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a Concise Survey of That Which Was Shaped into the Technique and Discipline We Know

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Pre-modern "algebra": A concise survey of that which was shaped into the technique and discipline we know

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Since long I owe Andur a schoolbook on the science of medieval Islam. Since that book has not yet been sketched, nor a fortiori written, I dedicate to him at least an article without footnotes, based on work undertaken in part under his benevolent administrative aegis and containing as many footnotes as any dusty pedant could want.

1 Conceptual keys

Contemporary "algebra" is an aggregate of practices, problem types and approaches that have only come to belong together through a historical process. If we wish to describe the early history of algebra we therefore have to make clear which of these practices etc. we discuss.

Elementary algebra today is the practice of solving equations "analytically", and this is the aspect of algebraic thought that is most conspicuous in most mathematical cultures until the outgoing sixteenth century.

An "equation" is the statement that some complex quantity (for instance, the area A of a square) defined in terms of one or more simple quantities (in the example, the side s), or the measure of this complex quantity, equals a certain number or (the measure of) another quantity. "Analysis", as formulated by Viète, is "the assumption of what is searched for as if it were given, and then from the consequences of this to arrive at the truly given" (to assume that s exists, whence $s \times s = A$, $s = \sqrt{A}$).

Since Viète, equations are written in symbols. This was not always the case. It is customary to distinguish (a) "rhetorical" algebra, in which everything is set out in full words; (b) "syncopated" algebra, in which standardized abbreviations or

signs are used, but the stenographic expression still represents language; (c) "symbolic algebra", in which the symbolic expression has proper value, and operations are performed directly on this level – as when $\frac{1}{1-x} = 2$ is multiplied by $1-x$ in order to yield $1 = 2 \cdot (1-x)$.

In particular with the advent of symbolization, equations can be trained as dealing with abstract number. In all applied equation algebra, however, the abstract numbers intended by the symbols *represent* other kinds of magnitudes – prices, velocities, population densities, etc.

Beyond being a technique for solving equations, algebra is also a theoretical discipline, dealing with the classification of equations, the principles used to solve them, the existence of and relations between solutions, etc. Such concerns are less frequent in pre-modern times. Finally, contemporary algebra encompasses group theory and its kin, which (*inter alia*) grew out of methods developed in traditional theoretical algebra, but which has left the concern with equations behind.

2 Antiquity

Egyptian texts from the early second millennium BCE present us with two basic elements of algebraic thought: a representation in terms of an abstract quantity or "heap"; and the use of a "false position" in an analytical argument, exemplified by the following problem with solution where an unknown quantity is posited falsely but conveniently to be 4 (in paraphrase):

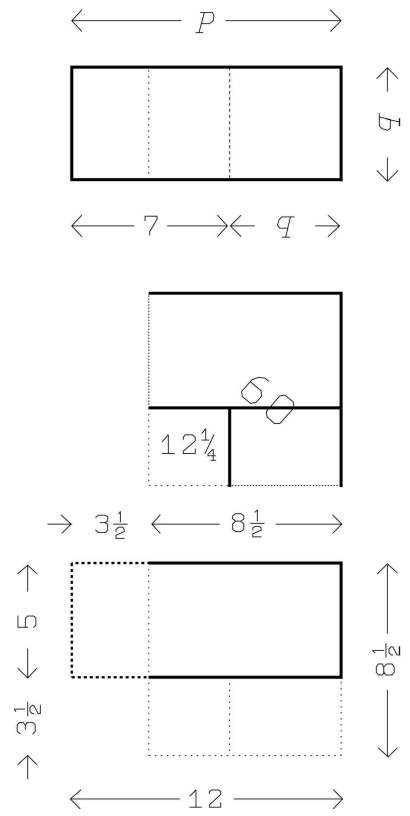
> A heap with its fourth part added produces 15. Assume for convenience that it is 4. Adding its fourth part gives 5. Since we should have 15/5 = 3 times as much, the quantity must instead by $3 \cdot 4 = 12$.

Arguments by false position were also used to solve homogeneous problems of the second degree.

Much more is offered by cuneiform texts from the Old Babylonian period (*c.* 2000–1600 BCE, the mathematical texts being from *c.* 1800–1600).

Firstly, the false position was widely used in first degree problems (as elsewhere until recent centuries). Secondly, a functionally abstract representation by means of measurable segments ("length", "width", "square side") and rectangular and square areas served to treat first- and second-degree problems about quantities of many kinds.

Second-degree problems were solved by means of "naive" cut-and-paste procedures. An example finds two numbers (say, p and q) whose product is 60 and whose difference is 7. The numbers are represented by the length and width of a rectangle with area 60 (see the figure). The excess of length over width is bisected and the outer half moved so as to contain together with the inner half a square



$3\frac{1}{2} \times 3\frac{1}{2}$; adding this small square produces a large square of area $72\frac{1}{4}$ and thus side $\sqrt{72\frac{1}{4}} = 8\frac{1}{2}$. Restoration of the piece that was moved shows that the width of the rectangle must be 5 and its length 12. The procedure is "naive" in the sense that no effort is made to prove that the procedure is correct – this is "seen" immediately.

This problem corresponds to our system $x - y = a$, $xy = b$; the same geometric procedure was used to solve problems about square areas and sides corresponding to $z^2 + m \cdot z = d$ and $z^2 - n \cdot z = e$, and a similar one for the geometric analogues of the system $x + y = a$, $xy = f$ and of the equation $p \cdot z - z^2 = g$. In non-normalized cases corresponding to the equation $r \cdot z^2 + s \cdot z = h$, a change of scale in one dimension was applied, corresponding to the transformation into $rz^2 + s \cdot (rz) = rh$. When linear conditions corresponding to $a \cdot x + b \cdot y = c$ are discussed, explicit terms for coefficients and contributions of the members may turn up. We also encounter two-step procedures corresponding to the change of variable $x' = ax + b$.

By having segments represent areas and volumes, the Babylonian calculators could formulate and solve biquadratic and certain other higher-degree problems. Irreducible third-degree problems were attacked by means of a variant of the false position combined with factorization – which presupposed that an easily factorizable solution was known to exist (all algebraic problems were scribe school problems constructed backwards from the solution, no single problem above the first degree had any practical use).

The description of problems and procedures often employed word signs heavily within the basically syllabic script. It has been claimed that these functioned as algebraic symbols, but since interpretation of the word signs depends on the total text, this cannot be the case (non-mathematical genres, indeed, used word signs just as much). However, certain standard phrases allowed the "nesting" of expressions, achieving part of what modern symbolism does by means of parentheses.

The original inspiration for this naive-geometric algebra appears to have come from "lay", that is, non-scribal practical geometers (surveyors etc.), among whom a small stock of geometric riddles appears to have circulated already in the late third millennium, remaining alive until the late Middle Ages: namely, to find

– one side of a rectangle from the area and the other side;
– the sides from the area and their sum or difference;
– the side of a square from the sum of the area and one or all four sides, or their difference;
– the sides of two squares from the sum of or difference between the areas, and the sum of or difference between the sides;

and a few others. Here, the same analytic naive-geometric procedures were used; but no coefficients appear beyond those occurring "naturally" (*the*, that is, *one* length, *one* width, *one* or *all four* square sides, etc.); nor did the technique serve for representation. In this original context, the technique is thus hardly to be characterized as "algebraic".

The Old Babylonian social system collapsed around 1600 BCE, and the scribe school disappeared together with advanced algebra. The surveyors' tradition survived, however, and inspired a revival of school algebra in Babylonia after 500 BCE; in the Hellenistic age, its stock of riddles swelled, as can be seen in unmistakeable borrowings in Demotic, Indian and Greek practitioners' mathematics. More important, however, was its influence in Greek theoretical mathematics, visible in particular in Euclid's *Elements* II, written somewhere during the third century BCE (the mathematics of book II is likely, however, to go back to the late fifth century).

Elements II.5–6 correspond to the algebraic identity $\left(\frac{x-y}{2}\right)^2 + xy = \left(\frac{x+y}{2}\right)^2$. Similarly, propositions 1–3 correspond to $p\cdot(q+r+\ldots+t) = pq+pr+\ldots+pt$, 4 to $(x+y)^2 = x^2+y^2+2xy$, proposition 7 to $x^2+y^2 = 2xy+(x-y)^2$, proposition

8 to $4 \cdot \frac{x+y}{2}\frac{x-y}{2} = x^2 - y^2$, propositions 9 and 10 to $x^2 + y^2 = 2 \cdot \left(\left[\frac{x+y}{2}\right]^2 + \left[\frac{x-y}{2}\right]^2 \right)$. Propositions 5 and 6, moreover, allow finding the sides of a rectangle from the area and, respectively, the sum of or difference between the sides, 8 allows finding the sides of two squares from the difference between their areas and the sum of or difference between the sides, 9 and 10 allow finding them from the sum of the areas and the difference between or sum of the sides.

Such implications of the theorems have been noticed since the Middle Ages. In the 1880s they were summed up by H. G. Zeuthen in the claim that the propositions constituted a *geometric algebra*, and the discovery around 1930 of Babylonian algebra (then interpreted as a numerical, not a geometrical technique) gave rise to Otto Neugebauer's further assumption that they represented a translation of Babylonian results into geometric language. The discovery of the geometric nature of the Babylonian technique and of the continuity of the surveyors' tradition allows a reformulation of this thesis: *Elements* II.1–10 constitute a theoretical investigation of the basis of the age-old technique, of the conditions under which the procedures are justified, and of the identities which underlie the solutions – the diagram on which II.6 is based is indeed identical with the one shown above, with the only difference that Euclid does not move areas around but constructs them and demonstrates their equality.

The presence of the surveyors' tradition in the Greek orbit is confirmed by the appearance of some of its riddles in characteristic phrasing in several manuscripts belonging to the pseudo-Heronian corpus (put together in modern times as *Geometrica*).

The same manuscripts contain problems related in structure to the traditional riddles (to find a right triangle whose perimeter equals its area) but indeterminate and meant to be solved in integer numbers, seemingly via factorization and use of identities corresponding to *Elements* II.

Indeterminate algebra searching for rational solutions constitutes the main body of Diophantos's *Arithmetic* (third century CE?); only book I, consisting of pure-number translations of the surveyors' and other traditional mathematical riddles, is in part determinate. The beginnings are simple (e.g., II.8, to split a square number into the sum of two square numbers), but soon matters become intricate (e.g., V.18, to find three numbers whose sum is a square, so that each added to the cube of their sum is also a square). The formulation is syncopated, making use of specific abbreviations for the unknown and its powers, for subtraction (the sum is made by juxtaposition) and for the square root; occasionally, operations take place at the level of symbols, thus approaching genuine symbolic algebra. Mathematicians of later ages have complained that Diophantos's procedures are opaque and do not reveal his underlying basic ideas.

Diophantos explains that his abbreviations belong to an established tradition within "theoretical arithmetic", which is confirmed by papyrological evidence.

Passages in Plato's writings (Republica 587d, Timaeus 31c–32a) suggest that this tradition goes back to the fifth century BCE and was carried by an environment of practical calculators, "logisticians" (but no evidence points toward a level much above Diophantos's book I combined with certain cubic problems).

The Indian Śulbasūtra's from the mid-first millennium BCE, containing rules for altar constructions fulfilling sophisticated mathematical conditions, contain solutions to non-homogeneous metro-geometric problems of the second degree; whether the solutions were found by any kind of algebraic argument is not clear, however. A few centuries later, on the other hand, it is likely that members of the Jaina community solved linear, quadratic and reducible higher-degree equations (the original texts are lost, but agreements between subjects cultivated in Jaina environments in late pre-Christian times and the contents of Mahāvīra's ninth-century *Gaṇita-sāra-saṅgraha* corroborate the assumption). Other evidence for early Indian algebra is a manuscript from Bakhshālī, probably a copy (with commentary) of a late ancient original, and the *Āryabhaṭīya* (499 CE). Taken together, these sources show:

– that the Near Eastern surveyors' tradition had reached India, both in the early and the Seleucid variant (Mahāvīra presents the respective methods in distinct chapters);
– that intricate second-degree problems were solved currently, but in versions that seem independent of the surveyors' riddles (they mostly deal with magnitudes and their square roots, not with magnitudes and their squares or products);
– that equations could be organized in schemes (combined with abbreviations) in which operations were made algorithmically, meaning that a transition to symbolic algebra (though very different from ours, and not allowing nesting whence less productive) had taken place;
– and that astronomers, for purposes of correlating planetary movements with each other, treated indeterminate linear equations.

Even the Chinese first-century CE *Nine Chapters on Arithmetic* betray some familiarity with the Near Eastern metro-geometric algebraic tradition, whose impact however was modest. Fully autochthonous is the creation of a technique (quite similar to our matrix manipulations, and thus another transition to symbolic algebra) for reducing systems of several linear equations (widely circulating riddles, which Diophantos was to treat with different techniques in *Arithmetica* I).

3 Algebras of the mature Middle Ages

The first surviving presentation of the technique from which our algebra developed and took its name is al-Khwārizmī's early ninth-century *Kitāb fi'l-jabr wa'l-*

muqābalah, "Book on restoration and opposition" – according to al-Khwārizmī's preface a brief introduction to an existing art. In the wake of al-Khwārizmī's work, "restoration" came to designate the addition of a subtracted member on both sides of an equation, and "opposition" subtraction on both sides; originally, "restoration" appears also to have encompassed multiplicative completion, and "opposition" to have designated the formation of an equation – or rather, perhaps, of its reduced form (which indeed would often ask for the subtraction of members occurring on both sides).

Pre-al-Khwārizmīan *al-jabr* consisted of two components, which may not have common origin. The core of *al-jabr* proper were fixed rules allowing the solution of equations dealing with a (monetary) possession (*māl*, becoming *census* in Latin), its square root (*jidhr*, becoming *radix*) and a number of dirhams (a coin); negative numbers not being considered, three rules were needed for the simple cases with two members and three for the "mixed" cases. The style (the square root of property) and certain linguistic clues suggest a connection to Indian algebra, most likely through a common ancestor. In *al-jabr*, however, these monetary riddles had become a general representation for second-degree problems.

The *al-jabr* rules went together with a technique for rhetorical transformation of equations, in which the unknown magnitude was spoken of as "a thing" (Arabic *šay'*, Latin *res*, Italian *cosa*), functioning like our *x*. Leonardo Fibonacci speaks of the technique as *regula recta* (a reference to its analytical nature), and treats it independently of *al-jabr* and with examples that suggest a link to elementary Greek "logisticians' algebra".

What made al-Khwārizmī's work pivotal was his introduction of (geometric) proofs for the *al-jabr* rules. The aim was no doubt to bring the presentation of the discipline in agreement with the already familiar Greek norms; the proofs themselves, however, were cut-and-paste proofs borrowed from the surveyors' tradition, only slightly adapted to Greek style.

This may have been a pedagogical advantage, but was deemed unsatisfactory by Thābit ibn Qurrah, a major translator of Greek texts in the second half of the ninth century and a prominent mathematician on his own account. In a small treatise he supplied new proofs for the rules based on *Elements* II.5–6 without even mentioning his predecessor.

The following major Arabic algebraist was Abū Kāmil (*c.* 850 – *c.* 930). He glued the reference to *Elements* II.5–6 to the naive diagrams in the proofs, but added others that produced the *māl* directly and not the root, showing thus that this quantity could be understood as an unknown in its own right (originally, the *māl* had certainly been *the* unknown – but that was long ago). Much in Abū Kāmil's treatise on algebra repeats and expands what al-Khwārizmī had done, but it goes beyond this model in the use of other monetary units as names for auxiliary variables (probably a borrowing from current practice), in its unconstrained use of

irrationals, and in the expanded operation with higher powers of the unknown in biquadratic and other reducible problems. In one section he calculates the sides of the regular pentagon and decagon, elsewhere he investigates indeterminate problems of the first and second degree.

Around 1000, al-Karajī produced more striking innovations. His handbook presenting practitioners with "the sufficient about reckoning" (*al-Kāfī*) suggests (*inter alia* through its pre-al-Khwārizmīan use of "restoration" and "opposition") that his starting point was the "low", not the "scientific" al-Khwārizmī-Abū-Kāmil tradition. His major works (*al-Fakhrī*, *al-Badī*) demonstrate familiarity with this tradition as well as with the newly translated Diophantos, but go further by systematizing the treatment of reducible higher-degree equations; by applying the Euclidean theory of irrational magnitudes to number (and expanding it); by formulating an arithmetic of polynomials (including division and root extraction); and, in indeterminate analysis, by formulating principles where Diophantos had only given solutions.

All of this was developed further around the mid-twelfth century by al-Samaw'al, who also extended the notion of "subtractive" magnitudes into a concept of negatives ("subtractive 2" can only be subtracted; but "$n-(-2)$" is meaningfully interpreted as $n+2$). In order to represent polynomials, he invented a schematic symbolization similar to what was used in Indian algebra.

Already around 1000, al-Bīrūnī and other astronomers had formulated the finding of the chord of a trisected angle (the kind of problem which the Greeks had solved by intersecting conic sections) as a cubic equation, solving it however by numerical, not by algebraic methods. In the context of a full classification of equations until the third degree (14 of which are irreducible cubics), al-Khayyāmī (*c.* 1100) made the reverse step and solved cubic equations by means of intersecting conic sections, identifying also the cases that were not solvable (in positive numbers) and some of those that have several solutions. Certain solutions of this kind had already been obtained by al-Khāzin (d. *c.* 965) and others, as al-Khayyāmī relates.

Developments of a different kind occurred in the Maghreb in the twelfth to fifteenth century, carried by a teacher-student network dense enough to be regarded as a "school" (and indeed organized as a teaching *system* and linked to mosque and madrasah teaching). Its algebra, as evidenced by its neglect of geometrical proofs, was basically in pre-al-Khwārizmīan style. Its essential innovation with regard to the "low" fundament was the development of abbreviations for both unknowns and their powers and for operations; seemingly, this systematic syncopation inspired parallel developments in Italian algebra, ultimately leading to the development of modern symbolic algebra.

In India, astronomers from Brahmagupta (598–*c.* 665) to Bhaskara II (1114–*c.* 1185) followed the lead of Āryabhata I, associating expositions of mathemat-

ics with astronomical treatises. The solution of indeterminate linear equations remained an important topic, but Brahmagupta also took up the study of the equations $Nx^2 \pm c = y^2$ (Pell equations), and showed how from one solution (found by trial and error) others can be produced. Bhaskara II formulated a general method. To judge from Brahmagupta's exposition, the pretext was sham astronomical computation, and his purpose the display of professional skill.

In China, the level of mathematics declined in the later first millennium. Between 1247 and 1304, however, a number of works introduce a sophisticated polynomial algebra, working with up to four variables and until degree 14, representing polynomials in a positional notation and solving equations by a procedure seemingly inspired by algorithms for root extraction (the "Horner-Ruffini method"): an approximate solution is found, a new equation for the remainder is derived, to which again an approximate solution is found, etc.

4 Latin Europe

Al-Khwārizmī's *Algebra* was translated twice into Latin in the twelfth century, first by Robert of Chester and next by Gherardo da Cremona. The riddles of the surveyors' tradition became available through Gherardo's translation of an Arabic work on mensuration, the *Liber mensurationum*, and to some extent through Plato of Tivoli's translation of Savasorda's *Collection on Mensuration*.

The echo was faint – the curriculum of the schools had no space for algebra. However, Gherardo's translations were used (at times copied *verbatim*) by Fibonacci in his *Liber abbaci* (1202, revised 1228) and *Practica geometrie* (1220) together with much material he had found in Islamic territory, Constantinople and Provence. The chapter of *Liber abbaci* dealing with "algebra et almuchabala" proves the rules for the mixed cases in ways reminding of Thābit's but possibly invented independently (one, "naive", seems inspired by *Elements* II.4 and does not copy al-Khwārizmī)0. The level and contents of problems are comparable to those of Abū Kāmil (who was only translated in the fourteenth century, without generating any response) and (in his use of the Euclidean theory of irrationals) of al-Karajī. Earlier in the work, as mentioned, rhetorical first-degree *thing*-algebra is used under the name *regula recta*.

Fibonacci was linked to the Italian urban patriciate and to the philo-Arabic Hohenstaufen court. The contemporary university mathematician Jordanus de Nemore (probably active in Paris somewhere between 1210 and 1240) responded differently to the challenge of Arabic algebra. Strongly attached to the metatheoretical ideals of Greek mathematics, and acknowledging that algebra dealt with *number*, he wrote a treatise *On given numbers* that was related to his *Elements of Arithmetic* much as Euclid's *Data* were related to the *Elements* of geometry. It

consists of theorems of the form "If certain arithmetical combinations of certain numbers [e.g., their difference and product] are given, then the numbers themselves are given". It thus does not teach the technique of solving equations (often it merely reduces a case to another case that is dealt with previously). Jordanus does not mention algebra at all; what he offers is a theoretical investigation of solvability. However, the theorems are illustrated by numerical illustrations, and these are often unmistakeably borrowed from the Arabic tradition. Readers who knew the latter thus got a hint that Jordanus meant to *replace* (Arabic) algebra by something which was theoretically better.

Jordanus's proofs are arithmetical and general, not based on numerical examples. This was possible because he represented numbers by letters (a technique he had probably developed when proving the validity of the algorithms for calculating with Arabic numerals in an earlier work, and which is also used in his *Elements of Arithmetic*). Since the outcome of every operation is designated by a new letter (thus "the quadruple of *d*" immediately becomes f.), this should not be mistaken for a symbolic algebra; the letters serve the same purpose as the segments used by Euclid in *Elements* VII–IX.

Fibonacci represents numbers in the same way in a few problems and alternative procedures that may have been added in the revision from 1228. In the fourteenth and fifteenth centuries, respectively, Oresme and Peurbach also refer to Jordanus's *Data* and betray to have understood the particular aim of his treatise; but apart from that it had no perceptible influence.

Traditionally it has been assumed that Fibonacci's *Liber abbaci* and *Practica geometrie* provided the foundation for that Italian "abbacus school" which emerged during the thirteenth century (a school where merchant youth was trained for two years in practical arithmetic); particularly it has been believed that abbacus algebra was derived from the *Liber abbaci*. Closer investigation of early abbacus books shows that this cannot be the case. In general, the *Liber abbaci* is instead an early (but overwhelming) exponent of a mathematical environment of abbacus-school type, which however may have been stronger in his times in the Ibero-Provençal area than in Italy (and may even have been shared by the commercial cultures of the western Mediterranean irrespective of religion). Specifically, even though the earliest Italian abbacus books contain material of a kind which can also be found in the *Liber abbaci* and which comes from shared sources, they contain no algebra at all. When eventually taking up the subject, the abbacus school adopted Arabic algebra via different channels.

The first influential abbacus treatise containing algebra was written by a certain Jacopo da Firenze in Montpellier in 1307; apart from the term *censo* for the second power, it shares nothing with the Latin predecessors. Containing no Arabisms, it must draw on a tradition that was already established in the Romance-speaking

world – most likely in Provençal-Catalan area, from which however no other evidence prior to the 1330s has been traced.

Jacopo's algebra is in the "low" Arabic style, as evidenced for instance by the absence of geometric proofs and by the way "restoration" and "opposition" are used (the latter appears as "putting equal to"). It contains correct rules (but no problems) for reducible cubics and quartics, and not the slightest trace of syncopation.

Within three to four decades, Jacopo's treatise together with new imports from the same area had spurred a surprising development (detailed verbal agreements demonstrate that Jacopo was in fact an important contributor, though supplementary inspiration seems to have arrived after a couple of decades). Binomials involving square roots were manipulated with great virtuosity, pure-number problems were created as illustrations of the higher-degree rules, and examples and non-valid rules for non-reducible cubics and quartics were produced and transmitted; they proliferated and remained alive throughout the fifteenth century. The reason for persistence of the latter mathematical scandal (as we would tend to view it) is double: the abbacus masters used them to impress their public and the municipal authorities that might employ them; and solutions contained intricate expressions involving roots, whence fallacies were difficult to expose.

Ongoing contacts to the Maghreb area may be responsible for the introduction of syncopation, for the increasing operation with subtractive quantities, and for the first hints of symbolization (fractions containing polynomials in the denominators and subjected to cross-multiplication, schemes for the arithmetic of polynomials); in any case, it seems certain that all of these were ultimately borrowed from the Maghreb. In the fifteenth century, geometrical proofs were gradually taken up – in part borrowed from Fibonacci, in part coming from al-Khwārizmī, in part from the surveyors' tradition, in part independent.

In 1494, algebra went into print, constituting part of Luca Pacioli's *Summa de arithmetica, geometria, proportioni: et proportionalita*. The wording of the rules and many problems are borrowed from the abbacus tradition, but geometrical proofs copied from Fibonacci form the theoretical basis. Moreover, Luca had discovered that the widely circulating solutions to the non-reducible higher-degree problems were false; restricts himself to giving solutions to the reducible cases; and points out that the others had not been solved *so far.*

The Ibero-Provençal abbacus-like tradition survived into the fifteenth and even the sixteenth century, now in interaction with the strong Italian tradition (but less dependent on it than has been assumed), yet until Nicolas Chuquet's *Triparty* (*c.* 1480, the culmination of the Provencal tradition) without leaving traces of any strong interest in algebra. Towards the mid-sixteenth century, Italian abbacus algebra was adopted by the German *Rechenmeister* under the name of *Coß* (from

Italian *cosa*). This was the source of Robert Recorde's treatment of "the coßike practise" in *The Whetstone of Witte* (1557).

5 The transition phase

Solution of the irreducible third- and fourth-degree cases became pivotal in the transition to modern algebra. The first step was made around 1515 by Scipione del Ferro, who discovered how to resolve the case "cube and roots equal to number". He communicated the rule to his pupils, one of whom used it in a public disputation with Niccolò Tartaglia in 1535 (though the solution was no longer fake, it served the same career purpose as before). Tartaglia managed to find the solution and was persuaded to disclose the rule to Gerolamo Cardano (according to his own account under oath of secrecy, according to Cardano's disciple Ludovico Ferrari without such conditions). When Cardano was informed about del Ferro's earlier discovery he felt free to publish (crediting both del Ferro and Tartaglia) in the *Ars magna* (1545).

Cardano not only published *the rule* for the case in question (and for the related case "cube equal to roots and number", which he may also have received from Tartaglia); he also gave geometric proofs, which he claims to have found himself (which is indeed quite plausible, once he knew the rule his training will have allowed him to recognize the way leading to the goal).

The proof for the latter case can be summarized as follows in modern symbolism: The equation is $x^3 = 3px + n$. We represent x^3 by a cube (see the figure), express x as a sum $x = u + v$, and dissect the cube into 5 pieces corresponding to the transformation $x^3 = (u+v)^3 = u^3 + v^3 + 3uv \cdot (u+v) = (u^3 + v^3) + 3uv \cdot x$ (one of the three pieces $uv \cdot x$ is shown separately). If $u^3 + v^3 = n$, $uv = p$ (whence $u^3 \cdot v^3 = p^3$) , $x = u + v$ will fulfil $x^3 = 3px + n$.

Now, the problem $r \cdot s = A$, $r + s = b$ was familiar in the abbacus tradition, from *Elements* II.5, and was part of the surveyors' stock since millennia. Its solution is

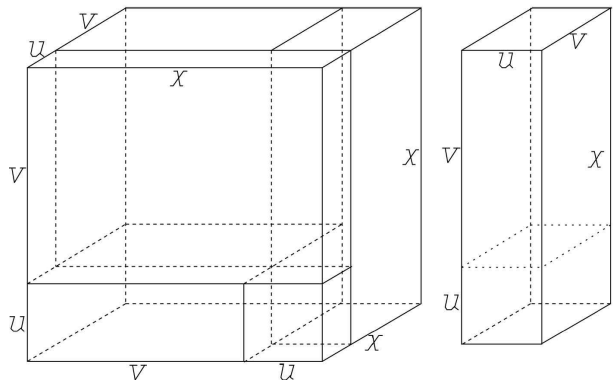
$$r = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - A} \ , \ t = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - A}. \tag{1}$$

Substituting p^3 for A, n for b, and finding u and v from u^3 and v^3, we get

$$x = u + v = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 - p^3}} + \sqrt[3]{\frac{n}{2} - \sqrt{\left(\frac{n}{2}\right)^2 - p^3}}. \tag{2}$$

This might have been nothing but an ingenious but traditional solution to a traditional problem, had it not been accompanied by other novelties. Firstly, *all* the

cubic cases are solved (the case $x^3 + px = n$ analogously, the others by transformation into this or the previous one), the necessary transformations being proved by Euclidean geometry.



Secondly, Cardano operated without difficulty with negative quantities and solutions (though considering them "fictitious"), which allowed him to clarify the relation between the sets of solutions to related equations and between different solutions to the same problem (and led him to the notion of coinciding solutions). Thirdly, he applied the theory of irrationals in order to find conditions which solutions would have to fulfil.

Also contained in the book is a discovery made by Ferrari: that the complement to be added to a quartic in order to transform it into a biquadratic (and thus resolve it) can be determined by means of a cubic equation.

A curiosity is Cardano's introduction of imaginary and complex solutions – regarded, respectively, as "a second kind of false" solution (the negatives being the first) and as "completely false". He may possibly have been provoked to think about these because they occur in some of the solutions to cubics (e.g., in the above, if $p^3 > (n/2)^2$); but he does not say so, and the example through which he introduces them is the corresponding second-degree problem, $r + t = 10$, $rt = 40$.

In 1572, Bombelli's *L'algebra* was published. Its declared aim was to put into intelligible shape what had so far been written confusedly on the subject – not least by Cardano.

Part of this clarification consisted in the introduction of new symbols. Cardano's style had been purely syncopated; for instance,

$$\sqrt[3]{42 + \sqrt{1700}} + \sqrt[3]{42 - \sqrt{1700}} - 2$$

appears as "R. V. cubica 42. p. R. 1700 p. R. V. cub. 42 m. 1700 m. 2" – "p." representing "più", "m." "meno", "R." "radice", and "V." indicating that the root is taken of two members. Bombelli introduces algebraic parentheses (written $\lfloor \ldots \rfloor$ and used for multiple nesting) and an arithmetical notation for powers, in which ^{0} represents our x^n. Both devices constitute steps toward symbolization. He had been preceded in part by Chuquet in the *Triparty*, but Chuquet's notations were uninfluential.

Bombelli also invented new geometrical constructions "in a plane surface" though "by means of instruments" (not compass and ruler only) that showed the existence of solutions to the cubics even in cases where Cardano's formulas gave them only in the form $\sqrt[3]{a+b\sqrt{-1}}+\sqrt[3]{a-b\sqrt{-1}}$. These solutions also made him take up the study of imaginary numbers (designating them "più di meno" and "meno di meno", respectively $a\sqrt{-1}$ and $-a\sqrt{-1}$) and their arithmetic. He does not refer to Cardano's modest beginnings, and successive work on complex numbers was derived from Bombelli, not Cardano.

With Cardano's and Bombelli's innovations we have reached the threshold to modern algebra. Viète and Descartes belong to the next chapter – of history, not of this volume.

Bibliographic notes

A collective work presenting Babylonian, ancient Egyptian, Greek, Islamic, Latin and later European algebra and algebraic modes of thought is:

> Scholz, E. (ed.) (1990). *Geschichte der Algebra. Eine Einführung*, Lehrbücher und Monographien zur Didaktik der Mathematik, Band 16, B. I. Wissenschaftsverlag, Mannheim etc. A revised English translation has been under way at Oxford University Press for so long that the project must probably be considered dead.

The latest large-scale presentation of ancient Egyptian mathematics, including its quasi-algebraic features is:

> Clagett, M. (1999). *Ancient Egyptian Science. A Source Book*, Vol. III. *Ancient Egyptian Mathematics*, Memoirs of the American Philosophical Society, 232, American Philosophical Society, Philadelphia.

An analysis of Old Babylonian algebra as a naive-geometric technique together with a relatively broad account of the surveyors' tradition is found in:

> Høyrup, J. (2002). *Lengths, Widths, Surfaces: A Portrait of Old Babylonian algebra and its kin*, Studies and Sources in the History of Mathematics and Physical Sciences, Springer, New York.

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Commentary: Apart from the Scholz-volume, which is simply the most recent general treatment of the topic (until the twentieth century), most of what is contained in the preceding list presents results and interpretations that go beyond the content of standard histories.