Existence, substantiality, and counterfactuality

observations on the status of mathematics according to Aristotle, Euclid, and others

Høyrup, Jens

Published in:
Filosofi og videnskabsteori på Roskilde Universitetscenter, 3. række : Preprints og reprints.

Publication date:
2000

Document Version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Høyrup, J. (2000). Existence, substantiality, and counterfactuality: observations on the status of mathematics according to Aristotle, Euclid, and others. *Filosofi og videnskabsteori på Roskilde Universitetscenter, 3. række : Preprints og reprints.*, (7).

ROSKILDE UNIVERSITETSCENTER
Faggruppen for filosofi og videnskabsteori

ROSKILDE UNIVERSITY
Section for philosophy and science studies

EXISTENCE, SUBSTANTIALITY, AND COUNTERFACTUALITY
Observations on the status of mathematics according to Aristotle, Euclid, and others

JENS HØYRUP

FILOSOFI OG VIDENSKABSTEORI PÅ ROSKILDE UNIVERSITETSCENTER

3. Række: Preprints og reprints

2000 Nr. 7

ISSN 0902-901X

In memory of Dirk Struik
30 September, 1894 to 21 October, 2000

Contents

Paper presented to the meeting on

Existence in mathematics

University of Roskilde
November 24–25, 2000

In current loose parlance, the view that mathematical objects *exist* is regarded as *Platonism*. It might therefore seem obvious that this existence was the basic assumption of ancient mathematics – not least if we accept Proclos's statement in his commentary to *Elements* I [ed. Friedlein 1873: 68^20f^, trans. Morrow 1970: 57] that Euclid "belonged to the persuasion of Plato and was at home in his philosophy".[1]

Actually, the ancient texts do not speak exactly in terms that can unambiguously be translated into "existence" (presupposing for the sake of the argument that this latter term possesses an unambiguous meaning) – at best they use forms and derivatives of the verb εἰμί, "to be", which (as pointed out in the ancient texts themselves) has a wide, and wider, range of meanings, though certainly including "being there". In order to decide the stance of the ancients with regard to the question of mathematical existence, we must therefore attempt a translation, not *de verbo ad verbum* but "conceptual network to conceptual network". We shall have to ask whether, and in which sense, something mathematical was supposed to exist by ancient Greek mathematicians and / or Greek philosophers discussing mathematics. The point of asking this question in a context which is not primarily concerned with ancient thought is that the outcome may elucidate not only ambiguities that inhere in the writings of the ancients but also ambivalences in more recent debates. I shall, however, abstain from drawing such wider conclusions myself on the present occasion.

In contemporary discussions, mathematical "existence" is often meant as a negation of unconstrained constructibility – what *exists* can be *discovered* but cannot be *invented*. If this perspective is combined with the view that mathematics is first of all concerned with coherent structures, the question to the ancients might be whether they considered mathematics *in toto* (or at least whole mathematical domains) as somehow existing or constrained by reality (and in case, by *which* reality). This is a problem to which we shall return; however, the explicit discussions in ancient writings which come closest to the question of existence are concerned not with mathematics as such or with domains but with *the objects* of mathematics, τὰ μαθηματικά, "the mathematicals" – numbers, ratios, lines, triangles, etc.

[1] I use three kinds of translations in the following: (i) some are borrowed from an existing translation, in which cases the translator is identified as here; (ii) others are similarly borrowed but corrected at points where I find the translation unduly free for my purpose or inconsistent with the remaining text – for these, the translator is identified in the same way, but corrected passages are put into superscript pointed brackets ⟨ ⟩ (inserted explanations and words from the original text evidently appear in [ ]); (iii) some shorter phrases I have finally translated myself (but obviously controlled against established translations); quotations with no identified translator belong to this category.

Aristotle

The surviving writings of the ancient mathematicians are not explicit on this account. This does not preclude a reading searching for hidden presuppositions, but such a reading is most likely to yield a result if seen in the light of the more outspoken discussions of the philosophers. I shall refrain, however, from a broad coverage of this topic and concentrate on the writings of *the philosopher*, that is, Aristotle.

The central Aristotelian concept in any discussion of what *is really there* is οὐσία, "being" or, in the received translation, "substance".[2] Actually, speaking of a "concept" may be a slight overstatement. Aristotle himself points out in *De anima* in consequence, even substance) constitutes a hierarchy, no single level: “utmost matter” (ὕλη ἔσχατ[η]), as substrate receiving the qualities of warmth, cold, humid and dry (“utmost form”), gives rise to the four elements, which seen in this perspective are substances; these, on their part, also function as matter which, united in proper proportion (λόγος – another level of form[4]) produce bone, flesh etc., a higher level of substance. But even flesh etc., growing naturally together (συμφύσει) are matter for a still higher level of substance, namely the head and other bodily parts. These then constitute the matter for a “substance in the strictest sense”, the living person (Socrates or Callias).

Elsewhere we find the term “material substance” (ουσία ὑλική) in what seems to be the sense of “non-utmost matter”.[5] This agrees without much difficulty with the passage that was just discussed. Not directly in harmony with this, nor however in blunt contradiction, is the usage of the *Categories* (trans. Cook in [Cook & Tredennick 1938]), according to which species and genera are “secondary substances” (δεύτεραι οὐσιαι), the individuals being “primary” (πρῶται ουσίαι) – individuals being “best spoken of” as substances, and among the universals species more properly than genera (2^{b}7–8,15–18). In contrast, *Metaphysics* Z, 1038^{b}9*ff* argues (echoed by I, 1053^{b}17) that the universal (τὸ καθόλου) cannot be a substance. For this reason (1039^{a}24*ff*), even Ideas cannot be (separable) substances, at least if species are constituted from a genus (functioning as substrate) and determined by the imposition of differentiae – but this whole discussion of the status of universals and forms is spurred by the preceding observation (1029^{a}27*ff*) that matter is even less a substance – characterized, as stated here (and again in Λ, 1070^{b}36–1071^{a}1), by separability and individuality – than both the form and the combination of form and matter.

All passages discussed so far regard sensible and movable “primary” substances and the matter and form that cause them to exist. In this domain, as we see, the term is supposed to cover primarily individual entities, and matter (at whatever level), form and universals only deserve it in restricted or secondary ways. Substances of this kind may be characterized mathematically (the oft-mentioned bronze sphere is an obvious instance – *Metaphysics* 1070^{a}3–4), but the mathematical characterization in itself is never discussed as a sensible substance. In *Metaphysics* H 1043^{b}34 it is said explicitly that *if* really numbers are substances, then at least not as claimed by “some” (probably Pythagoreans) as aggregates of (possibly sensible) monads. Below we shall meet a reference to “sensible circles” which are told *not* to be those of mathematics.

Another category of substance can neither be perceived by the senses nor

---

[4] This specification is borrowed from the kindred passage in *De anima* I, 410^{a}2 – cf. also *Metaphysics* N, 1092^{b}18–22.

[5] *Metaphysics* H, 1044^{a}15; Θ, 1049^{a}36; and M, 1077^{a}36.

moved – it is immovable and “intelligible” (νοητός), it can only be reached through the intellect, the νοῦς. In *Metaphysics* Λ, 1069ª34–36 we are told that this separate category is postulated by some, either comprising both Forms and mathematicals (either as distinct subcategories or as one indistinct class) or consisting of mathematicals alone. Later in book Λ, however, only the Prime Mover (and, possibly, the movers of the celestial spheres) are counted as intelligible substances – Aristotle evidently does not share the views presented a bit earlier. Book Z, 1028ᵇ20–27 [trans. Tredennick 1933: I, 315] is more explicit:

> Thus Plato posited the Forms and the ⟨mathematicals⟩ as two kinds of substance, and as a third the substance of sensible bodies; and Speusippos assumed still more kinds of substances, starting with “the One,” and positing principles for each kind: one for numbers, another for magnitudes, and then another for the soul. In this way he multiplies the kind of substance. Some [the followers of Xenocrates, successor to Speusippos as head of the Academy] again hold that the Forms and numbers have the same nature, and that other things – lines and planes – are dependent upon them; and so on back to the substance of ⟨the heaven⟩ and sensible things.

Even to Aristotle, however, the mathematicals are, or are sometimes, intelligible. In *Metaphysics* Z, 1035ᵇ33–1036ª12 [trans. Tredennick 1933: I, 361–363] we find the following reflections:[6]

> A part, then, may be part of the form (by form I mean essence), or of the concrete whole composed of form and matter, or of the matter itself. But only the parts of the form are parts of the formula, and the formula refers to the universal; for “circle” is the same as “essence of circle”, and “soul” the same as “essence of soul”. But when we come to the concrete thing, *e.g. this* circle – which is a particular individual, either sensible or intelligible (by intelligible circles I means those of mathematics, and by “sensible” those which are of bronze or wood) – of these individuals there is no definition; we apprehend them by intelligence [μετὰ νοήσεως] or perception and when they have passed from the sphere of actuality [ἐκ τῆς ἐντελεχείας] it is uncertain whether they exist [εἰσὶν] or not, but they are always spoken of and apprehended by the universal formula. But the matter is in itself unknowable [ἄγνωστος]. Some matter is sensible and some intelligible; sensible, such as bronze and wood and all movable matter; intelligible, that which is present in sensible things not *qua* sensible, ⟨such as⟩ the ⟨mathematicals⟩.

Mathematical circles are thus two different things: the word may refer to the universal defined by a formula or λόγος, for instance the one which is quoted in Plato’s (or ps-Plato’s) *Seventh Letter* [trans. Bury 1929: 533], “that which is everywhere equidistant from the extremities to the centre” or it may designate a particular specimen, distinct (it seems) from the actual drawing but still coming into actual existence and passing away from it (together with the drawing, we must presume);

---

[6] “Essence” corresponds everywhere in the quotation to τὸ τί ἦν εἶναι, cf. note 2; “formula” corresponds to λόγος, cf. p. 3.

we may think of one of the two distinct circles that serve in the construction of *Elements* I.1.[7] The final passage "But the matter ... mathematicals" could be taken to refer to the problem that the existence of distinct individuals belonging to the same species presupposes that they originate by the imposition of form upon matter; in this reading, Aristotle concludes that the matter (ὕλη) of the distinct circles is unknowable (ἄγνωστος) but intelligible.[8] Alternatively, the mathematicals themselves are regarded as matter, which is "present in" (ὑπάρχουσα) sensible objects (the term for presence having strong connotations of "belonging properly to" and of subordination, which indeed fits matter better than form). The latter reading is strongly supported by the parallel "such as bronze ... such as the mathematicals" (οἶον χαλκὸς ... οἶον τὰ μαθηματικά) and by the choice of grammatical case.[9] It is

---

[7] Such particular specimens are of course localizable, and to assume that Aristotle should refer to them might seem to contradict the assertion made in N, 1092ᵃ19–20 – *viz* that mathematicals (including in particular mathematical solids) have no position; but the context (a discussion of generating principles) shows that here the category of mathematicals which coincide with their essence is meant.

[8] This is how the passage is interpreted by Ian Mueller [1970: 163]. Further on (pp. 166*f*), Mueller mentions (accepting the view) that an ancient commentary ascribed to Alexander of Aphrodisias understands the matter in question be mere extension – cf. also [Heath 1949: 224]. Since *Physics* IV, 209ᵃ21 says explicitly that place (τόπος) is not the matter of anything, the idea is not without problems – in particular because it has just been explained (208ᵇ27–28) that the vacuum which certain thinkers accept is "place deprived of bodies", which shows us that Aristotle's "place" encompasses what we would call "space". To this comes, as pointed out by Thomas Anderson [1969: 22 n. 59], the difficulty that numbers can hardly be supposed to have extension as their matter (instead, as we shall see in note 23, the observation that numbers and geometrical shapes are always numbers and shapes of *something*, leads him to suggest that the intelligible matter may be indeterminate, unspecified substance).

To modern ears it may sound strange that something is intelligible but unknowable, but the two terms belong at different levels. The former epithet refers to an ontological dichotomy – indeed, "sensible matter" cannot be reached by the senses. The latter refers to the shared characteristic of everything not submitted to measure and order (thus not least utmost matter); its use can be elucidated by a passage from the *Rhetoric* (1408ᵇ21–28, trans. [Freese 1926: 383]) about the form of prose composition: it should not be metrical, nor however without rhythm, "for that which is unlimited (ἄπειρον) is unpleasant and unknowable. Now all things are limited by number, and the number belonging to the form of diction is rhythm, of which the metres are the divisions".

[9] At least three other passages from the *Metaphysics* refer to mathematicals as matter, in one obliquely, in the others explicitly but in different senses. The first is M, 1078ᵃ29–31, where we find that the geometricians are right in claiming that what they discuss is something real (ὄντα); being (τὸ ὄν), indeed, is of two kinds – either entelechy, "full actuality" (which the mathematicals have just been argued not to be) or material (ὑλικῶς). The second is N, 1092ᵇ17–18, where it is said in connection with a discussion of the composition of flesh and

also confirmed by the context – the passage sums up a preceding discussion in which it is stated among other things [trans. Tredennick 1933: I, 355–359] that

> [1035^{a}9] This is why the formula of the circle does not contain that of the segments, whereas the formula of the syllable does contain that of the letters; for the letters are parts of the formula of the form [of the syllable]; they are not matter; but the segments are parts in the sense of matter in which the form is induced. [... 1035^{a}17] For even if the line is divided and resolved into its halves, or if the man is resolved into bones and ⟨sinews⟩ and flesh, it does not follow that they are composed of these as part of their essence but as their matter; and these are parts of the concrete whole, but not of the form, or that to which the formula refers. [... 1035^{a}26] All things which are concrete combinations of form and matter (*e.g.* "the snub" or the bronze circle) can be resolved into form and matter, and the matter is a part of them. [... 1035^{a}32] For this reason the clay statue can be resolved into clay, and the sphere into bronze, and Callias into flesh and bones, and the circle too into segments, because it is something which is combined with matter. For we use the same name the absolute circle and for the particular circle, since there is no special name for the particular circles.
>
> [... 1935^{b}7] But the formula of the right angle is not divisible into the formula of an acute angle, but *vice versa*; since in defining the acute angle we use the right angle, because "the acute angle is less than a right angle." It is the same with the circle and the semicircle; for the semicircle is defined by means of the circle. And the finger is defined by means of the whole body; for a finger is a particular kind of part of a man. Thus such parts as are material, and into which the whole is resolved as into matter, are posterior to the whole [...].

Semicircles, line segments and acute angles are certainly mathematicals (and the text points out explicitly not to speak exclusively of the sensible circles etc.). One sense in which mathematicals can be matter is thus as components of other mathematicals (this does not preclude that it may also be possible in other senses, but Aristotle does not tell how). That the reflections are really concerned with mathematicals and not with sensible shapes alone – but with *individual*, not universal mathematicals – is confirmed by a later passage from the book, namely 1036^{b}33–1037^{a}5 [trans. Tredennick 1933: I, 367–369]:

> And with respect to ⟨the mathematicals⟩, why are the formulae of the parts not parts of

---

bone from the elements in a specific ratio (see p. 3) that "clearly, numbers are neither substance not the cause of forms; the ratio, indeed, is the substance, and the number matter". The third is H, 1045^{a}34–36 [trans. Tredennick 1933: I, 425], "some matter is intelligible and some sensible, and part of the formula is always matter and part actuality; *e.g.*, the circle is a plane figure". The last passage clearly corresponds to the view of a genus as a substrate and a kind of quasi-matter which, when receiving *differentiae specificae* functioning as quasi-form, brings forth the more substantial species (*Metaphysics* Δ, 1016^{a}24–32; Z, 1039^{a}24–27), and thus speaks of the universal circle; the ratio, in contrast, appears as a parallel to the individual man, and must thus be seen as an individual ratio between particular numbers (which on their part correspond to the bodily parts).

the formulae of the whole; *e.g.*, why are the formulae of the semicircles not parts of the formulae for the circle? for they are not sensible. Probably this makes no difference; because there will be matter even of some things which are not sensible. Indeed there will be matter in some sense in everything which is not essence or form considered independently, but a particular thing. Thus the semicircles will be parts not of the universal circle but of the particular circles, as we said before [namely in the passage quoted above] – for some matter is sensible, and some intelligible.

Metaphysics M, 1077ᵃ33–36 asks in what sense lines (referred to as the primary generators of mathematicals) can be substances.[10] Not by being "some kind of form or shape" (εἶδος καὶ μορφή τις) like the soul, nor as matter, as the body is; indeed, "it certainly does not appear that [anything] can be composed of lines or planes or points", which would be the case if lines were a kind of material substance. Superficially read, the passage looks like a rejection of the view of mathematicals as intelligible matter. Aristotle's own inference, however, is that lines cannot be "material substance" for (potentially animate, and thus sensible) *bodies*, as are flesh and bone for the living being. When summarizing, he concludes (1077ᵇ12, trans. [Tredennick 1933: I, 187])

(a) that the ⟨mathematicals⟩ are not more substantial than corporeal objects; *(b)* that they are not prior in point of existence to sensible things, but only in formula; and *(c)* that they cannot in any way exist in separation.

This is thus a rejection of Pythagorean and related derivations of sensible reality from the mathematicals, and has no bearing on the status of mathematicals as intelligible matter for other intelligible entities (mathematicals, or whatever else might be imagined).

A discussion in book B is part of the same confrontation. Already in book A, 987ᵇ15–18 [trans. Tredennick 1933: I, 45] we find that the mathematicals not only constitute a separate category along with forms and sensible things for Plato, but that they stand between (μεταξύ) these, differing "from sensible things in being eternal and immutable, and from the Forms in that there are many similar ⟨of them⟩, whereas each Form is itself unique". In book B, 997ᵇ13–998ᵃ19 [trans. Tredennick 1933: I, 113–117] Aristotle makes it clear that he finds the notion absurd:

[...] if anyone posits Intermediates distinct from Forms and sensible things, he will have many difficulties; because obviously not only will there be lines apart from both Ideal [τ' αὐτάς] and sensible lines, but it will be the same with each of the other classes. Thus since astronomy is one of the mathematical sciences, there will have to be a heaven besides the sensible heaven, and a sun and moon, and all the other heavenly bodies.

[10] This follows shortly after a cross-reference to the discussion of the impossible independent existence of mathematicals in book B to which we shall turn imminently. The cross-referencing between books A, B, Z and M ensures that it is legitimate to combine the evidence they offer.

But how are we to believe this. Nor is it reasonable that the heaven should be immovable; but that it should move is utterly impossible.[11] It is the same with the objects of optics and the mathematical theory of harmony; these too, cannot exist apart from sensible objects. Because if there are intermediate objects of sense and sensations, clearly there will also be animals intermediate between the Ideal animals and the perishable animals.

One might also raise the question with respect to what kind of objects we are to look for these sciences. For if we are to take it that the only difference between mensuration and geometry is that the one is concerned with things which we can perceive and the other with things which we cannot, clearly there will be a science parallel to medicine (and to each of the other sciences), intermediate between Ideal medicine and the medicine which we know. Yet how is this possible? for then there would be a class of healthy things apart from those which are sensible and from the Ideally healthy. Nor, at the same time, is it true that mensuration is concerned with sensible and perishable magnitudes; for then it would perish as they do. Nor, again, can astronomy be concerned with sensible magnitudes or with this heaven of ours; for as sensible lines are not like those of which the geometrician speaks (since there is nothing which is straight and curved in that sense; the [sensible] circle touches the ruler not at a point but [along a line] as Protagoras used to say in refuting the geometricians), so the paths and orbits of our heaven are not like those which astronomy discusses, nor have the 〈points〉 [σημεῖα] of the astronomer the same nature as the stars.

Some, however, say that these so-called Intermediates between Forms and sensible objects do exist: not indeed separately from the sensibles, but in them. It would take too long to consider in detail all the impossible consequences of this theory, but it will be sufficient to observe the following. On this view it is not logical that only this should be so; clearly it would be possible for the Forms also to be in sensible things; for the same argument applies to both. Further, it follows necessarily that two solids must occupy the same space; and that the forms cannot be immovable, being present in sensible things, which move. And in general, what is the object of assuming that Intermediates exist, but only in sensible things? The same absurdities as before will result: there will be a heaven besides the sensible one, only not apart from it, but in the same place; which is still more impossible.

Various observations beyond the rejection of Intermediates can be made regarding this passage. Firstly, that much of the argument is built around "the more physical of the mathematical sciences" (τὰ φυσικώτερα τῶν μαθημάτον – *Physics* II, 194^{a}8): astronomy, optics and harmonics. Since Aristotle does ascribe to these a particular status among the *mathemata,* he might have discarded them as not directly relevant, which he fails to do. Secondly, the parallel to medicine strengthens the impression that the relation between the mathematical sciences (and thus also the mathematicals) and the sensible world is understood to be fundamental. Thirdly, that Aristotle feels entitled to assume that the presence of intelligible mathematicals *in* sensibles assumed by some of those against whom he argues is meant spatially. *Perhaps* we should

[11] Namely because movement belongs solely with the sensible, and thus not with a merely intelligible intermediate heaven.

understand this “presence in” as an identification of the intelligible sphere with the actual surface of the bronze sphere. That such an identification is in any case *not* accepted by Aristotle is clear from the endorsement of what Protagoras says about the actual comportment of lines and circles understood as external limits of sensible rulers and wheels. The relation between mathematicals and sensibles, though indubitably pivotal, is no such simple connection. Nor is it a Platonic “participation” (μέθεξις), since this is said in book A, 987^{b}13–14 to be nothing but a formulation in other words of the Pythagorean view of reality as an imitation (μίμησις) of numbers[12] (deprived moreover of clear meaning), and in 991^{a}22–23 to be “empty talk and poetical metaphors”.

These scattered observations should allow us to approach the main investigation of the possible existence of mathematicals as unchangeable and eternal substance, which is undertaken in *Metaphysics* M, chapters I–III – the chapters on which Edward Hussey’s fairly recent discussion [1992] of Aristotle’s view of the mathematicals concentrates. Many points made in these chapters repeat what we have already discussed (and a few were already quoted) – thus the (cross-referenced) dismissal of the Platonist and similar theses about the status of Forms and mathematicals as independent substances either separate from sensibles or to be found “in” them.

Before discussing what is said we should take note of what is *not* said. Even in these chapters, the mathematicals are never told to be substances;[13] however large the range of categories that on one or the other occasion are designated thus (be it in a secondary or lesser sense, as with universals, forms and matter), this range is not large enough to encompass numbers, ratios and circles. This is made explicit (concerning the special case of lines) in chapter II, 1077^{a}32–36 [trans. Tredennick 1933: II, 185–187] – (cf. p. 8):

---

[12] In N, 1090^{a}21–24 [trans. Tredennick 1933: II, 277] we find the more familiar version of the Pythagorean doctrine – namely that the Pythagoreans,

> observing that many attributes of numbers apply to sensible bodies, assumed that real things are numbers; not that numbers exist separately, but that real things are composed of numbers. But why? Because the attributes of numbers are to be found in a musical scale, in the heavens, and in many other connections.

Since the account in book A is specific and refers to a precise terminology (“μίμησις”), we may take it to render something which at least some Pythagoreans would sometimes say, if not necessarily to express a generally held Pythagorean view of the nature of number and material reality.

[13] The only exception I have found is *Metaphysics* M, 1092^{b}13, where *the ratio* – the least substantial of Aristotle’s mathematicals, we might say – is said to be the οὐσία and number its matter; but this is probably too particular to allow the conclusion beyond the ensuing refusal of substantial status *for number*.

> [...] body is a kind of substance, since it already in some sense possesses completeness (τέλειος); but in what sense are lines substances? Neither as being a kind of form or shape, nor as being body, like the body; for it does not appear that anything can be composed of either lines or of planes or of points, whereas if they were a kind of material substance it would be apparent that things can be so composed.

The refusal to classify the mathematicals as substances even in the most diluted sense already locates them at the same level as the accidental characteristics (πάθη) of substances proper, that is, of entities that have full and separate existence. And the text indeed goes on as follows (1077^{b}1–17, trans. [Tredennick 1933: II, 187]):

> Let it be granted that [lines etc.] are prior in formula; yet not everything which is prior in formula is also prior in substantiality. Things are prior in formula from whose formulae the formulae of other things are compounded. And these characteristics are not indissociable [οὐχ ἅμα ὑπάρχει]. For if attributes, such as "moving" or "white," do not exist apart from their substances, "white" will be prior in formula to "white man," but not in substantiality; for it cannot exist in separation, but always exists conjointly with the concrete whole [ἅμα τῷ συνόλῳ ἐστίν] – by which I mean "white man." Thus it is obvious that neither is the result of abstraction [ἀφαίρεσις, "taking away"] prior, nor the result of adding a determinant [πρόσθεσις[14]] posterior – for the expression "white man" is the result of adding a determinant to "white".
>
> Thus we have sufficiently shown (*a*) that the ⟨mathematicals⟩ are not more substantial than corporeal objects;[15] (*b*) that they are not prior in point of existence to sensible things but only in formula; and (*c*) that they cannot in any way exist in separation. And since we have seen that they cannot exist in sensible things, it is clear that either they do not exist at all, or they exist only in a certain way, and therefore not absolutely; for "exist" has several senses.

This does not assert that there is no difference between mathematicals and properties like colour, and probably it is not meant to convey that message; but if the argument is to possess any validity it must presuppose that they belong to ontologically comparable categories.

Chapter III gets closer to positive assertions about the mathematicals. Its beginning reads thus (1077^{b}18–1078^{a}9, trans. [Tredennick 1933: II, 187–191]):

> The general propositions in mathematics are not concerned with objects which exist separately apart from magnitudes and numbers; they are concerned with magnitudes and numbers, but not with them as possessing magnitude or being divisible.[16] It is

[14] Literally "putting unto", the opposite of "taking away" – no "determinant" is spoken of in the Greek text.

[15] The whole passage is indeed directed against the Pythagorean, Platonic and related persuasions.

[16] As examples of such "general propositions" we may first of all think of Euclid's "common notions"; but *Posterior Analytics* I (74^{a}18–25, trans. [Tredennick 1960: 51]) offers a more

clearly possible that in the same way propositions and logical proofs may apply to sensible magnitudes; not *qua* sensible, but *qua* ⟨being such⟩ [ᾗ τοιαδί]. For just as there can be many propositions about things merely *qua* movable, without any reference to the ⟨suchness⟩ [τῶν τοιούτων] of each one or their ⟨accidents⟩ [τῶν συμβεβηκότων], and it does not necessarily follow from this either that there is something movable which exists in separation from sensible things or that there is a distinct movable nature in sensible things; so too there will be propositions and sciences which apply to movable things, not *qua* movable but *qua* corporeal only; and again *qua* planes only and *qua* lines only, and *qua* divisible, and *qua* indivisible only. Therefore since it is true to say in a general sense not only that things which are separable but that things which are inseparable exist, *e.g.*, that movable things exist, it is also true to say in a general sense that ⟨the mathematicals⟩ exist, and ⟨indeed such as they [the mathematicians] say⟩. And just as it is true to say generally of the other sciences that they deal with a particular subject – not with that which is accidental to it (*e.g.* not with "white" if "the healthy" is white, and the subject of the science is "the healthy"), but with that which is the subject of the particular science; with the healthy if it treats of things *qua* healthy, and with man if *qua* man – so this is also true of geometry. If the things of which it treats are accidentally sensible, although it does not treat of them *qua* sensible, it does not follow that the mathematical sciences treat of sensible things – nor, on the other hand, that they treat of other things which exist independently apart from these.

Many ⟨accidents⟩ are essential [καθ'ἀυτὰ] properties of things possessing a particular characteristic; *e.g.*, there are ⟨affections⟩ [πάθη] peculiar to an animal *qua* female or *qua* male, although there is no such thing as female or male in separation from animals.[17] Hence there are also attributes which are peculiar to things merely *qua* lines or planes.

---

sophisticated instance, which is indeed taken by Heath [1949: 223] as the clearest example of the category:

> [...] the law that *proportionals alternate* might be supposed to apply to numbers *qua* numbers, and similarly to lines, solids and periods of time; as indeed it used to be demonstrated of these subjects separately. It could, of course, have been proved of them all by a single demonstration, but since there were no single term to denote the common quality of numbers, lengths, time and solids, and they differ in species from one another, they were treated separately; but now the law is proved universally; for the property did not belong to them *qua* lines or *qua* numbers, but *qua* possessing this special quality which they are assumed to possess universally.

[17] Richard Hope [1960: 275] understands this period differently:

> So there are many traits that things have because they are what they are: a living being has the attributes of being female or male, yet there is no female or male being separate from living beings.

Read in this way (which seen in isolation seems to agree better with the Greek text), the point would thus be analogous to the observation that bodies may have different shapes, but being bodies they need to have *some* shape. However, the following period (which Hope translates in the same vein as Tredennick, "there are peculiar attributes that things have when taken only as lengths or as planes") agrees better with Tredennick's reading.

This is remarkable not only for stating the legitimacy of an investigation which leaves out certain characteristics as irrelevant and concentrates on those which are the concern of the science in question *as if* these could occur in isolation but also for a remarkable symmetry between types of characteristics. In the "textbook version" of Aristotle's ontology, genuine substances come about when form is impressed upon matter; this provides these substances with necessary attributes. Besides these, they may receive other attributes which, however, are *accidental*; this is the basis for the distinction between explanations based on *causes*, answers to questions "why", and a mathematical description à la Archimedes or Ptolemy. In the present passage, as we see, it is supposed to depend on the perspective of the investigation – whether a natural object is examined by physics or geometry – which characteristics are accidental and which not. When contemplated by a geometer, the sensibility of a bronze sphere – its most fundamental characteristic according to the standard ontology – becomes an "accident".[18]

After a passage treating of the exactness of various types of knowledge (to which we shall return on p. 15), Aristotle closes the reflections on the abstraction from (literally, removal of) what is accidental according to the perspective of the science in question with these words (1078^{a}14–31, trans. [Tredennick 1933: II, 191–193]):

> The same principle applies to both harmonics and optics, for neither of these sciences studies objects *qua* sight or *qua* sound, but *qua* lines and numbers; yet the latter are affections peculiar to the former. The same is also true of mechanics.
>
> Thus if we regard objects independently of their ⟨accidents⟩ and investigate any aspect of them as so regarded, we shall not be guilty of any error on this account, any more than when we draw a diagram on the ground and say that a line is a foot long when it is not; because the error is not in the premisses. The best way to conduct an investigation in every case is to take that which does not exist in separation and consider it separately; which is just what the arithmetician or the geometrician does. For man, *qua* man, is one indivisible thing;[19] and the arithmetician assumes man to be one indivisible thing,[20] and then considers whether there is any ⟨accident⟩ of man *qua* indivisible. And the geometrician considers man neither *qua* man nor *qua* indivisible, but *qua* something solid [στερεόν]. For clearly ⟨what⟩ would have belonged to "man"

---

[18] As a matter of fact, the same dependence on the perspective of what is accidental is asserted in book A, 980^{b}20. Here it is accidental that Socrates is a man if the observation is made by a physician. The incipient shift of the distinction between the necessary and the accidental from ontology to epistemology is thus no mere effect of the need to rationalize the existence of mathematics in *Metaphysics* M.

[19] If divided (mentally or by a butcher's saw), indeed, man is no longer man but a plurality of bodily parts (either conceptually or in reality). Cf. p. 3.

[20] Namely when identifying a collection of *n* persons with the number *n*. We remember the (Pythagorean and later) definition of number as a "collection of units".

> even if man were somehow not indivisible can belong to man irrespectively of his humanity or indivisibility. Hence for this reason the geometricians are right in what they maintain, and treat of ⟨being things⟩ [ὄντα]; ⟨and being things⟩ exist. For ⟨being is twofold⟩, either complete reality or matterlike.[21]

A similar point of view is expressed in more detail in the same terminology in book I, 1061^{a}29*ff* [trans. Tredennick 1933: II, 67–69]:

> [...] the mathematician makes a study of abstractions (for in his investigations he first abstracts everything that is sensible, such as weight and lightness, hardness and its contrary, and also heat and cold and other sensible contrarieties, leaving only quantity an continuity – sometimes in one, sometimes in one, sometimes in two and sometimes in three dimensions – and their affections *qua* quantitative and continuous, and does not study them with respect to any other thing; and in some cases investigates the relative positions of things and the properties of these, and in others their commensurability or incommensurability, and in others their ratios; yet nevertheless we hold that there is one and the same science of all these things, viz. geometry) [...].

Two passages from other works should be drawn in at this point. One is *Posterior Analytics* I, 74^{a}39–74^{b}4 [trans. Tredennick 1960: 53]. In connection with the question how to attain universal knowledge when we have so far only achieved knowledge of particular cases (cf. note 16), Aristotle observes that

> the property of having angles equal to the sum of two right angles will apply to "bronze isosceles triangle"; and it will still apply when "bronze" and "isosceles" are removed

– but not if "triangle" is removed. We notice that the matter and the geometrical property are treated in full parallel, in agreement with the "matterlike" existence of mathematical properties and with the discussion of mathematicals as "intelligible matter" in *Metaphysics* Z.

The other is *Physics* II, 193^{b}24–36 [trans. Wicksteed & Cornford 1957: I, 119], which is often mentioned as a parallel to the discussion of abstraction or removal in *Metaphysics* M. Here we read that

> we have next to consider how the mathematician differs from the ⟨natural philosopher⟩ [φυσικός]; for natural bodies have surfaces and occupy spaces, have lengths and present points, all which are subjects of mathematical study. And then there is the connected question whether astronomy is a separate science ⟨or part of natural philosophy⟩; for if the student of nature is concerned to know what the sun and moon are, it were strange if he could avoid ⟨their accidents⟩; especially as we find that the writers on nature have, as a fact, discoursed on the shape of the moon and sun and ⟨on⟩ whether the earth and the cosmos⟩ is spherical or otherwise.
>
> ⟨The mathematician, then, also deals with these [i.e., surfaces, spaces, lengths, points], but not⟩ *qua* boundaries of natural bodies, nor ⟨does he consider the accidents indicated as accidents of such substances⟩. Therefore he separates them; for they are ⟨separable

[21] Cf. note 9.

according to thought from motions, and it makes no difference, nor does separation lead to any falsity〉.

As we might perhaps expect in a work dealing centrally with natural philosophy ("physics"), attributes belonging to "nature" and regarded by mathematics are treated on a less equal footing here than in *Metaphysics* M – appurtenance according to nature seems to be ontological, whereas the mathematicians' abstraction is epistemological, only performed "according to thought" (τῇ νοήσει) – and it is no longer produced by "removal" but by "separation".[22] From the absolute perspective of this work, the properties regarded by the mathematician – in geometry, the shape – remain "accidents". But geometrical shapes remain properties of physical bodies, properties that can be isolated from these by some process of abstraction, and they have no genuinely separate existence.

We must conclude that this is the general view of Aristotle regarding the status of the mathematicals; geometrical shape is a property of bodies, as number is a property of collections of units; even ratio is a ratio between numbers of something or quantities, and thus a property of a relation between such concrete collections or quantities; this is indeed stated in *Metaphysics* N, 1092^{b}18–23. His only vacillation is between the asymmetrical position of the *Physics* (which may also be that of the early *Metaphysics* N, though that is not sure) and the symmetrical stance of *Metaphysics* M: (i) the *Physics* distinguishes between an ontologically based suppression of the accidents of shape which may be undertaken by the natural philosopher, and the abstraction in thought from "physical" properties undertaken by the mathematician; (ii) *Metaphysics* M, in contrast, grounds removals neither on an ontological distinction between natural necessity and accidents of shape or number nor on individual thought, but on the distinctive view or approach *of the investigating science* – somehow intermediate between the absolute ontological necessity and (legitimate but still arbitrary) personal choice.

In terms of the distinction of *Metaphysics* Z, 1035^{b}33–1036^{a}12 (above, p. 5), this understanding of the mathematicals as properties of substances would first of all concern the individual circles and their kin, which would be properties of the drawn individual circles. The circle in the sense of λόγος or "what it is to be a circle" would then be a universal, similar to the "secondary substances" of the *Categories*.

[22] This difference of vocabulary is hardly random: a proper elimination of characteristics that are regarded as essential might sound too strange. Instead, the *Physics* (which according to cross-references is likely to be earlier than the *Metaphy*
sics-M passages) makes use of a terminology closer to the discussion of the Platonic stance.

This seems a reassuring conclusion,[23] but only until this passage from *Metaphysics* B returns to our mind (cf. fuller quotation on p. 10):

> Nor, again, can astronomy be concerned with sensible magnitudes or with this heaven of ours; for as sensible lines are not like those of which the geometrician speaks (since there is nothing which is straight and curved in that sense; the [sensible] circle touches the ruler not at a point but [along a line] as Protagoras used to say in refuting the geometricians), so the paths and orbits of our heaven are not like those which astronomy discusses, nor have the points of the astronomer the same nature as the stars.

If the circular and the straight are properties of the boundaries of physical bodies, merely regarded in abstraction from the physical properties of these bodies, how can that which a circle *regarded mathematically* has in common with a touching straight line be a mere point and that which they have in common *without mental removal* be a whole line segment?

The dilemma is not restricted to mathematics; it has a strict parallel in the difficulty which we encounter if we interpret an Aristotelian "nature" through the modern notion of a natural law. We may look at this passage from the *Politics* (1253^{a}2–5, trans. [Rackham 1944: 9]):

> [...] therefore it is clear that the city-state is ⟨by nature⟩, and that man is by nature a political animal, and a man that is citiless by nature and not merely by fortune is ⟨truly either a petty man or beyond measure⟩.

An Aristotelian nature is, indeed, a final cause, that after which the entity for which it is a cause strives, not necessarily that which it achieves. This is explained in *Physics* II, (199^{a}7–199^{b}26, trans. [Hardie & Gaye 1930]):

> [...] action for an end is present in things which come to be and are by nature.
>
> Further, where a series has a completion, all the preceding steps are for the sake of that. Now surely as in intelligent action, so in nature; and as in nature, so it is in each action, if nothing interferes. Now intelligent action is for the sake of an end; therefore the nature of things also is so. Thus if a house, e.g., had been a thing made by nature, it would have been made in the same way as now by art; and if things made by nature were made also by art, they would come to be in the same way as by nature. Each step then in the series is for the sake of the next; and generally art partly completes what nature cannot bring to a finish, and partly imitates her. If, therefore, artificial products

---

[23] Or perhaps not fully reassuring, since it can be argued to entail that the "abstraction of the mathematician ends up not with pure quantity but with quantified substance" [Anderson 1969: 17]. Therefore, as the sensible circle has the sensible bronze circle has its substrate, the mathematical circle (at least the individual circle) must have unspecified physical substance as its substrate and, in this sense, its matter. This would be a parallel to the view of a genus as quasi-matter and the *differentiae specificae* quasi-form (cf. note 9), and would therefore present us with no difficulties if only Aristotle had ever hinted at something of the kind. Unfortunately he never does but leaves it to Thomas Aquinas to make the point – cf. [Anderson 1969a].

> are for the sake of an end, so clearly also are natural products. [...].
>
> [...] And since 'nature' means two things, the matter and the form, of which the latter is the end, and since all the rest is for the sake of the end, the form must be the cause in the sense of 'that for the sake of which'.
>
> Now mistakes come to pass even in the operations of art: the grammarian makes a mistake in writing and the doctor pours out the wrong dose. Hence clearly mistakes are possible in the operations of nature also . If then in art there are cases in which what is rightly produced serves a purpose, and if where mistakes occurred there was a purpose in what was attempted, only it was not attained, so must it be also in natural products, and monstrosities will be failures in the purposive effort. [...]
>
> [...]
>
> [...] For those things are natural which, by a continuous movement originated from an internal principle, arrive at some completion: the same completion is not reached from every principle; nor any chance completion, but always the tendency in each toward the same end, if there is no impediment.
>
> [...] But when an event takes place always or for the most part, it is not incidental or by chance. In natural products the sequence is invariable, if there is no impediment.

Next follows a discussion of "necessity", where we read the following (200^a^15–19):

> Necessity in mathematics is in a way similar to necessity in things which come to be through the operation of nature. Since a straight line is what it is, it is necessary that the angles of a triangle should equal two right angles. But not conversely; though if the angles are *not* equal to two right angles, then the straight line is not what it is either.

Beyond what this passage asserts directly, it tells us that it is legitimate to think of the mathematical properties of things in parallel to those properties that constitute their "nature": not something which is necessarily so but as something which is so to the extent "there is no impediment". For further illustration, we may turn to the passage from *Metaphysics* M which was omitted between two quotations on p. 18 (1078^a^9–13, trans. [Tredennick 1933: II, 191]):

> And in proportion as the things which we are considering are prior in formula and simpler, they admit of greater exactness (ακριβής); for simplicity implies exactness. Hence we find greater exactness where there is no magnitude, and the greatest exactness where there is no motion; or if motion is involved, where it is primary, because this is the simplest kind; and the simplest kind of primary motion is uniform motion.

Since it has just been stated (1077^b^1, above, p. 12) "that [lines etc.] are prior in formula" without being thereby "prior in substantiality", it appears that mathematical properties of substances may indeed surpass "in exactness" those substances of which they are properties, just as these may be surpassed by their forms in perfection.[24]

---

[24] This point, and this analogy, is probably the explanation of what Hussey [1992: 125 n.40] regards as a "puzzling and isolated passage", namely *Posterior Analytics* I, 79^a^6–10 [trans. Tredennick 1960: 91]:

> Of this kind [viz, studied by more than one science] are all objects which, while having

Other sciences should study *the nature* of their object, not those shortcomings of individual specimens that are produced by accidental impediments that cripple this nature; in the same way, mathematics should deal with the mathematicals in their precision (on which condition they are only intelligible), not impeded by the shortcomings of the sensible substances of which they are properties.[25]

We may see this as an implied acceptance of the separate existence of ideal mathematicals; but we must recognize that this was not how Aristotle saw things, nor apparently the readers of his epoch.

Euclid's postulates: counterfactual existence claims?

The preceding scrutiny of Aristotle has shed some indirect light on the Pythagorean and Platonic views of the nature of the mathematicals. We might go further in this direction and look at later writings belonging to the Platonic and neo-Pythagorean currents – beginning with Nicomachos, according to whose *Introduction* (I.VI, trans. [Bertier 1978: 59*f*]) that number which orders everything by nature in the cosmos is able to do so because it is founded upon "le nombre préexistant dans la penseée du dieu créateur", and going on with other writers at various levels. One reason for not proceeding in this direction (apart from limitations of time and space) is that this esoterical number and its associates have little to do with the mathematicals that were treated in mathematics; as Aristotle observes in *Metaphysics* N, 1090^{b}27-35 [trans. Tredennick 1933: II, 281] about the particular "ideal numbers", "no mathematical theorem applies to them, unless one tries to interfere with the principles of mathematics and invent particular theories of one's own"; further, those who invented "two kinds of number, the Ideal and the mathematical as well, neither have explained nor can explain in any way how mathematical number will exist and of what it will be composed".

If we are interested in understanding ancient *mathematics*, it may therefore be more illuminating to confront the results of the preceding with the *Elements* – that

> a separate substantial existence, yet exhibit certain specific forms. For the mathematical sciences are concerned with forms; they do not confine their demonstrations to a particular substrate. Even if geometrical problems ⟨treat of⟩ a particular substrate, at least they do ⟨not do so *qua* treating of a substrate⟩.

Obviously, since the pertinence of "more than one science" is a consequence of the possession of the "specific forms" referred to, these "forms" cannot be the nature that give the objects in question their "substantial existence"; they (and thus the "forms" with which the mathematical sciences are concerned) must be something which shares fundamental characteristics of forms without being forms in the ontological sense. The passage should probably rather be read in the light of *Metaphysics* M than in continuation of *Physics* II.

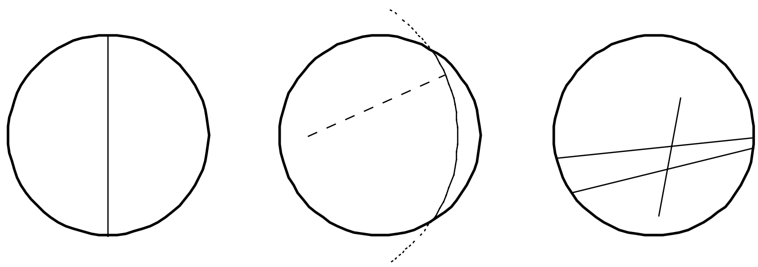
[25] We notice that this eliminates Anderson's dilemma as described in note 23.

ancient mathematical treatise which comes closest to the model for scientific work delineated in the *Posterior Analytics*. The point on which we shall focus is the set of postulates, things which are required (ed. [Heiberg 1883: I, 8], trans. [Heath 1926: 154]):

[1] ⟨It is requested⟩ to draw a straight line from any point to any point,
[2] ⟨and⟩ to produce a finite straight line continuously in a straight line,
[3] ⟨and⟩ to describe a circle with any centre and distance,
[4] ⟨and⟩ that all right angles are equal to one another,
[5] ⟨and⟩ that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

There is no doubt that definitions were familiar in mathematics already in Plato's and Aristotle's times; at least one of Euclid's common notions is also quoted by Aristotle as an example of the category of axioms. It is much less certain how far back the postulates should be dated, and how they relate to Aristotle's notions of hypotheses and postulates.[26] I shall therefore leave the question of chronology and inventor aside and just argue from the postulates as they are found in the text.[27]

A feature which as far as I have noticed has never been emphasized is that postulates 2–3 and 5 are counterfactual according to ancient standard cosmologies – including



Postulates 2–3 and 5 related to a finite cosmos.

[26] See the convenient summary in [McKirahan 1992: 133–137]. To this may be added, however, that the first postulate seems to have been known to Aristotle in a formulation close to what we find in the *Elements* – *Physics* III, 207b29–31 [trans. Hardie & Gaye 1930] explains that mathematicians "do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish".

[27] More precisely, in the Heiberg text and in most of the manuscripts. Some medieval manuscript traditions include a supplementary postulate, namely that two lines cannot include a space – thus "Adelard I", ed. [Busard 1983: 32]). Others make this a common notion – thus the translation due to Gerard of Cremona [ed. Busard 1984: 3]. The former variant is referred to by Simplicios in his commentary as quoted by al-Nayrīzī [ed., trans. Heiberg 1893: 25], who adds that it is not found in old manuscripts; the latter was known to Proclos (*In primum Euclidis Elementorum librum commentarii* 196$^{21–23}$, trans. [Morrow 1970: 154]), who points out that this is no notion common to more mathematical sciences but exclusively geometric.

Aristotle's.[28] If the cosmos is a sphere with a finite radius, it will still be possible to connect any two points with a straight line, since all points will have to fall within the sphere. But a line going from one pole of the firmament to the other cannot be produced; if we chose a point with a small distance δ from the firmament as our centre and $2R–2\delta$ as radius (R being the radius of the firmament and cosmos), then our attempted circle will be a short arc and no "figure" (something *enclosed* by a boundary or boundaries – def. 14) – or, if we take the firmament as part of the boundary, then two circles of the kind will not meet, and the demonstration of prop. 1 fails. Even the fifth postulate becomes untenable – actually, what we get (presupposing the adequate metric) is one of Felix Klein's models for non-Euclidean geometry.

This can be understood in one of three ways. We may assume that Euclid the mathematician rejected the prevalent cosmology of the philosophers, or followed a philosophy which held the cosmos to be without limit; we may look at the list of postulates as part of a Wittgensteinian language game; and we may try to make sense of it within the Platonic or Aristotelian framework.

Since we know nothing of Euclid as a person, the first possibility cannot be totally excluded; but this would at most explain why he wrote as he did, not why his work was so widely accepted and called forth no objections from later commentators on this account – neither from Ptolemy, who according to Proclos was critical of taking the fifth postulate as a postulate, nor from Proclos the Platonist himself, nor from Simplicios the Aristotelian. We may add that the most renowned infinite cosmology – that of Epicuros – was coupled to an understanding of geometry that was probably much too naive to appeal to anybody versed in mathematics.

Remain the Wittgensteinian (or, better perhaps, "naive-Wittgensteinian") interpretation, and the correlation with Aristotelian and Platonic views.

An appeal to the notion of a language game is less out of the way than it might seem at first. It is an old observation that several of Euclid's definitions define little.To explain a point as "that which has no part" might identify it, for instance, with Parmenides's, Speusippos's or Plotinus's *One* or with Allah (as was indeed done

[28] Simplicios [ed., Latin trans. Besthorn & Heiberg 1898: I, 15] comes so close that he would certainly have mentioned the cosmological problem if it had been familiar or had come to his own well-trained Aristotelian mind. In his commentary to the principles of the *Elements* he refers both to the common-sense protest that one cannot continue a line over the sea (apparently thinking on drawings made on the Rhodian or some other shore) and the philosophical objection that "the infinite does not exist", ascribing both to what is "commonly held" (the philosophical objection is of course somewhat off the point, since the postulate does not ask for this actual but only for the potential infinite – cf. also the quotation from *Physics* III quoted in note 26).

by medieval Islamic theologians). But if we already know that we are practising geometry, it becomes meaningful to explain that a dot (be it στιγμή, "a mark", be it σημεῖον, "a sign") is henceforth a dot so small that it has no parts, and that a stroke (γραμμή) is henceforth presupposed to be a length so narrow that it has no breadth at all; as Aristotle points out (*Topics* VI, 143^{b}28–29), this definition presupposes that there are lengths without as well as lengths provided with a width.

We may look at the postulates in the same light, and say that they provide the foundation that is indispensable if we want to play the game of geometry (that is, play it in agreement with the intuitions that are either more or less inborn or produced by a century's professional practising of the discipline).[29] Since it is doubtful whether Aristotle knew these postulates, they seem indeed to have been introduced as things which one *has* to require if the game is to be played.

In the case of the fifth postulate, it is rather obvious from the sources that this is what had happened. In *Prior Analytics* II, 64^{b}34–65^{a}9 [trans. Tredennick 1938: 485–487], Aristotle discusses circular reasoning in deductive systems – proving A from B, and B from Γ, even though Γ is itself only accepted because it is a consequence of A. As an example he refers to

> those persons who [...] think that they are drawing parallel lines; for they do not realize that they are making assumptions which cannot be proved unless the parallel lines exist.

The use of the present tense "do" (ποιοῦσιν) shows that this was still a flaw of geometry as practised at least by some of Aristotle's contemporaries. Aristotle also tells the way out: to take as an axiom (αξιόω) that which is proposed; but since he does not tell that this is done in geometry, nor *a fortiori* how the axiom or postulate should run, we may be fairly sure that no such decision had gained general acceptance.

A similar process seems to lie behind postulate 4. The so-called "geometric algebra" of *Elements* II.1-10 appears to be a "critical" re-elaboration of a set of geometrical riddles that had circulated among Near-Eastern surveyors since c. 2000 BCE, and was solved by them by a "naive" cut-and-paste geometry, in which the correctness of procedures could be "seen" immediately.[30] Here, as in pre-Greek surveying in general, the concept of a quantified angle did not exist; obviously, Babylonian as well as Egyptian surveyors distinguished practically right-angled from

---

[29] In Cicero's words (*Academica* II.116, trans. [Rackham 1933: 617]), they are "first principles of mathematics which must be granted before [the geometricians] are able to advance an inch". Simplicios answers similarly to the "commonly held" objections (see note 28).

[30] The best known reflection of this "surveyors' proto-algebra" is the Old Babylonian school algebra, whose geometric character is argued in [Høyrup 1990]. The relation to the geometry of *Elements* II is examined in [Høyrup 1998].

acute and obtuse corners, but all evidence at hand suggests that their understanding of the matter can be summed up in a pun: for them, a "right angle" was the opposite of a "wrong angle", that is, an angle whose legs did not determine an area.

Elements II.1–10 is a theoretical "critique" of the procedures and solutions in the sense that they put them on a secure foundation, proving the equality of areas instead of "moving them around". In order to do this, the author or authors of these proofs (probably working in the outgoing fifth century BCE) had to clarify their notion of angles. According to the *Republic* (510C), the concepts of the right, the obtuse and the acute angle were at first taken as self-evident and above discussion (and perhaps in no need of definition); but in *Metaphysics* Z, 1035^{b}8–9 and M, 1084^{b}7 we see that Aristotle already knew something like the definitions given in the *Elements* (quoting in the former passage the definition of the acute from the right angle in words that are almost the same as Euclid's, see above, p. 8). That the equality of all right angles does not follow from the definition is likely to be a secondary discovery, certainly not yet known to Plato and probably not to Aristotle; but at some point between Plato and Euclid the fallacy has been noticed and induced somebody to require explicitly that all right angles be equal, since the game could not be played otherwise.

A "Wittgensteinian" interpretation thus makes sense, and does not require that we ascribe to Euclid and his contemporaries any familiarity with the *Philosophical Investigations*. All that is asked for is that intellectual autonomy of a particular profession or scientific practice of which Aristotle's view of separate sciences with each its own principles is one rationalization, and the language-game formulation another. The discrepancy between what Euclid really does and what is prescribed by Aristotle is sufficiently large to warrant a rather a-philosophical interpretation of the Euclidean text – Euclid, in other words, may have regarded the conflict between the postulates and cosmology as a problem for cosmologists alone. In terms of "existence", such a view would be compatible with a view close to what is regarded today as "naive Platonism" in didactical discussion – *of course* the objects of mathematics exist (if somebody should get the quaint idea to ask), and *of course* they are ideal and not to be identified with the diagrams we draw and the collections of material objects which we amass.

Though compatible with the "Wittgensteinian" explanation of the use of apparently counterfactual postulates, however, this a-philosophical stance does not follow from it. No less compatible is the Aristotelian view as interpreted above, in particular in the symmetrical version of *Metaphysics* M. In such a view, the cosmological constraints might be seen as "impediments" which would interfere with the full unfolding of the essence of lines ("what it means to be a line"), circles, etc. The counterfactuality of the postulates would only concern the material objects of which the mathematicals are properties, and lines and circles might therefore

with *Physics* II that what goes on is a mental separation, there is no hint in Aristotle's formulations that this separation could have made in a way that resulted in a different mathematics, once we have chosen to concentrate on *mathematical* properties and to disregard the rest.[31]

It is certainly possible to project part of the "doctrine of odd and even" of *Elements* IX.21–34 onto an arithmetic modulo 2. But there is, firstly, not a single word in the sources to suggest that the ancients saw this sequence of theorems as "an (alternative) arithmetic" and not simply as a part of arithmetic; secondly, some of the theorems in question are pointless if not meaningless if understood through this projection.[32] To see this as an "arithmetic constructed by the Pythagoreans", as done by Imre Toth [1998: 166*f*], is thus misleading if meant as an interpretation of ancient thought.[33]

Those who doubted that mathematics as known was compulsory seem to have rejected theoretical mathematics wholesale, not just *this* mathematics. In the above quotations from Aristotle, we have already encountered two instances (both directed against geometry): Protagoras's appeal to the sensible line and circle, and the censure of geometers who claim that a line drawn on the ground is one foot long even if it is not.[34] In Cicero's *Academica* (a convenient compendium of post-Platonic scepticist

---

[31] More precisely, since mathematical sciences are plural: once we have chosen the perspective of arithmetic, only one arithmetic can result; and if we apply that of geometry, the geometry that results is equally predetermined. Cf. *Metaphysics* M as quoted on p. 15 (1078^{a}24–26, trans. [Tredennick 1933: II, 193]),

> the arithmetician assumes man to be one indivisible thing, and then considers whether there is any accident of man *qua* indivisible. And the geometrician considers man neither *qua* man nor *qua* indivisible, but *qua* something solid.

[32] Thus proposition 30 [trans. Heath 1926: II, 417], "If an odd number measure an even number, it will also measure the half of it" – which translates into "if 1 measure 0 [which it always does], it will also measure half of it [which can be 1 as well as 0]). Even more absurd, of course, is the imposition of a concept of "0(mod. 2)" onto a thinking which does not know "0".

[33] Toth's statement is made in connection with a broad argument meant to demonstrate that Plato and Aristotle were aware of the possibility to construct a consistent non-Euclidean geometry (which, as he sees it, they rejected on other grounds, thus assimilating their position to that of Saccheri and not that of Bolyai and Lobachevsky). Unfortunately for the thesis (set forth by Toth in a number of publications since 1966), the textual evidence on which it is based turns out on close scrutiny to be either misquoted or, at best, interpreted freely and out of context – see [Høyrup 2000] (reproduced below as appendix).

[34] Since the point is made both in *Metaphysics* M, 1078^{a}19–20 and in *Metaphysics* N, 1089^{a}23–26 (probably written at a considerably earlier moment), it seems likely that somebody had

opinions) we find that Epicurus held all geometry to be false and the sun to be no larger than it looks (II.106 and II.82, respectively); Cicero himself (arguing as a spokesman of the scepticism of the post-Platonic Old Academy) accepts the compelling force of geometrical reasoning.

All in all the ancients were thus convinced that mathematics, to the extent it was at all accepted as a separate field of theoretical study, was one and compelling (apart from its split into distinct disciplines). If this absence of free constructibility is what we really mean by asking for existence, mathematics was certainly held to exist.

---

formulated this objection in earnest.

Appendix: Review of

Imre Toth, *Aristotele e i fondamenti assiomatici della geometria. Prolegomeni alla comprensione dei frammenti non-euclidei nel «Corpus Aristotelicum» nel loro contesto matematico e filosofico.* (Temi metafisici e problemi del pensiero antico. Studi e testi, 56). Milano: Vita e Pensiero, ²1998.

Forthcoming in *Zentralblatt für Mathematik und ihre Grenzgebiete*.

From 1966 onward, Imre Toth argued in a number of publications that the mathematicians at Plato's Academy tried to prove, first directly and next indirectly, an equivalent to Euclid's fifth postulate (namely *Elements* I.29, cf. below). In this way, as he saw it, a counterpart of Saccheri's quasi-non-Euclidean geometry was created – a coherent deductive chain of propositions based on the remaining postulates and axioms and on the negation of the (equivalent of the) fifth postulate. He mainly argued from a set of passages in the Aristotelian corpus, which in his reading showed that a whole body of theorems belonging to such a chain was known.

This not fully unprecedented but still revolutionary thesis was mostly received with taciturn reservation, for which reason Toth is now publishing a book size discussion of the Aristotelian passages (which may have appeared, but which the reviewer has not seen); the volume under review is a kind of broad prolegomenon and philosophical commentary to the matter. The book falls in two parts, the first of which is meant as a survey of the origin of the search for axiomatics at the Academy (referring mostly to Platonic and Aristotelian texts and organized around the non-Euclidean problem); the second part is a protracted essay located in the boundary region between the philosophy of mathematics (centred on Euclidean, non-Euclidean and absolute geometry) and non-chronological history.

Unfortunately, the book is unconvincing when submitted to a close reading and checked against the sources. This follows in part from the essay style, where misquotations, reformulations and oblique allusions to the sources outweigh precise references, in part from what the reviewer cannot help seeing as distorted interpretations. Many of the philosophical reflections and observations are stimulating; however, the philosophical stance, in as far as it can be safely extricated from the poetical apparel, seems inconsistent. For reasons of space, a few illustrations of these objections will have to suffice (all quotations from the book and from Greek sources are translated into English by the reviewer).

1. On p. 585 (and already, slightly less sharply, on p. 564), it is stated that Aristotle gives *absolute* priority to the object which is known over the knowledge about the object. The actual claim of the passage referred to (*Categories* $7^{b}23$–27) is that the object will *generally* have ontological priority, and that our knowledge will come into being

together with its object "in few or no cases".
2. *Elements* I.29 is referred to repeatedly, often obliquely; however, the first time its content is explained (p. 100) it misquoted as "if two straight lines are parallel, then they are co-orthogonal". Actually, the proposition deals with the angles that are produced if a pair of parallels is cut by a third line; co-orthogonality follows without difficulty, but is not mentioned at all by Euclid. As a result, the discussion of a purported fallacy in the proof (rejection of the elliptic geometry but not of the hyperbolic possibility) on pp. 465*ff* is wholly off the point – all Euclid does when two symmetrically located and thus equivalent angles are supposed to be unequal is to assume that a specified one of the two is taken to be larger.
3. The supposedly most striking proof that Aristotle knew about (quasi-)non-Euclidean geometry is a passage from the *Eudemian Ethics* (1222^{b}15–37, repeated in the post-Aristotelian epitome *Magna moralia* 1187^{a}36–^{b}3). It is asserted (in these words on p. 584, but equivalently elsewhere) that "in order to illustrate the concept of preferential choice, Aristotle does not cite an example drawn from the domain of ethical or political praxis. Unexpectedly, even surprisingly: the only example comes from the domain of geometry. And it is the *alternative* between a *Euclidean* and a *non-Euclidean* triangle". What actually goes on is very different: Aristotle wants to illustrate in a simple way (referring for deeper explanation to the *Analytics*) the relation between basic principles (*archai*) and their consequences. The example is that the sum of the angles in a quadrangle (4 right angles) is a consequence of the sum of the angles of the triangle; if the latter were to change (*metaballō̄* – thus *not* "if it were different"), for instance into three right angles, then even the former would change (*viz* into 6 right angles).

This geometric observation is in need of no axiomatic network; it follows from the drawing of a diagonal in the quadrangle. Aristotle does not explain this, but obviously expects the derivation to be something simple which his audience (not familiar with the technicalities of the *Analytics*) understands. Moreover, is it explained that *if* the sum π of the angles of the triangle did not follow from other reasons (which it is thus supposed to do), then this would have the role of a first principle.
4. On p. 526 it is stated that *Metaphysics* 1052^{a}4–7 asserts that "*Euclidicity* and *non-Euclidicity* are invariant properties, immutable, of each its own universe, since it cannot happen that one triangle be Euclidean and another non-Euclidean or – to speak in terms of time, which anyhow brings the same result – that a triangle may sometimes be Euclidean and sometimes non-Euclidean". What Aristotle actually says is simply that "if we assume that the triangle does not change, then we shall not assume that at some times it possesses two right angles and at some times not (for this would mean that it changed)".

Many other examples are of the same kind. Several chapters are spun over the assumption that Plato's *Cratylus* deals with the question whether the internal coherence of non-Euclidean geometry suffices for making it true (supposed to be what Cratylus really means when claiming that the sounds or letters of words determine their meaning), or mathematical truth has to be guaranteed in a different way (assumed to be what Socrates means to demonstrate when destroying Cratylus's phono-semantics by counterexamples).

One passage remains which to a first reading might seem to lend some support to the thesis, namely *On the Heavens* 281^{b}2–7, which is taken on p. 109 to "assert the existence of squares with commensurate diagonal" (similarly *passim*), and on p. 539 to present the impossibility "of a triangle to have the sum of the angles equal to 2R" as an example of a merely "hypothetical impossibility". But even though this agrees with current translations, the non-Euclidean implications are dubious. The passage distinguishes things that are false *haploos*, "taken in isolation", from those which are false "from a foundation" (*ex hypotheseōs*); the most plausible reading of the passage is thus simply that the incommensurability of the diagonal and the sum of the angles of the triangle are not independent or primary facts but consequences of prior principles (cf. what was said above on *Eudemian Ethics* 1222^{b}15–37).

It remains that *Prior Analytics* 65^{a}4–9 criticizes "those who suppose they draw parallels" using the sum of the angles of the triangle, which sum on its part is only established on the assumption that parallels can be drawn; and that Plato (*Republic* 533C) values mathematics less than dialectic in the education of the guardians because the reasoner in mathematics does not understand the starting point or *archē*, while dialectic is supposed to get beyond this kind of unproved first principle or "hypothesis"; but control of Toth's impressive body of textual hints and references left the reviewer unconvinced that this indubitable awareness of the conditions of axiomatic thinking (which Toth is not the first to recognize) led to the creation of any kind of quasi-non-Euclidean geometry.

As to the apparent philosophical inconsistencies (which may however be mere consequences of polemically intended eclecticism), one example shall suffice. Mostly, "Euclidean and non-Euclidean knowledge" are supposed to possess no truth value within the absolute geometry encompassing both, to "describe no preexisting object", and to constitute only "linguistic objects" once articulated (in these words p. 564); but a passage blaming Aristotle for not understanding that auxiliary lines that can be added to a diagram in a proof are present in the diagram actually, not only potentially (p. 520) asserts that all auxiliary lines and (in less clearcut words) all geometric figures are timelessly present in "actual being"; since the auxiliary line in question cannot be constructed in an elliptic geometry, this claim of absolute existence must concern the three geometries separately. (Most likely, a precise formulation of the vague statement would entail paradoxes similar to those familiar from set theory).

References

Anderson, Thomas C. 1969. “Intelligible Matter and the Objects of Mathematics in Aristotle”. *The New Scholasticism* **43** (1969), 1–28.

Anderson, Thomas C. 1969a. “Intelligible matter and the Objects of Mathematics in Aquinas”. *The New Scholasticism* **43** (1969), 555–576.

Bertier, Janine (ed.trans.), 1978. Nicomachus Gerasenus, *Introduction arithmétique*. (Histoire des doctrines de l’antiquité classique, 2). Paris: Vrin.

Besthorn, R. O. & J. L. Heiberg (eds), 1893. *Codex Leidensis 399, 1. Euclidis Elementa ex interpretatione al-Hadschdschadschii cum commentariis al-Narizii*. Arabice et latine. 3 vols. København: Gyldendalske Boghandel, 1893–1932.

Bonitz, Hermann, 1955. *Index aristotelicus*. Berlin: Akademie-Verlag ([1]1870).

Bury, R. G. (ed. trans.), 1929. Plato, *Timaeus. Critias. Cleitophon. Menexenus. Epistles*. (Loeb Classical Library). Cambridge, Mass.: Harvard University Press / London: Heinemann.

Busard, H. L. L. (ed.), 1983. *The First Latin Translation of Euclid’s ‘Elements’ Commonly Ascribed to Adelard of Bath*. (Studies and Texts, 64). Toronto: Pontifical Institute of Mediaeval Studies.

Busard, Hubert L. L. (ed.), 1984. *The Latin Translation of the Arabic Version of Euclid’s* Elements *Commonly Ascribed to Gerard of Cremona*. (Asfār, 2). Leiden: Brill.

Cook, Harold P. (ed., trans.), Aristotle, *The Categories*, *in* Harold P. Cook & Hugh Tredennick (eds, trans.), 1938. Aristotle, *The Categories. On Interpretation. Prior Analytics*. (Loeb Classical Library). London: Heinemann / Cambridge, Mass.

Freese, John Henry (ed. trans.), 1926. Aristotle, *The “Art” of Rhetoric*. (Loeb Classical Library). Cambridge, Mass.: Harvard University Press / London: Heinemann.

Friedlein, Gottfried (ed.), 1873. Procli Diadochi *In primum Euclidis Elementorum librum commentarii*. Leipzig: Teubner.

Hardie, R. P. & R. K. Gaye (eds, trans.), 1930. Aristotle, *Physica*, *in* Aristotle, *Works* (ed. W. D. Ross), vol. II. Oxford: Clarendon Press.

Heath, Thomas L. (ed. trans.), 1926. *The Thirteen Books of Euclid’s Elements*, Translated with Introduction and Commentary. 2nd revised edition. 3 vols. Cambridge: Cambridge University Press / New York: Macmillan ([1]1908).

Heath, Thomas L. 1949. *Mathematics in Aristotle*. Oxford: Oxford University Press.

Heiberg, J. L. (ed. trans.), 1883. Euclidis *Elementa*. 5 vols. (Euclidis Opera omnia, vol. I-V). Edidit et latine interpretatus est I. L. Heiberg. Leipzig: Teubner, 1883–1888.

Hett, W. S. (ed. trans.), 1936. Aristotle, *On the Soul. Parva Naturalia. On Breath*. (Loeb Classical Library). Cambridge, Mass.; Harvard University Press / London: Heinemann.

Hope, Richard (ed. trans.), 1960. Aristotle, *Metaphysics*. With an analytical index of technical terms. Ann Arbor: University of Michigan Press / Rexdale, Canada: John Wiley.

Høyrup, Jens, 1990. “Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought”. *Altorientalische Forschungen* **17**, 27–69, 262–354.

Høyrup, Jens, 1998. “On a Collection of Geometrical Riddles and Their Role in the Shaping of Four to Six ‘Algebras’”. *Filosofi og Videnskabsteori på Roskilde Universitetscenter*. 3. Række: *Preprints og Reprints* 1998 Nr. 2. Forthcomining *Science in Context* (2001).

Høyrup, Jens, 2000. [Review of Imre Toth, *Aristotele e i fondamenti assiomatici della geometria. Prolegomeni alla comprensione dei frammenti non-euclidei nel «Corpus Aristotelicum» nel loro contesto matematico e filosofico*. (Temi metafisici e problemi del pensiero antico. Studi e testi, 56). Milano: Vita e Pensiero, [2]1998]. Forthcoming in *Zentralblatt für Mathematik und ihre Grenzgebiete*.

Hussey, Edward, 1992. “Aristotle on Mathematical Objects”, pp. 105–133 *in* I. Mueller (ed.), *Peri Tōn Mathēmatōn*. Edmonton, Alberta: Academic Printing and Publishing. (= *Apeiron* **24**:4 (1991))

McKirahan, Richard D. Jr., 1992. *Principles and Proofs: Aristotle’s Theory of Demonstrative Science*. Princeton: Princeton University Press.

Morrow, Glenn R. (ed. trans.), 1970. Proclus, *A Commentary on the First Book of Euclid’s Elements*. Translated with Introduction and Notes. Princeton, New Jersey: Princeton University Press.

Mueller, Ian, 1971. “Aristotle on Geometrical Objects”. *Archiv für Geschichte der Philosophie* **52**, 156–171.

Rackham, H. (ed. trans.), 1933. Cicero, *De natura deorum*. *Academica*. With an English Translation. (Loeb Classical Library). Cambridge, Mass.: Harvard University Press / London: Heinemann.

Rackham, H. (ed. trans.), 1944. Aristotle, *Politics*. (Loeb Classical Library). Cambridge, Mass.: Harvard University Press / London: Heinemann.

Toth, Imre, 1998. *Aristotele e i fondamenti assiomatici della geometria. Prolegomeni alla comprensione dei frammenti non-euclidei nel «Corpus Aristotelicum» nel loro contesto matematico e filosofico*. (Temi metafisici e problemi del pensiero antico. Studi e testi, 56). Milano: Vita e Pensiero ([1]1997).

Tredennick, Hugh (ed, trans.), 1933. Aristotle, *The Metaphysics*. 2 vols. (Loeb Classical Library). Cambridge, Mass.: Harvard University Press / London: Heinemann, 1933, 1935.

Tredennick, Hugh (ed., trans.), 1938. Aristotle, *Prior Analytics*, *in* Harold P. Cook & Hugh Tredennick (eds, trans.), 1938. Aristotle, *The Categories*. *On Interpretation*. *Prior Analytics*. (Loeb Classical Library). London: Heinemann / Cambridge, Mass.

Tredennick, Hugh, & E. S. Forster (eds, trans.), 1960. Aristotle, *Posterior Analytics* and *Topica*. (Loeb Classical Library). Cambridge, Mass.: Harvard University Press / London: Heinemann.

Wicksteed, Philip H., & Francis M. Cornford (ed., trans.), 1957. Aristotle, *The Physics*. 2 vols. (Loeb Classical Library). Cambridge, Mass.: Harvard University Press / London: Heinemann, 1957 ([1]1929), 1934.