65). Kant's Schematism Reconsidered, *Review of Metaphysics* **18**: 267–92.

Sieg, W. (2000). Towards Finitis Proof Theory, *in* V. F. Hendricks, S. A. Pedersen and K. F. Jørgensen (eds), *Proof Theory: History and Philosophical Significance*, Vol. 292 of *Synthese Library*, Kluwer Academic Publishers, Dordrecht.

Spector, C. (1962). Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics, *Proceedings of the Symposia in Pure Mathematics* **5**: 1–27.

Strawson, P. (1966). *The Bounds of Sense*, Methuen, London.

Tait, W. (1981). Finitism, *Journal of Philosophy* **78**: 524–56.

Tait, W. (2005). *The Provenance of Pure Reason*, Oxford University Press, Oxford, New York.

Tait, W. (n.d.). Remarks on finitism, *in* W. Sieg, R. Sommer and C. Talcott (eds), *Reflections on the Foundations of Mathematics: Essays in Honor of Solomon Feferman*, Vol. 15 of *LNL*, Association for Symbolic Logic.

Tiles, M. (1991). *Mathematics and the Image of Reason*, Routledge, London.

Tiles, M. (2004). Kant: From General to Transcendental Logic, *in* D. M. Gabbay and J. Woods (eds), *Handbook of the History of Logic*, Elsevier, Amsterdam, pp. 85–130.

Warren, D. (1998). Kant and the Apriority of Space, *The Philosophical Review* **107**(2): 179–224.

Young, J. M. (1992). Construction, Schematism, and Imagination, *in* C. J. Posy (ed.), *Kant's Philosophy of Mathematics*, Kluwer, pp. 159–175.

Zach, R. (2001). *Hilbert's Finitism*, PhD thesis, University of California, Berkeley.