The `Unknown Heritage´

Trace of a Forgotten Locus of Mathematical Sophistication

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The “Unknown Heritage”

Trace of a forgotten locus of mathematical sophistication

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FILOSOFI OG VIDENSKABSTEORI PÅ ROSKILDE UNIVERSITETSCENTER

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In memoriam
Marshall Clagett
and
David Pingree

Abstract

The "unknown heritage" is the name usually given to a problem type in whose archetype a father leaves to his first son 1 monetary unit and $^1/_n$ (n usually being 7 or 10) of what remains, to the second 2 units and $^1/_n$ of what remains, etc. In the end all sons get the same, and nothing remains.

The earliest known occurrence is in Fibonacci's *Liber abbaci*, which also contains a number of much more sophisticated versions, together with a partial algebraic solution for one of these and rules for all which do not follow from his algebraic calculation. The next time the problem turns up is in Planudes's late 13th-c. *Calculus according to the Indians, Called the Great*. After that the simple problem type turns up regularly in Provençal, Italian and Byzantine sources. It seems never to appear in Arabic or Indian writings, although two Arabic texts (one from c. 1190) contain more regular problems where the number of shares is given; they are clearly derived from the type known from European and Byzantine works, not their source. The sophisticated versions turn up again in Barthélemy de Romans' *Compendy de la praticque des nombres* (c. 1467) and, apparently inspired from that, in the appendix to Nicolas Chuquet's *Triparty* (1484). Apart from a single trace in Cardano's *Practica arithmetice et mensurandi singularis*, the sophisticated versions never surface again, but the simple version spreads for a while to German practical arithmetic and, more persistently, to French polite recreational mathematics.

Close analysis of the sources shows that Barthélemy cannot have drawn his familiarity with the sophisticated rules from Fibonacci. It also suggests that the simple version is originally either a classical, strictly Greek or a medieval Byzantine invention, and that the sophisticated versions must have been developed before Fibonacci within an environment (located in Byzantium, Provence, or possibly in Sicily?) of which all direct traces has been lost, but whose mathematical level must have been quite advanced.

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A starting point

In the final collection of mixed problems in the Vatican manuscript of Jacopo da Firenze's *Tractatus algorismi* (Vat. Lat. 4826),[1] we find the following (fol. 54^{v}–55^{r}):

> Io vo a uno giardino, et giongho a'ppede de una melarancia. Et coglione una. Et poi coglio el decimo del rimanente. Poi vene un altro dopo me, et coglene doy, et anchora el decimo de rimanente. Poi vene un altro et coglene 3, et anchora el decimo de rimanente. Poi vene un altro et coglene 4 et el decimo de rimanente. Et così venghono molti. Poi quello che vene da sezzo, cioè dercto, coglie tucte quelle che retrova. Et non ve ne trova né più né meno che abiamo auti li altri. Et tanto ne colze l'uno quante l'altro. Et tanti homini quanti erano, tante melarancie ebbe per uno. Vo' sapere quanti homini forono, et quante melarancie colseno per uno, et quante ne colzeno fra tucti quanti. Fa così, tray uno de 10, resta 9, et 9 homini forono, et 9 melarancie colseno per uno. Et colzero in tucto 81 melarancie. Et se la voli provare, fa così.

			El primo ne colze	I, restano
80.	El decimo	è octo,	et ày che illo n'ebbe	9, restano
72.	El secondo	2, restano 70,	el decimo è 7, et ebe ne	9, restano
63.	El terzo	3, restano 60,	el decimo è 6, et ebe ne	9, restano
54.	El quarto	4, restano 50,	el decimo è 5, et ebe ne	9, restano
45.	El quinto	5, restano 40,	el decimo è 4, et ebe ne	9, restano
36.	El sexto	6, restano 30,	el decimo è 3, et ebe ne	9, restano
27.	El sectimo	7, restano 20,	el decimo è 2, et ebe ne	9, restano
18.	Ell'octavo	8, restano 10,	el decimo è 1, et ebe ne	9, restano
9.	El nono, cioè quello da sezzo, colze quelle 9, né più né meno, che non ve n'erano più. Siché vedi che ella è bene facta. Et sta bene. Et così se fano le simiglianti ragioni.			

[1] This treatise was written in Montpellier in 1307. In spite of its Latin title and incipit, it is written in Tuscan (the orthography being somewhat coloured by the Provençal linguistic environment).

Two other manuscripts claim to contain the same treatise, Florence, Ricc. 2236 (undated) and Milan, Trivulziana, Ms. 90 (c. 1410) (see [Van Egmond 1980: 148, 166]; Van Egmond's dating of the Florence copy is misleading, since it merely repeats the date of Jacopo's original as it appears in the incipit). The Vatican manuscript is from c. 1450 but a meticulous copy of a meticulous copy, and linguistic and textual as well as mathematical homogeneity shows the Vatican manuscript to be quite close to the common archetype for all three manuscripts, whereas the other two descend from an abbreviated adaptation, probably adjusted to the curriculum of an abbacus school – see [Høyrup 2006: 7]. The final collection of supplementary problems is absent from the Florence and Milan manuscripts, as are the chapters on algebra.

In translation:[2]

> I go to a garden, and come to the foot of an orange. And I pick one of them. And then I pick the tenth of the remainder. Then comes another after me, and picks two of them, and again the tenth of the remainder. Then comes another and picks 3 of them, and again the tenth of the remainder Then comes another and picks 4 of them and the tenth of the remainder. And thus come many. Then the one who comes last, that is, behind, picks all that which he finds left. And finds by this neither more nor less than we others got. And one picked as much as the other. And as many men as there were, so many oranges each one got. I want to know how many men there were, and how many oranges they picked (each) one, and how many they picked all together. Do thus, detract one from 10, 9 is left, and there were 9 men, and 9 oranges (each) one picked. And they picked in all 81 oranges. And if you want to verify it, do thus,

		the first picked I of them,	left
80.	The tenth	is eight, and you have that this one got	9, left
72.	The second	2, left 70, the tenth is 7, and he got	9, left
63.	The third	3, left 60, the tenth is 6, and he got	9, left
54.	The fourth	4, left 50, the tenth is 5, and he got	9, left
45.	The fifth	5, left 40, the tenth is 4, and he got	9, left
36.	The sixth	6, left 30, the tenth is 3, and he got	9, left
27.	The seventh	7, left 20, the tenth is 2, and he got	9, left
18.	The eighth	8, left 10, the tenth is 1, and he got	9, left
9.	The ninth, that is, the last one, picked these 9, neither more nor less, as there were no more. So that you see that it is well done. And it goes well. And thus are done the similar computations.		

A modern reader encountering a problem of this kind for the first time is usually stunned. As Euler says about it in his didactical *Élémens d'algebre* [1774: 489], "this question is of a quite particular nature, and therefore deserves attention".[3] As we see, the rule works – still in Euler's words, "it fortunately happens that ..." – and the rule holds for any aliquot part $\phi = {}^1\!/_n$. Moreover, as we shall see, if only the absolutely defined contributions form an arithmetical progression and ϕ is any fraction and not too large it still works, in the sense that one can still find an initial amount T such that all shares except the last are equal.

[2] As all translations in the following where no translator is identified, this one is due to the present author.

[3] [Tropfke / Vogel et al 1980: 582–588] discusses it under the general heading of *Schachtelaufgaben*, "[nested] box problems", together with problems with the structure

$$(\ldots((x+a_1)\phi_1+a_2)\phi_2+\ldots)\phi_{n+1}+a_n = R\ ,$$

admitting however that it is of "a particular kind". Actually, the mathematical structure is wholly different. Normal box problems are easily solved by stepwise reverse calculation; in the present case, this is impossible.

Jacopo probably did not know why his rule functioned – when he knows, he is fond of giving pedagogical explanations, and here he only presents the complete calculation as a verification. However, the original inventor must have known why, one does not stumble on the structure in question by accident.

We cannot know where the idea came from,[4] but the arrangement of dots in Figure 1 (reduced for convenience to $\phi = {}^{1}\!/_{6}$) is a possibility:

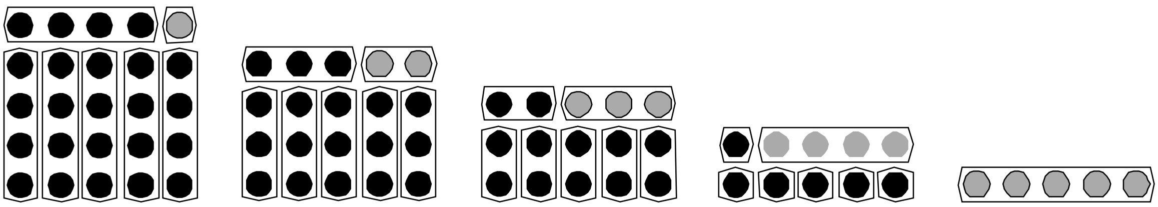

Figure 1

If we remove 1 small (grey) dot from a square pattern of $n\times n$ dots, what is left can be grouped as n+1 strips of n–1 (black) dots. Removal of one of these strips (${}^{1}\!/_{n+1}$ of what is left) leaves a rectangular system of $n\times(n–1)$ dots. Removing 2 small (grey) dots from this rectangle leaves n+1 strips of n–2 (black) dots, and removing one of these strips (still ${}^{1}\!/_{n+1}$ of the remainder) leaves a rectangle $n\times(n–2)$ dots, etc. In symbols, and for p = 0, 1, 2, ...,

$$(n–p)\times n = (n–[p+1])\times n+(p+1)+(n–[p+1]).$$

This is obviously an argument of the same kind as those based on pebble counters or *psephoi* used in early classical arithmetic. Contemporary readers accustomed to working on paper with a square grid may prefer the version in Figure 2, in which the summary to the right shows that the square is divided

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[4] A direct arithmetical *solution* is possible, but it could never give rise to the idea. It only works because the overdetermined problem does possess a solution, and it cannot be generalized to similar but different situations; moreover, it only finds the sole *possible* solution without showing that this is indeed a solution:

> Since the last visitor of the garden (say, no. N) leaves nothing, the remainder r_N of which he takes the fraction ${}^{1}\!/_{d}$ must be 0 (if not, $(1–{}^{1}\!/_{d})r_N$ would be left over. But since each visitor picks as many apples as his number before taking ${}^{1}\!/_{d}$ of the remainder, no. N gets N apples, and so therefore do all the others. But the second-last visitor (no. N–1) only picks N–1 apples before taking the fraction ${}^{1}\!/_{d}$ of the remainder $r_{N–1}$. Therefore this fraction must be 1 (he has already picked N–1, but should have N), and in consequence $r_{N–1}$ is N+1. ${}^{N+1}\!/_{d}$ is thus 1, whence N must be d–1.

No source or historian's discussion I have looked at contains the least hint that its author had seen this.

into the sum of two triangular numbers, one of which – namely 1+2+...+n – consists of the absolutely and the other – namely (n–1)+(n–2)+...+1 – of the relatively defined contributions.

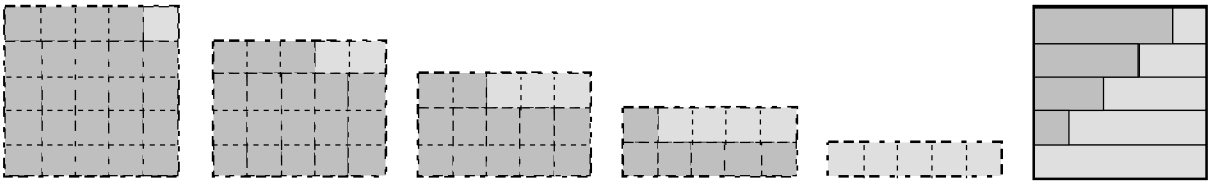

Figure 2

Leonardo Fibonacci

We shall return to the reasons that this argument may indeed be the one from which the problem was constructed. Initially, however, we shall have to look at other texts where problems of this kind turn up – beginning with the earliest source for them, Leonardo Fibonacci's *Liber abbaci* from 1228 [ed. Boncompagni 1857: 279–281]. Fibonacci first presents his reader with two versions dealing with an unknown heritage distributed to an unknown number of heirs (this, not fruit-picking, is the habitual dress for the problem), next with a sequence of structurally similar but more sophisticated pure-number problems.[5]

Fibonacci's first inheritance version shares the structure of Jacopo's fruit-picking problem (apart from the fraction being $^1/_7$ and the number of sons thus 6, each receiving 6 bezants). The method is also similar. However, Fibonacci does not give the information that the amount which each son receives equals the total number of sons, although his explanation presupposes it (which allows us to conclude that his source for the problem was even closer to Jacopo):

> For the seventh which he gave to every one you retain 7; from which detract 1, 6 remain; and so many were his sons; which 6 multiplied by itself makes 36; and so many were his bezants.

In the second inheritance problem, each son receives first $^1/_7$ of what is at disposal and afterwards respectively 1 bezant, 2 bezants, etc.; it is then stated (but no argument given) that 6 sons get 7 bezants each. The reader must be expected to identify 7 as the denominator of the fraction, and 6 as 7–1. Finally Fibonacci explains that if the absolutely defined contributions in the two cases had been instead 3 bezants, 6 bezants, etc., the number of sons would still have been 6,

[5] A full French translation of this part of the *Liber abbaci* is found in [Spiesser 2003: 711–718]. [Sigler 2002: 399–401] contains an English translation.

and the total possession 3×36 bezants and 3×42 bezants, respectively.

Even in the case where the fraction is taken first, a "proof" by means of pebble counters is possible – see Figure 3. Here, a number n×(n+1) is split into two triangular numbers of order n, one of which represents the successive absolutely defined, the other the relatively defined contributions.

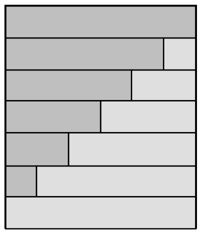

Figure 3

In the ensuing pure-number versions, the fractions and absolute contributions are more intricate. In order to facilitate the further discussion we shall henceforth designate by (α,ε | ϕ) the type where absolutely defined contributions α+εi (i = 0, 1, ...) are taken first, and a fraction ϕ of the remainder afterwards; (ϕ | α,ε) designates the type where a fraction ϕ of what is at disposal is taken first and absolutely defined contributions α+εi (i = 0, 1, ...) afterwards. In this notation, Fibonacci's problems are the following (the inheritance problems are in the left column, the other columns contain the number problems)

(1,1 | $^1/_7$) ($^1/_7$ | 1,1) (3,3 | $^1/_7$) ($^1/_7$ | 3,3)	(1,1 | $^2/_{11}$) (4,4 | $^2/_{11}$) ($^2/_{11}$ | 1,1) ($^2/_{11}$ | 4,4)	(2,3 | $^6/_{31}$) ($^6/_{31}$ | 2,3)	(3,2 | $^5/_{19}$) ($^5/_{19}$ | 3,2)

The problems in the second column (where α = ε) are treated by the same rules as those of the first column, in the sense that the fraction $^2/_{11}$ is tacitly dealt with as $^1/_{5½}$. The trick is not explained, however, we only find the prescription (for the first problem)

> Divide 11 by 2, which are above 11, 5$^1/_2$ result; from which take away 1, 4$^1/_2$ remain; and so many were the shares; which multiplied together, were 20$^1/_4$ for the divided number.

For the problem (2,3 | $^6/_{31}$) in the third column, the solution is found by means of the *regula recta*, that is, first-degree rhetorical algebra in which the unknown is referred to as *a thing*. Fibonacci puts the number to be divided equal to this *thing*, and finds by successive computation the first two shares, which he knows to be equal. Resolving the resulting equation he finds the number to be T = 56$^1/_4$, the number of shares to be N = 4$^1/_2$, and each share Δ = 12$^1/_2$. He has thus found the only possible solution, but his algebraic computation does not show that the subsequent shares will be as required, that is, that this *is* indeed a solution. Fibonacci makes no hint at this deficiency, but he performs a complete calculation step by step (similar to Jacopo's) which verifies that the first four shares are $12\,^1/_2$, after which $6\,^1/_4$ remains for the final $^1/_2$-share. Finally Fibonacci claims to "extract" the following rule from the calculation[6] ($\phi = {}^p/_q$):

$$(1^a) \qquad T = \frac{[(\varepsilon-\alpha)\,q+(q-p)\,\alpha]\cdot(q-p)}{p^2}\ ,$$

$$(1^b) \qquad N = \frac{(\varepsilon-\alpha)\,q+(q-p)\,\alpha}{\varepsilon p}\ ,$$

$$(1^c) \qquad \Delta = \frac{\varepsilon\,(q-p)}{p}\ .$$

At closer inspection, the rule turns out *not* to be extracted. If one follows the algebraic calculation step by step, it leads to

$$(2^a) \qquad T = \frac{q^2(\alpha+\varepsilon)-(q-p)q\alpha-(q-p)p\alpha-(\alpha+\varepsilon)pq}{p^2}$$

which (by means which were at Fibonacci's disposal) can be transformed into

$$(2^{a}*) \qquad T = \frac{[q\,(\alpha+\varepsilon)-(p+q)\,\alpha]\cdot(q-p)}{p^2}$$

but not in any obvious way into the rule which Fibonacci pretends to extract – if anything, attempts at further manipulation would rather lead to the reduction

$$(3^a) \qquad T = \frac{[\varepsilon q-\alpha p]\cdot(q-p)}{p^2}\ .$$

The implication appears to be that Fibonacci adopted a rule whose fundament he did not know, and that he pretended it to be a consequence of his own (correct but partial) solution.[7]

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[6] Obviously Fibonacci uses the specific numbers belonging to the problem when stating the rule, but since he identifies each number in the rule by pointing to its role in the computation, the symbolic formulae map his rule precisely and unambiguously.

[7] This case of minor fraud is not without parallel in Fibonacci's works. In the *Pratica geometrie* [ed. Boncompagni 1862: 66], Fibonacci copies from Gherardo da Cremona's translation of Abū Bakr's *Liber mensurationum* [ed. Busard 1968: 94] a fallacious solution to a rectangle problem $\ell-w = \alpha$, $\sqsubset\!\sqsupset(\ell,w) = \beta$ (the words are so close that Fibonacci's copying is beyond question, here as in several other places). Afterwards Fibonacci undertakes an explication by means of algebra (which Abū Bakr does not give in this case even though he does so in others). When arriving to the point where the mistake becomes evident (but where Fibonacci appears not to know how it has come about nor how to repair it) he concludes the exposition with the words "et cetera".

This inference is corroborated by what happens when Fibonacci treats the problem $(3{,}2 \mid {}^{5}\!/_{19})$. Here, α cannot be subtracted from ε (the outcome is negative), and therefore Fibonacci (who knew well how to make elementary operations with negative numbers even though he did not fully accept them) replaces (1) by

$$(4^{a}) \qquad T = \frac{[(q-p)\alpha-(\alpha-\varepsilon)q]\cdot(q-p)}{p^{2}} ,$$

$$(4^{b}) \qquad N = \frac{(q-p)\alpha-(\alpha-\varepsilon)q}{\varepsilon p} ,$$

$$(4^{c}) \qquad \Delta = \frac{\varepsilon\,(q-p)}{p} .$$

If Fibonacci himself had reduced the algebraic solution (2^{a}), why should he have chosen an expression which is neither fully reduced nor valid for all cases? Neither (2^{a}) nor (2^{a}*) nor (3^{a}) depends on whether $\alpha<\varepsilon$ or $\alpha>\varepsilon$.

For the case $({}^{6}\!/_{31} \mid 2{,}3)$, Fibonacci just states and applies these rules

$$(5^{a}) \qquad T = \frac{[(\varepsilon-\alpha)\,q+(q-p)\alpha]\cdot q}{p^{2}} ,$$

$$(5^{b}) \qquad N = \frac{(\varepsilon-\alpha)q+(q-p)\alpha}{\varepsilon p} ,$$

$$(5^{c}) \qquad \Delta = \frac{\varepsilon q}{p} ,$$

and for $({}^{5}\!/_{19} \mid 3{,}2)$

$$(6^{a}) \qquad T = \frac{[(q-p)\alpha-(\alpha-\varepsilon)q]\cdot q}{p^{2}} ,$$

$$(6^{b}) \qquad N = \frac{(q-p)\alpha-(\alpha-\varepsilon)q}{\varepsilon p} ,$$

$$(6^{c}) \qquad \Delta = \frac{\varepsilon q}{p} .$$

Once again, if (1^{a}) had really resulted from the algebraic solution, why should (5) and (6) be set forth without being derived from the pertinent algebraic operations (which are evidently not the same as before)?

We must conclude that not only what we shall henceforth call the "simple versions" of the problem (Jacopo's, and those in the first column of the scheme on p. 5, those where $\varepsilon = \alpha$ and where ϕ is an aliquot part) and their rules were “around” but also the much more sophisticated versions and rules in columns 2–4 of the scheme. The question then presents itself, *where?*

As is well known, most of the “recreational” problems found in the *Liber abaci* and in the various abbacus treatises are widely disseminated, turning up in Indian, Persian and Arabic problem collections, some also in the *Greek Anthology*, in Ananias of Shirak’s collection, or in the Carolingian *Propositiones ad acuendos iuvenes*, some even in ancient or medieval Chinese treatises. Not so in the present case. [Tropfke / Vogel et al 1980: 587*f*] and [Singmaster 2000] only list Byzantine and (Christian-)Occidental occurrences, and I have not been able to find parallel examples in sources from elsewhere, whether published before or after 1980. (Two Arabic “corrected” versions and their implications are discussed below, p. 18.)

Maximos Planudes

Three Byzantine occurrences are known: one in Maximos Planudes’s late thirteenth-century *Calculus According to the Indians, Called the Great* [ed., trans. Allard 1981: 191–194]; another one in a problem collection from the early fourteenth century [ed., trans. Vogel 1968: 102–105]; the last one in Elia Misrachi’s book on arithmetic from c. 1500 ([ed., trans. Wertheim 1896: 59*f*]. The cases treated are (1,1 | $^1/_7$) (all three) and (1,1 | $^1/_{10}$) (Elia Misrachi alone). All follow the simple rule we know from Jacopo and Fibonacci, and in so far they are uninformative. It may be observed, however, that the fourteenth-century problem deals with apples served at lunch, not with a heritage – Jacopo was thus not quite alone in deviating from the inheritance dress.[8] More important is that Planudes – whose testator dies before he has finished his will, which Planudes takes to explain that the number of heirs is unknown – brings the problem as an illustration of the following arithmetical observation (almost a theorem):[9]

When a unit is taken away from any square number, the left-over is measured by

[8] There is no reason to conclude from the common fruit theme that Jacopo and the Byzantine text were connected, in particular since the general settings (garden / lunch) are different. “Box problems” (see note 3) about apples were common; though roughly contemporary, the two authors (or their sources) probably made independent but analogous changes of the usual dress (Jacopo repeatedly uses familiar dresses for problem types with which they usually do not go together).

[9] I try to make a very literal translation, conserving all quasi-logical particles even when they offend the modern ear; a somewhat less literal French translation accompanies Allard’s edition of the Greek text.

> two numbers multiplied by each other, one smaller than the side of the square by a unit, the other larger than the same side by a unit. As for instance, if from 36 a unit is taken away, 35 is left. This is measured by 5 and 7, since the quintuple of 7 is 35. If again from 35 I take away the part of the larger number, that is the seventh, which is then 5 units, and yet 2 units, the left-over, which is then 28, is measured again by two numbers, one smaller than the said side by two units, the other larger by a unit, since the quadruple of 7 is 28. If again from the 28 I take away 3 units and its seventh, which is then 4, the left-over, which is then 21, is measured by the number which is three units less than the side and by the one which is larger by a unit, since the triple of 7 is 21. And always in this way.

This description does not refer explicitly to counters, but it is noteworthy that the whole passage fits the above geometric explanation of Jacopo's problem to the slightest detail. Without support by either symbolic algebra or a geometric representation it is also difficult to see that the "theorem" holds for "any square number", and only the geometric diagram makes it evident that the procedure will continue in such a way that exactly nothing remains in the end.[10] It is also to be observed that the quasi-theorem and the illustrating problem come exactly at the point where Planudes goes beyond Indian calculus.

In the next section (which closes the treatise) Planudes treats the problem to "find a figure equal in perimeter to another figure and a multiple of it in area" – that is, for a given n to find two rectangles[11] $\sqsubset\!\sqsupset(a,b)$ and $\sqsubset\!\sqsupset(c,d)$ such that $a+b = c+d$, $n{\cdot}ab = cd$ (a, b, c and d being tacitly assumed to be integers). Two solutions are given, the second being stated to be Planudes's own invention – which implies that the first solution was not (as indeed we shall see). In this borrowed solution, the following choice is made (n being taken to be 4):

$$a = n-1 \qquad\qquad b = (n^3-1)-(n-1)$$
$$c = n^2-1 \qquad\qquad d = (n^3-1)-(n^2-1)$$

Planudes maintains that this solution is only valid for $n = 4$, 3 and 5. This is not true, Planudes must either have calculated badly or relied on bad information. In any case, he proposes the following alternative of his own (where t is arbitrary):

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[10] A corresponding calculations in symbols based on the corresponding sequence of identities $n{\cdot}(n-p+1) = (n+1){\cdot}(n-p)$ *can* of course show it, but with much less ease. A purely verbal argument like that of Planudes and unsupported by a diagram would hardly give the idea.

[11] Actually, χωρίον, here translated "figure", may have the more specific meaning "rectangular area".

$$a = t \qquad\qquad b = n\cdot(n+1)\cdot t$$
$$c = (n+1)\cdot t \qquad\qquad d = n^2\cdot t$$

As Allard [1981: 235] points out, the second solution coincides with the first if t is replaced by n–1. Planudes is not likely to have noticed this, but it may explain how he guessed his own scheme for the correct solution for $n = 3$.

The statement of the problem and the first solution are found in almost exactly the same words in the pseudo-Heronic *Geometrica* Ch. 24 [ed., trans. Heiberg 1912: 414–417], cf. [Sesiano 1998: 284–286]. The manuscripts ("S" and "V") from which this section of the conglomerate is taken are of Byzantine date (the eleventh respectively fourteenth century), and the use of the late form πολυπλασιάζω instead of the classical πολλαπλασιάζω points to an origin of the text certainly no earlier than the second century CE, perhaps considerably later. The shape of the problem, however, is ancient, not medieval: even though it is not found in Diophantos's *Arithmetic*, the stylistic similarity is unmistakeable. The problem is likely to come from that already existing tradition of "theoretical arithmetic" within which Diophantos tells to have found his names and abbreviations for powers of the unknown [ed. Tannery 1893: I, 4].

This does not prove that even Planudes's "theorem" for the inheritance problem goes back to Antiquity, but the vicinity and the absence of a claim that he invented it himself suggests it to have been at least traditional.

In his edition of Elia Misrachi's text, Wertheim [1896: 60] suggests that the problem might be inspired by one which is found in a late fourteenth- or early fifteenth-century Byzantine manuscript (the cod. Cizensis) containing also Nicomachos's *Introduction* and Philoponos's scholia to that work (for which reason Wertheim may have thought it ancient, even if he does not say so). This problem [ed. Hoche 1866: 153*f*] deals with the legacy of a father with three sons and three daughters, who has disposed as follows:[12]

– The first son puts into the chest as many coins as it already contains and then takes 250 coins;
– then the second son does as much;
– then the third son does as much;
– then the first daughter puts into the chest as many coins as she finds there, and takes 125 coins;
– then the second daughter does as much;
– finally, the third daughter does as much, after which nothing remains.

The text gives the solution (originally, the chest contained $232+{}^1\!/_3+{}^1\!/_{12}+{}^1\!/_{192}$ gold

[12] I am grateful to C. M. Taisbak for assisting me in the interpretation of the text.

coins) but does not explain how it is reached.

Beyond the occurrence solely in a manuscript from c. 1400, other reasons speak against an early dating of the problem. Firstly, the term for the coin is the medieval χρύσινος (known only from the fourth century CE onward – the ancient form is χρυσίον); secondly, according to Taisbak, the syntax is Byzantine and not ancient. The present problem might therefore well be a secondary derivation from the problem type we have dealt with so far – a reduction to the normal "box problem" type allowing a solution by stepwise reverse calculation. In any case, the striking feature of equal shares is absent from it (indeed, the youngest son and the youngest daughter get the greatest shares); the basic unknown-heritage problem could therefore at most have borrowed *the dress* of an unknown heritage: the mathematical structure must have been an independent discovery.

The mathematics of the full problem

Before we go on with the analysis of further sources, it may be convenient to have an exhaustive mathematical analysis at hand; it should be kept in mind that this is a mathematical analysis, and not an interpretation of any source.

Let us assume that a total T is distributed into shares (δ_1, δ_2, ..., δ_n, ...) in this way:

– The first share δ_1 receives a_1, and furthermore a fraction ϕ of what is left after a_1 has been given.
– The second share δ_2 receives a_2, and furthermore a fraction ϕ of what is left after subtraction of the first share and of a_2.
  . . .
– The n-th share δ_n receives a_n, and furthermore a fraction ϕ of what is left after subtraction of the preceding shares and of a_n.
  . . .

We want to find the condition imposed on the sequence a_1, a_2, ..., a_n, ... by the request that $\delta_1 = \delta_2 = ... = \delta_n = ... = \Delta$ (admitting that the last share may be fractional; furthermore, we ask for the value of the total T, of the value Δ of the single share, and of the number N of shares.[13]

[13] If we go beyond the mathematics of the thirteenth and fourteenth centuries and admit negative numbers, we may instead investigate for instance three sequences $S(n)$, $a(n)$, *and* $U(n)$, coupled through the conditions

$$U(n) = S(n)–a(n)\ ,\quad S(n+1) = S(n)–a(n)–\phi U(n)\ ,$$

with n running through the domain of all integers (negative as well as positive, ϕ being

Before the n-th take, S_n is at disposition ($S_1 = T$). The n-th share is then

$$\delta_n = a_n+\phi(S_n-a_n) = \phi S_n+(1-\phi)a_n \ .$$

After it has been removed, the remainder is

$$S_{n+1} = S_n-\delta_n = (1-\phi)S_n-(1-\phi)a_n = (1-\phi)\cdot(S_n-a_n) \ ,$$

and the n+1-th share becomes

$$\delta_{n+1} = \phi S_{n+1}+(1-\phi)a_{n+1} \ .$$

Since we have required that $\delta_n = \delta_{n+1} = \Delta$, which implies that $S_n-S_{n+1} = \Delta$, we find that

$$(1-\phi)\cdot(a_{n+1}-a_n) = \phi(S_n-S_{n+1}) = \phi\Delta \ ,$$

whence also $a_{n+1}-a_n$ must be constant and equal to $\varepsilon = {}^{\phi}/_{1-\phi}\cdot\Delta$. The absolutely defined contributions must therefore constitute an arithmetical progression, $a_n = \alpha+(n-1)\cdot\varepsilon$

For a given set of values for $T = S_1$, ϕ and $\alpha = a_1$ follows

$$\Delta = \delta_1 = \alpha+\phi(T-\alpha) = \phi T+(1-\phi)\cdot\alpha \ ,$$

$$\varepsilon = a_2-a_1 = {}^{\phi}/_{1-\phi}\Delta = {}^{\phi}/_{1-\phi}(\phi T+[1-\phi]\alpha) \ .$$

If the resulting Δ does not exceed ${}^1/_2 T$, this gives us at least 2 full shares; the sequence can be constructed stepwise until the remainder becomes less than Δ (a strict proof of this asks for complete induction, but it should be possible to dispense with that tedium here).

However, the texts do not start from given values of T, ϕ and α but from ϕ, α and ε. From this they find T, Δ and N. We may do as much. From $\varepsilon = {}^{\phi}/_{1-\phi}\cdot\Delta$ follows

(7[c]) $$\Delta = {}^{1-\phi}/_{\phi}\varepsilon \ .$$

But since $\Delta = \delta_1 = \phi S_1+(1-\phi)a_1 = \phi T+(1-\phi)\cdot\alpha$,

$$\phi T = (1-\phi)\cdot({}^{\varepsilon}/_{\phi}-\alpha) \ ,$$

(7[a]) $$T = {}^{1-\phi}/_{\phi}\cdot({}^{\varepsilon}/_{\phi}-\alpha) \ ,$$

and finally

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an arbitrary real number), and ask for the condition that $\delta(n) = S(n)-S(n+1)$ be constant. Further investigation of the properties of this system might perhaps present us with some interesting mathematics (though I doubt it), but it would lead us away from the problem of our texts.

The wider class of coupled progressions does contain interesting objects. For instance, self-references are removed from the “Fibonacci series” if it is dissolved into three cyclically coupled sequences S, T, and U, where

$$U(i) = S(i)+T(i) \ , \quad S(i+1) = T(i)+U(i) \ , \quad T(i+1) = U(i)+S(i+1) \ .$$

This observation, and the fact that the side-diagonal-algorism for a square consists by its very nature of two coupled progressions S and D,

$$S(i+1) =, \quad D(i+1) = 2S(\mathrm{i})+D(i) \ ,$$

suggests a link to continued fractions.

$$(7^b) \qquad N = \frac{T}{\Delta} = \frac{\frac{1-\phi}{\phi}\left(\frac{\varepsilon}{\phi}-\alpha\right)}{\frac{1-\phi}{\phi}\cdot\varepsilon} = \frac{1}{\phi}-\frac{\alpha}{\varepsilon} = \frac{\varepsilon-\phi\alpha}{\phi\,\varepsilon} \ .$$

The condition that at least two full shares can be found (that is, $N \geq 2$, all parameters taken to be positive) is then that

$$(8) \qquad \phi \leq \frac{1}{2+\frac{\alpha}{\varepsilon}} \ ,$$

which is clearly fulfilled in all examples we have seen.

In order to compare with Fibonacci's rules, we put $\phi = {}^{p}/_{q}$. Thereby the formulae become

$$(9^c) \qquad \Delta = \frac{q-p}{p}\varepsilon \ ,$$

$$(9^a) \qquad T = \frac{(q-p)\cdot(\varepsilon q-\alpha p)}{p^2} \ ,$$

$$(9^b) \qquad N = \frac{\varepsilon q-\alpha p}{\varepsilon p} \ .$$

We may also express ϕ as ${}^{1}/_{d}$, in agreement the trick Fibonacci used to treat cases in the second column (p. 5), for instance $(1,1 \mid {}^{2}/_{11})$. Then the formulae look much simpler:

$$(10^c) \qquad \Delta = (d-1)\cdot\varepsilon \ ,$$
$$(10^a) \qquad T = (d-1)\cdot(d\varepsilon-\alpha)$$
$$(10^b) \qquad N = d-{}^{\alpha}/_{\varepsilon} \ ,$$

From (10^b) we see that if ϕ is an aliquot part (and d thus integer), N is integer if and only if ε divides α. For other cases we see from (9^b), presupposing that ${}^{p}/_{q}$ is reduced to minimal terms and thus that p and q are mutually prime, that

$$N\varepsilon = \frac{\varepsilon q}{p}-\alpha \ .$$

If α and ε are integer (as they always are in the texts), this can only be fulfilled if p divides ε, $\varepsilon = \mu p$. Inserting this we see that

$$N\mu p = \mu q-\alpha \ ,$$

whence

$$\alpha = \mu\cdot(q-Np) \ .$$

For a given value of ϕ reduced to minimal terms ${}^{p}/_{q}$, all types leading to an integer solution for N are thus $(q,p \mid {}^{p}/_{q})$ and those types which can be obtained by trivial means from it by multiplying all the absolutely defined contributions

$q+ip$ by the same constant μ and/or by starting from a different point in the sequence $\mu q+i\cdot(\mu p)$ (taking care that N remain in the requested domain, e.g., $N \geq 2$.[14] Since no sources contain a problem with non-integer d leading to an integer value for N, this was probably not known to the medieval calculators

All of this concerns the situation where the absolutely defined contributions are taken first and the fraction of the remainder afterwards. All calculations are similar in the case where the fraction is taken first. The corresponding formulae become:

$$\Delta = T+\alpha \ , \tag{12^c}$$

$$T = \frac{\varepsilon-\alpha\phi}{\phi^2} \ , \tag{12^a}$$

$$N = \frac{\varepsilon-\alpha\phi}{\varepsilon\phi} \ . \tag{12^b}$$

$$\Delta = \frac{\varepsilon q}{p} \ , \tag{13^c}$$

$$T = \frac{q(\varepsilon q-\alpha p)}{p^2} \ , \tag{13^a}$$

$$N = \frac{\varepsilon q-\alpha p}{\varepsilon p} \ . \tag{13^b}$$

$$\Delta = d\varepsilon \ , \tag{14^c}$$

$$T = d^2\varepsilon-d\alpha \ , \tag{14^a}$$

$$N = d-{}^{\alpha}\!/\!_{\varepsilon} \ . \tag{14^b}$$

Since the formulae for N are the same, the condition for the number of shares being integer and being at least 2 are unchanged.

After this modern reconstruction one may ask how corresponding calculations could be made with the tools at hand in late Antiquity or the Middle Ages. Problems of the types $(\varepsilon,\varepsilon\,|\,{}^1\!/\!_d)$ and $({}^1\!/\!_d\,|\,\varepsilon,\varepsilon)$ can of course be solved by means of counters for any integer value of ε, not only for $\varepsilon = 1$, just by taking the value of each counter to be ε instead of 1; so can problems of the types $(n\varepsilon,\varepsilon\,|\,{}^1\!/\!_d)$ and $({}^1\!/\!_d\,|\,n\varepsilon,\varepsilon)$ – the procedures are exactly the same as in the previous case, just with omission of the first n–1 steps; anybody familiar with the operations on the

[14] For $\phi = {}^6\!/\!_{31}$, (α,ε) may thus be any one of the sets (1,6), (7,6), (13,6) and (19,6) or their multiples $(\mu, 6\mu)$ etc.

square pattern might discover that.[15] Even for cases where d is non-integer and / or ε does not divide α it is possible to construct pebble justifications once the solution is known (on the condition of accepting fractional pebbles); but it is difficult to see how such pebble proofs could be found if one did not know the solution already.[16] What then?

The first step will be to show that the equality of shares implies that the

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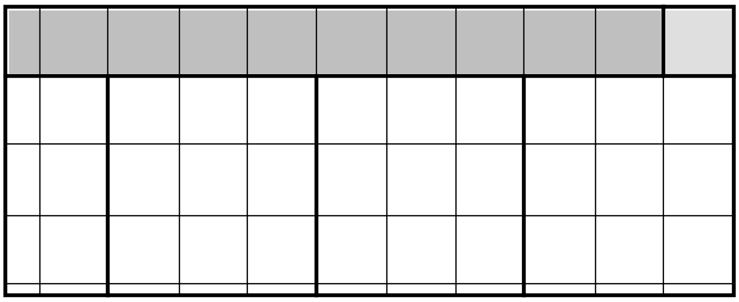
[15] On the other hand, anybody familiar just with the *rule* for the case $(1,1\,|\,{}^1/_d)$ might also observe that the solution to the case $(n,1\,|\,{}^1/_d)$ is obtained from the former case by omission of the first n–1 heirs. Solutions to the case $(\varepsilon,\varepsilon\,|\,{}^1/_d)$ is of course obtained from that for $(1,1\,|\,{}^1/_d)$ by simple proportionality, no new proof being needed; the same holds for the relation between the cases $(n\varepsilon,\varepsilon\,|\,{}^1/_d)$ and $(n,1\,|\,{}^1/_d)$.

[16] For the relatively simple case $d = 4^1/_2$, $\alpha = 1$, $\varepsilon = 3$ (whence $\Delta = 3^1/_2 \cdot 3 = 10^1/_2$, $n = 4^1/_2 - {}^1/_3 = 4^1/_6$, the square-grid diagram corresponding to the pebble justification looks as follows:



Each row is equal in area to Δ, and the number of rows is $N = 4^1/_6$. If we remove α = 1 in the first row, $9^1/_2$ are left. The lower $3^1/_6$ rows can be divided into three columns with area $3 \cdot 3^1/_6 = 9^1/_2$, and a narrow column with area $1^1/_2 \cdot 3^1/_6 = 4^3/_4 = {}^1/_2 \cdot 9^1/_2$. The $9^1/_2$ left over in the upper row is thus, as it should be, ${}^1/_d$ of the remainder. When it is removed, we are left with the lower $3^1/_6$ rows. $\alpha+\varepsilon = 4$ is removed from the upper of these, leaving $6^1/_2$ in the same row and $3^1/_2$ times $6^1/_2$ in the following; etc.

After having gone through this operation I suppose that the reader, firstly, will find it unlikely that somebody should invent this diagram unless it be done (as here) from the already known result; and, secondly, will doubt that Fibonacci's formulae (or those we shall encounter below in the *Compendy de la praticque des nombres*) were derived from such diagrammatic considerations. One could ask for no better example of an *a posteriori* synthesis which is of no help whatsoever in the reconstruction of a corresponding analysis.

I also expect the reader to find new sympathy for Plato's insistence (*Republic* 525d–526a, ed., trans. [Shorey 1930: 162–165]) that it is a bad habit to transfer to the realm of theoretical arithmetic that breaking of the unit with which shopkeepers were conversant. "Visual" mathematics, seductive as it is in simple cases, becomes much more difficult than formal calculation as soon as intricacies arise; symbolic algebra conquered for good reasons.

absolutely defined contributions constitute an arithmetical progression. A possible means for showing this is used amply in the *Liber abbaci*, namely the line diagram (but not used for these problems). Let us first try (Figure 4) the more intricate case where the absolutely defined contribution is taken first; for convenience I shall use letter symbols, but pointing and words could do the same:

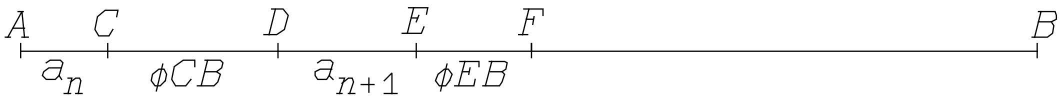


Figure 4

AB represents S_n, that is, the amount that is at disposition when the *n*-th share is to be taken, *n* being arbitrary (but possible).[17] This share is *AD*, consisting of $AC = a_n$ and $CD = \phi CB$. The following share is *DF*, consisting of $DE = a_{n+1}$ and $EF = \phi EB$. Since $AD = DF = \Delta$, $CB = CD+DB$, and $EB = EF+FB$, we find that

$$a_{n+1}–a_n = \phi(CB–EB) = \phi(CD–EF)+\phi(DB–FB) = \phi(a_{n+1}–a_n)+\phi\Delta \ ,$$

whence

$$(1–\phi)\cdot(a_{n+1}–a_n) = \phi\Delta$$

and further (in order to avoid a formal algebraic division) the proportion

$$\Delta : (a_{n+1}–a_n) = (1–\phi) : \phi \ .$$

By means, for instance, of Euclid's *Data*, prop. 2 [trans. Taisbak 2003: 254], "If a given magnitude [here Δ] have a given ratio [here $(1–\phi):\phi$] to some other magnitude [here $a_{n+1}–a_n$], the other is also given in magnitude" (or applying simply the rule of three), we find that $a_{n+1}–a_n$ has the same value irrespective of the step where we are. In consequence, the absolutely defined contributions have to constitute an arithmetical progression.

If the fraction is taken first, we may use the line diagram in Figure 5:

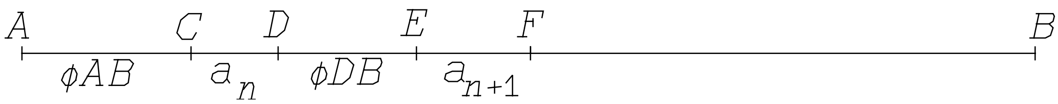


Figure 5

[17] The reason Fibonacci offered no proof of this kind may be that the structures of secondary logic ("for any ...", "for all ...", etc.) were not integrated in his mathematical standard language and therefore did not offer themselves readily for the construction of proofs. The present line-diagram proof, if made during or before his times, is likely not to have looked at an arbitrary step but to have started from the first and then given an argument by quasi-induction. Fibonacci, making the calculation in numbers that change from step to step, could not generalize his result.

In this case, $\phi AB+a_n = \phi DB+a_{n+1}$, and therefore $a_{n+1}-a_n = \phi(AB-DB) = \phi\Delta$, which again means that the absolutely defined contributions must form an arithmetical progression.

In both cases, once we are so far it is legitimate to construct the rules from the equality of the first two shares only. This can be done by somewhat laborious but simple first-degree algebra – Fibonacci shows one way to do it, but there are alternatives. It can also be done by means of two false positions (see note 19), and probably by still other methods. Possibly, one might reconstruct the way that was actually followed in order to find the rules from the detailed make-up of these. I have not been shrewd enough to do so.

Fourteenth-century abbacus writings

In its basic inheritance shape, the problem turns up in quite a few fourteenth-century abbacus treatises. The earliest of these is the *Livero de l'abbecho* [ed. Arrighi 1989: 116].[18] Here we find a problem of type $(7,1\,|\,{}^1\!/_{24})$ dealing with a heritage consisting of an unknown number of sheep. The rule that is given is that "we should strike off one from the fraction, and ${}^1\!/_{23}$ remain, and we shall strike off 7 from 24, and 17 remain" (which gives 17 sons and 23 sheep for each son). These rules are clearly not derived from Fibonacci's rule (4^b) for the intricate case, which would give the number of sons as $(24-1)\cdot7-(7-1)\cdot24$. Instead they may come from the observation that the outcome corresponds to that of the distribution $(1,1\,|\,{}^1\!/_{24})$, the six first shares being omitted; but the mathematical quality of the rest of the treatise does not make it likely that the compiler was

[18] On the words of its compiler, this treatise purportedly written "secondo la oppenione de maiestro Leonardo de la chasa degli figluogle Bonaçie da Pisa" has been believed to be extracted from the *Liber abbaci*, and from internal evidence it has been supposed to be from 1288–1290. The internal evidence consists of loan documents which turn out to be copied from elsewhere (whether original documents or an earlier abbacus treatise cannot be decided), for which reason the real date must be somewhat later (hardly much, the language seems rather archaic). As regards the link to Fibonacci, the treatise does contain a number of advanced problems borrowed from the *Liber abbaci*; but these are external decoration, the stem of the treatise is independent of Fibonacci (the *Liber abbaci* as well as any other work he may have written, as revealed by linguistic analysis), and repeatedly the compiler reveals not to understand what he copies from Fibonacci [Høyrup 2005a]. The problem about the unknown heritage is located in a final collection of mixed questions, some of which are taken from the *Liber abbaci* and others not.

The problem type is *not* represented in the *Columbia algorism* [ed. Vogel 1977], which now appears to be the earliest extant abbacus text (from c. 1280 or not much later, albeit the manuscript we possess is a fourteenth-century copy), cf. [Høyrup 2005a: 27 n.5].

able to get that idea on his own.

Paolo Gherardi's *Libro di ragioni*, written in Montpellier in 1327, contains a problem of type $(1,1 \mid {}^{1}\!/_{10})$ [ed. Arrighi 1987: 37*f*]; the story deals with a father who gives 1 mark of gold and $^{1}\!/_{10}$ of the gold that remains in his box to the first son, etc. The numbers are thus like those of Jacopo, but already the return to the traditional inheritance story shows that Jacopo is not the source – or at least not the only source.

The *Libro de molte ragioni* [ed. Arrighi 1973: 199], a conglomerate from Lucca from c. 1330, has another inheritance story with the same numbers $(1,1 \mid {}^{1}\!/_{10})$, sufficiently different (both at the level of the story and in the formulation of the rule) to exclude any direct link.

In the *Istratti di ragioni* [ed. Arrighi 1964: 140*f*] – a problem collection written down in c. 1440 but claiming to go back to Paolo dell'Abbaco (c. 1340) and in any case likely to contain material from this age – we find two variants, namely $(1000,1000 \mid {}^{1}\!/_{10})$ and $({}^{1}\!/_{6} \mid 10,10)$. The former (about *bizanti*) is solved by the usual rule (the denominator of the fraction minus 1), the latter (about *fiorini*) by a double false position (using only the equality of the first two shares).[19]

Arabic pseudo-kin

Due to the kind assistance of Mahdi Abdeljaouad (personal communication), I have come to know about two Arabic problems obviously inspired from the simple version of the problem type we are discussing. Both replace it by something closer to the orthodox "box problem" (though not changing it as radically as the late Byzantine analogue discussed on p. 10), yet without taking

[19] The formulation runs thus:

> We shall find a number such that, when $^{1}\!/_{6}$ is detracted and then 10, and from the remainder again $^{1}\!/_{6}$ and then 20, one [detraction] is as much as the other; and therefore posit that this number be 60, seize $^{1}\!/_{6}$ of 60, it is 10, and 10 more, you get 20; you have when you detract 20 from 60, 40 remain, and now seize $^{1}\!/_{6}$ of 40, which is $6^{2}\!/_{3}$, and 20 more, you get $26^{2}\!/_{3}$. So that you see that he has $6^{2}\!/_{3}$ more than the first. And now posit another number, and let us posit that it is 120, and therefore seize $^{1}\!/_{6}$ of 120, it is 20, and 10 more, you get 30. You have that the remainder is 90, now seize $^{1}\!/_{6}$ of 90, which is 15, and 20 more, you get 35. You have that to the second falls 5 more than to the first; so that you will say: and for 20, 5 more. And now follows the rule you have heard several times in this book,

according to which the true total heritage is $\frac{6\frac{2}{3}\cdot 120-5\cdot 60}{6\frac{2}{3}-5}$ = 300 *fiorini*. The single share is then found as $^{1}\!/_{6}\cdot 300+10 = 60$, and the number of sons as $300/60 = 5$.

advantage of the change.

One comes from Ibn al-Yāsamīn's *Talqīḥ al-afkār fī'l ʿamali bi rušūm al-ghubâr* ("Fecundation of thoughts through use of *ghubār* numerals") – written in Marrakesh in c. 1190. It runs as follows:[20]

> An inheritance of an unknown amount. A man has died and has left at his death to his six children an unknown amount. He has left to one of the children one dinar and the seventh of what remains, to the second child two dinars and the seventh of what remains, to the third three dinars and the seventh of what remains, to the fourth child 4 dinars and the seventh of what remains, to the fifth child 5 dinars and the seventh of what remains, and to the sixth child what remains. He has required the portions be identical. What is the sum?
>
> The solution is to multiply the number of children by itself, you find 36, it is the unknown sum. This is a rule that recurs in all problems of the same type.

The other is found in the *al-Maʿūna fī ʿilm al-ḥisāb al-hawāʾī* ("Assistance in the science of mental calculation") written by Ibn al-Hāʾim (1352–1412, Cairo, Mecca & Jerusalem, and familiar with Ibn al-Yāsamīn's work):[21]

> An amount of money has been diminished by one dirham and the seventh [of what remains]; by two dirhams, and then the seventh of what remains; then three dirhams and the seventh of what remains; then four dirhams and the seventh of what remains; then five dirhams and the seventh of what remains. In the end, six remain.
>
> Take the square of the six that remain, it is the amount which was asked for.

The number of portions is thus given in both versions; none the less, both still use the same rule as the "Christian" version of the simple problem. As we observe, Ibn al-Yāsamīn omits the information that the last portion is determined according to the same rule as the preceding ones, whereas Ibn al-Hāʾim does not require the portions to be equal. Both informations are indeed superfluous.

We also observed that Ibn al-Hāʾim's version is not overdetermined; it can be solved backwards step by step, in this way:

The fifth portion is $5+{}^1/_7A$, where $A+5$ is what is left after the taking of the fourth portion; but this remainder is also the sum of the fifth and sixth portions. Hence,

$$A+5 = 6+5+{}^1/_7A,$$

from which follows $A = 7$. The fourth portion is $4+{}^1/_7B$, where $B+4$ is what is left after the taking of the third portion; but this is also the sum of the fourth, fifth and sixth portions; etc.

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[20] My translation from Mahdi Abdeljaouad's French translation.

[21] Still my translation from the French.

Obviously, a similar backward calculation could be made for varying fractions and for absolutely defined contributions that are not in arithmetical progression. However, the rule is only valid for a constant fraction $^1/_{N+1}$, where N is the given number of portions, and if the absolutely defined contributions are $1+(i–1)$. There is hence no doubt that Ibn al-Hāʾim's problem descends from the "Christian" problem and results from an attempt to assimilate it to a more familiar structure.

Ibn al-Yāsamīn's problem *is* overdetermined, but the evident way to solve it would still be a backward calculation: if S is what is left when the fifth share is to be taken, the fifth share is $5+{}^1/_7(S–5)$, and the sixth share is what is left, i.e., $S–5–{}^1/_7(S–5)$. From their equality follows that S is 12, each share thus 6, and the total $6{\cdot}6$. The rule, once again, is valid but not naturally adapted to the actual problem.

The conclusion is that mathematicians from the Maghreb or al-Andalus[22] had come to know about the problem type already before the *Liber abbaci* was written; but their use of a rule which is better adapted to the "Christian" version of the problem shows that this latter version with its unknown value of N was not derived from the "Islamic" box-problem versions but was indeed the original form. Whether Ibn al-Hāʾim knew the problem from the Maghreb mathematicians or through other channels cannot be decided at present. In any case, the aberrant character of the two Arabic problems are strong evidence that Fibonacci and Planudes did not get *their* problem from the Arabic world – if it was known and accepted there, why should our two authors need to make it more familiar by making N a given magnitude? Ibn al-Yāsamīn confirms that the problem type which inspired him was indeed familiar (in a place that might inspire him and where he expected to find readers) before the *Liber abbaci* was thought of.

Provence and Barthélemy de Romans

The problem type $(1,1\,|\,{}^1/_8)$ turns up in a manuscript of the *Trattato di tutta l'arte dell'abacho* from 1340 (Rome, Accademia Nazionale dei Lincei, Cors. 1875, fol. 85^v). The rule is once again that the number of sons is found by subtracting one from the denominator – "if he had said $^1/_9$ to them, you would subtract one from 9, but because he said $^1/_8$, subtract one from 8, 7 remains, and 7 were the sons". It is likely but not certain that the author picked up the problem in

[22] Ibn al-Yāsamīn's "all problems of the same type" seems to prove that he was not the only mathematician in his area to know about them. He had been active in Morocco and in Muslim Spain; he may have encountered the derived problem type in either place.

Avignon, where the original was written around 1334.[23] In any case the genre is well represented in treatises written in Provence in the early fourteenth century (Jacopo in 1307, Paolo Gherardi in 1327), being absent only from the *Liber habachi* [ed. Arrighi 1987], written around 1310, almost certainly in Provence and almost certainly *not* by Paolo Gherardi.[24] It is absent from most other fourteenth- and fifteenth-century treatises from the Ibero-Provençal area I know about – thus from the Castilian *Libro de arismética que es dicho alguarismo* [ed. Caunedo del Potro & Córdoba de la Llave 2000: 133–213], from Francesc Santcliment's *Summa de l'art d'aritmètica* from 1482 [ed. Malet 1998]; from Francés Pellos's *Compendion de l'abaco* from 1492 [ed. Lafont & Tournière 1967]; and (as far as can be concluded from the description in [Sesiano 1984b]) from the "Pamiers algorism".[25] However, it is represented in the mid–fifteenth-century *Traicté de la praticque d'algorisme* by four problems – according to the description in [Spiesser 2003: 154] of the types $(1,1\,|\,{}^{1}\!/_{10})$, $(\varepsilon,\varepsilon\,|\,{}^{1}\!/_{10})$, $({}^{1}\!/_{10}\,|\,1,1)$ and $({}^{1}\!/_{10}\,|\,\varepsilon,\varepsilon)$; in Barthélemy de Romans' *Compendy de la praticque des nombres*;[26] and in the problem collection

[23] For this dating, see [Cassinet 2001]. The problem is not in what appears to be a draft autograph of the treatise (Florence, Biblioteca Nazionale Centrale, fond. prin. II,IX.57), but since this draft does not represent the finished treatise its author may well have added even the actual problem afterwards (other material with no parallel in the main draft but in the same hand as the main treatise has been added in the Lincei manuscript; when metrologies are referred to in these problems, they are the same as in the main treatise, and of Provençal rather than Tuscan type).

[24] The date being rather late and the orthography purely Tuscan, it is not certain whether we should count as genuinely Provençal an occurrence of a problem $(1,1\,|\,{}^{1}\!/_{7})$ in Francesco Bartoli's *Memoriale*, written down in Avignon before 1425 and copied from unidentified abbacus material [ed. Sesiano 1984a: 138]. We may notice, however, that Bartoli's problem shares with Paolo Gherardi's version (and with no other) that everything is measured in weight units of gold, not in coin (here ounces, in Gherardi marks of gold).

Bartoli's rule is the usual one – that subtraction of 1 from 7 gives both the number of sons and the amount each one receives; maybe the Papal courtly environment is the reason that his testator is a count.

[25] It is also absent from two twelfth-century Latin works prepared in Iberian area where it could have been expected to turn up if it had been known, the *Liber augmenti et diminutionis* [ed. Libri 1838: I, 304–369] and the *Liber mahamaleth* (at least in as far as can be determined from the description of the latter work in [Sesiano 1988].

[26] Barthélemy probably wrote this treatise around 1467, but what we possess is a revised redaction from 1476 due to Mathieu Préhoude – see [Spiesser 2003: 26, 30]. Barthélemy himself presents his work as an extension of an earlier treatise from his own hand (possibly the just-mentioned *Traicté de la praticque d'algorisme*) aimed at giving his readers

which Nicolas Chuquet attached to his *Triparty en la science des nombres*.

No known source ever treated the genre as fully as Barthélemy de Romans' *Compendy*. Maryvonne Spiesser [2003] not only offers an edition of the text (pp. 391–423) and a translation into modern French (pp. 543–579) but also a substantial commentary (pp. 139–156), of which I shall take advantage so as to concentrate on what is important in the present context; page references to the treatise refer to Spiesser's edition.

In general, Barthélemy prefers to present first the general principles of a matter, and afterwards the examples. Thus also to some extent here, but with the proviso that this part of the text falls in two major sections, each of which contains general principles and examples.

Barthélemy gives the genre a name not known from earlier sources and probably his own invention, *progressions composees*;[27] he also gives a name to the quantity $^{1}/_{\phi} = {}^{q}/_{p} = d$, the *vray denominateur* or "true denominator". Since this entity was used by Fibonacci in a way that suggests the idea not to be his own and since the name is close at hand it is less certain that even this term was Barthélemy's invention.

Barthélemy starts by distinguishing between *deux manieres*, "two modes", in the first of which the absolutely defined contributions (*les nombres de la progression*) are taken first and the fraction of what remains (*la partie ou les parties que l'on veut du demourant*) afterwards; in the second, the "part or the parts" are taken first, and afterwards "the numbers that make the progression" from what remains. Then the "true denominator" is explained and exemplified, and it is pointed out that in the first mode, *four numbers* are fundamental: the true denominator (d), "the number that is one less than the denominator" (d–1), "the number which makes the progression" (ε) and the "number by which the progression starts" (α); he does not forget to say that the latter two may be equal, but they should none the less be treated as different. He also points out that three hidden numbers are sought for, "the number that can be divided by this progression" (T), "how much there will be in each place" (Δ) and "how many places there will be in the progression" (N); he claims as a general fact that T >

profounder understanding.

[27] Firstly, the topic is never grouped together with arithmetical progressions in other sources; secondly, there are some suggestions in Barthélemy's text that he might be accustomed to find it together with the double false position, in agreement with the occasional use of this method to solve the problems – see below.

$\Delta > N$.[28]

Thereby he has come to the enunciation of a "general rule" for progressions of the first kind:

(15^c^) $$\Delta = (d-1)\cdot\varepsilon\ ,$$

(15^a^) $$T = ([d-1]\varepsilon-\alpha)\cdot d+\alpha\ ,$$

(15^b^) $$N = T/\Delta\ .$$

(15^c) coincides with (10^c), and (15^a) easily reduces to (10^a), whereas Fibonacci's (4^a) reduces to $([d-1]\alpha-[\alpha-\varepsilon]d)\cdot(d-1)$ if we introduce into it the true denominator *d*. The rule is illustrated by three examples of types $(3,3\,|\,{}^1\!/_7)$, $(2,3\,|\,{}^2\!/_{11})$ and $(3,2\,|\,{}^3\!/_{13})$. The first example is told to deal with the division of a number in agreement with the progression – in the end it turns out that a division among *N* "men" is thought of; the two others only speak about "making a progression". We notice that in the first problem, $\alpha = \varepsilon$, in the second $\alpha > \varepsilon$, in the third $\alpha < \varepsilon$. This principle is pointed out by Barthélemy. He also observes, however, that the first deals with "one part", the second with "two parts", the third with "three parts"; this is wholly unimportant as long as the "true denominator" is used, and could be a reminiscence of the similar distinction (though only between "a part" and "parts") in Boethian arithmetic.

Then Barthélemy points out that the problems where $\alpha = \varepsilon$ "can be done by another practice, for which this is the appurtenant rule":

(16^b) $$N = d-1\ ,$$

(16^c) $$\Delta = (d-1)\cdot\varepsilon\ ,$$

(16^a) $$T = N^2\cdot\varepsilon\ ,$$

This rule is then applied to a final example of the first mode, $(3,3\,|\,{}^2\!/_9)$, and it is pointed out that the outcome would have been the same if rule (15) had been applied. From Barthélemy's words and argument it is fairly obvious that he did not arrive at the specific rule by reducing the general one; but is seems likely that he himself formulated *as a rule* a practice that he had only encountered in the shape of particular problems (since the inheritance problems are all of this type, many with $\varepsilon = 1$ but others with $\varepsilon = 10$, $\varepsilon = 100$ or $\varepsilon = 1000$, this is quite possible). He does not bother the reader with any argument that one set of rules can be derived from the other by reduction, and the formulation of a such an argument would indeed be quite cumbersome in the absence of algebraic symbolism (provided Barthélemy had the idea, which is far from certain –

[28] As we have seen, this is not strictly true – if $\alpha = \varepsilon = 1$, $N = \Delta$. But for all other integer positive values of α and ε (the only ones considered by Barthélemy and our other authors) it is true for acceptable values of *d*.

mathematical intuitions are rarely more than one step in advance of that which established familiar terminology and concepts can grasp).

For the "second mode" this rule, valid for the case $\alpha = \varepsilon$, is given first:

$$N = d-1 \,, \tag{17^b}$$

$$\Delta = d\cdot\varepsilon \,, \tag{17^c}$$

$$T = (d-1)\cdot d\cdot\varepsilon \,, \tag{17^a}$$

which is then applied to the cases ($^1/_7$ | 2,2) and ($^2/_{11}$ | 3,3). Nothing is said about this rule corresponding to a practice, but that may be because the corresponding general rule has not yet been presented – indeed, when all the rules with appurtenant examples have been explained, they are spoken of as *les praticques precedants*. In any case there is no doubt that this is the counterpart of the simplified rule (16) for the case (ε,ε | d).

There may be a good reason for giving separately the rule for the case $\alpha = \varepsilon$. Afterwards, indeed, separate rules are given for the cases $\alpha < \varepsilon$ and $\alpha > \varepsilon$ – and these rules *have* to be stated separately, because they are of the same type as Fibonacci's (5) and (6) though not exactly the same – respectively

$$T = \frac{[(\varepsilon-\alpha)\,q+(q-p)\,\alpha]\cdot q}{p^2} \,, \tag{18^a}$$

$$N = \frac{(\varepsilon-\alpha)\,q+(q-p)\,\alpha}{\varepsilon p} \,, \tag{18^b}$$

$$\Delta = \frac{\varepsilon q}{p} \,, \tag{18^c}$$

and

$$T = \frac{[(q-p)\,\alpha-(\alpha-\varepsilon)\,q]\cdot q}{p^2} \,, \tag{19^a}$$

$$N = \frac{(q-p)\,\alpha-(\alpha-\varepsilon)\,q}{\varepsilon p} \,, \tag{19^b}$$

$$\Delta = \frac{\varepsilon q}{p} \,. \tag{19^c}$$

The examples are ($^1/_7$ | 3,5), ($^2/_9$ | 3,5), ($^6/_{31}$ | 2,3), ($^1/_6$ | 5,3), ($^2/_{11}$ | 5,2) and ($^5/_{19}$ | 5,3).

The totally different approaches to the two modes, one by means of the true denominator and the other one (except when $\varepsilon = \alpha$) by means of p and q, suggests that all the rules presented here are borrowed (this is also suggested by Maryvonne Spiesser [2003: 152]). The discrepancy between Barthélemy's treatment of the two modes makes it implausible that the *Liber abbaci* was his source.[29]

What comes from this point (p. 402) onward is likely to be Barthélemy's own original contribution. First he offers a systematic exposition of the principles of rules (18) and (19) together with their counterparts for the first mode (almost coinciding with (1) and (4) as set forth in the *Liber abbaci*) and summarizes everything in a single rule (even here, the ambiguities of a verbal expression makes him insert an example ($^6/_{31}$ | 3,3)); next, "for the practice of this rule and in order to see rapidly how one should make the necessary multiplications for

[29] Maryvonne Spiesser [2003: 156] finds it to be a "very plausible" hypothesis that the *Liber abbaci* was the direct source – a conclusion which I endorsed in [Høyrup 2005b] because of the lack of evidence for alternatives. Since the present broader investigation of the question shows that non-Fibonacci solutions even to the sophisticated versions *must* have circulated, this argument can no longer be considered valid. Spiesser takes the shared occurrence of uncommon fractions like $^5/_{19}$ and $^6/_{31}$ as supplementary evidence for an intimate connection; however, Barthélemy's range of non-aliquot-part fractions ($^2/_{11}$, $^3/_{13}$, $^2/_9$, $^6/_{31}$, $^5/_{19}$, $^3/_{11}$, $^4/_{15}$, $^5/_{21}$, $^6/_{25}$, $^4/_{27}$) goes far beyond what we find in Fibonacci ($^2/_{11}$, $^5/_{19}$, $^6/_{31}$) but remains within the same vaguely defined family – all denominators are odd, most are prime, the values fall between 0.148 and 0.272 (all but one between 0,181 and 0.272), the numerators being evidently larger than 1; no denominator except 13 occurs in more than one fraction. If we restrict ourselves to those of Barthélemy's fractions which appear in the part of his text discussed so far, that is, the part which seems to build upon borrowed rules and therefore perhaps also on borrowed examples ($^2/_9$, $^2/_{11}$, $^3/_{13}$, $^5/_{19}$, $^6/_{31}$), the characteristics are even more narrowly defined: all values fall between, 0.181 and 0.263, no denominator appears more than once and all denominators are prime. Strikingly, *all but two* non-reducible fractions with denominators below 37 which fulfil these (partly mathematical, partly aesthetic) criteria are used – the exceptions being $^5/_{23}$ and $^7/_{29}$. If both Fibonacci and Barthélemy drew on a fund of problems defined by these criteria, simple statistics shows us that the coincidences are not striking: if Fibonacci were to select 3 from the list of 7 possible fractions, the probability that all three would fall within the range of 5 values used by Barthélemy is $\binom{5}{3}\div\binom{7}{3} \approx 28\%$. The uniformity of the possibly borrowed examples in Barthélemy's text shows that such aesthetic and mathematical criteria were efficient (his own probably added examples, though widening the limits of the permissible a bit, also confirms that the criteria were felt, since his deviations from the canon that is implicit in the first part are quite modest).

Further, *if* Barthélemy had really borrowed from Fibonacci problems with ϕ equal to $^2/_{11}$, $^5/_{19}$ and $^6/_{31}$, one should also expect him to have borrowed the appurtenant sets (α,ε) – but this only happens in 1 of 9 instances (1 of 7 if we count pairs (α,ε | ϕ) and (ϕ | α,ε) with coinciding parameters as a single instance), namely for the case ($^6/_{31}$ | 2,3). Given how often the set (α,ε) = (2,3) is used, this is once again no more than could be expected from a random distribution.

the three numbers that should be divided by the three dividers to get the three hidden numbers”, he shows “how the necessary numbers can be put into a diagram”, as shown in Figure 6 – at first in general form, with the numbers described in technical verbal terms defined by Barthélemy (here replaced by our usual symbols).

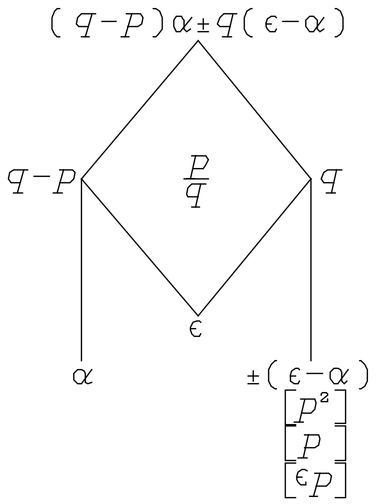

Figure 6

A new sequence of numerical examples follows in which the diagram is used, all in pairs representing the two modes.[30] At a certain point (p. 413) he shows how the diagram applies to the rules based on a “true denominator”. He explains that the three numbers in bottom (not counting those in [ ]) are integers and the others actually fractions, a denominator equal to p being tacitly understood, and that there is only one divisor (*viz* εp, which reduces to ε, p and p^2 being both reduced to 1). The exposition corresponds to what is shown here in Figure 7, and so do the diagrams used in the subsequent numerical examples.[31]

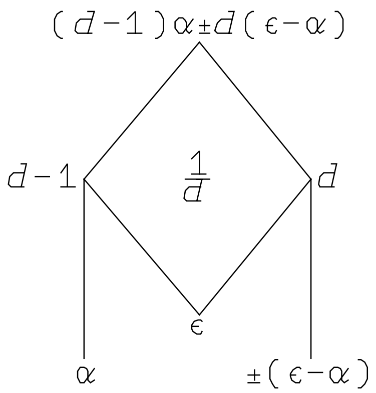

Figure 7

The whole treatment of division according to progressions is made under the general heading of “two false positions”, whose rule is simply stated (p. 390) as *plus et plus, meins et meins, sustrayons. Plus et meins, adjoustons* – “More and more, less and less, we shall subtract. More and less, we shall add”. The meaning is that if both initial guesses lead to an excess or to a deficit, the rule with addition is to be used. If one leads

[30] The three divisors written in [ ] in the diagram – sometimes as here to the right, sometimes to the left – are not in the general diagram but only in the particular examples.

[31] Maryvonne Spiesser [2003: 148] finds that “the author gets lost and loses us in an exposition that seems to lead nowhere” in this change between two representations of the problem. Once we have accepted that both sets of rules offered in the first part of the chapter are inherited, one might rather find the present discussion to be a praiseworthy (and, on the conditions of the terminological difficulties, mathematically blameless) verification that the two approaches are equivalent. This time Barthélemy does not satisfy himself with a control that the two ways lead to the same numerical result (as earlier on, when the equivalence of rules (15) and (16) were argued, see p. 23, and as commonly done in the abbacus tradition).

to an excess and the other to a deficit, the variant with subtraction should be used. The rule itself (weighing the two guesses in inverse proportion to their error) is not presented, instead Barthélemy goes directly first (briefly) to “simple” (that is, arithmetical) and then to the composite progressions discussed here.

On p. 420 Barthélemy returns to the topic of the heading and legitimizes it by a claim that distributions according to progressions cannot be made by means of the rule of three or a single false position but only by a double false position. As regards the proportional distribution according to a given arithmetical progression this is evidently false. However, Barthélemy asks for something different, namely the starting point α of an arithmetical progression $\alpha+(\alpha+\varepsilon)+...+(\alpha+4\varepsilon)$ with given sum (e.g., 60) and given ε (e.g., 3), and then he is right. After that he submits the composite progressions to the double false. His method is not the one used in the *Istratti di ragioni* (see above, p. 18) and not independent of the rules that he has already set forth (and hence it presents no alternative to these). Indeed, Δ is first found by (15^{c}) or (17^{c}), depending on the mode; afterwards, two guesses for T are used, and for each the first share ($\alpha+\phi\cdot(T-\alpha)$ or $\phi T+a$, depending on the mode) is calculated; from the two errors the true value of T can then be determined. In order to show how convenient guesses depend on the value of ϕ, two examples follow – $(2,3\,|\,{}^{2}/_{7})$, for which the guesses are $T_1 = \alpha = 2$, $T_2 = \alpha+q = 9$, and $({}^{1}/_{4}\,|\,5,3)$, with guesses $T_1 = q = 4$, $T_2 = 2q = 8$).[32]

The first, general part of the discussion of the use of the double false position is illustrated by a truncated version of the diagram (Figure 8), containing what

[32] Thereby, the complete list of Barthélemy’s examples is:

$(3,3\,\|\,{}^{1}/_{7})$	$({}^{1}/_{7}\,\|\,3,5)$	$({}^{6}/_{31}\,\|\,3,3)$	$(2,3\,\|\,{}^{5}/_{21})$	$({}^{3}/_{11}\,\|\,3,3)$
$(2,3\,\|\,{}^{2}/_{11})$	$({}^{2}/_{9}\,\|\,3,5)$	$(3,3\,\|\,{}^{3}/_{11})$	$({}^{5}/_{21}\,\|\,2,3)$	$(3,5\,\|\,{}^{6}/_{31})$
$(3,2\,\|\,{}^{3}/_{13})$	$({}^{6}/_{31}\,\|\,2,3)$	$({}^{3}/_{11}\,\|\,3,3)$	$(5,3\,\|\,{}^{6}/_{25})$	$({}^{6}/_{31}\,\|\,3,5)$
$(3,3\,\|\,{}^{2}/_{9})$	$({}^{1}/_{6}\,\|\,5,3)$	$(2,2\,\|\,{}^{4}/_{15})$	$({}^{6}/_{25}\,\|\,5,3)$	$(5,3\,\|\,{}^{6}/_{25})$
$({}^{1}/_{7}\,\|\,2,2)$	$({}^{2}/_{11}\,\|\,5,2)$	$({}^{4}/_{15}\,\|\,2,2)$	$(3,2\,\|\,{}^{4}/_{27})$	$({}^{6}/_{25}\,\|\,5,3)$
$({}^{2}/_{11}\,\|\,3,3)$	$({}^{5}/_{19}\,\|\,5,3)$	$(3,5\,\|\,{}^{6}/_{31})$	$({}^{4}/_{7}\,\|\,3,2)$	$(2,3\,\|\,{}^{2}/_{7})$
		$({}^{6}/_{31}\,\|\,3,5)$	$(3,3\,\|\,{}^{3}/_{11})$	$({}^{1}/_{4}\,\|\,5,3)$

The two columns to the left contain what is likely to be borrowed material, the three to the right what he probably constructed himself in order to illustrate the general rule and the use of the diagram.The somewhat wider limits for the choice of ϕ was already discussed; everywhere, we notice, α and ε are chosen among the numbers 2, 3 and 5.

is needed for the determination of Δ. Already for this reason – but also because of the rather pointless introduction of an alternative that is no proper alternative, we must presume Barthélemy to be responsible for the chimaera in question. However, the precedent of the *Istratti di ragioni* makes it plausible that the use of the rule of double false for such problems was known; this would also explain why Barthélemy dealt with the topic under a heading with which it has preciously little to do, and where the fragile connection that does exist is only shown in the very end.

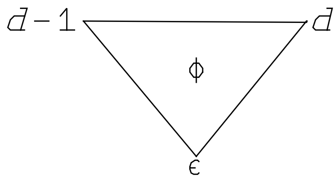

Figure 8

Chuquet

Apart from Barthélemy, nobody dedicates as much space to the genre as does Chuquet. The place where he does so is in the problem collection attached to his *Triparty* from 1484. The problems, as listed in [Marre 1881: 448–451], are of the following types:

$(1,1 \mid {}^{1}/_{10})$ $(2,1 \mid {}^{1}/_{7})$ $(2,3 \mid {}^{1}/_{8})$	$(2,3 \mid {}^{2}/_{11})$ $(3,2 \mid {}^{3}/_{13})$ $({}^{1}/_{7} \mid 2,2)$ $({}^{2}/_{11} \mid 3,3)$	$({}^{1}/_{7} \mid 3,5)$ $({}^{2}/_{9} \mid 3,5)$ $({}^{6}/_{31} \mid 2,3)$	$({}^{1}/_{6} \mid 5,3)$ $({}^{2}/_{11} \mid 5,2)$ $({}^{5}/_{19} \mid 5,3)$

The first problem in the left column is the one we encountered repeatedly, one of the two paradigmatic types – the other being $(1,1 \mid {}^{1}/_{7})$; the second problem belongs to the same kind as the one found in the *Livero de l'abbecho*. The third still has an integer denominator and looks simple, but this appearance is already deceitful: the parameters lead to a non-integer value of Δ. None of these are found in Barthélemy's text. The rest are identical with problems in the *Compendy* which Barthélemy is likely to have borrowed. Of these presumably borrowed problems only one is omitted by Chuquet – namely $(3,3 \mid {}^{2}/_{9})$; moreover, Chuquet brings them in exactly the same order as Barthélemy. This can only have one of two explanations: either Chuquet copied from Barthélemy, or both build (with or without written intermediaries) on the same written source – no oral tradition would conserve the order of 10 problems intact when this order is not dictated by some inner principle. Given that Chuquet stops exactly at the point in Barthélemy's list where the latter appears to begin his own contributions, a shared source might seem to be the most likely explanation. On the other hand, Chuquet was familiar with other parts of the *Compendy* – he refers to Barthélemy by name when discussing his solution to a problem coming shortly before the composite progressions [ed. Marre 1881: 442], and Chuquet may have chosen to stop where Barthélemy goes into a "theoretical" exposition which he did not agree with. All in all, the shared source is a superfluous hypothesis which should fall victim to Occam's razor.

Indeed, Chuquet's treatment of the material also differs from Barthélemy's. Firstly, all Chuquet's problems are dressed in the traditional way, as dealing with a father distributing the unknown contents of a chest to an unknown number of children; even when N is not integer, Chuquet speaks of it as "the number of children". Secondly, he appears to enunciate only one rule,[33] after the second problem:

> Multiply the number which is 1 less than the denominator of the common part by the number which makes the progression. Which multiplication [i.e., product] you put aside, because it is the number of deniers which each one shall receive. Then subtract from this multiplication the number which the first one takes when he goes to the box, that is the number by which the progression begins. And multiply the remainder by the denominator of the common part, to which multiplication join the number by which the progression begins, because the addition [i.e., sum] is the number of deniers in the box. Which number divide by the multiplication which was put aside, that is, by the share which each one gets, and you have the number of children.

In symbols once more:

(15^c) $$\Delta = (d-1)\cdot\varepsilon \ ,$$

$(15^{a)})$ $$T = ([d-1]\cdot\varepsilon-\alpha)\cdot d+\alpha \ ,$$

(15^b) $$N = T/\Delta \ ,$$

that is, Barthélemy's "general rule" for the first case (above, p. 23); Chuquet, however, speaks of d simply as the denominator, not as any "true denominator", and at this place in his text only integer values for d have in fact occurred.

What can be concluded is, firstly, that Chuquet knew the genre not only from the *Compendy* but also from elsewhere; secondly, that he was not very fond of Barthélemy's ways to transform it into some kind of coherent *theory* – as we know, he had his own ways.[34] He actually closes the sequence by the remark

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[33] "Appears" because Marre's transcription is incomplete, leaving out the calculations; however, since Marre includes one rule he would probably have included others if they had been there. This inference was confirmed to me by Stéphane Lamassé (personal communication), who has inspected the manuscript.

[34] There is indeed a fundamental difference between Barthélemy's and Chuquet's aims. Barthélemy's schemes are similar in spirit to the schemes used in Indian medieval

[ed. Marre 1881: 451] *Toutes telles raisons facilement se peuent faire par la rigle des premiers*, "all such calculations can easily be done by the rule of algebra".

As we have seen, it is not quite easy to make a genuine complete algebraic solution. Whether Chuquet thought of making it is uncertain; he may well have been satisfied with incomplete solutions like the one offered by Fibonacci.

The Aftermath in Italy

The "unknown heritage" did not disappear after Chuquet, its appeal caused it to be repeated in several Italian problem collections in the outgoing fifteenth and the sixteenth century.[35]

One of these collections is Filippo Calandri's *De arimethrica opusculum* from 1491, republished in 1518. Here [Calandri 1518: i 5] we find an inheritance problem of type ($^1/_{10}$ | 1000,1000), with mere indication of the answer. Another one is Francesco Ghaligai's *Summa de arithmetica* from 1521 (later editions under the title *Praticha d'arithmetica*), which has the problems (1000,1000 | $^1/_7$) and ($^1/_7$ | 1000,1000) and gives the usual simple rules [Ghaligai 1572: 65^r].[36] Both deal with an inheritance; for later use we observe that Ghaligai's testator is a *padre di famiglia*, a paterfamilias, and that the equality of the shares is only discovered by his children after his death – two details which are not found in any of the examples mentioned so far except in part with Chuquet, who has a *père de*

mathematics, schemes which Nesselmann [1842: 302] saw as a kind of genuine *symbolic* algebra but which do not allow embedding and therefore can express only that which is already known as an algorithm – Barthélemy's transformation of the scheme when he replaces $^p/_q$ with $^1/_d$ is the maximal flexibility it allows and already strains it. Chuquet's use of underlining with parenthesis-function and his arithmetization of the designation of roots and powers of the unknown, on the other hand, is a first step in the development of productive symbolization (the term "productive" understood as in linguistics).

[35] Among the abbacus works which I have looked through *without* finding it, Piero della Francesca's *Trattato d'abaco* [ed. Arrighi 1970], Benedetto da Firenze's *Tractato d'abbacho* [ed. Arrighi 1974] and Luca Pacioli's *Summa* [1494] should be mentioned. It is also absent from Pedro Nuñez' *Libro de algebra* [1567].

[36] "Do thus, always subtract 1 from 7, that is $^1/_7$, 6 remains, and so many were the sons, which 6 multiply by itself, it makes 36, and this multiply by s. 1000, it makes s. 36000, and so much money was in the box; and in order to know how much is due to one, divide s. 36000 by 6, s. 6000 results"; and "subtract again 1 from 7 that have signified $^1/_7$, 6 remain, and so many were the sons, then multiply 6 by 7, it makes 42, ...".

famille.[37]

Even the various rules for the sophisticated cases must still have been accessible in Italy (though perhaps in corrupt versions) well into the sixteenth century in ways we do not know about – in the *Practica arithmetice et mensurandi singularis* [1539: fol. FF ii^{r}], Cardano deals with the case ($^{1}/_{7}$ | 100,100) not according to the usual rule but in agreement with this one:[38]

(20^{b}) $$N = q–p\ ,$$

(20^{a}) $$T = \frac{[(q–p)q]\cdot\alpha}{p^2}\ .$$

(20^{a}) would result if $\alpha = \varepsilon$ was inserted in Fibonacci's rule (5^{a}) though with a different order of the factors, which in itself makes the *Liber abbaci* an unlikely source. However, (20$^{b)}$ is a mistake[39] for

(20^{b}*) $$N = \frac{q-p}{p}\ .$$

This mistake makes it utterly implausible that Cardano should have used Fibonacci's work directly. Nor, as we see, can he have used any of the two recently printed works where the problem type is present – Filippo Calandri's and Ghaligai's.

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[37] Later on in the sixteenth century, Tartaglia presents the simple problem both in the *Quesiti et inventioni* [1546: 98$^{r–v}$] ((1,1 | $^{1}/_{8}$), saying that it had been proposed to him in 1524 by one fra Raphaelle) and in the *General trattato* [1556: I, 245^{v}–246^{f}] (1,1 | $^{1}/_{6}$) – told here about a merchant who finds a purse and distributes the *ducati* it contains to his sons). In both works, the rule is said to be that the subtraction of 1 from the denominator gives the number of sons as well as the amount each one receives; also in both works, the outcome of variations of the denominator ($^{1}/_{7}$ being the alternative in the former work, $^{1}/_{7}$ and $^{1}/_{13}$ in the latter) is explained.

[38] The story is singular: "Some dying man left sons and *aurei*, not knowing how many, and ordered that when the first returned, he should receive $^{1}/_{7}$ of the total and 101 [sic, error for 100] more, and the second ...". The equality of the shares is only discovered as the sons have returned, implicitly thus after the death of the testator.

[39] Since the division by 1^2 is dutifully performed in (20^{a}), we are really entitled to speak of a mistake.

Elsewhere

In a personal communication, Maria do Céu Silva has kindly informed me about two sixteenth-century Portuguese occurrences of the simple version. The first is in Gaspar Nicolás' (or Nycolas') *Tratado da pratica Darismetyca* from [1519: fol. 59^v–60^v], the other in Bento Fernandes' *Tratado da arte de arismética* from [1555: fol. 102^r]. Both deal with an inheritance, and the formulations suggest them to be mutually independent. Nicolás deals with the case $(1{,}1 \mid {}^{1}\!/_{7})$, with the alternative $(1{,}1 \mid {}^{1}\!/_{9})$, Fernandes with $(1{,}1 \mid {}^{1}\!/_{14})$. Both make a complete verification of the result similar to what was offered by Jacopo and by none of the later Italian authors we have considered, and Nicolás even introduces it with the same phrase,[40] but already the inheritance dress shows that Jacopo is not their source. Nor are they based directly on any of the occurrences discussed above, but both share characteristic phrases with the *Trattato di tutta l'arte dell'abacho* (above, p. 20) – phrases which are also somewhat similar to what we find with Chuquet.[41] Fernandes shares with Ghaligai the idea that the equality of the shares is discovered after the death of the testator.[42]

These similarities suggest that the Portuguese writers draw on an Ibero-Provençal rather than Italian traditions (for this problem – in other respects it is highly probable that Fernandes drew on Italian material [do Céu Silva 2006]. The German occurrences of the problem are more likely to be based on Italian inspiration.

The first of these is among the supplementary problems which Friedrich Amann inserted in the *Algorismus Ratisbonensis*, ms. Clm 14908 [ed. Vogel 1954:

[40] Jacopo, "se la voli provare", Nicolás, "se qyuseres prouar". Fernandes has "como podeis prouar".

[41] Where *Trattato de tutta l'arte* starts "A man has his sons, I do not know how many [non so quanti], and gives them *denari*, I do not say how many", Nicolás' problem runs "There is a man who has sons, I do not say how many [nam dygo quantos], and he also has *cruzados*, I do not say how many". Chuquet, in the same vein but not quite the same, tells that "there is a paterfamilias, who has children, one does not know [on nescet] the number. And there is in his chest a sum of *deniers*, of which one does not know the amount [le compte]".

Where the *Trattato* tells about the absolutely defined contributions that they "grow [crescie] for each one florin", Fernandes state that "grow [vay crecendo] for each son one cruzado". Chuquet has "en augmentant tousious la porcion de ses enfanss de 1 denier".

[42] As we remember, Ghaligai shares the "paterfamilias" with Chuquet, who however only lets the children "discover" the equality of their shares (in fact, Chuquet does not speak of a testament but of money distributed from a chest).

64*f*] in 1461.[43] Friedrich (like Chuquet) does not speak of an inheritance but of a distribution of money (florins) from a *wechselpanck*. He gives two examples, one of type (1,1 | $^1/_{10}$) (sons) and one of type (1,1 | $^1/_6$). He gives the usual rule, but after the second problem he adds the rule for (a slightly corrupted version of) the problem type ($^1/_6$ | 0,1), namely $N = \Delta = d$ – correct but not found in any other source.

In 1467–68, Magister Gottfried Wolack held a lecture in Erfurt University which is the earliest public presentation of abbacus mathematics we know about in Germany (unless we count the copying of manuscripts of the *Algorismus ratisbonensis* as such); its Latin manuscript appears to have had a certain influence.[44] As a "tenth rule called 'of equality of parts'" he presents a problem of type (1,1 | $^1/_{10}$) [ed. Wappler 1900: 52*f*], must either be a slightly paraphrasing translation of Friedrich Amann's first problem or build on a close source for this problem; Wolack's rule is also formulated in very similar words.

Since Johannes Widmann knew Wolack's manuscript very well [Wappler 1900: 54*f*], Wolack *could* be behind the appearance of the same type in Johannes Widmann's *Behend und hüpsch Rechnung vff allen Kauffmanschaften* from 1489 (partial facsimile reproduction in [Tropfke / Vogel et al 1980: 589]. Widmann's formulation, however, is quite different from what we find in Amann and Wolack – Widmann starts by explaining that the intention of the testator was to give the same to all his children. Widmann, on his part, is certainly the direct or indirect source for Christoff Rudolff [ed. Stifel 1615: 416] – Widmann's unusual initial explanation and other particulars are borrowed. But Rudolff (whose aim it is to show the efficiency of *coss*, algebra) does not refer to a rule, instead he offers an algebraic solution (based on the equality of the first two shares, and as usually not controlling the validity of the solution).[45]

After Rudolff and Stifel, no German author seems to have been interested

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[43] For the description of the various manuscripts of the *Algorismus* and the dating of this particular part of the relevant manuscript, see [Vogel 1954: 10–12, 14]. For the identification of the frater Fridericus who wrote the manuscript with Friedrich Amann (and not with Friedrich Gerhart), see [Gerl 1999].

[44] At least in 1900, three manuscripts existed (Leipzig, Dresden, Munich); moreover, it was studied by Johannes Widmann, who may even have used it for his teaching. See [Wappler 1900: 47, 54*f*].

[45] This is at least what is found in Michael Stifel's "improved and expanded" edition (1553); I have not been able to inspect Rudolff's original from 1545, but [Tropfke / Vogel et al 1980: 588] signals no difference between the two versions; in both editions, the problem is no. 110.

in the unknown heritage. In France it had a more persistent success after having been taken up by mathematically interested humanists (as the Germans, they stick to the simple versions). The earliest examples I know about are in Buteo's *Logistica* from 1560 and Bachet's *Problemes plaisans et delectables qui se font par les nombres* from 1612 (I used second edition from 1624). Buteo as well as Bachet and Ozanam, the latter in 1694, take up some of the typical Ibero-Provençal formulations (not the same!), suggesting that the whole French branch did not depend on Italian inspiration.

Buteo [1560: 286–288], unprecedented but quite reasonably, thinks the testator must be a *vir logisticus*, a calculator; his testament is of the type ($^1/_6$ | 100,100). Quite exceptionally, the first share is that of the youngest heir; the equality of shares is discovered only after the testator has passed away – suggesting that the heirs / readers are supposed to expect the youngest to have received the least, with only 100 *aurei* beyond the $^1/_6$ which everybody gets (no writer for merchants and merchant sons had ever expected such mathematical naivety!).

Buteo, well versed in much more than abbacus mathematics and the abbacus norm for what constituted an adequate explanation, starts by pointing out that if each had received only $^1/_6$, the number of heirs would have been 6; under the actual circumstances, however,

> the rule is that you always remove a unit from the name of the fraction, which is 6, 5 remains for the number of sons.[46] And hundred *aurei* in addition can be nothing but the sixth of the share. This will therefore be 600 *aurei*. Multiply 600 by 5, the number of sons, it results that there were 3000 *aurei* in the money.

As we see, no argument is given for the rule $N = d–1$; the assertion "hundred *aurei* ... can be nothing but" uses that $\Delta = {}^1/_5 T = {}^1/_6 T+100$ (whence $\Delta = {}^5/_6\Delta+100$, and therefore $100 = \Delta–{}^5/_6\Delta = {}^1/_6\Delta$). Finally it is added that the fraction cannot exceed $^1/_3$, because there can be no less than two heirs, and that the denominator of the fraction always exceeds the number of heirs by a unit.

Bachet's problem [1624: 221–226] is of the type (1,1 | $^1/_7$), dealing with "a man who is going to die"; the equality is discovered after his death. After stating the rule ($N = \Delta = 7–1$) he gives a proof that it works, very similar to that of

[46] The vocabulary shows Buteo to be rooted ideologically in the particular environment of French lawyer humanism – *arithmetica* is regarded as vulgar / vernacular for *logistica* (the title of the work), an aliquot part is a *particula* instead of *pars*, its denominator *particulae nomen* instead of *denominatio*, the number one to be detracted is a *monas* and no *unitas*, an amount of money (or the chest containing the money?) is *as*(!). Molière's *précieuses ridicules* had spiritual grandfathers who were taken very seriously in their times (and afterwards) – but Buteo, prudish as a linguist, was a good mathematician.

Planudes but using (as elsewhere in the work) the particular letter formalism developed by Jordanus de Nemore.[47] After the proof Bachet points out that one may choose a different denominator (if only the same fraction is used for all children, and if only the numerator is 1 – otherwise, the problem is told to be impossible) or take different absolute contributions, if only (in our terminology) $\alpha = \varepsilon$; then N is still d–1, but Δ becomes $\alpha\cdot(d–1)$. The proof of the corresponding rule is left to the reader.

After that, the rule for the case ($^1/_7$ | 1,1) is stated, and it is said that the proof is analogous. Bachet goes through the generalization to cases ($^1/_d$ | α,α), and once more states (in our terminology) that d must be integer and $\alpha = \varepsilon$.

Already closer to the Enlightenment and its use of science as polite leisure is Ozanam's *Récréations mathématiques et physiques* from 1694. The genre is represented once [Ozanam 1778: I, 185], namely by the type (10000,10000 | $^1/_7$). The testator, as with Chuquet and Ghaligai, is a *père de famille*; as with Ghaligai, Buteo and Bachet, the equality of the shares is discovered after the death of the testator.

Ozanam does not state the usual rule, nor any other. Instead, his explanation runs as follows:

> One finds, by the analysis, that the possession of the father was 360000 *livres*; that there were six children, and that each of them received 60000 *livres*. Indeed, the first taking 10000, the remainder of the possession is 350000 *livres*, the seventh part of which is 50000 which, with 10000, makes 60000 *livres*.The first child having taken his share, 300000 *livres* remain; from which sum, when the second has taken 20000 *livres*, the remainder is 280000, the seventh part of which is 40000 which, with the above 20000, still makes 60000 *livres*. And so on.

It is possible (but barely) that the calculation which follows upon the phrase "indeed" (*en effet*) is meant to represent the "analysis" referred to initially (which would evidently be a misuse of this high-flown concept but might sound well in the ears of that public upon which Ozanam depended for his living); it is also possible that he did perform some kind of analysis or thought of Bachet's proof (which indeed is no analysis but a synthesis *a posteriori*) but did not find it adequate for the same public; most likely the term is an empty claim. In any case it presents us with no evidence that Ozanam understood the matter better

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[47] Bachet may have known it from Lefèvre d'Étaples' edition of Jordanus's *Arithmetica demonstrata* [1514]. The formalism should not be mistaken for an algebraic symbolism, since each operation leads to the introduction of a new letter. In the present case, B is thus 7, B–1 becomes A, A–1 becomes C, $A\cdot A$ becomes F, $B\cdot C$ becomes G, etc. The symbolism allows generality of the argument, not algebraic manipulation.

than, say, Jacopo da Firenze.

As mentioned initially, Euler deals with "this question [which] is of a quite particular nature and therefore deserves attention" in the *Élémens d'algebre* [1774: 488–491]. Unlike all writers on the topic since the fifteenth century except Bachet, Euler gives a mathematical argument for the solution. In a problem of type (100,100 | $^1/_{10}$), he introduces the variables z (our T) and x (our Δ), concluding that the successive remainders are z, $z–x$, $z–2x$, ..., $z–5x$, ..., finds the successive shares according to the prescription, and detects that the successive differences between these are "fortunately" all equal to $100–^{x–100}/_{10}$. Since they *should* be 0, he finds $x = 900$ (etc.).

Euler certainly *could* make a theoretically complete and coherent analysis which did not to appeal to the good luck of a strongly overdetermined problem – but apparently he could not do it in a way that would fit an elementary treatise.

Theoretically complete analyses (still only of the simple version) turn up in the nineteenth century. Labosne [1859: 158] gives one in his paraphrase of Bachet, but there are others. Most illuminative is perhaps the treatment of the matter which is offered by Pierre Louis Marie Bourdon in his *Élémens d'algèbre* from 1817 (a university textbook). Bourdon starts [1831: 66–71] by an easier version, almost the same as the box-problem version proposed by Ibn al-Hāʾim (see above, p. 19, 43): The number of children is given (3), the fraction is an abstract $^1/_n$, the absolutely defined contributions (assigned before the fraction) are the equally abstract a, $2a$ and $3a$.[48] Afterwards [Bourdon 1831: 71–73] comes the problem (a,a | $^1/_n$), which Bourdon points out to be overdetermined; as all algebraic predecessors, Bourdon constructs an equation from the equality of the first two shares; afterwards, he shows the validity of the solution he obtains by an algebraic version of Planudes's quasi-induction – no impressive advance in an abundant half-millennium.

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[48] It is not told explicitly, as by Ibn al-Hāʾim, that the last share consists of *nothing but* the absolutely defined contribution; but since nothing remains after the taking of $^1/_n$, this should be evident. Since the calculation runs over more than five pages (whereas my complete backward calculation of Ibn al-Hāʾim's six-child version could be made on a A6-sheet of paper), this is hardly a proof of the superiority of Bourdon's algebra.

Whence?

We may have given up the Comtean belief in general guaranteed progress. None the less, we are accustomed to believe in over-all progress in mathematical insight since the thirteenth century, caused by at least three factors:

- The general intellectual climate engendered by increased schooling and literacy at all levels;
- the recovery and digestion of the ancient mathematical legacy;
- the creation of new tools, first of all symbolic analysis.

The story surrounding the unknown heritage is a strange exception to this rule of progress, though admittedly concerned with a trifle which cannot change the overall picture significantly.

Indeed, our very first source for the genre – the *Liber abbaci* – also shows it in its fullest bloom, in the double sense of possessing already all the rules even for the sophisticated versions and of presenting a partial algebraic solution for one of these (showing it could be made for all cases). In the fifteenth century, Barthélemy also knew the rules for simple as well as sophisticated versions but offered no reasoned solution (apart from one depending on the rules); the *Istratti*, from the same century but probably going back to c. 1340, offer a partial solution of one of the simple cases by means of a double false position; dealing with a simple case, Euler does as well as Fibonacci on one of the simple cases, and uses a method which would also work for the sophisticated ones (although Euler does not say so and does not mention these) – but like Fibonacci's, Euler's approach only "fortunately happens" to work for the overdetermined problem. Well before Euler, Cardano had demonstrated to know some mutilated version of the full rules but not the reason they worked – which indeed *his* version of the rules would not have done for non-integer *d*.

Bodily organs which over time are gradually reduced by the combined force of mutations and selection are known as rudiments – and rudiments point back to a situation where their counterparts were fully efficient. Speaking of "efficiency" when we deal with a useless mathematical riddle may be unwarranted, but Fibonacci's and Barthélemy's rules are much too complicated to have been found by trial and error. Those who found them must have been good at mathematics – *very* good indeed, given that they found them without having symbolic calculation at their disposal. Whatever technique they used must have been quite refined, and thus carried by a competent environment – which should allow us to characterize them in a vague sense not only as "very good at mathematics" but as "very good mathematicians".

This leaves us with a first difficult question: Since the problem type appears to be unknown in the Arabic world (except for a clearly derivative, distorted import) and left no traces in pre-1500 Spain which we know about, *where* and *in what epoch* should we search for this environment and for these very good mathematicians?

All we can safely conclude is that they must be anterior to Fibonacci and Ibn al-Yāsamīn, and that Fibonacci had access to their results. Among the places where Fibonacci declares [ed. Boncompagni 1857: 1] to have pursued the study of abbacus matters – in his boyhood Bejaïa, and afterwards "Egypt, Syria, Greece, Sicily and Provence" – only Greece (i.e., Byzantium) and Provence fall outside the Arabic orbit with certainty, while Sicily had a mixed Arabic-Byzantine heritage).[49] The frequency with which the problem turns up in writings from early fourteenth-century Provence and the links between these, Chuquet and the Portuguese writers suggests that the encounter could have happened here, while Planudes's presentation of a proof that might reflect the original invention of the problem suggests the simple version to have been transmitted within the Byzantine orbit; Ibn al-Yāsamīn's problem makes it plausible that the problem was present somewhere in the Western Mediterranean before 1190. However, neither Planudes nor the writings from fourteenth-century Provence contain any trace of the sophisticated variants, which *could* suggest Sicily to be their cradle but proves nothing. Barthélemy's familiarity with two complete sets of rules could seem to speak in favour of Provence as an important focus, not least because Italian sources from the 140 years that separate him from Fibonacci tell us nothing about the sophisticated types. Nor do the Italian sources give any information, however, about the way Cardano acquired his partial knowledge of the sophisticated rules, which none the less he did acquire.[50]

All in all, the most certain result we get from the analysis is a general admonition that known written sources may perhaps provide us with an adequate picture of what went on in mathematics in the Christian cathedral school and incipient university environment and of that level of Arabic and Byzantine mathematics that was linked to madrasa learning, recognized scholarship and astronomy; but they do not thereby provide us with anything

[49] Fibonacci is also known to have drawn verbatim on the scholarly translations into Latin from the twelfth century even though he does not mention them (see, for instance, above, note 7), but no Latin source of this kind appears to be relevant for the question.

[50] However, the faint echo of Chuquet and the Portuguese we find in Cardano's story ("not knowing", etc.) may imply that this apparent objection is not really one.

like a complete canvas of what went on in mathematics. Even if we limit our interest to advanced matters, much remains to be known – if it *can* be known at all.

Who used pebbles?

The formulation of "a first difficult question" promises that there will be at least one more question. The first question asked for the environment where the sophisticated versions of our problem were formulated and solved; the second one is a return to the question of the first origin of the simple version.

Planudes's "theorem" corroborates the hypothesis that the invention was based on pebble counters placed in a square pattern. It constitutes no absolute proof, but let us take the hypothesis for granted for a while. Should we then make the further inference that we are confronted, if not necessarily with a Pythagorean discovery then at least with a discovery belonging within the circuit of early Greek theoretical arithmetic? This is the second question.

Prima facie, the answer need not be affirmative. Pebble arguments were certainly used within that environment – but not exclusively, as we shall see. Evidence that the general Greek public (and not only some closed Pythagorean circle) could be supposed to be familiar with them in the early decades of the fifth century BCE is offered by Epicharmos Fragment B 2 (ed. [Diels 1951: I, 196], a passage from a comedy fragment dated c. 475 BCE or earlier), which refers to the representation of an odd number ("or, for that matter, an even number") by a collection of ψῆφοι, pebble counters, as something trivially familiar.

Evidence that might seem to link the simple versions of our problem to Pythagoreanism is an observation made by Iamblichos in his commentary to Nicomachos's *Introduction*[51], and by various modern editors and commentators to Greek arithmetical writings[52] – namely that 10×10 laid out as a square and counted "in horse-race" as shown in Figure 9 demonstrates that

$$10\times10 = (1+2+...+9)+10+(9+...+2+1) \,,$$

whence

$$10\times10+10 = 2S_{10} \,,$$

S_n being the triangular number of order *n*. Rearranging and generalizing we get

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[51] Ed. [Pistelli 1975: $75^{25\text{–}27}$], cf. [Heath 1921: 113*f*].

[52] The diagram described by Iamblichos is identical with what we find in J. Dupuis's edition of Theon of Smyrna's *Expositio* [1892: 69 n. 14] and in Ivor Bulmer Thomas's commentary to an excerpt from Nicomachos [1939: I, 96 n. a].

$$S_n = \sum_{i=1}^{n} i = \frac{n^2+n}{2}$$

instead of the usual alternatives derived from the pairwise coupling as (1+n)+(2+[n–1])+...,

$$S_n = \frac{n\cdot(n+1)}{2} = \frac{n}{2}\cdot(n+1) = n\cdot\frac{n+1}{2} .$$

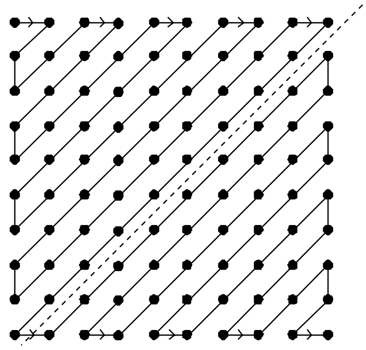

Figure 9

That the sum of two consecutive triangular numbers is a square number can be found in other authors close to the neo-Pythagorean and Platonizing current;[53] it is unlikely that anybody interested in figurate numbers should miss the point. The expression of the sum S_{10} in a way that depends on this observation is more interesting. In order to see that we shall leave the Greek cultural area for a moment.

In the cuneiform tablet AO 6484[54] (a mixed anthology text dated to the early second century BCE, thus to the Seleucid epoch), we find among other things summations of series "from 1 to 10". In obv. 1–2, $1+2+...+2^9$ is found, in obv. 3–4 $Q_{10} = 1+4+...+10^2$ is determined. The latter follows the formula

$$Q_{10} = \sum_{i=1}^{10} i^2 = (1\cdot\tfrac{1}{3}+10\cdot\tfrac{2}{3})\cdot 55 ,$$

which can be interpreted as a special case of the formula

$$Q_\mathrm{n} = \sum_{i=1}^{n} i^2 = (1\cdot\tfrac{1}{3}+n\cdot\tfrac{2}{3})\cdot S_n ,$$

S_n being still the triangular number of order n. The determination of the factor $1\cdot\frac{1}{3}+n\cdot\frac{2}{3}$ is described in precise detail; we may therefore be confident that the unexplained number 55 was indeed found as S_{10} in an earlier problem of the original text from which the anthology has borrowed its two summations.

P. British Museum 10520,[55] a Demotic papyrus probably of early Roman date begins by stating that "1 is filled up twice to 10", that is, by asking for the sums

$$S_{10} = \sum_{i=1}^{10} i \quad \text{and} \quad P_{10} = \sum_{i=1}^{10} S_i$$

and answering from the correct formulae

[53] For example, Theon of Smyrna, *Expositio* I.xxviii, ed., trans. [Dupuis 1892: 68*f*].

[54] Ed. [Neugebauer 1935: I, 96–99].

[55] Ed., trans. [Parker 1972].

$$S_n = \frac{n^2+n}{2} \quad \text{and} \quad P_n = \left(\frac{n+2}{3}\right)\cdot\left(\frac{n^2+n}{2}\right) .$$

This does not overlap with the series dealt with in AO 6484, but the four summations are sufficiently close in style to be reckoned as members of a single cluster. Moreover, the cuneiform formula for Q_n follows from the Demotic formula for P_n when combined with the observation that $i^2 = T_i+T_{i-1}$.

The two texts just cited postdate the Epicharmos fragment by centuries. Their use of a formula apparently derived from a pebble-based argument might in principle represent a borrowing of results obtained by early Greek arithmeticians. However, the total absence from the same texts of anything else which reminds of Greek theoretical mathematics makes such a borrowing unlikely. Independent adoption of the same type of Greek material in Egypt and in Mesopotamia is also hard to imagine, given the general absence of such borrowings from both the Seleucid cuneiform and the Demotic mathematical traditions.

Another piece of evidence also speaks against a Greek invention. The determination of

$$Q_{10} = 1^2+2^2+...+10^2 \quad \text{as} \quad (1\cdot\tfrac{1}{3}+10\cdot\tfrac{2}{3})\cdot\sum_{i=1}^{10} i$$

turns up again in the pseudo-Nicomachean *Theologumena arithmeticae* (X.64, ed. [de Falco 1975: 86], trans. [Waterfield 1988: 115]), in a quotation from the mid–third-century bishop and computist Anatolios of Alexandria (in a passage dealing with the many wonderful properties of the number 55). Anatolios, however, gives the sum in abbreviated form, as “sevenfold” $\sum_{i=1}^{10} i$, that is, in a form from which the correct Seleucid formula cannot be derived; this in itself does not prove that earlier Greek arithmeticians did not know better; but it shows that the Seleucid-Demotic cluster cannot derive from the form in which the formula was known to Anatolios. In addition, the absence of the formula from any earlier Greek source derived from the theoretical or Pythagorean tradition (including Theon of Smyrna and Nicomachos) suggests that the learned Anatolios has picked it up elsewhere.

All in all, the only argument in favour of a Greek theoreticians’ discovery of these summation formulae is that their shape points with high certainty toward a derivation or proof based on pebbles, and only if this observation is combined with the axiom that no mathematics not inspired by the Greeks can have been based on proofs. If this axiom is given up, we may conclude the other way around: that (heuristic) proofs based on pebbles were no Greek or Pythagorean invention but part of the heritage which the Greeks adopted from the cultures of the Near East – most likely from that practitioners’ melting pot of which the various shared themes and formulae of Seleucid (or earlier Babylonian) and Demotic mathematics bear witness.[56] If this is true, and *if* the inheritance problem was inspired by pebble arithmetic, then the idea might, according to the arguments given so far, just as well have arisen in the wider Near Eastern area as in a Greek environment.

However, an argument *ex silentio* supports a Greek invention.[57] Such arguments are usually weak, but the present one is not without force. Triangular and square numbers and the corresponding pyramid numbers P_n and Q_n turn up together (and always together with the sum $\sum_{i=1}^{n} i^3 = T_n^2$) in Indian sources and in al-Karajī's *Fakhrī*.[58] Higher polygonal numbers, on the other hand, are absent from these sources (of which the Indian ones, Āryabhaṭa as well as Brahmagupta and Bhāskara II, are more systematic than can be expected from the surviving random fragments of clay tablets and papyri), although they normally go together with the triangular and square numbers and their pyramids in Greek and Greek-derived writings. This difference makes it natural to suppose that the higher polygonal numbers represent a Greek theoretical elaboration, whereas triangular and square numbers and their pyramids are part of a shared Near Eastern heritage which was to spread widely.

The Seleucid and Demotic mathematical sources also contain a number of quasi-algebraic geometric problems; even these spread widely, at least to India (more precisely to Jaina mathematics as we known it through Mahāvīra), Arabic practical geometry and Greco-Roman agrimensors.[59] The total absence of anything similar to our inheritance problem therefore speaks against its presence in the shared heritage of Near Eastern calculators.

Admittedly, the problem is also absent from such Greek and Greek-derived sources where it might have been expected to turn up – the arithmetical epigrams of *Anthologia Graeca* XIV [ed. Paton 1979: V, 25–107] and Ananias of Shirak's problem collection [ed. Kokian 1919]. But absent from these – probably because they were too difficult – are also a number of problem types which we know

[56] See [Høyrup 2002].

[57] Cf. also above, p. 10, on the apparently "traditional" character of Planudes's problem and proof.

[58] See [Clark 1930: 37] (Āryabhata), [Colebrooke 1817: 290–294] (Brahmagupta), [Colebrooke 1817: 51–57] (Bhāskara II), and [Woepcke 1853: 61] (*Fakhrī*).

[59] A detailed exploration of this theme would lead much too far, but see [Høyrup 2001, 2002, 2004].

from their traces in Diophantos's *Arithmetica* I and elsewhere to have been known in the Greek world – the "purchase of a horse" etc.[60] Like these, the "unknown heritage" may simply have been too difficult to be included. But the invention might also be medieval – the fact that Byzantine mathematical *scholarship* was not at the level of ancient theoretical mathematics – see, e.g., [Tihon 1988] – does not prove that mathematical intelligence was absent from all strata of Byzantine society.

All answers to our second question remain hypothetical, but it appears that the most plausible hypothesis is that the simple version of the problem type was invented either in Greek Antiquity or in medieval Byzantium (and perhaps transmitted from there to Sicily or Provence for further sophistication). However, any discovery of the genuine problem type (not the box version) in a medieval Indian, Persian or Arabic source would force us to evaluate probabilities anew.

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[60] The undressed "purchase of a horse" is *Arithmetica* I.24–25. Further evidence that such problems must have been known is offered by the "bloom of Thymarides" presented by Iamblichos in his commentary to Nicomachos's (*Introduction* [ed. Pistelli 1975: 62–67], cf. [Heath 1921: 94*f*]), and by a passage in Plato's *Republic* (333b-c, ed., trans. [Shorey 1930: I, 332*f*]), in which the purchase *in common* of a horse is said to be an occasion in which one needs an expert – but apparently not the purchase effectuated by a single buyer, which excludes that veterinarian expertise is thought of.

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