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The Maslov index in weak symplectic functional analysis

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Abstract We recall the Chernoff-Marsden definition of weak symplectic structure and give a rigorous treatment of the functional analysis and geometry of weak symplectic Banach spaces. We define the Maslov index of a continuous path of Fredholm pairs of Lagrangian subspaces in continuously varying Banach spaces. We derive basic properties of this Maslov index and emphasize the new features appearing.

Keywords Closed relations, Fredholm pairs of Lagrangians, Maslov index, spectral flow, symplectic splitting, weak symplectic structure.

2010 Mathematics Subject Classification   Primary 53D12; Secondary 58J30

1 Introduction

1.1 Our setting and goals

First, we recall the main features of finite-dimensional and infinite-dimensional strong symplectic analysis and geometry and argue for the need to generalize from strong to weak assumptions.

1.1.1 The finite-dimensional case

The study of dynamical systems and the variational calculus of N-particle classical mechanics automatically lead to a symplectic structure in the phase space $X = \mathbb{R}^{6N}$ of position and

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impulse variables: when we trace the motion of N particles in 3-dimensional space, we deal with a bilinear (in the complex case sesquilinear) anti-symmetric (in the complex case skew-symmetric) and non-degenerate form $\omega\colon X \times X \to \mathbb{R}$. The reason for the skew-symmetry is the asymmetry between position and impulse variables corresponding to the asymmetry of differentiation. To carry out the often quite delicate calculations of mechanics, the usual trick is to replace the skew-symmetric form ω by a skew-symmetric matrix J with $J^2 = -I$ such that

$$\omega(x,y) \;=\; \langle Jx,y\rangle \quad \text{for all } x,y \in X, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in X.

For geometric investigations, the key concept is a Lagrangian subspace of the phase space. For two continuous paths of Lagrangian subspaces, an intersection index, the *Maslov index* is well-defined. It can be considered as a re-formulation or generalization of counting conjugate points on a geodesic. In Morse Theory, this number equals the classical Morse index, i.e., the number of negative eigenvalues of the Hessian (the second variation of the action/energy functional). This Morse Index Theorem (cf. M. Morse [30]) for geodesics on Riemannian manifolds was extended by W. Ambrose [1], J.J. Duistermaat [22], P. Piccione and D.V. Tausk [34,35], and the second author [43,44]. See also the work of M. Musso, J. Pejsachowicz, and A. Portaluri on a Morse index theorem for perturbed geodesics on semi-Riemannian manifolds in [31] which has in particular lead N. Waterstraat to a K-theoretic proof of the Morse Index Theorem in [39].

For a systematic review of the basic vector analysis and geometry and for the physics background, we refer to V.I. Arnold [2] and M. de Gosson [25].

1.1.2 The strong symplectic infinite-dimensional case

As shown by K. Furutani and the first author in [7], the finite-dimensional approach of the Morse Index Theorem can be generalized to a separable Hilbert space when we assume that the form ω is bounded and can be expressed as in (1) with a bounded operator J, which is skew-self-adjoint (i.e., $J^* = -J$) and not only injective but invertible. The invertibility of J is the whole point of a *strong* symplectic structure. Then, without loss of generality, one can assume $J^2 = -I$ like in the finite-dimensional case (see Lemma 1 below), and many calculations of the finite-dimensional case can be preserved with only slight modifications. The model space for strong symplectic Hilbert spaces is the von Neumann space $\beta(A) := \operatorname{dom}(A^*)/\operatorname{dom}(A)$ of *natural* boundary values of a closed symmetric operator A in a Hilbert space X with symplectic form given by Green's form

$$\omega(\gamma(u),\gamma(v)) :=\; \langle A^*u,v\rangle - \langle u,A^*v\rangle \quad \text{for all } u,v \in \operatorname{dom}(A^*), \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in X and $\gamma\colon \operatorname{dom}(A^*) \to \beta(A)$ is the trace map. A typical example is provided by a linear symmetric differential operator A of first order over a manifold M with boundary Σ. Here we have the minimal domain $\operatorname{dom}(A) = H^1_0(M)$ and the maximal domain $\operatorname{dom}(A^*) \supset H^1(M)$. Note that the inclusion is strict for $\dim M > 1$. Recall that $H^1_0(M)$ denotes the closure of $C^\infty_0(M\setminus\Sigma)$ in $H^1(M)$. For better reading we do not mention the corresponding vector bundles in the notation of the Sobolev spaces of vector bundle sections.

As in the finite-dimensional case, the basic geometric concept in infinite-dimensional strong symplectic analysis is the Lagrangian subspace, i.e., a subspace which is isotropic and co-isotropic at the same time. Contrary to the finite-dimensional case, however, the common definition of a Lagrangian as a *maximal* isotropic space or an isotropic space of *half* dimension becomes inappropriate.

In order to define the Maslov index in the infinite-dimensional case as intersection number of two continuous paths of Lagrangian subspaces, one has to make the additional assumption that corresponding Lagrangians make a Fredholm pair so that, in particular, we have finite intersection dimensions.

In [23], A. Floer suggested to express the spectral flow of a curve of self-adjoint operators by the Maslov index of corresponding curves of Lagrangians. Following his suggestion, a multitude of formulae was achieved by T. Yoshida [41], L. Nicolaescu [32], S. E. Cappell, R. Lee, and E. Y. Miller [18], the first author, jointly with K. Furutani and N. Otsuki [8,9] and P. Kirk and M. Lesch [27]. The formulae are of varying generality: Some deal with a fixed (elliptic) differential operator with varying self-adjoint extensions (i.e., varying boundary conditions); others keep the boundary condition fixed and let the operator vary. An example for a path of operators is a curve of Dirac operators on a manifold with fixed Riemannian metric and Clifford multiplication but varying defining connection (background field). See also the results by the present authors in [13] for varying operator and varying boundary conditions but fixed maximal domain and in [14] (in preparation) also for varying maximal domain. Recently, M. Prokhorova [36] considered a path of Dirac operators on a two-dimensional disk with a finite number of holes subjected to local elliptic boundary conditions and obtained a beautiful explicit formula for the spectral flow (respectively, the Maslov index).

1.1.3 Beyond the limits of the strong symplectic assumption

Weak (i.e., not necessarily strong) symplectic structures arise on the way to a spectral flow formula in the full generality wanted: for continuous curves of, say linear formally self-adjoint elliptic differential operators of first order over a compact manifold of dimension ≥ 2 with boundary and with varying maximal domain (i.e., admitting arbitrary continuous variation of the coefficients of first order) and with continuously varying regular (elliptic) boundary conditions, see [14]. An interesting new feature for the comprehensive generalization is the following "technical" problem: For regular (elliptic) boundary value problems (say for a linear formally self-adjoint elliptic differential operator A of first order on a compact smooth manifold M with boundary Σ), there are three canonical spaces of boundary values: the above mentioned von Neumann space $\beta(A) = \operatorname{dom}(A^*)/\operatorname{dom}(A)$, which is a subspace of the distributional Sobolev space $H^{-1/2}(\Sigma)$; the space of boundary values $H^{1/2}(\Sigma) \simeq H^1(M)/H^1_0(M)$ of the operator domain $H^1(M)$; and the most familiar and basic $L^2(\Sigma)$.[1] As in (2), Green's form induces symplectic forms on all three section spaces which are mutually compatible.

More precisely, Green's form yields a strong symplectic structure not only on $\beta(A)$, but also on $L^2(\Sigma)$ by

$$\omega(x,y) := -\langle Jx, y\rangle_{L^2(\Sigma)} .$$

Here J denotes the principal symbol of the operator A over the boundary in inner normal direction. The multiplicative operator induced by J is invertible (= injective and surjective,

[1] In the tradition of geometrically inspired analysis, we think mostly of *homogeneous* systems when talking of elliptic boundary value problems. Our key reference is the monograph [11] by K. P. Wojciechowski and the first author and the supplementary elaborations by J. Brüning and M. Lesch in [16]. For a more comprehensive treatment, emphasizing *non-homogeneous* boundary value problems and assembling all relevant section spaces in a huge algebra, we refer to the more recent article [38] by B.-W. Schulze.

i.e., with bounded inverse) since A is elliptic. For the induced symplectic structure on the Sobolev space $H^{1/2}(\Sigma)$ the corresponding operator J' is *not* invertible for $\dim\Sigma \geq 1$, see Remark 2b in Section 2.1 below. So, for $\dim\Sigma \geq 1$ the space $H^{1/2}(\Sigma)$ becomes only a *weak* symplectic Hilbert space, to use a notion introduced by P.R. Chernoff and J.E. Marsden [19, Section 1.2, pp. 4-5].

An additional incitement to investigate weak symplectic structures comes from a stunning observation of E. Witten (explained by M.F. Atiyah in [3] in a heuristic way). He considered a weak (and degenerate) symplectic form on the loop space $\mathrm{Map}(S^1, M)$ of a finite-dimensional closed orientable Riemannian manifold M and noticed that a (future) thorough understanding of the infinite-dimensional symplectic geometry of that loop space "should lead rather directly to the index theorem for Dirac operators" (l.c., p. 43). Of course, restricting ourselves to the linear case, i.e., to the geometry of Lagrangian subspaces instead of Lagrangian manifolds, we can only marginally contribute to that program in this paper.

1.2 Main results and plan of the paper

In this paper we shall deal with the preceding *technical* problem. To do that, we generalize the results of J. Robbin and D. Salamon [37], S.E. Cappell, R. Lee, and E.Y. Miller [17], K. Furutani, N. Otsuki and the first author in [8, 9] and of P. Kirk and M. Lesch in [27]. We give a rigorous definition of the Maslov index for continuous curves of Fredholm pairs of Lagrangian subspaces in a fixed Banach space with varying weak symplectic structures and *continuously* varying symplectic splittings and derive its basic properties. Part of our results will be formulated and proved for relations instead of operators to admit wider application.

Throughout, we aim for a *clean* presentation in the sense that results are proved in suitable generality. We wish to show clearly the minimal assumptions needed in order to prove the various properties. We shall, e.g., prove purely algebraic results algebraically in symplectic vector spaces and purely topological results in Banach spaces whenever possible - in spite of the fact that we shall deal with symplectic Hilbert spaces in most applications.

The routes of [8, 9] and [27] are barred to us because they rely on the concept of strong symplectic Hilbert space. Consequently, we have to replace some of the familiar reasoning of symplectic analysis by new arguments. A few of the most elegant lemmata of strong symplectic analysis can not be retained, but, luckily, the new weak symplectic set-up will show a considerable strength that is illustrative and applicable also in the conventional strong case.

In Section 2, we give a thorough presentation of weak symplectic functional analysis. Basic concepts are defined in Subsection 2.1. A new feature of weak symplectic analysis is the lack of a canonical symplectic splitting: for *strong* symplectic Hilbert space, we can assume $J^2 = -I$ by smooth deformation of the metric, and obtain the canonical splitting $X = X^+ \oplus X^-$ into mutually orthogonal closed subspaces $X^\pm := \ker(J \mp iI)$ which are both invariant under J. That permits the representation of all Lagrangian subspaces as graphs of unitary operators from X^+ to X^- (see Lemma 2), which yields a transfer of contractibility from the unitary group to the space of Lagrangian subspaces. Moreover, that representation is the basis for a functional analytical definition of the Maslov index. For *weak* symplectic Hilbert or Banach spaces, the preceding construction does not work any longer and we must assume that a symplectic splitting is given and fixed (its existence follows, however, from Zorn's Lemma). Given an elliptic differential operator A of first order over a manifold M with boundary Σ, however, we have a natural symplectic splitting of the symplectic spaces of sections over Σ, both in the strong and weak symplectic case, see Remark 3a, Equation 11.

In Subsection 2.2, we turn to Fredholm pairs of Lagrangian subspaces to prepare for the counting of intersection dimensions in the definition of the Maslov index. Here another new feature of weak symplectic analysis is that the Fredholm index of a Fredholm pair of Lagrangian subspaces does not need to vanish. On the one hand, this opens the gate to new interesting theorems. On the other hand, the re-formulation of well-known definitions and lemmata in the weak symplectic setting becomes rather heavy since we have to add the vanishing of the Fredholm index as an explicit assumption.

As a side effect of our weak symplectic investigation, we hope to enrich the classical literature with our new purely algebraic conditions for isotropic subspaces becoming Lagrangians, in Lemma 4 and Propositions 1 and 2.

At present, the homotopy types of the full Lagrangian Grassmannian and of the Fredholm Lagrangian Grassmannian remain unknown for weak symplectic structures. We give a list of related open problems in Subsection 2.3 below. To us, however, it seems remarkable that a wide range of familiar geometric features can be re-gained in weak symplectic functional analysis — in spite of the incomprehensibility of the basic topology.

In Subsection 2.4, we lay the next foundation for a rigorous definition of the Maslov index by investigating continuous curves of operators and relations that generate Lagrangians in the new wider setting. Referring to the concepts of our Appendix, we define the spectral flow of such curves.

In Section 3 we finally come to the intersection geometry. In Subsection 3.1, we show how to treat varying weak symplectic structures in a fixed Banach space with *continuously* varying symplectic splittings and define the Maslov index for continuous curves of Fredholm pairs of Lagrangian subspaces in this setting. We obtain the full list of basic properties of the Maslov index as listed by S.E. Cappell, R. Lee, and E.Y. Miller in [17]. We can not claim that this new Maslov index is always independent of the splitting projections. However, for strong symplectic Banach space the independence will be proved in Proposition 6. That establishes the coincidence with the common definition of the Maslov index.

In Subsection 3.2, in our general context, we establish the relation between real symplectic analysis (in the tradition of classical mechanics) on the one side, and the more elegant complex symplectic analysis (as founded by J. Leray in [28]) on the other side.

In Subsection 3.3, we pay special attention to questions related to the embedding of symplectic spaces, Lagrangian subspaces and curves into larger symplectic spaces. Our investigations are inspired by the extremely delicate embedding questions between the two strong symplectic Hilbert spaces $\beta(A)$ and $L^2(\Sigma)$ as studied by K. Furutani, N. Otsuki and the first author in [9]. One additional reason for our interest in embedding problems is our observation of Remark 2c, that each weak symplectic Hilbert space can naturally be embedded in a strong symplectic Hilbert space, imitating the embedding of $H^{1/2}(\Sigma)$ into $L^2(\Sigma)$.

In Appendix A.1 and A.2, we recall the basic knowledge and fix our notations regarding gaps between closed subspaces in Banach space, uniform properties, closed linear relations and their spectral projections. Then, in Appendix A.3, we give a rigorous definition of the spectral flow for admissible families of closed relations. Our discussion of continuous operator families in Subsection 2.4 and the whole of Section 3 is based on that definition.

The main results of this paper were achieved many years ago by the authors and informally disseminated in [12]. Through all the years, our goal was to establish a truly general spectral flow formula by applying the weak symplectic functional analysis. But here we met a technical gap in the argumentation: Only recently we found the correct sufficient conditions for continuous variation of the Cauchy data spaces (or, alternatively stated, the continuous variation of the pseudo-differential Calderón projection) for curves of elliptic operators in joint work with G. Chen and M. Lesch [6]. Now that gap is bridged, a full general spectral flow formula is obtained in [14] and the relevance of weak symplectic functional analysis has become sufficiently clear for a regular publication of our results.

2 Weak symplectic functional analysis

2.1 Basic symplectic functional analysis

We fix our notation. To keep track of the required assumptions, we shall not always assume that the underlying space is a Hilbert space but permit Banach spaces and — for some concepts — even just vector spaces. For easier presentation and greater generality, we begin with *complex* symplectic spaces.

Definition 1 Let X be a complex Banach space. A mapping

$$\omega\colon X \times X \longrightarrow \mathbb{C}$$

is called a (*weak*) *symplectic form* on X, if it is sesquilinear, bounded, skew-symmetric, and non-degenerate, i.e.,
(i) $\omega(x,y)$ is linear in x and conjugate linear in y;
(ii) $|\omega(x,y)| \leq C\|x\|\|y\|$ for all $x,y \in X$;
(iii) $\omega(y,x) = -\overline{\omega(x,y)}$;
(iv) $X^\omega := \{x \in X \mid \omega(x,y) = 0 \text{ for all } y \in X\} = \{0\}$.
Then we call (X,ω) a *(weak) symplectic Banach space*.

There is a purely algebraic concept, as well.

Definition 2 Let X be a complex vector space and ω a form which satisfies all the assumptions of Definition 1 except (ii). Then we call (X,ω) a *complex symplectic vector space*.

Definition 3 Let (X,ω) be a complex symplectic vector space.
(a) The *annihilator* of a subspace λ of X is defined by

$$\lambda^\omega := \{y \in X \mid \omega(x,y) = 0 \quad \text{for all } x \in \lambda\}.$$

(b) A subspace λ is called *symplectic*, *isotropic*, *co-isotropic*, or *Lagrangian* if

$$\lambda \cap \lambda^\omega = \{0\}, \quad \lambda \subset \lambda^\omega, \quad \lambda \supset \lambda^\omega, \quad \lambda = \lambda^\omega,$$

respectively.
(c) The *Lagrangian Grassmannian* $\mathscr{L}(X,\omega)$ consists of all Lagrangian subspaces of (X,ω).

Definition 4 Let (X,ω) be a symplectic vector space and X^+, X^- be linear subspaces. We call (X,X^+,X^-) a *symplectic splitting* of X, if $X = X^+ \oplus X^-$, the quadratic form $-i\omega$ is positive definite on X^+ and negative definite on X^-, and

$$\omega(x,y) = 0 \quad \text{for all } x \in X^+ \text{ and } y \in X^-. \tag{3}$$

Remark 1 (a) By definition, each one-dimensional subspace in real symplectic space is isotropic, and there always exists a Lagrangian subspace. However, there are complex symplectic Hilbert spaces without any Lagrangian subspace. That is, in particular, the case if $\dim X^+ \neq \dim X^-$ in $\mathbb{N} \cup \{\infty\}$ for a single (and hence for all) symplectic splittings.
(b) If $\dim X$ is finite, a subspace λ is Lagrangian if and only if it is isotropic with $\dim\lambda = \frac{1}{2}\dim X$.
(c) In symplectic Banach spaces, the annihilator λ^{ω} is closed for any subspace λ. In particular, all Lagrangian subspaces are closed, and we have for any subspace λ the inclusion

$$\lambda^{\omega\omega} \supset \overline{\lambda}. \tag{4}$$

(d) Let X be a vector space and denote its (algebraic) dual space by X'. Then each symplectic form ω induces a uniquely defined injective mapping $J\colon X \to X'$ such that

$$\omega(x,y) \;=\; (Jx,y) \qquad \text{for all } x,y \in X, \tag{5}$$

where we set $(Jx,y) := (Jx)(y)$.

If (X,ω) is a symplectic Banach space, then the induced mapping J is a bounded, injective mapping $J\colon X \to X^*$ where X^* denotes the (topological) dual space. If J is also surjective (so, invertible), the pair (X,ω) is called a *strong symplectic Banach space*. As mentioned in the Introduction, we have taken the distinction between *weak* and *strong* symplectic structures from Chernoff and Marsden [19, Section 1.2, pp. 4-5].

If X is a Hilbert space with symplectic form ω, we identify X and X^*. Then the induced mapping J is a bounded, skew-self-adjoint operator (i.e., $J^* = -J$) on X with $\ker J = \{0\}$ and can be written in the form $J = \begin{pmatrix} iA_+ & 0 \\ 0 & -iA_- \end{pmatrix}$ with $A_\pm > 0$ bounded self-adjoint (but not necessarily invertible, i.e., $A_\pm^{-1}$ not necessarily bounded). As in the strong symplectic case, we then have that $\lambda \subset X$ is Lagrangian if and only if $\lambda^\perp = J\lambda$.

The proof of the following lemma is straightforward and is omitted.

Lemma 1 *Any strong symplectic Hilbert space $(X, \langle\cdot,\cdot\rangle, \omega)$ (i.e., with invertible J) can be made into a strong symplectic Hilbert space $(X, \langle\cdot,\cdot\rangle', \omega)$ with $J'^2 = -I$ by smooth deformation of the inner product of X into*

$$\langle x,y\rangle' \;:=\; \langle \sqrt{J^*J}x, y\rangle$$

without changing ω.

Remark 2 (a) In a strong symplectic Hilbert space many calculations become quite easy. E.g., the inclusion (4) becomes an equality, and all Fredholm pairs of Lagrangian subspaces have vanishing index, see below Definition 5, Equations (12)-(14).
(b) From the Introduction, we recall an important example of a weak symplectic Hilbert space: Let A be a formally self-adjoint linear elliptic differential operators of first order over a smooth compact Riemannian manifold M with boundary Σ. As mentioned in the Introduction, we have (we suppress mentioning the vector bundle)

$$H^{1/2}(\Sigma) \simeq H^1(M)/H^1_0(M) \tag{6}$$

with uniformly equivalent norms. Green's form yields a strong symplectic structure on $L^2(\Sigma)$ by

$$\{x,y\} \;:=\; -\langle Jx,y\rangle_{L^2(\Sigma)}\,. \tag{7}$$

Here J denotes the principal symbol of the operator A over the boundary in inner normal direction. It is invertible since A is elliptic. For the induced symplectic structure on $H^{1/2}(\Sigma)$ we define J' by

$$\{x,y\} \;=\; -\langle J'x,y\rangle_{H^{1/2}(\Sigma)} \qquad \text{for } x,y\in H^{1/2}(\Sigma).$$

Let B be a formally self-adjoint elliptic operator B of first order on Σ. By Gårding's inequality, the $H^{1/2}$ norm is equivalent to the induced graph norm. This yields $J' = (I+|B|)^{-1}J$. Since B is elliptic, it has compact resolvent. So, $(I+|B|)^{-1}$ is compact in $L^2(\Sigma)$; and so is J'. Hence J' is not invertible. In the same way, any dense subspace of $L^2(\Sigma)$ inherits a weak symplectic structure from $L^2(\Sigma)$.
(c) Each weak symplectic Hilbert space $(X,\langle\cdot,\cdot\rangle,\omega)$ with induced injective skew-self-adjoint J can naturally be embedded in a strong symplectic Hilbert space $\left(X',\langle\cdot,\cdot\rangle',\omega'\right)$ with invertible induced J' by setting $\langle x,y\rangle' := \langle |J|x,y\rangle$ as in Lemma 1 and then completing the space. This imitates the situation of the embedding of $H^{1/2}(\Sigma)$ into $L^2(\Sigma)$. It shows that the weak symplectic Hilbert space $H^{1/2}(\Sigma)$ with its embedding into $L^2(\Sigma)$ yields a model for all weak symplectic Hilbert spaces. In Section 3.3, we shall elaborate on the embedding weak $\hookrightarrow$ strong a little further.

The following lemma is a key result in symplectic analysis. The representation of Lagrangian subspaces as graphs of unitary mappings from one component X^+ to the complementary component X^- of the underlying symplectic vector space (to be considered as the induced complex space in classical real symplectic analysis, see, e.g., K. Furutani and the first author [7, Section 1.1]) goes back to J. Leray [28]. We give a simplification for complex vector spaces, first announced in [43]. Of course, the main ideas were already contained in the real case. The Lemma is essentially well-known and will be obtained in the more general setting below: (i) is clear; (ii) will follow from Lemma 3; and (iii) from Proposition 2.

Lemma 2 *Let (X,ω) be a strong symplectic Hilbert space with $J^2=-I$. Then*

(i) the space X splits into the direct sum of mutually orthogonal closed subspaces

$$X \;=\; \ker(J-iI)\oplus\ker(J+iI),$$

which are both invariant under J;

(ii) there is a 1-1 correspondence between the space $\mathscr{U}^J$ of unitary operators from $\ker(J-iI)$ *to* $\ker(J+iI)$ *and $\mathscr{L}(X,\omega)$ under the mapping $U\mapsto\lambda:=\mathfrak{G}(U)$ (= graph of U);*

(iii) if $U,V\in\mathscr{U}^J$ and $\lambda:=\mathfrak{G}(U)$, $\mu:=\mathfrak{G}(V)$, then (λ,μ) is a Fredholm pair (see Definition 5b) if and only if $U-V$, or, equivalently, $UV^{-1}-I_{\ker(J+iI)}$ is Fredholm. Moreover, we have a natural isomorphism

$$\ker(UV^{-1}-I_{\ker(J+iI)})\simeq\lambda\cap\mu\,. \tag{8}$$

The preceding method to characterize Lagrangian subspaces and to determine the dimension of the intersection of a Fredholm pair of Lagrangian subspaces provides the basis for defining the Maslov index in strong symplectic spaces of infinite dimensions (see, in different formulations and different settings, the quoted references [7], [9], [24], [27], and Zhu and Long [45]).

Surprisingly, it can be generalized to weak symplectic Banach spaces in the following way.

Lemma 3 *Let (X,ω) be a symplectic vector space with a symplectic splitting (X,X^+,X^-).*
(a) Each isotropic subspace λ can be written as the graph

$$\lambda \;=\; \mathfrak{G}(U)$$

of a uniquely determined injective operator

$$U\colon\ \mathrm{dom}(U) \longrightarrow X^-$$

with $\mathrm{dom}(U) \subset X^+$. *Moreover, we have*

$$\omega(x,y) \;=\; -\omega(Ux,Uy) \quad \textit{for all } x,y \in \mathrm{dom}(U). \tag{9}$$

(b) If X is a Banach space, then $X^\pm$ are always closed and the operator U defined by a Lagrangian subspace λ is closed as an operator from X^+ to X^- (not necessarily densely defined).
(c) For a closed isotropic subspace λ in a strong symplectic Banach space X, we have $\mathrm{dom}(U)$ *and* $\mathrm{im}\,U$ *are closed. Moreover, if λ is Lagrangian, then* $\mathrm{dom}(U) = X^+$ *and* $\mathrm{im}\,U = X^-$*; i.e., the generating U is bounded and surjective with bounded inverse.*

Proof a. Let $\lambda \subset X$ be isotropic and $v_+ + v_-, w_+ + w_- \in \lambda$ with $v_\pm, w_\pm \in X^\pm$. By the isotropic property of λ and our assumption about the splitting $X = X^+ \oplus X^-$ we have

$$0 \;=\; \omega(v_+ + v_-, w_+ + w_-) \;=\; \omega(v_+,w_+) + \omega(v_-,w_-). \tag{10}$$

In particular, we have

$$\omega(v_+ + v_-, v_+ + v_-) \;=\; \omega(v_+,v_+) + \omega(v_-,v_-) \;=\; 0$$

and so $v_- = 0$ if and only if $v_+ = 0$. So, if the first (respectively the second) components of two points $v_+ + v_-, w_+ + w_- \in \lambda$ coincide, then also the second (respectively the first) components must coincide.

Now we set

$$\mathrm{dom}(U) \;:=\; \{x \in X^+ \mid \exists y \in X^- \text{ such that } x + y \in \lambda\}.$$

By the preceding argument, y is uniquely determined, and we can define $Ux := y$. By construction, the operator U is an injective linear mapping, and property (9) follows from (10).
b. By Definition 4 of a symplectic splitting, Equation (3) we have $X^- \subset (X^+)^\omega$. Now let $x_+ + x_- \in (X^+)^\omega$ with $x_\pm \in X^\pm$. Then $\omega(x_+ + x_-, x_+) = \omega(x_+,x_+) = 0 \iff x_+ = 0$ since $-i\omega$ is positive definite on X^+. That proves $X^- = (X^+)^\omega$, and correspondingly $X^+ = (X^-)^\omega$. As noticed in Remark 1c, annihilators are always closed. This proves the first part of (b). Now let λ be a Lagrangian subspace and let U be the uniquely determined injective operator $U\colon\ \mathrm{dom}(U) \to X^-$ with $\mathrm{dom}(U) \subset X^+$ and $\mathfrak{G}(U) = \lambda$. By Definition 3b we have $\lambda = \lambda^\omega$, hence λ is closed as an annihilator and so is the graph of U, i.e., U is closed.
c. Let $\lambda = \mathfrak{G}(U)$. Let $\{x_n\}$ be a sequence in $\mathrm{dom}(U)$ convergent to $x \in X^+$. Since X is strong, we see from (9) that the sequence $\{Ux_n\}$ is a Cauchy sequence and therefore is also convergent. Denote by y the limit of $\{Ux_n\}$. Since λ is closed, we have $x \in \mathrm{dom}\,U$ and $y = Ux$. Thus $\mathrm{dom}(U)$ is closed. We apply the same argument to $\mathrm{dom}(U^{-1}) \subset X^-$, relative to the inner product $i\omega$ and obtain that $\mathrm{im}\,U$ is closed. This proves the first part of (c).

Now assume that λ is a Lagrangian subspace. Firstly we show that U is densely defined in X^+. Indeed, if $\overline{\mathrm{dom}(U)} \neq X^+$, there would be a $v \in V$, $v \neq 0$, where V denotes the orthogonal complement of $\mathrm{dom}(U)$ in X^+ with respect to the inner product on X^+ defined by $-i\omega$. Clearly $(\operatorname{dom}(U))^{\omega} = V + X^-$. So, $V = \overline{(\operatorname{dom}(U))^{\omega} \cap X^+}$. Then $v + 0 \in \lambda^{\omega} \setminus \lambda$. That contradicts the Lagrangian property of λ. So, we have $\overline{\operatorname{dom}(U)} = X^+$.

We have shown that $\operatorname{dom}(U)$ is closed and dense. Hence $\operatorname{dom}(U) = X^+$. Now the boundedness of U follows from the closedness of $\mathfrak{G}(U)$. Applying the same arguments to $\operatorname{dom}(U^{-1}) \subset X^-$ relative to the inner product $i\omega$ yields $\operatorname{im} U = \operatorname{dom}(U^{-1}) = X^-$ and U^{-1} is bounded.

Remark 3 (a) Note that the symplectic splitting is not unique. Its existence can be proved by Zorn's Lemma. In our applications, the geometric background provides natural splittings. Let A be an elliptic differential operator of first order, acting on sections of a Hermitian vector bundle E over the Riemannian manifold M with boundary Σ. Then the symplectic Hilbert space structures of $L^2(\Sigma; E|_{\Sigma})$ and $H^{1/2}(\Sigma; E|_{\Sigma})$ of (7) and (6) are compatible and their symplectic splitting is defined by the bundle endomorphism (the principal symbol of A in inner normal direction) $J\colon E|_{\Sigma} \to E|_{\Sigma}$ in the following way:

$$H^{\pm} := H^{1/2}(\Sigma; E^{\pm}|_{\Sigma}) \quad \text{and} \quad L^{\pm} := L^2(\Sigma; E^{\pm}|_{\Sigma})$$
$$\text{with } E^{\pm}|_{\Sigma} := \text{ lin. span of} \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \text{eigenspaces of } iJ. \tag{11}$$

Note that L^+, L^- change continuously if J changes continuously. For varying splittings see also the discussion below in Section 3.
(b) The symplectic splitting and the corresponding *graph* representation of isotropic and Lagrangian subspaces must be distinguished from the splitting in complementary Lagrangian subspaces which yields the common representation of Lagrangian subspaces as *images* in the real category (see Lemma 11 below).

2.2 Fredholm pairs of Lagrangian subspaces

A main feature of symplectic analysis is the study of the *Maslov index*. It is an intersection index between a path of Lagrangian subspaces with the *Maslov cycle*, or, more generally, with another path of Lagrangian subspaces.

Before giving a rigorous definition of the Maslov index in weak symplectic functional analysis (see below Section 3) we fix the terminology and give several simple criteria for a pair of isotropic subspaces to be Lagrangian.

We recall:

Definition 5 (a) The space of (algebraic) *Fredholm pairs* of linear subspaces of a vector space X is defined by

$$\mathscr{F}^2_{\mathrm{alg}}(X) := \{(\lambda, \mu) \mid \dim \lambda \cap \mu < +\infty \text{ and } \dim X/(\lambda + \mu) < +\infty\} \tag{12}$$

with

$$\operatorname{index}(\lambda, \mu) := \dim \lambda \cap \mu - \dim X/(\lambda + \mu). \tag{13}$$

(b) In a Banach space X, the space of (topological) *Fredholm pairs* is defined by

$$\mathscr{F}^2(X) := \{(\lambda, \mu) \in \mathscr{F}^2_{\mathrm{alg}}(X) \mid \lambda, \mu \text{ and } \lambda + \mu \subset X \text{ closed}\}. \tag{14}$$

Remark 4 Actually, in Banach spaces the closedness of $\lambda+\mu$ follows from its finite codimension in X in combination with the closedness of λ,μ (see [8, Remark A.1] and [26, Problem 4.4.7]). So, the set of algebraic Fredholm pairs of Lagrangian subspaces of a symplectic Banach space X coincides with the set $\mathscr{F}\mathscr{L}^2(X)$ of topological Fredholm pairs of Lagrangian subspaces of X.

We begin with a simple algebraic observation.

Lemma 4 *Let (X,ω) be a symplectic vector space with transversal subspaces λ,μ. If λ,μ are isotropic subspaces, then they are Lagrangian subspaces.*

Proof From linear algebra we have

$$\lambda^{\omega}\cap\mu^{\omega} \;=\; (\lambda+\mu)^{\omega} \;=\; \{0\},$$

since $\lambda+\mu=X$. From

$$\lambda\subset\lambda^{\omega},\mu\subset\mu^{\omega} \tag{15}$$

we get

$$X \;=\; \lambda^{\omega}\oplus\mu^{\omega}. \tag{16}$$

To prove $\lambda^{\omega}=\lambda$ (and similarly for μ), we consider an $x\in\lambda^{\omega}$. It can be written in the form $x=y+z$ with $y\in\lambda$ and $z\in\mu$ because of the splitting $X=\lambda\oplus\mu$. Applying (15) and the splitting (16) we get $y=x$ and so $z=0$, hence $x\in\lambda$.

With a little work, the preceding lemma can be generalized from direct sum decompositions to (algebraic) Fredholm pairs. At first we have

Lemma 5 *Let V,W be two vector spaces and $f\colon V\times W\to\mathbb{C}$ be a sesquilinear mapping. Assume that $\dim W<+\infty$. If for each $v\in V$, the condition $f(v,w)=0$ for all $w\in W$ implies $v=0$, then we have $\dim V\le\dim W$.*

Proof Let $\widetilde{W}$ be the space of conjugate linear functionals on W. Let $\widetilde{f}\colon V\to\widetilde{W}$ be the induced map of f defined by $(\widetilde{f}(v))(w):=f(v,w)$. Then $\widetilde{f}$ is linear. Our condition implies that $\widetilde{f}$ is injective. Thus we have $\dim V\le\dim\widetilde{W}=\dim W$.

Corollary 1 *Let (X,ω) denote a symplectic vector space.*
(a) For any finite-codimensional linear subspace λ, we have $\dim\lambda^{\omega}\le\dim X/\lambda$.
(b) For any finite dimensional linear subspace μ, we have $\mu^{\omega\omega}=\mu$ and $\dim\mu=\dim X/\mu^{\omega}$.

Proof a. Define $f\colon\lambda^{\omega}\times(X/\lambda)\to\mathbb{C}$ by $f(x,y+\lambda):=\omega(x,y)$ for all $x\in\lambda^{\omega}$ and $y\in X$. Then f satisfies the condition in Lemma 5. So our result follows.
b. Define $g\colon(X/\mu^{\omega})\times\mu\to\mathbb{C}$ by $g(x+\mu^{\omega},y):=\overline{\omega(x,y)}$ for all $x,y\in\mu$. Then g satisfies the condition in Lemma 5. So we have $\dim X/\mu^{\omega}\le\dim\mu$. By (a) we have $\dim\mu^{\omega\omega}\le\dim X/\mu^{\omega}$. Since $\mu\subset\mu^{\omega\omega}$, our result follows.

Proposition 1 *Let (X,ω) be a symplectic vector space and $(\lambda,\mu)\in\mathscr{F}^2_{\mathrm{alg}}(X)$. If λ,μ are isotropic subspaces with $\operatorname{index}(\lambda,\mu)\ge 0$, then λ and μ are Lagrangian subspaces of X,*

$$\operatorname{index}(\lambda,\mu)\overset{(i)}{=}0,\quad(\lambda+\mu)^{\omega}\overset{(ii)}{=}\lambda\cap\mu,\quad\text{and }(\lambda+\mu)^{\omega\omega}\overset{(iii)}{=}\lambda+\mu.$$

Proof Set $\widetilde{X} := (\lambda+\mu)/(\lambda\cap\mu)$ with the induced form

$$\widetilde{\omega}([x+y],[\xi+\eta]) \;:=\; \omega(x+y,\xi+\eta) \quad \text{for } x,\xi\in\lambda \quad \text{and} \quad y,\eta\in\mu,$$

where $[x+y] := x+y+\lambda\cap\mu$ denotes the class of $x+y$ in $\frac{\lambda+\mu}{\lambda\cap\mu}$. The aim is to show that $\widetilde{X}$ is a symplectic vector space. During the proof of this fact the claimed equalities (i)-(iii) will be obtained.

Since λ,μ are isotropic, we have $\omega(x+y+z,\xi+\eta+\zeta)=\omega(x+y,\xi+\eta)$ for any $z,\zeta\in\lambda\cap\mu$. So $\widetilde{\omega}$ is well-defined and inherits the algebraic properties from ω.

To show that $(\widetilde{X})^{\widetilde{\omega}}=\{0\}$, we observe

$$(\lambda+\mu)^{\omega} \;=\; \lambda^{\omega}\cap\mu^{\omega}\supset\lambda\cap\mu\,. \tag{17}$$

By Corollary 1a, we have

$$\dim(\lambda+\mu)^{\omega}\le\dim X/(\lambda+\mu)\le\dim(\lambda\cap\mu).$$

Here the last inequality is just the non-negativity of the Fredholm index as defined in (13). This proves (i), namely

$$\dim(\lambda+\mu)^{\omega} \;=\; \dim X/(\lambda+\mu) \;=\; \dim(\lambda\cap\mu). \tag{18}$$

Combining (18) with (17) yields (ii), namely

$$\lambda\cap\mu \;=\; \lambda^{\omega}\cap\mu^{\omega} \;=\; (\lambda+\mu)^{\omega}. \tag{19}$$

By Corollary 1b, we have

$$\dim X/(\lambda+\mu) \;=\; \dim\lambda\cap\mu \;=\; \dim X/(\lambda\cap\mu)^{\omega} \;=\; \dim X/(\lambda+\mu)^{\omega\omega}.$$

Thus we have proved (iii), namely $\lambda+\mu=(\lambda+\mu)^{\omega\omega}$.

To finish our proof that $\widetilde{\omega}$ is non-degenerate, one checks that

$$\Big(\frac{\lambda+\mu}{\lambda\cap\mu}\Big)^{\widetilde{\omega}} \;=\; \frac{(\lambda+\mu)^{\omega}}{\lambda\cap\mu}\,. \tag{20}$$

With (19) that proves that $\big(\dfrac{\lambda+\mu}{\lambda\cap\mu}\big)^{\omega}=\{0\}$, hence $\widetilde{X}=\dfrac{\lambda+\mu}{\lambda\cap\mu}$ is a true symplectic vector space for the induced form $\widetilde{\omega}$. It is spanned by the transversal isotropic subspaces

$$\frac{\lambda+\mu}{\lambda\cap\mu} \;=\; \frac{\lambda}{\lambda\cap\mu}\oplus\frac{\mu}{\lambda\cap\mu}\,.$$

By Lemma 4, the spaces $\dfrac{\lambda}{\lambda\cap\mu}, \dfrac{\mu}{\lambda\cap\mu}$ are Lagrangian subspaces.

It remains to prove that λ,μ itself are Lagrangian subspaces of X. Clearly $\lambda\subset\lambda^{\omega}\cap(\lambda+\mu)$. Now consider $x\in\lambda$ and $y\in\mu$ with $x+y\in\lambda^{\omega}$. Then

$$[x+y]\in\Big(\frac{\lambda}{\lambda\cap\mu}\Big)^{\widetilde{\omega}} \;=\; \frac{\lambda}{\lambda\cap\mu}$$

by the Lagrangian property of $\frac{\lambda}{\lambda\cap\mu}$. It follows that $x+y\in\lambda$, hence

$$\lambda^{\omega}\cap(\lambda+\mu) \;=\; \lambda \text{ and similarly } \mu^{\omega}\cap(\lambda+\mu) \;=\; \mu\,. \tag{21}$$

Combined with the fact that

$$\lambda^{\omega}\subset(\lambda\cap\mu)^{\omega} \;=\; (\lambda+\mu)^{\omega\omega} \;=\; \lambda+\mu,$$

the inclusion $\lambda\supset\lambda^{\omega}$ follows and so the Lagrangian property of λ (and similarly of μ).

Remark 5 For related topological (unsolved) questions see below Subsection 2.3.

We close this subsection with the following characterization of Fredholm pairs.

Proposition 2 *Let (X,ω) be a symplectic Banach space and let (X,X^+,X^-) be a symplectic splitting. Let λ,μ be isotropic subspaces. Let U,V denote the generating operators for λ,μ in the sense of Lemma 3. We assume that*

$$V: X^+ \to X^- \quad \text{is bounded and bounded invertible}. \tag{22}$$

Then
(a) The space μ is a Lagrangian subspace of X.
(b) Moreover,

$$(\lambda,\mu) \in \mathscr{F}^2(X) \iff UV^{-1} - I_{X^-} \text{ is a Fredholm operator with domain } V(\operatorname{dom} U).$$

(c) In this case, $U-V$ is a (closed, not necessarily bounded) Fredholm operator with domain dom U *and*

$$\operatorname{index}(\lambda,\mu) \;=\; \operatorname{index}(UV^{-1} - I_{X^-}).$$

(d) In particular, $U-V$ (and thus $UV^{-1}-I_{X^-}$) is closed if λ is closed and V is bounded (as assumed above).

Note 1 Our assumption (22) is needed for (a). For (b) and (c) (in (23) below) it is only required that $\operatorname{dom}(U) \subset \operatorname{dom}(V)$. For (d) we need only that V is bounded.
For (b), (c) and (d) recall from Definition 5b) that we require of a pair in $\mathscr{F}^2(X)$ to consist of *closed* subspaces.

Proof a. Since $\mu = \mathfrak{G}(V)$ is an isotropic subspace of X, the space $\mu' := \mathfrak{G}(-V)$ is also isotropic. We show that μ,μ' are transversal in X. Then by Lemma 4, μ (and μ') are Lagrangian subspaces. First, from the injectivity of V, we have $\mu \cap \mu' = \{0\}$.

Next, let $x+y$, or, more suggestively, $\begin{pmatrix} x \\ y \end{pmatrix}$ denote an arbitrary point in X with $x \in X^+$ and $y \in X^-$. Since V is bounded with bounded inverse, we have $y \in \operatorname{im} V$ and $z,w \in \operatorname{dom} V$, where

$$z \;:=\; \frac{x+V^{-1}y}{2} \quad \text{and} \quad w \;:=\; \frac{x-V^{-1}y}{2}.$$

Then $z+w = x$ and $z-w = V^{-1}y$, so

$$\begin{pmatrix} x \\ y \end{pmatrix} \;=\; \begin{pmatrix} z \\ Vz \end{pmatrix} + \begin{pmatrix} w \\ -Vw \end{pmatrix}.$$

This proves $X = \mu \oplus \mu'$.
b and c. Let $\lambda = \mathfrak{G}(U)$ and $\mu = \mathfrak{G}(V)$ with V bounded and bounded invertible. Let $P_\pm$ denote the projections of $X = X^+ \oplus X^-$ onto $X^\pm$. Then

$$\lambda \cap \mu \;=\; \left\{ \begin{pmatrix} x \\ Vx \end{pmatrix} \mid x \in \operatorname{dom}(U) \text{ and } Ux \;=\; Vx \right\}.$$

So, P_- induces an algebraic and topological isomorphism between $\lambda \cap \mu$ and $\ker\big(UV^{-1} - I_{X^-}\big)$.

Now we determine

$$\begin{aligned}\lambda+\mu &= \left\{\begin{pmatrix} x\\ Ux\end{pmatrix}+\begin{pmatrix} y\\ Vy\end{pmatrix} \mid x\in \operatorname{dom}(U), y\in X^+\right\}\\ &= \left\{\begin{pmatrix} x'\\ Vx'\end{pmatrix}+\begin{pmatrix} 0\\ z\end{pmatrix} \mid x'\in X^+ \text{ and } z\in \operatorname{im}(UV^{-1}-I_{X^-})\right\}\\ &= \mu\oplus \operatorname{im}(UV^{-1}-I_{X^-}).\end{aligned} \tag{23}$$

The last direct sum sign comes from the invertibility of V: It induces $\mu\cap X^-=\{0\}$ and, similarly, $\mu+X^-=X$. From that we obtain the direct sum decomposition $X=\mu\oplus X^-$ with projections Π_μ and Π_- onto the components. So, Π_- yields an algebraic and topological isomorphism of $\lambda+\mu$ onto $\operatorname{im}(UV^{-1}-I_{X^-})$. In particular, we have $\lambda+\mu$ closed in X if and only if $\operatorname{im}(UV^{-1}-I_{X^-})$ is closed in X^- and

$$X/(\lambda+\mu)\simeq X^-/\operatorname{im}(UV^{-1}-I_{X^-})$$

with coincidence of the codimensions.
d. Let $x_n\to x$ and $y_n\to y$ be such that $(U-V)x_n\to y$. Since V is bounded, $Vx_n\to Vx$. Then $Ux_n\to Vx+y$. Since λ is closed, U is closed. Thus $x\in\operatorname{dom}(U)$ and $Ux=Vx+y$. Hence $(U-V)x=y$.

2.3 Open topological problems

2.3.1 Fredholm pairs of Lagrangian subspaces with negative index?

Proposition 1 shows that Fredholm pairs of Lagrangian subspaces in symplectic vector spaces cannot have positive index. In contrast to the strong case, one may expect that we have pairs with negative index in weak symplectic Hilbert space. By now, however, this is an open problem.

2.3.2 Characterization of Lagrangian subspaces by the canonical symmetry property of the projections?

The delicacy of Lagrangian analysis in weak symplectic Hilbert space may also be illuminated by addressing the orthogonal projection onto a Lagrangian subspace. In a strong symplectic Hilbert space with unitary J, the range of an orthogonal projection is Lagrangian if and only if the projections P and $I-P$ are conjugated by the operator J in the way

$$I-P \;=\; JPJ^*\,,$$

which is familiar from characterizing elliptic self-adjoint pseudo-differential boundary conditions for elliptic differential of first order, see [11, Proposition 20.3]. In weak symplectic analysis, J maps the range $\operatorname{im}P$ onto a dense subset of $\ker P$, but there the argument stops.

2.3.3 Contractibility of the space of Lagrangian subspaces?

There are two further differences between the weak and the strong case, namely regarding the topology: while the Lagrangian Grassmannian $\mathscr{L}(X,\omega)$ inherits contractibility from the space of unitary operators in separable Hilbert spaces by Lemma 2(ii), more refined arguments will be needed to prove the contractibility in the weak case, if it is true at all.

2.3.4 *Bott periodicity of the homotopy groups of the space of Fredholm pairs of Lagrangian subspaces?*

Next, consider the space $\mathscr{F}\mathscr{L}_\lambda(X)$ of all Lagrangian subspaces which form a Fredholm pair with a given Lagrangian subspace λ. Its topology is presently also unknown in the weak case, whereas we have

$$\pi_1\big(\mathscr{F}\mathscr{L}_\lambda(X)\big) \cong \mathbb{Z}$$

in strong symplectic Hilbert spaces X (see [8, Corollary 4.3] and the generalization to Bott periodicity in [27, Equation (6.2) with Lemma 6.1 and Proposition 6.5]).

2.4 Curves of *unitary* operators that are *admissible* with respect to the positive half-line

We begin with some observations on inner product spaces and refer to the Appendix A.2 for a rigorous definition of the basic concepts of linear relations.

Lemma 6 *Let (X,h_X), (Y,h_Y), (Z,h_Z) denote three inner product spaces, A, B linear relations between X and Y, and C a linear relation between X and Z.*
(a) Assume that C is a linear operator, $\operatorname{dom}(A) \subset \operatorname{dom}(C)$*, and* $h_Y(y,y) \le h_Z(Cx,Cx)$ *for all* $(x,y) \in A$*. Then A is a linear operator.*
(b) Assume that B is a linear operator, $\operatorname{dom}(A) = \operatorname{dom}(C) \subset \operatorname{dom}(B)$*, and*

$$h_Y(y,y) + h_Z(z,z) \le h_Y(Bx,Bx) \tag{24}$$

for all $(x,y) \in A$ *and* $(x,z) \in C$*. Then A and C are linear operators and* $\ker(B-A) \subset \ker C$*.*

Proof a. Let $y \in \ker A$, i.e., $(0,y) \in A$. By our assumption we have $h_Y(y,y) \le h_Z(C0,C0) = 0$. Since h_Y is positive definite, we have $y = 0$.
b. By (a) A and C are linear operators. Let $x \in \ker(B-A)$. Then $Bx = Ax$. By (24) we have $h_Z(Cx,Cx) \le 0$. Since h_Z is positive definite, we have $Cx = 0$, i.e., $x \in \ker C$.

Let X be a complex Banach space. We apply the following notations:

$$\begin{aligned} \mathscr{C}(X) &:= \text{ closed operators on } X \text{ with dense domain,}\\ \mathscr{B}(X) &:= \text{ bounded linear operators } X \to X, \text{ and}\\ \mathscr{C}\mathscr{F}(X) &:= \text{ closed (not necessarily bounded) Fredholm operators on } X. \end{aligned}$$

The topology on $\mathscr{C}\mathscr{F}(X)$ is defined in the Appendix.

We assume that X is an inner product space with a fixed inner product (i.e., a sesquilinear, symmetric positive definite form) $h\colon X \times X \to \mathbb{C}$ which is bounded

$$|h(x,y)| \le c\|x\|\|y\| \quad \text{for all } x,y \in X.$$

Definition 6 An operator $A \in \mathscr{C}(X)$ will be called *unitary* with respect to h, if

$$h(Ax,Ay) = h(x,y) \quad \text{for all } x,y \in \operatorname{dom}(A).$$

Remark 6 (a) Note that h induces a uniformly smaller norm than $\|\cdot\|$ on X which makes X into a Hilbert space if and only if X becomes complete for this h-induced norm.
(b) The concept of h-unitary extends trivially to closed operators with dense domain in one Banach space equipped with an inner product, and range in a second Banach space, possibly with a different inner product. In this sense, for any Lagrangian subspace the generating operator $U \in \mathscr{C}(X^+,X^-)$ (established in Lemma 3) is (h^+,h^-)-unitary with $h^\pm = \mp i\omega|_{X^\pm}$.

Like for unitary operators in Hilbert spaces, the following lemma shows that a unitary operator with respect to h has no eigenvalues outside the unit circle.

Lemma 7 *Let $A \in \mathscr{C}(X)$ be unitary with respect to h and $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$. Then* $\ker(A - \lambda I) = \{0\}$.

Proof Let $x \in \ker(A - \lambda I)$, so $Ax = \lambda x$ and

$$h(x,x) \;=\; h(Ax,Ax) \;=\; |\lambda|^2 h(x,x).$$

Since $|\lambda| \neq 1$, we get $h(x,x) = 0$ and so $x = 0$ since h is positive definite.

For a certain subclass of unitary operators with respect to h we show that they have discrete spectrum close to 1. Consequently, they are admissible with respect to the positive half-line ℓ (in the sense of Definition 12 of Appendix A.3) and so permit the definition of spectral flow through ℓ for continuous families (Appendix A.3). Here the co-orientation of ℓ is upward.

Proposition 3 *(a) Let X be a Banach space with bounded inner product h. Let $A \in \mathscr{C}(X)$ be h-bounded, i.e., an operator satisfying*

$$h(Ax,Ay) \;\leq\; h(x,y) \quad \textit{for all } x,y \in \operatorname{dom}(A).$$

We assume $A - I \in \mathscr{C}\mathscr{F}(X)$ of index 0. If either A is h-unitary or A is bounded, then there is a bounded neighborhood $N \subset \mathbb{C}$ of 1 with closure $\bar{N}$ such that

$$\sigma(A) \cap \bar{N} \subset \{1\}, \quad \dim P_N(A) \;=\; \dim\ker(A - I).$$

(b) Let $\mathfrak{m}$ be an open submanifold of $\ell := (0,+\infty)$ and A be unitary with respect to h. If A is admissible with respect to $\mathfrak{m}$ in the sense of Definition 12a of the Appendix A.3, then $\sigma(A) \cap \mathfrak{m} \subset \{1\}$.
(c) Let $\{h_s\}_{0 \leq s \leq 1}$ be a family of inner products on X. Let $A_s \in \mathscr{C}(X)$ be unitary with respect to h_s. We assume that the family $\{A_s\}$ is continuous. We denote $h_0 =: h$ and $A_0 =: A$ and choose N like in (a). Then for $s \ll 1$ the spectrum part $\sigma(A_s) \cap N$ consists of eigenvalues of finite algebraic multiplicity and we have

$$\sigma(A_s) \cap N \subset S^1\,.$$

(d) Let $\{h_s\}$ and A_s be as in (c). Then there exists $\varepsilon \in (0,1)$ such that the family $\{A_s\}$ is a family of admissible operators that is spectral continuous near $\mathfrak{m}_\varepsilon := (1-\varepsilon, 1+\varepsilon)$ in the sense of Definitions 12a and 13a.
(e) Let $\{h_s\}$ and A_s be as in (c). Let $\mathfrak{m} \ni 1$ be an open submanifold of $\ell = (0,+\infty)$. If the family $\{A_s\}$ is a family of admissible operators that is spectral continuous near $\mathfrak{m}$, we can define the spectral flow

$$\operatorname{sf}_\ell\{A_s\} := \operatorname{sf}_{\mathfrak{m}}\{A_s\}.$$

Proof *a*. Since $\ker(A - I)$ is finite-dimensional, we have an h-orthogonal splitting

$$X \;=\; \ker(A - I) \oplus X_1$$

with closed X_1 . (Take $X_1 := \Pi(X)$ with $\Pi(x) := x - \sum_{j=1}^n h(x,e_j)e_j$, where $\{e_j\}$ is an h-orthonormal basis of $\ker(A - I)$). We notice that $\ker(A - I) \subset \operatorname{dom}(A)$, so

$$\operatorname{dom}(A) \;=\; \ker(A - I) \oplus (\operatorname{dom}(A) \cap X_1). \tag{25}$$

Then the operator A can be written in block form

$$A = \begin{pmatrix} I_0 & A_{01} \\ 0 & A_{11} \end{pmatrix}, \tag{26}$$

where I_0 denotes the identity operator on $\ker(A-I)$.

Since A is h-bounded, by Lemma 6b we have $\ker(A_{11}-I_1) \subset \ker A_{01}$. Here I_1 denotes the identity operator on X_1. So we have

$$\ker(A_{11}-I_1) \subset \ker(A_{11}-I_1) \cap \ker A_{01} \cap X_1 \ = \ \ker(A-I) \cap X_1 \ = \ \{0\}.$$

Now we distinguish two cases. If A is h-unitary, let $y \in \operatorname{dom}(A) \cap X_1$ and $x \in \ker(A-I)$. Then

$$h(x,Ay) \ = \ h(Ax,Ay) \ = \ h(x,y) \ = \ 0$$

by (25). So, the range $\operatorname{im}(A|_{\operatorname{dom}(A)\cap X_1})$ is h-orthogonal to $\ker(A-I)$ and, hence, contained in X_1. Hence $A_{01}=0$. We observe that $A-I$ is closed as bounded perturbation of the closed operator A; it follows that the component A_{11} and the operator $A_{11}-I_1$ are closed in X_1. That proves that $A_{11}-I_1$ has a bounded inverse.

If, on the other side, A is bounded, then both A_{01} and A_{11} are bounded and we have

$$\begin{aligned} \operatorname{index}(A_{11}-I_1) \ &= \ \operatorname{index}\big(\operatorname{diag}(I_0, A_{11}-I_1)\big) \\ &= \ \operatorname{index}\big((A-I)\operatorname{diag}(I_0,I_1)\big) \\ &= \ \operatorname{index}(A-I) + \operatorname{index}\big(\operatorname{diag}(I_0,I_1)\big) \ = \ 0. \end{aligned}$$

By $\ker(A_{11}-I_1)=\{0\}$ we have $A_{11}-I_1$ surjective. By the Closed Graph Theorem, it follows that $(A_{11}-I_1)^{-1}$ is bounded and so $A_{11}-I_1$ has a bounded inverse.

We conclude that in both cases A_1 has no spectrum near 1. From the decomposition (26) we get $\sigma(A)=\sigma(I_0)\cup\sigma(A_1)$ with $\sigma(I_0)=\{1\}$. So, if $1\in\sigma(A)$ it is an isolated point of $\sigma(A)$ of multiplicity $\dim\ker(A-I)$.

b. Since A is admissible with respect to $\mathfrak{m}$, there exists a bounded open subset N of $\mathbb{C}$ such that $\sigma(A)\cap\mathfrak{m}=\sigma(A)\cap N$ and $\dim\operatorname{im}P_N(A)<+\infty$. Then $P_N(A)AP_N(A)$ defined on the finite dimensional vector space $\operatorname{im}P_N(A)$ is unitary with respect to $h|_{\operatorname{im}P_N(A)}$. Thus we have $\sigma(P_N(A)AP_N(A))\subset S^1$ and

$$\sigma(A)\cap\mathfrak{m}=\sigma(A)\cap N=\sigma(P_N(A)AP_N(A))\subset S^1\cap\mathfrak{m}\subset S^1\cap\ell=\{1\}.$$

c. From our assumption it follows that $\sigma(A)\cap\partial N=\emptyset$ and, actually, $\sigma(A_s)\cap\partial N=\emptyset$ for s sufficiently small. Then

$$\left\{P_N(A_s) \ := \ -\frac{1}{2\pi i}\int_{\partial N}(A-\lambda I)^{-1}d\lambda\right\}$$

is a continuous family of projections. From [26, Lemma I.4.10] we obtain

$$\dim\operatorname{im}P_N(A_s) \ = \ \dim\operatorname{im}P_N(A) \ < \ +\infty \quad \text{and} \quad P_N(A_s)A_s \ \subset \ A_sP_N(A_s),$$

and from [26, Lemma III.6.17] we get $\sigma(A_s)\cap N=\sigma(P_N(A_s)A_sP_N(A_s))$. Since all operators $P_N(A_s)A_sP_N(A_s)$ are unitary with respect to $h_s|_{\operatorname{im}P_N(A_s)}$, it follows $\sigma(P_N(A_s)A_sP_N(A_s))\subset S^1$.

d. By (c) and Lemma 18 of Appendix A.3, for any $t\in[0,1]$, there exists $\varepsilon(t)\in(0,1)$ and $\delta(t)>0$ such that $\{A_s\}$, $s\in(t-\delta(t),t+\delta(t))\cap[0,1]$ is a family of admissible operators that is spectral continuous near $\mathfrak{m}_{\varepsilon(t)}$. The open cover $\{(t-\delta(t),t+\delta(t))\}$, $t\in[0,1]$ of $[0,1]$ has

a finite subcover $\{(t_k-\delta(t_k), t_k+\delta(t_k))\}$, $k=1,\dots,n$. Set $\varepsilon=\min\{\varepsilon(t_k); k=1,\dots,n\}$. Then $\{A_s\}$, $s\in(t_k-\delta(t_k), t_k+\delta(t_k))\cap[0,1]$ is a family of admissible operators that is spectral continuous near $\mathfrak{m}_\varepsilon$, and $\{A_s\}$, $s\in[0,1]$ is a family of admissible operators that is spectral continuous near $\mathfrak{m}_\varepsilon$.
e. By (d), such $\mathfrak{m}$ do exist. By (b), $\mathrm{sf}_{\mathfrak{m}}\{A_s\}$ does not depend on the choice of $\mathfrak{m}$. Thus our concept of the spectral flow relative $\ell=(0,+\infty)$ is well-defined.

Thus, it follows that any h-unitary operator A with $A-I$ Fredholm of index 0 has the same spectral properties near $|\lambda|=1$ as unitary operators in Hilbert space with the additional property that 1 is an isolated point of the spectrum of finite multiplicity.

This now permits us to define the Maslov index in weak symplectic analysis.

3 Maslov index in weak symplectic analysis

Now we turn to the geometry of curves of Fredholm pairs of Lagrangian subspaces in weak symplectic Banach spaces. We show how the usual definition of the Maslov index can be suitably extended and derive basic and more intricate properties.

3.1 Definition and basic properties of the Maslov index

Our data for defining the Maslov index are a *continuous* family $\{(X,\omega_s,X_s^+,X_s^-)\}$ of weak symplectic Banach spaces with *continuous* splitting and a *continuous* family $\{(\lambda_s,\mu_s)\}$ of Fredholm pairs of Lagrangian subspaces of $\{(X,\omega_s)\}$ of index 0. Our first task is defining the involved "continuity".

Definition 7 Let X be a fixed complex Banach space and $\{\omega_s\}$ a family of weak symplectic forms for X. Let (X,ω_s,X_s^+,X_s^-) be a family of symplectic splittings of (X,ω_s) in the sense of Definition 4.
(a) The family $\{(X,\omega_s,X_s^+,X_s^-)\}$ will be called *continuous* if the family of forms $\{\omega_s\}$ is continuous, and the families $\{X_s^\pm\}$ are continuous as closed subspaces of X in the gap topology. Equivalently, we may demand that the family $\{P_s\}$ of projections

$$P_s\colon\, x+y \;\mapsto\; x, \qquad \text{for } x\in X_s^+ \text{ and } y\in X_s^-\,,$$

is continuous.
(b) Let $\{(X,\omega_s,X_s^+,X_s^-)\}$, $s\in[a,b]$ be a continuous family of symplectic splittings with induced inner products $h_s^\pm=\mp\omega|_{X^\pm}$. Let $\{(\lambda_s,\mu_s)\}$ be a continuous curve of Fredholm pairs of Lagrangian subspaces of index 0. Let $U_s\colon\,\mathrm{dom}(U_s)\to X_s^-$, resp. $V_s\colon\,\mathrm{dom}(V_s)\to X_s^-$ be closed (h_s^+,h_s^-)-unitary operators with $\mathfrak{G}(U_s)=\lambda_s$ and $\mathfrak{G}(V_s)=\mu_s$. We define the *Maslov index* of the curve $\{\lambda_s,\mu_s\}$ with respect to P_s by

$$\mathrm{Mas}\{\lambda_s,\mu_s;P_s\} \;:=\; \mathrm{sf}_\ell\Big\{\begin{pmatrix}0 & U_s\\ V_s^{-1} & 0\end{pmatrix}\Big\}, \tag{27}$$

where V^{-1} denotes the algebraic inverse of the closed injective operator V and $\ell:=(0,+\infty)$ and with upward co-orientation. The spectral flow sf_ℓ is defined in the sense of Proposition 3e.

Remark 7 Let $\{(X,\omega_s,X_s^+,X_s^-)\}$ be a continuous family. A curve $\{\lambda_s\}$ of Lagrangian subspaces is continuous (i.e., $\{\lambda_s=\mathfrak{G}(U_s)\}$ is continuous as a curve of closed subspaces of X), if and only if the family $\{S_{s,s_0}\circ U_s\circ S_{s,s_0}^{-1}\}$ is continuous as a family of closed, generally unbounded operators in the space $\operatorname{im}P_{s_0}$. Here U_s denotes the generating operator $U_s\colon \operatorname{dom}U_s\to X_s^-$ with $\mathfrak{G}(U_s)=\lambda_s$ (see Lemma 3); $s_0\in[0,1]$ is chosen arbitrarily to fix the domain of the family; and

$$S_{s,s_0}\colon \operatorname{im}P_s\longrightarrow \operatorname{im}P_{s_0}$$

is a bounded operator with bounded inverse which is defined in the following way (see also [26, Section I.4.6, pp. 33-34]):

$$S_{s,s_0} \;:=\; S'_{s,s_0}(I-R)^{-1/2} \;=\; (I-R)^{-1/2}S'_{s,s_0}\,,$$

where

$$R \;:=\; (P_s-P_{s_0})^2 \qquad\text{and}\qquad S'_{s,s_0} \;:=\; P_{s_0}P_s+(I-P_{s_0})(I-P_s).$$

The main result of our paper is

Theorem 1 *The Maslov index of Definition 7b is well-defined.*

Proof By a series of lemmas below, we check that the family of block matrices on the right side of (27) satisfies the condition of Proposition 3e. Then our theorem follows. Note that we do not need the continuity of ω_s for our chain of arguments.

Lemma 8 *Let (X,ω) be a weak symplectic Banach space. Let Δ denote the diagonal (i.e., the canonical Lagrangian) in the product symplectic space $X\boxplus X:=(X,\omega)\oplus(X,-\omega)$, and λ,μ Lagrangian subspaces of (X,ω). Then*

$$(\lambda,\mu)\in\mathscr{F}\mathscr{L}^2(X) \quad\Longleftrightarrow\quad (\lambda\boxplus\mu,\Delta)\in\mathscr{F}\mathscr{L}^2\big(X\boxplus X\big)$$

and

$$\operatorname{index}(\lambda,\mu) \;=\; \operatorname{index}(\lambda\boxplus\mu,\Delta),$$

where $\lambda\boxplus\mu:=\{(x,y)\mid x\in\lambda,y\in\mu\}$.

Proof Clearly $\lambda\boxplus\mu$ and Δ are Lagrangian subspaces of $X\boxplus X$. Since

$$(\lambda\boxplus\mu)\cap\Delta \;=\; \{(x,x)|x\in\lambda\cap\mu\} \;\simeq\; \lambda\cap\mu,$$

these spaces have the same dimension. Set

$$\Delta' \;:=\; \{(x,-x)|x\in X\} \quad\text{and}\quad \Delta'_{\lambda+\mu} \;:=\; \{(x,-x)|x\in\lambda+\mu\}.$$

Then we have

$$\begin{aligned}\lambda\boxplus\mu+\Delta \;&=\; \{(x,y)+(\xi,\xi)|x\in\lambda,y\in\mu,\xi\in X\}\\ &=\; \{(\frac{x-y}{2},\frac{y-x}{2})+(\frac{x+y}{2}+\xi,\frac{x+y}{2}+\xi)|x\in\lambda,y\in\mu,\xi\in X\} \;=\; \Delta+\Delta'_{\lambda+\mu}.\end{aligned}$$

So the following holds:

$$\frac{X\boxplus X}{\lambda\boxplus\mu+\Delta} \;=\; \frac{\Delta\oplus\Delta'}{\Delta\oplus\Delta'_{\lambda+\mu}} \;\simeq\; \frac{\Delta'}{\Delta'_{\lambda+\mu}} \;\simeq\; \frac{X}{\lambda+\mu},$$

and they have the same dimension. Lemma 8 is proved.

Lemma 9 *Let (X,ω,X^+,X^-) be a weak symplectic Banach space with symplectic splitting. Set $h^\pm := \mp i\omega|_{X^\pm}$. Let Δ denote the diagonal in the symplectic space $X \boxplus X$. Let λ and μ be two Lagrangian subspaces of X with generator U, V respectively. Then we have:*
(a) The pair (λ,μ) is Fredholm of index 0 if and only if $\begin{pmatrix} 0 & U \\ V^{-1} & 0 \end{pmatrix} - I$ is of index 0.
(b) The matrix $\begin{pmatrix} 0 & U \\ V^{-1} & 0 \end{pmatrix}$ is $(h^- \oplus h^+)$-unitary.

Proof *a*. Let $P\colon X \to X^+$ denote the projection corresponding to the splitting. Let $(\lambda,\mu) \in \mathscr{F}\mathscr{L}^2(X)$. Then we have

$$\begin{pmatrix} 0 & U \\ V^{-1} & 0 \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & V^{-1} \end{pmatrix} \begin{pmatrix} 0 & I_{X^+} \\ I_{X^-} & 0 \end{pmatrix},$$

and

$$\widetilde{\mathfrak{G}} \begin{pmatrix} U & 0 \\ 0 & V^{-1} \end{pmatrix} = \lambda \boxplus \mu, \quad \text{and} \quad \widetilde{\mathfrak{G}} \begin{pmatrix} 0 & I_{X^-} \\ I_{X^+} & 0 \end{pmatrix} = \Delta, \tag{28}$$

where $\widetilde{\mathfrak{G}}$ denotes the graph of closed operators from $\operatorname{im}\mathscr{P}$ to $\operatorname{im}(I-\mathscr{P})$ with $\mathscr{P} := P \boxplus (I - P)$. Consequently, $\begin{pmatrix} 0 & U \\ V^{-1} & 0 \end{pmatrix} - I$ is a Fredholm operator of index 0. Conversely, by Equation (28), we can derive that (λ,μ) is a Fredholm pair of index 0 if $\begin{pmatrix} 0 & U \\ V^{-1} & 0 \end{pmatrix} - I$ is a Fredholm operator of index 0.
b. For all $x_1, x_2 \in X^+$ and $y_1, y_2 \in X^-$, we have

$$\begin{aligned}
&(h^- \oplus h^+)\left(\begin{pmatrix} 0 & U \\ V^{-1} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ x_1 \end{pmatrix}, \begin{pmatrix} 0 & U \\ V^{-1} & 0 \end{pmatrix}\begin{pmatrix} y_2 \\ x_2 \end{pmatrix}\right) \\
&\quad = (h^- \oplus h^+)\left(\begin{pmatrix} Ux_1 \\ V^{-1}y_1 \end{pmatrix}, \begin{pmatrix} Ux_2 \\ V^{-1}y_2 \end{pmatrix}\right) \\
&\quad = h^-(Ux_1, Ux_2) + h^+(V^{-1}y_1, V^{-1}y_2) \\
&\quad = h^+(x_1,x_2) + h^-(y_1,y_2) = (h^- \oplus h^+)\left(\begin{pmatrix} y_1 \\ x_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ x_2 \end{pmatrix}\right).
\end{aligned}$$

Lemma 10 *The family $\left\{\begin{pmatrix} 0 & U_s \\ V_s^{-1} & 0 \end{pmatrix}\right\}$ in (27) is a continuous family of closed operators such that $\begin{pmatrix} 0 & U_s \\ V_s^{-1} & 0 \end{pmatrix} - I$ is Fredholm of index 0 for each $s \in [a,b]$.*

Proof The proof of the above Lemma 9a shows that the graph of $\left\{\begin{pmatrix} 0 & U_s \\ V_s^{-1} & 0 \end{pmatrix}\right\}$ is a continuous family of closed subspaces of $X \oplus X$. Then the family $\left\{\begin{pmatrix} 0 & U_s \\ V_s^{-1} & 0 \end{pmatrix}\right\}$ is a continuous family of closed operators. By Lemma 9a we have that $\begin{pmatrix} 0 & U_s \\ V_s^{-1} & 0 \end{pmatrix} - I$ is Fredholm of index 0 for each $s \in [a,b]$.

That ends the proof of Theorem 1.

The proof of Lemma 9a leads to the following important result:

Proposition 4 *Let $\{X, \omega_s\}$, $s \in [a,b]$ be a continuous family of weak symplectic forms for X with a continuous family of symplectic splittings $(X, \omega_s, X_s^+, X_s^-)$ in the sense of Definition 7a and a corresponding family of projections $\{P_s \colon X \to X_s^+\}$. Let $\{\lambda_s, \mu_s\}$, $s \in [a,b]$ be a continuous curve in $\mathscr{F}\mathscr{L}^2(X)$. We denote the generating operators by U_s, respectively V_s.*
(a) If V_s is bounded and has a bounded inverse for each $s \in [0,1]$, then we have

$$\operatorname{Mas}\{\lambda_s, \mu_s; P_s\} \; = \; \operatorname{sf}_\ell\{U_s V_s^{-1}\}, \tag{29}$$

where $\ell := (0, +\infty)$ with upward co-orientation. The spectral flow is defined in the sense of Proposition 3e.
(b) We have

$$\begin{aligned}
\operatorname{Mas}\{\lambda_s \boxplus \mu_s, \Delta; \mathscr{P}_s\} \; &= \; \operatorname{Mas}\{\lambda_s, \mu_s; P_s\} && (30)\\
&= \; \operatorname{Mas}\{\mu_s, \lambda_s; I - P_s\}, \qquad \textit{in } (X, -\omega_s), && (31)\\
&= \; \operatorname{Mas}\{\Delta, \lambda_s \boxplus \mu_s; I - \mathscr{P}_s\}, \quad \textit{in } (X, -\omega_s) \boxplus (X, \omega_s), && (32)
\end{aligned}$$

where $\mathscr{P}_s := P_s \boxplus (I - P_s)$.

Proof *a*. By our assumption, we have

$$\dim\ker(z^2 I - U_s V_s^{-1}) = \dim\ker\left(zI - \begin{pmatrix} 0 & U_s \\ V_s^{-1} & 0 \end{pmatrix}\right)$$

for all $z \in \mathbb{C} \setminus \{0\}$. By Proposition 3 and Lemma 9, both the total algebraic multiplicities of the spectrum of the matrices $U_s V_s^{-1}$ and $\begin{pmatrix} 0 & U_s \\ V_s^{-1} & 0 \end{pmatrix}$ near 1 are $\dim\ker(I - U_s V_s^{-1})$, and each spectrum of them with finite algebraic multiplicity is on S^1. Then (a) follows from the definition of the Maslov index and Proposition 3.
b. Let $\widetilde{\mathfrak{G}}$ denote the graph of closed operators from $\operatorname{im} \mathscr{P}_s$ to $\operatorname{im}(I - \mathscr{P}_s)$. By Equations (27), (28) and (29) we have

$$\begin{aligned}
\operatorname{Mas}\{\lambda_s \boxplus \mu_s, \Delta; \mathscr{P}_s\} \; &= \; \operatorname{Mas}\Big\{\widetilde{\mathfrak{G}}\begin{pmatrix} U_s & 0 \\ 0 & V_s^{-1} \end{pmatrix}, \widetilde{\mathfrak{G}}\begin{pmatrix} 0 & I_{X_s^-} \\ I_{X_s^+} & 0 \end{pmatrix}; \mathscr{P}_s\Big\}\\
&= \; \operatorname{sf}_\ell\Big\{\begin{pmatrix} U_s & 0 \\ 0 & V_s^{-1} \end{pmatrix}\begin{pmatrix} 0 & I_{X_s^+} \\ I_{X_s^-} & 0 \end{pmatrix}\Big\}\\
&= \; \operatorname{sf}_\ell\Big\{\begin{pmatrix} 0 & U_s \\ V_s^{-1} & 0 \end{pmatrix}\Big\} \; = \; \operatorname{Mas}\{\lambda_s, \mu_s; P_s\}.
\end{aligned}$$

So (30) is proved.

For the symplectic space $(X, -\omega_s)$ with symplectic splitting $(X, -\omega_s, X_s^-, X_s^+)$, the generating operators of λ_s, μ_s are U_s^{-1}, V_s^{-1} respectively. Note that

$$\begin{pmatrix} 0 & V_s^{-1} \\ U_s & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{X_s^+} \\ I_{X_s^-} & 0 \end{pmatrix}\begin{pmatrix} 0 & U_s \\ V_s^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & I_{X_s^+} \\ I_{X_s^-} & 0 \end{pmatrix}.$$

By the definition of the Maslov index we have

$$\operatorname{Mas}\{\mu_s, \lambda_s; I - P_s\}_{\text{in } (X, -\omega)} \; = \; \operatorname{sf}_\ell\left\{\begin{pmatrix} 0 & V_s^{-1} \\ U_s & 0 \end{pmatrix}\right\} \; = \; \operatorname{sf}_\ell\left\{\begin{pmatrix} U_s & 0 \\ 0 & V_s^{-1} \end{pmatrix}\right\} \; = \; \operatorname{Mas}\{\lambda_s, \mu_s; P_s\}.$$

So (31) is proved. (32) follows from (31) and (30).

From the properties of our general spectral flow, as observed at the end of our Appendix, we get all the basic properties of the Maslov index (see S. E. Cappell, R. Lee, and E. Y. Miller [17, Section 1] for a more comprehensive list).

Proposition 5 *(a) The Maslov index is* invariant under homotopies *of curves of Fredholm pairs of Lagrangian subspaces with fixed endpoints. In particular, the Maslov index is invariant under* re-parametrization *of paths.*
(b) The Maslov index is additive under catenation*, i.e.,*

$$\mathrm{Mas}\{\lambda_1 * \lambda_2, \mu_1 * \mu_2; P_s * Q_s\} \;=\; \mathrm{Mas}\{\lambda_1, \mu_1; P_s\} + \mathrm{Mas}\{\lambda_2, \mu_2; Q_s\},$$

where $\{\lambda_i(s)\}, \{\mu_i(s)\}$, $i = 1,2$ *are continuous paths with* $\lambda_1(1) = \lambda_2(0)$, $\mu_1(1) = \mu_2(0)$ *and*

$$(\lambda_1 * \lambda_2)(s) \;:=\; \begin{cases} \lambda_1(2s), & 0 \le s \le \frac{1}{2}, \\ \lambda_2(2s-1), & \frac{1}{2} < s \le 1, \end{cases}$$

and similarly $\mu_1 * \mu_2$ *and* $\{P_s\} * \{Q_s\}$.
(c) The Maslov index is natural *under symplectic action: let* $\{(X', \omega'_s)\}$ *be a second family of symplectic Banach spaces and let*

$$L_s \in \mathrm{Sp}(X, \omega_s; X', \omega'_s) \;:=\; \{L \in \mathscr{B}(X, X') \mid L \textit{ invertible and } \omega'_s(Lx, Ly) = \omega_s(x,y)\},$$

such that $\{L_s\}$ *is a continuous family of bounded operators. Then* $\{X' = L_s(X_s^+) \oplus L_s(X_s^-)\}$ *is a continuous family of symplectic splittings of* $\{(X', \omega'_s)\}$ *inducing projections* $\{Q_s\}$*, and we have*

$$\mathrm{Mas}\{\lambda_s, \mu_s; P_s\} \;=\; \mathrm{Mas}\{L_s\lambda_s, L_s\mu_s; Q_s\}.$$

(d) The Maslov index vanishes*, if* $\dim(\lambda_s \cap \mu_s)$ *is constant for all* $s \in [0,1]$.
(e) Flipping. *We have*

$$\mathrm{Mas}\{\lambda_s, \mu_s; P_s\} + \mathrm{Mas}\{\mu_s, \lambda_s; P_s\} \;=\; \dim \lambda_0 \cap \mu_0 - \dim \lambda_1 \cap \mu_1.$$

We can not claim that the Maslov index, $\mathrm{Mas}\{\lambda_s, \mu_s; P_s\}$ is always independent of the splitting projection P_s in general Banach spaces. However, we have the following result.

Proposition 6 *Let* $\{(X, \omega_s)\}$ *be a continuous family of* strong *symplectic Banach spaces and let* $\{X = X_{s,\rho}^+ \oplus X_{s,\rho}^-\}$ *be two continuous families of symplectic splittings in the sense of Definition 7a with projections* $P_{s,\rho}: X \to X_{s,\rho}^+$ *for* $s \in [0,1]$ *and* $\rho = 0,1$. *Let* $\{(\lambda_s, \mu_s)\}$ *be a continuous curve of Fredholm pairs of Lagrangian subspaces of* $\{(X, \omega_s)\}$. *Then*
(a) index $(\lambda_s, \mu_s) = 0$ *for all* $s \in [0,1]$*; and*
(b) $\mathrm{Mas}\{\lambda_s, \mu_s; P_{s,0}\} = \mathrm{Mas}\{\lambda_s, \mu_s; P_{s,1}\}$.

Note 2 Commonly, one assumes $J^2 = -I$ in strong symplectic analysis and defines the Maslov index with respect to the induced decomposition. In view of Lemma 1, the point of the preceding proposition is that the Maslov index is independent of the choice of the metrics.

Proof a. Using $-i\omega_s$, we make (X, ω_s) into a symplectic Hilbert space and deform the metric such that $J_s^2 = -I$. Clearly, the dimensions entering into the definition of the Fredholm index do not change under the deformation. So, we are in the well-studied standard case.
b. We recall that our two families of symplectic splittings define two families of Hilbert structures for X defined by

$$\langle x_{s,\rho}^+ + x_{s,\rho}^-, y_{s,\rho}^+ + y_{s,\rho}^- \rangle_{s,\rho} \;:=\; -i\omega_s(x_{s,\rho}^+, y_{s,\rho}^+) + i\omega_s(x_{s,\rho}^-, y_{s,\rho}^-)$$

for $x^+_{s,\rho}, y^+_{s,\rho} \in H^+_{s,\rho}, x^-_{s,\rho}, y^-_{s,\rho} \in H^-_{s,\rho}$, and $\rho = 0,1$. For any $\rho \in [0,1]$ we define

$$\langle x,y\rangle_{s,\rho} \ := \ (1-\rho)\langle x,y\rangle_{s,0} + \rho\langle x,y\rangle_{s,1}.$$

Then all $(X, \langle\cdot,\cdot\rangle_{s,\rho})$ are Hilbert spaces.

Define $J_{s,\rho}$ by $\omega_s(x,y) = \langle J_{s,\rho}x,y\rangle_{s,\rho}$ and let $X^{\pm}_{s,\rho}$ denote the positive (negative) space of $-iJ_{s,\rho}$ and $P_{s,\rho}$ the orthogonal projection of X onto $X^+_{s,\rho}$. Then the two-parameter family $\{J_{s,\rho}\}$ is a continuous family of invertible operators, $\{P_{s,\rho}\}$ is continuous, and $\{H^+_{s,\rho}\}$ is continuous. So $\operatorname{Mas}\{\lambda_s,\mu_s;P_{s,\rho}\}$ is well-defined. By Proposition 5a and 5b, we have

$$\begin{aligned}\operatorname{Mas}\{\lambda_0,\mu_0;P_{0,\rho};0\le\rho\le1\} + \operatorname{Mas}\{\lambda_s,\mu_s;P_{s,0};0\le s\le1\}&\\ = \ \operatorname{Mas}\{\lambda_s,\mu_s;P_{s,1};0\le s\le1\} + \operatorname{Mas}\{\lambda_1,\mu_1;P_{1,\rho};0\le\rho\le1\}.&\end{aligned}$$

By Proposition 5d, we have

$$\operatorname{Mas}\{\lambda_0,\mu_0;P_{0,\rho};0\le\rho\le1\} \ = \ \operatorname{Mas}\{\lambda_1,\mu_1;P_{1,\rho};0\le\rho\le1\} \ = \ 0.$$

So we obtain

$$\operatorname{Mas}\{\lambda_s,\mu_s;P_{s,0};0\le s\le1\} \ = \ \operatorname{Mas}\{\lambda_s,\mu_s;P_{s,1};0\le s\le1\}.$$

3.2 Comparison with the real (and strong) category

For a fixed strong symplectic Hilbert space X, choosing one single Lagrangian subspace λ yields a decomposition $X = \lambda \oplus J\lambda$. This decomposition was used in [7, Definition 1.5] (see also [9, Theorem 3.1] and [24, Proposition 2.14]) to give the first functional analytic definition of the Maslov index, though under the somewhat restrictive (and notationally quite demanding) assumption of *real* symplectic structure. Up to the sign, our Definition 7b is a true generalization of that previous definition. More precisely:

Let (H,ω) be a real symplectic Hilbert space with

$$\omega(x,y) \ = \ \langle Jx,y\rangle,\ J^2 \ = \ -I,\ J^t \ = \ -J.$$

Clearly, we obtain a symplectic decomposition $H^+ \oplus H^- = H \otimes \mathbb{C}$ with the induced complex strong symplectic form $\omega_{\mathbb{C}}$ by

$$H^{\pm} \ := \ \{(I \mp iJ)\zeta \mid \zeta \in H\}.$$

Definition 8 We fix one (real) Lagrangian subspace $\lambda \subset H$.
a) Then there is a real linear isomorphism $\varphi\colon H \cong \lambda\otimes\mathbb{C}$ defined by $\varphi(x+Jy) := x+iy$ for all $x,y \in \lambda$.
b) For $A = X+JY : H \to H$ with $X,Y : H \to H$ real linear and

$$X(\lambda)\subset\lambda,\ Y(\lambda)\subset\lambda, \quad \text{and} \quad XJ \ = \ JX,\ YJ \ = \ JY, \tag{33}$$

we define

$$\varphi_*(A) \ := \ \varphi\circ A\circ\varphi^{-1} \ = \ X+iY, \quad \overline{A}_\lambda \ := \ X-JY, \quad A^{t_\lambda} \ := \ X^t + JY^t,$$

where X^t, Y^t denote the real transposed operators.
c) Let μ be a second Lagrangian subspace of H and let $\widetilde{V} : H \to H$ with $\widetilde{V}J = J\widetilde{V}$ be a real generating operator for μ with respect to the orthogonal splitting $H = \lambda\oplus J\lambda$, i.e., $\mu = \widetilde{V}(J\lambda)$ and $\varphi_*(\widetilde{V})$ is unitary. Then we define the complex generating operator for $\mu\otimes\mathbb{C}$ with respect to λ by $S_\lambda(\widetilde{V}) := \varphi_*(\widetilde{V})\varphi_*\big(\widetilde{V}^{t_\lambda}\big)$.

Note 3 The complex generating operator for $\mu\otimes\mathbb{C}$ with respect to λ was defined by J. Leray in [28, Section I.2.2, Lemma 2.1] and elaborated in the references given at the beginning of this subsection.

Lemma 11 *Let (λ,μ) be any pair of Lagrangian subspaces of H (in the real category). Let $\widetilde{V}\colon H\to H$ with $\widetilde{V}J = J\widetilde{V}$ be a real generating operator for μ with respect to the orthogonal splitting $H=\lambda\oplus J\lambda$. Let $U,V\colon H^+\to H^-$ denote the unitary generating operators for $\lambda\otimes\mathbb{C}$ and $\mu\otimes\mathbb{C}$, i.e., we have*

$$\lambda\otimes\mathbb{C} \;=\; \mathfrak{G}(U) \;\text{ and }\; \mu\otimes\mathbb{C} \;=\; \mathfrak{G}(V).$$

Then we have $VU^{-1} = -\overline{S_\lambda(\widetilde{V})}$, where $S_\lambda(\widetilde{V})$ denotes the complex generating operator for $\mu\otimes\mathbb{C}$ with respect to λ, as introduced in the preceding Definition.

Proof At first we give some notations used later. For $\zeta = x+Jy\in H$ with $x,y\in\lambda$, we define $\overline{\zeta}_\lambda := \varphi^{-1}\big(\overline{\varphi(\zeta)}\big) = x-Jy$. Moreover, for $A = X+JY\colon H\to H$ with $X,Y\colon H\to H$ real linear with (33), we define $\widetilde{S}_\lambda(A) := AA^{t_\lambda}$. Then we have $S_\lambda(A) = \varphi_*\big(\widetilde{S}_\lambda(A)\big)$.

Now we give explicit descriptions of U and V. It is immediate that U takes the form

$$\begin{array}{rccc} U: & H^+ & \longrightarrow & H^- \\ & (I-iJ)\zeta & \mapsto & (I+iJ)\overline{\zeta}_\lambda\ . \end{array}$$

By the definition of $\widetilde{V}$, we have

$$\mu \;=\; \widetilde{V}(J\lambda) \;=\; \{2\widetilde{V}Jx + 2i\widetilde{V}Jy \mid x,y\in\lambda\}.$$

We shall find $V\colon (I-iJ)\zeta\mapsto(I+iJ)\zeta_1$ with $\zeta,\zeta_1\in H$ such that $\mathfrak{G}(V) = \mu\otimes\mathbb{C}$, i.e., we shall find ζ_1 to $\zeta = x+Jy$ such that

$$(I-iJ)\zeta + (I+iJ)\zeta_1 \;=\; 2\widetilde{V}Jx + 2i\widetilde{V}Jy \quad \text{ for all } x,y\in\lambda. \tag{34}$$

Comparing real and imaginary part of (34) yields $\zeta+\zeta_1 = 2\widetilde{V}Jx$ and $-iJ(\zeta-\zeta_1) = -i\widetilde{V}Jy$, so

$$\zeta \;=\; \widetilde{V}(Jx-y) \quad\text{ and }\quad \zeta_1 \;=\; \widetilde{V}(Jx+y).$$

From the left equation we obtain $\overline{\zeta}_\lambda = -\overline{\widetilde{V}}_\lambda(Jx+y)$. Since $\varphi_*(\widetilde{V})$ is unitary, we obtain from the right side

$$\zeta_1 \;=\; \widetilde{V}(Jx+y) \;=\; -\widetilde{V}\overline{\widetilde{V}}_\lambda^{-1}\overline{\zeta}_\lambda \;=\; -\widetilde{V}\widetilde{V}^t\overline{\zeta}_\lambda \;=\; -\widetilde{S}_\lambda(\widetilde{V})\overline{\zeta}_\lambda.$$

This gives

$$\begin{array}{rccc} V: & H^+ & \longrightarrow & H^- \\ & (I-iJ)\zeta & \mapsto & -(I+iJ)\widetilde{S}_\lambda(\widetilde{V})\overline{\zeta}_\lambda. \end{array}$$

So for all $z_1 := (I+iJ)\zeta_1$ with $\zeta_1\in H$, we have

$$\begin{aligned} VU^{-1}z_1 &= -(I+iJ)\widetilde{S}_\lambda(\widetilde{V})\zeta_1 \\ &= -\widetilde{S}_\lambda(\widetilde{V})(I+iJ)\zeta_1 \\ &= -\widetilde{S}_\lambda(\widetilde{V})(I-iJ)\overline{\varphi(\zeta)} \\ &= -\overline{\varphi_*(\widetilde{S}_\lambda(\widetilde{V}))}(I-iJ)\overline{\varphi(\zeta)} \\ &= -\overline{S_\lambda(\widetilde{V})}(I+iJ)\zeta_1 \\ &= -\overline{S_\lambda(\widetilde{V})}z_1. \end{aligned}$$

That is, $VU^{-1} = -\overline{S_\lambda(\widetilde{V})}$.

With the preceding notation, we recall from [7, Definition 1.5] the definition of the Maslov index

$$\mathrm{Mas}_{\mathrm{BF}}\{\mu_s,\lambda\} \;:=\; \mathrm{sf}_{\ell'}\{S_\lambda(\widetilde{V}_s)\} \tag{35}$$

of a continuous curve $\{\mu_s\}$ of Lagrangian subspaces in real symplectic Hilbert space H which build Fredholm pairs with one fixed Lagrangian subspace λ. Here $\ell' := (-1-\varepsilon,-1+\varepsilon)$ with downward orientation.

Corollary 2

$$\mathrm{Mas}\{\lambda\otimes\mathbb{C},\mu_s\otimes\mathbb{C}\} \;=\; -\mathrm{Mas}_{\mathrm{BF}}\{\mu_s,\lambda\}.$$

Proof Let ℓ,ℓ' denote small intervals on the real line close to 1, respectively -1 and give ℓ the co-orientation from $-i$ to $+i$ and ℓ' vice versa. We denote by ℓ^- the interval ℓ with reversed co-orientation. Then by our definition in (27), Proposition 4a, elementary transformations, the preceding lemma, and the definition recalled in (35):

$$\begin{aligned}\mathrm{Mas}\{\lambda\otimes\mathbb{C},\mu_s\otimes\mathbb{C}\} \;&=\; -\,\mathrm{sf}_\ell\{UV_s^{-1}\} \;=\; -\,\mathrm{sf}_{\ell^-}\{V_sU^{-1}\}\\ &=\; -\,\mathrm{sf}_{\ell^-}\{-\overline{S_\lambda(\widetilde{V}_s)}\} \;=\; -\,\mathrm{sf}_\ell\{-S_\lambda(\widetilde{V}_s)\}\\ &=\; -\,\mathrm{sf}_{\ell'}\{S_\lambda(\widetilde{V}_s)\} \;=\; -\mathrm{Mas}_{\mathrm{BF}}\{\mu_s,\lambda\}.\end{aligned}$$

3.3 Invariance of the Maslov index under embedding

We close this section by discussing the invariance of the Maslov index under embedding in a larger symplectic space, assuming a simple regularity condition.

Lemma 12 *Let $\{(X,\omega_s,X_s^+,X_s^-)\}$ be a continuous family of symplectic splittings for a (complex) Banach space X and let $\{(\lambda_s,\mu_s)\in\mathscr{FL}^2(X,\omega_s)\}$ be a continuous curve with* index $(\lambda_s,\mu_s)=0$ *for all $s\in[0,1]$. Let Y be a second Banach space with a linear embedding $Y\hookrightarrow X$ (in general neither continuous nor dense). We assume that*

$$\widetilde{\omega}_s \;:=\; \omega_s|_{Y\times Y} \quad\text{ and }\quad Y_s^\pm \;:=\; X_s^\pm\cap Y$$

yields also a continuous family $\{(Y,\widetilde{\omega}_s,Y_s^+,Y_s^-)\}$ of symplectic splittings. Moreover, we assume that $\dim(\lambda_s\cap\mu_s)-\dim(\lambda_s\cap\mu_s\cap Y)$ *is constant and $(\lambda_s\cap Y,\mu_s\cap Y)\in\mathscr{FL}^2(Y,\widetilde{\omega}_s)$ of index* 0 *for all s, and that the pairs define also a continuous curve in Y. Then we have*

$$\mathrm{Mas}\{\lambda_s,\mu_s;P_s\} \;=\; \mathrm{Mas}\{\lambda_s\cap Y,\mu_s\cap Y;\widetilde{P}_s\},$$

where P_s and $\widetilde{P}_s$ denote the projections of X onto X_s^+ along X_s^- and the projections of Y onto Y_s^+ along Y_s^- respectively.

The lemma is an immediate consequence of Lemma 19 of the Appendix.

Acknowledgements We would like to thank Prof. K. Furutani (Tokyo), Prof. M. Lesch (Bonn) and Prof. R. Nest (Copenhagen) for inspiring discussions about this subject and BS H. Larsen (Roskilde) for drawing the figures of the Appendix. Not least we thank the referee for careful reading and an heroic effort comprising 191 thoughtful comments, corrections, and helpful suggestions which led to many improvements of the presentation. The referee clearly went beyond the call of duty, and we are indebted.

A Spectral flow

The spectral flow for a one parameter family of linear self-adjoint Fredholm operators was introduced by M. Atiyah, V. Patodi, and I. Singer [4] in their study of index theory on manifolds with boundary. Since then other significant applications have been found. Later this notion was made rigorous for curves of bounded self-adjoint Fredholm operators in J. Phillips [33] and for gap-continuous curves of self-adjoint (generally unbounded) Fredholm operators in Hilbert spaces in [10] by the Cayley transform. The notion was generalized to the higher dimensional case in X. Dai and W. Zhang [21] for Riesz-continuous families, and to more general operators in [40,43,45].

For manifolds with singular metrics, there may appear linear relations (cf. C. Bennewitz [5] and M. Lesch and M. Malamud [29]). It is well known that many statements on *relations* can be translated into those on the resolvents in the realm of *operator* theory, see, e.g., B. M. Brown, G. Grubb, and I. G. Wood [15]. It seems to us, however, that this translation can not always be made globally, i.e., not for a whole curve of relations.

In this Appendix we shall provide a rigorous definition of the *spectral flow* of *spectral-continuous* curves of *admissible* closed linear relations in Banach spaces relative to a co-oriented real curve $\ell \subset \mathbb{C}$. (All the preceding terms will be explained).

A.1 Gap between subspaces

Let $\mathscr{S}(X)$ denote the set of all closed subspaces of a Banach space X.

The gap topology

The *gap* between subspaces $M,N \in \mathscr{S}(X)$ is defined by

$$\hat{\delta}(M,N) \;:=\; \max\{\delta(M,N),\delta(N,M)\}, \tag{1}$$

where $\delta(M,N) := \sup\{\operatorname{dist}(x,N) \mid x \in M, \|x\| = 1\}$, $\delta(M,\{0\}) := 1$ for $M \neq \{0\}$, and $\delta(\{0\},N) := 0$. The sets $U(M,\varepsilon) = \{N \in \mathscr{S}(X) \mid \delta(M,N) < \varepsilon\}$, where $M \in \mathscr{S}(X)$ and $\varepsilon > 0$, form a basis for the so-called *gap topology* on $\mathscr{S}(X)$. This is a complete metrizable topology on $\mathscr{S}(X)$ [26, Section IV.2.1].

Let X be a Hilbert space. Then the gap between closed subspace M,N is a metric for $\mathscr{S}(X)$ and can be calculated by

$$\hat{\delta}(M,N) \;=\; \|P_M - P_N\|, \tag{2}$$

where P_M, P_N denote the orthogonal projections of X onto M,N respectively, [26, Theorem I.6.34].

We have the following lemma.

Lemma 13 *Let X be a Hilbert space, and Y be a closed linear subspace of X. Then the mapping $M \mapsto M+Y$ induces a bijection from the space $\mathscr{S}(X,Y)$ of closed linear subspaces of X containing Y onto the space $\mathscr{S}(X/Y) = \mathscr{S}(Y^{\perp})$ of closed linear subspaces of X/Y, which preserves the metric.*

Proof We view X/Y as $Y^{\perp}$. Let $M,N \subset Y^{\perp}$ be two closed subspaces and P_M, P_N be the orthogonal projections onto M, N respectively. Then we have

$$\hat{\delta}(M+Y,N+Y) \;=\; \|P_{M+Y} - P_{N+Y}\| \;=\; \|P_M - P_N\| \;=\; \hat{\delta}(M,N).$$

Uniform properties

In general, the distances $\delta(M,N)$ and $\delta(N,M)$ can be very different and, even worse, behave very differently under small perturbations. However, for finite-dimensional subspaces of the same dimension in a Hilbert space we can estimate $\delta(M,N)$ by $\delta(N,M)$ in a uniform way.

Lemma 14 *Let X be a Hilbert space and M,N be two subspaces with* $\dim M = \dim N = n \in \mathbb{N}$. *If $\delta(N,M) < \frac{1}{\sqrt{n}}$, then we have*

$$\delta(M,N) \leq \frac{\sqrt{n}\,\delta(N,M)}{1-\sqrt{n}\,\delta(N,M)}. \tag{3}$$

Proof Let $y_1,\dots,y_n$ be an orthonormal basis of N. Let $x_k \in M$ denote the vectors with $\|x_k - y_k\| = \mathrm{dist}(y_k,M)$. Then $\|x_k - y_k\| \le \delta(N,M)$.

For any $a_1,\dots,a_n \in \mathbb{C}$, set $x = \sum_{k=1}^n a_k x_k$. Then we have

$$\begin{aligned} \|x\| &= \|\sum_{k=1}^n a_k y_k + \sum_{k=1}^n a_k(x_k - y_k)\| \ge \|\sum_{k=1}^n a_k y_k\| - \sum_{k=1}^n |a_k|\|x_k - y_k\| \\ &\ge (\sum_{k=1}^n a_k^2)^{\frac{1}{2}} - \sum_{k=1}^n |a_k|\,\delta(N,M) \ge (1-\sqrt{n}\,\delta(N,M))(\sum_{k=1}^n a_k^2)^{\frac{1}{2}}. \end{aligned} \tag{4}$$

If $x = 0$, by (4) we have $a_k = 0$. Thus $x_1,\dots,x_n$ are linearly independent and therefore they form a basis of M.

For any $x = \sum_{k=1}^n a_k x_k \in M$ with $\|x\| = 1$, let $y = \sum_{k=1}^n a_k y_k$. By (4) we have

$$\|x-y\| = \|\sum_{k=1}^n a_k(x_k - y_k)\| \le \sum_{k=1}^n |a_k|\,\delta(N,M) \le \frac{\sqrt{n}\,\delta(N,M)}{1-\sqrt{n}\,\delta(N,M)}.$$

Hence we have (3).

Clearly, taking the sum of two closed subspaces is not a continuous operation in general, but becomes continuous when fixing the dimension of the intersection and keeping the sum closed.

The following Lemma is well-known and the proof is omitted.

Lemma 15 *Let X,Y be two Hilbert space and $A_s \in \mathscr{B}(X,Y)$ be a norm-continuous family of semi-Fredholm bounded operators. If $\dim\ker A_s$ is constant, then $\ker A_s \in \mathscr{S}(X)$ and $\mathrm{im} A_s \in \mathscr{S}(Y)$ are continuous families of closed subspaces (continuous in the gap topology).*

We recall the notion of semi–Fredholm pairs: Let $M,N \in \mathscr{S}(X)$. The pair M,N is called *(semi-)Fredholm* if $M+N$ is closed in X, and both of (one of) the spaces $M\cap N$ and $\dim X/(M+N)$ are (is) finite dimensional. In this case, the *index* of (M,N) is defined by

$$\mathrm{index}\,(M,N) := \dim M\cap N - \dim X/(M+N) \in \mathbb{Z}\cup\{-\infty,\infty\}. \tag{5}$$

Note that by [8, Remark A.1] (see also [26, Problem 4.4.7]), $X/(M+N)$ of finite dimension implies $M+N \in \mathscr{S}(X)$.

Proposition 7 *Let X be a Hilbert space and $n \in \mathbb{N}$. Denote by $\mathscr{S}\mathscr{F}^2_{1,n}(X)$ (respectively $\mathscr{S}\mathscr{F}^2_{2,n}(X)$) the set of semi-Fredholm pairs (M,N) of closed subspaces with $\dim M\cap N = n$ (respectively $\dim X/(M+N) = n$). Then the following four natural mappings $\varphi_{k,l}\colon \mathscr{S}\mathscr{F}^2_{l,n}(X) \to \mathscr{S}(X)$, $k,l = 1,2$ are continuous:*

$$\varphi_{1,l}(M,N) := M\cap N, \qquad \varphi_{2,l}(M,N) := M+N.$$

Proof (Communicated by R. Nest) Let $(M,N) \in \mathscr{S}(X)\times\mathscr{S}(X)$. Let P_M and P_N denote the orthogonal projections of X onto M and N respectively. Then we have

$$\mathrm{im} P_M + \mathrm{im} P_N = \mathrm{im}((I-P_N)P_M) + \mathrm{im} P_N\,.$$

So $M+N$ is closed if and only if $\mathrm{im}((I-P_N)P_M)$ is closed, and the kernel of $(I-P_N)P_M \in \mathscr{B}(\mathrm{im} P_M, \ker P_N)$ is $M\cap N$. By Lemma 15, the maps $\varphi_{k,1}$, $k = 1,2$ are continuous. Recall that taking orthogonal complements is continuous. Then $\varphi_{k,2}$ is continuous by the fact that

$$\varphi_{k,2}(M,N) = \big(\varphi_{3-k,1}(M^\perp,N^\perp)\big)^\perp, \quad k = 1,2.$$

A.2 Closed linear relations

This subsection discusses some general properties of closed linear relations. For additional details, see Cross [20].

Basic concepts of closed linear relations

Let X,Y be two vector spaces. A *linear relation* A between X and Y is a linear subspace of $X\times Y$. As usual, the *domain*, the *range*, the *kernel* and the *indeterminant part* of A are defined by

$$\begin{aligned}\mathrm{dom}(A) &:= \{x\in X \mid \text{ there exists } y\in Y \text{ such that } (x,y)\in A\},\\ \mathrm{im}A &:= \{y\in Y \mid \text{ there exists } x\in X \text{ such that } (x,y)\in A\},\\ \ker A &:= \{x\in X \mid (x,0)\in A\},\\ A(0) &:= \{y\in Y \mid (0,y)\in A\},\end{aligned}$$

respectively.

Let X,Y,Z be three vector spaces. Let A,B be linear relations between X and Y, and C a linear relation between Y and Z. We define $A+B$ and CA by

$$A+B := \{(x,y+z)\in X\times Y \mid (x,y)\in A,(x,z)\in B\}, \tag{6}$$

$$CA := \{(x,z)\in X\times Z \mid \exists y\in Y \text{ such that } (x,y)\in A,(y,z)\in C\}. \tag{7}$$

Definition 9 Let X,Y be two Banach spaces. A *closed linear relation* between X,Y is a closed linear subspace of $X\times Y$. We denote by $\mathscr{C}LR(X,Y)=\mathscr{S}(X\times Y)$ and $\mathscr{C}LR(X)=\mathscr{S}(X\times X)$.

Note that a linear relation A between X,Y is a graph of a linear operator if and only if $A(0)=\{0\}$. In this case we shall still denote the corresponding operator by A. After identifying an operator and its graph, we have the inclusions

$$\mathscr{B}(X,Y)\subset\mathscr{C}(X,Y)\subset\mathscr{C}LR(X,Y)$$

with the notations of Section 2.4.

Let A be a linear relation between X,Y. The *inverse* A^{-1} of A is a always defined. It is the linear relation between Y,X defined by

$$A^{-1}=\{(y,x)\in Y\times X;(x,y)\in A\}. \tag{8}$$

Definition 10 Let X,Y be two Banach spaces and $A\in\mathscr{C}LR(X,Y)$.

(i) A is called *Fredholm*, if $\dim\ker A<+\infty$, $\mathrm{im}A$ is closed in Y and $\dim(Y/\mathrm{im}A)<+\infty$. In this case, we define the *index* of A to be

$$\mathrm{index}\,A \;=\; \dim\ker A-\dim(Y/\mathrm{im}A). \tag{9}$$

(ii) A is called *bounded invertible*, if $A^{-1}\in\mathscr{B}(Y,X)$.

Lemma 16 *(a) A is Fredholm, if and only if the pair $(A,X\times\{0\})$ is a Fredholm pair of closed subspaces of $X\times Y$. In this case,* $\mathrm{index}\,A=\mathrm{index}\,(A,X\times\{0\})$.
(b) A is bounded invertible, if and only if $X\times X$ is the direct sum of A and $X\times\{0\}$.

Proof Our results follow from the fact that

$$A\cap(X\times\{0\}) \;=\; \ker A\times\{0\}, \text{ and } A+(X\times\{0\}) \;=\; (\{0\}\times\mathrm{im}A)+(X\times\{0\}).$$

Spectral projections of closed linear relations

Definition 11 Let X be a Banach space and $A\in\mathscr{C}LR(X)$. Let ζ be a complex number. ζ is called a *regular point* of A if $A-\zeta I$ is bounded invertible. Otherwise ζ is called a *spectral point* of A. We denote the set of all spectral points of A by $\sigma(A)$ and the set of all regular points of A by $\rho(A)$. The *resolvent* of A is defined by

$$R(\zeta,A) \;=\; (A-\zeta I)^{-1}, \quad \zeta\in\rho(A). \tag{10}$$

Let X be a Banach space, and $A\in\mathscr{C}LR(X)$. Let $N\subset\mathbb{C}$ be a bounded open subset. Assume that $\sigma(A)\cap\partial N$ does not contain an accumulation point of $\sigma(A)\cap N$. Then there exists an open subset $N_1\subset N$ such that

$$\overline{N_1}\subset N,\ \partial N_1\in C^1,\ \sigma(A)\cap N_1 \;=\; \sigma(A)\cap N, \quad \text{and} \quad \sigma(A)\cap\partial N_1 \;=\; \emptyset, \tag{11}$$

and the spectral projection

$$P_N(A) \;:=\; -\frac{1}{2\pi i}\int_{\partial N_1}(A-\zeta I)^{-1}d\zeta \tag{12}$$

is well-defined and does not depend on the choice of N_1. We have the following lemma (cf. [26, Theorem III.6.17]):

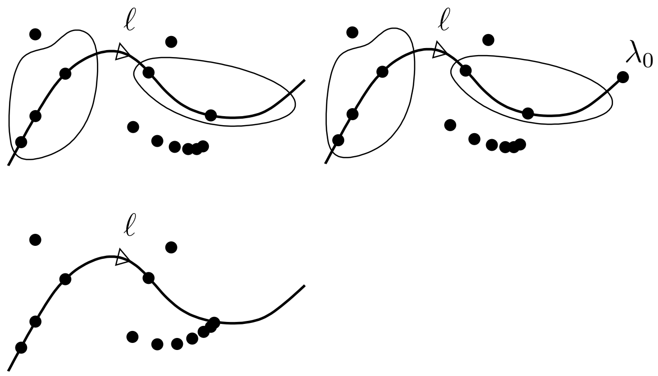

Fig. 1 Upper left: Closed linear relation with admissible spectrum with respect to ℓ. Upper right: Admissible spectrum with $\lambda_0 \in \overline{\ell} \setminus \ell$. Bottom: Non-admissible spectrum since $\sigma(A) \cap N \neq \sigma(A) \cap \ell$ and $\dim \operatorname{im} P_N(A) = +\infty$, each contradicting (16)(i) and (ii)

Lemma 17 *(a) We have*

$$P_N(A)A \subset AP_N(A) \;=\; P_N(A)AP_N(A) + \{0\} \times A(0), \tag{13}$$

where the composition is taken in the sense of (7).
(b) We have

$$P_N(A)AP_N(A) \;=\; -\frac{1}{2\pi i}\int_{\partial N_1} \zeta (A - \zeta I)^{-1} d\zeta. \tag{14}$$

(c) If we view $P_N(A)AP_N(A)$ as a linear relation on $\operatorname{im}(P_N(A))$*, then we have $P_N(A)AP_N(A) \in \mathscr{B}\big(\operatorname{im}(P_N(A))\big)$, and*

$$\sigma(A) \cap N \;=\; \sigma\big(P_N(A)AP_N(A)\big). \tag{15}$$

Proof Let $z \in N \setminus N_1$ be a regular point. Then we have

$$P_N(A)R(z,A) = R(z,A)P_N(A) = -\frac{1}{2\pi i}\int_{\partial N_1} (z-\zeta)^{-1}(A - \zeta I)^{-1} d\zeta.$$

Since $R(z,A)$ is bounded and $0 \neq (z-\zeta)^{-1}$ for all $\zeta \in \sigma(A) \cap N_1$, we have $\ker R(z,A) = A(0) \subset \ker(P_N(A))$. Then our results follow form the corresponding results for $R(z,A)$.

A.3 Spectral flow for closed linear relations.

At first we give the definition of admissible relations.

Definition 12 (Cf. Zhu [42, Definition 1.3.6], [43, Definition 2.1], and [45, Definition 2.6]). Let $\ell \subset \mathbb{C}$ be a C^1 real 1-dimensional submanifold which has no boundary and is co-oriented (i.e., with oriented normal bundle). Let X be a Banach space and $A \in \mathscr{C}LR(X)$ be a closed linear relation.
(a) We call A *admissible* with respect to ℓ, if there exists a bounded open subset N of $\mathbb{C}$ (called *test domain*) such that (see also Fig. 1)

$$\text{(i) } \sigma(A) \cap N \;=\; \sigma(A) \cap \ell \quad \text{and} \quad \text{(ii) } \dim \operatorname{im} P_N(A) < +\infty. \tag{16}$$

Then $P_N(A)$ does not depend on the choice of such a test domain N. We set

$$P_\ell(A) \;:=\; P_N(A) \quad \text{and} \quad \nu_\ell(A) \;:=\; \dim \operatorname{im} P_N(A). \tag{17}$$

For fixed ℓ and X we shall denote the space of all ℓ-admissible closed linear relations in X by $\mathscr{A}_\ell(X)$.

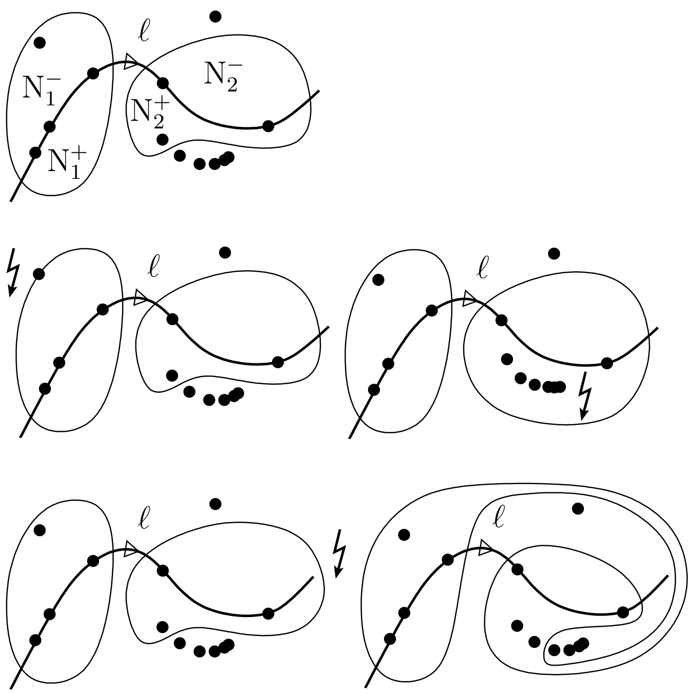

Fig. 2 Top: Admissible test domain triple (N,N^+,N^-). Middle and bottom: Non-admissible test domain triples. Middle left: $\sigma(A)\cap\partial N\neq\emptyset$. Middle right: $\dim P_N(A)=+\infty$. Bottom left: $\overline{N^0}=\overline{N}\cap\ell$ not satisfied. Bottom right: $N\cap\ell$ not connected while N connected

(b) Let $A\subset\mathscr{A}_\ell(X)$. Let $N\subset\mathbb{C}$ be open and bounded with C^1 boundary. We set $N^0:=N\cap\ell$ and assume

$$\overline{N^0}\subset\ell,\quad \sigma(A)\cap\ell\subset N,\quad \sigma(A)\cap\partial N=\emptyset,\quad\text{and}\quad \dim\operatorname{im}P_N(A)<+\infty. \tag{18}$$

Moreover, we require that ∂N intersects ℓ transversely and that each connected component of N has connected intersection with ℓ. Then

- $\overline{N^0}=\overline{N}\cap\ell$,
- the positive and negative parts $N^\pm$ of N with respect to the co-orientation of ℓ are well-defined, and
- we have a disjoint union $N=N^+\cup N^0\cup N^-$.

We shall call the resulting triple $(N;N^+,N^-)$ *admissible* with respect to ℓ and A, and write $(N;N^+,N^-)\in\mathscr{A}_{\ell,A}$. Clearly the set $\mathscr{A}_{\ell,A}$ is non-empty. See also Fig. 2.

Note 4 To prove $\overline{N^0}=\overline{N}\cap\ell$, we notice $\overline{N^0}\subset\ell$. So we have $\overline{N^0}\subset\overline{N}\cap\ell$. Since ∂N intersects ℓ transversely, we have $\partial N\cap\ell\subset\overline{N^0}$. Then $\overline{N}\cap\ell\subset\overline{N^0}$. That yields $\overline{N^0}=\overline{N}\cap\ell$.

Now we are able to define spectral-continuity and the spectral flow. Our data are a co-oriented curve $\ell\subset\mathbb{C}$, a family of Banach spaces $\{X_s\}_{s\in[a,b]}$ and a family $\{A_s\}_{s\in[a,b]}$ of closed linear relations on X_s.

Definition 13 (a) We shall call the family $\{A_s\in\mathscr{A}_\ell(X_s)\}$, $s\in[a,b]$ *spectral continuous* near ℓ at $s_0\in[a,b]$, if

(i) there is an $\varepsilon(s_0)>0$ and a triple $(N;N^+,N^-)$ such that

$$(N;N^+,N^-)\in\mathscr{A}_{\ell,A_s}\quad\text{for all } |s-s_0|<\varepsilon(s_0),$$

(ii) for all triple $(N';N'^+,N'^-)\in\mathscr{A}_{\ell,A_{s_0}}$ with $N'\subset N$ and $N'^\pm\subset N^\pm$, we have

$$(N';N'^+,N'^-)\in\mathscr{A}_{\ell,A_s}\quad\text{for all } |s-s_0|\ll 1,$$

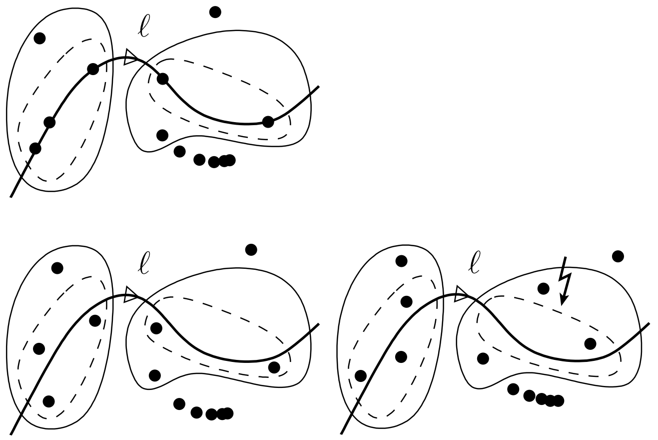

Fig. 3 Neighborhoods of the spectra of a spectral-continuous family near ℓ at s_0 : The same test domain triple (N,N^+,N^-) (solid line) works at s_0 in the upper figure, at $s_0-\varepsilon$ in bottom left, and at $s_0+\varepsilon$ in bottom right. The sub-triple (N',N'^+,N'^-) (encircled by the broken line) will also work at s_0 and for $s_0-\varepsilon$, but only for $s_0+\varepsilon'$ with $\varepsilon' \ll \varepsilon$

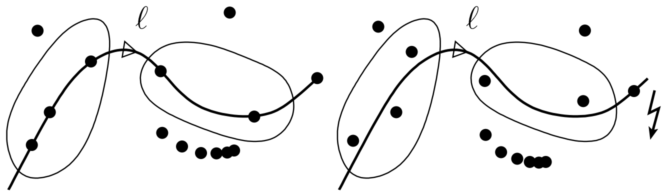

Fig. 4 A curve of closed linear relations with admissible spectra may fail to become spectral-continuous near ℓ due to a spectral point $\lambda_0 \in \bar{\ell}\setminus\ell$ for s_0 (left), which moves inward on ℓ for $s=s_0\pm\varepsilon$ (right)

(iii) for all triple $(N';N'^+,N'^-)$ and subinterval K of $(s_0-\varepsilon(s_0),s_0+\varepsilon(s_0))$ with $N'\subset N$, $N'^{\pm}\subset N^{\pm}$, and $(N';N'^+,N'^-)\in\mathscr{A}_{\ell,A_s}$ for all $s\in K$, we have $\dim\operatorname{im}P_{N'}(A_s)$ and $\dim\operatorname{im}P_{N^{\pm}\setminus N'^{\pm}}(A_s)$ do not depend on $s\in K$. See also Fig. 3 and Fig. 4.

We shall call the family $\{A_s\}\in\mathscr{A}_\ell(X_s)$, $s\in[a,b]$ *spectral-continuous* near ℓ, if it is spectral-continuous near ℓ at s_0 for all $s_0\in[a,b]$.

(b) Let $\{A_s\}\in\mathscr{A}_\ell$, $s\in[a,b]$ be a family of admissible operators that is spectral-continuous near ℓ. Then there exists a partition

$$a \;=\; s_0\le t_1\le s_1\le \ldots s_{n-1}\le t_n\le s_n \;=\; b \tag{19}$$

of the interval $[a,b]$, such that $s_{k-1},s_k\in(t_k-\varepsilon(t_k),t_k+\varepsilon(t_k))$, $k=1,\ldots,n$. Let $(N_k;N_k^+,N_k^-)$ be like a $(N;N^+,N^-)$ in (a) for t_k such that $s_{k-1},s_k\in(t_k-\varepsilon(t_k),t_k+\varepsilon(t_k)),k=1,\ldots,n$. Then we define the *spectral flow* of $\{A_s\}_{a\le s\le b}$ through ℓ by

$$\mathrm{sf}_\ell\big\{A_s;a\le s\le b\big\} \;:=\; \sum_{k=1}^{n}\Big(\dim\operatorname{im}\big(P_{N_k^-}(A_{s_{k-1}})\big)-\dim\operatorname{im}\big(P_{N_k^-}(A_{s_k})\big)\Big). \tag{20}$$

When ℓ is a bounded open submanifold of $i\mathbb{R}$ containing 0 with co-orientation from left to right, we set

$$\mathrm{sf}\big\{A_s;a\le s\le b\big\} \;:=\; \mathrm{sf}_\ell\big\{A_s;a\le s\le b\big\}.$$

From our assumptions it follows that the spectral flow is independent of the choice of the partition (19) and admissible $(N_k;N_k^+,N_k^-)$, hence it is well-defined. From the definition it follows that the spectral flow through ℓ is path additive under catenation and homotopy invariant. For details of the proof, see [33] and [45].

Lemma 18 *Let $\ell \subset \mathbb{C}$ be as in Definition 12a and X be a Banach space. Let $\{A_s \in \mathscr{C}LR_\ell(X)\}$, $a \le s \le b$ be a continuous family and $A := A_{s_0} \in \mathscr{A}_\ell(X)$ with $s_0 \in [a,b]$. Let $\mathfrak{m}$ be a bounded open submanifold of ℓ such that $\overline{\mathfrak{m}} \subset \ell$. If $\sigma(A_s) \cap \ell \subset \mathfrak{m}$ for all $s \in [a,b]$, we have:*

(a) There exists an $\varepsilon > 0$ such that $A_s \in \mathscr{A}_\ell(X)$ for all $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$.

(b) The family $\{A_s\}$ is spectral continuous near ℓ at s_0.

Proof *a*. Since ℓ is co-oriented and $A \in \mathscr{A}_\ell(X)$, there exists a bounded open subset N of $\mathbb{C}$ such that $\sigma(A) \cap N = \sigma(A) \cap \ell$ and $\dim \operatorname{im} P_N(A) < +\infty$. Since $\mathfrak{m}$ is a bounded open submanifold of ℓ, $\overline{\mathfrak{m}} \subset \ell$ and $\sigma(A) \cap \ell \subset \mathfrak{m}$, we can choose N such that ∂N is C^1, $N \cap \ell = \mathfrak{m}$ and $\sigma(A) \cap \partial N = \emptyset$. Since $\{A_s \in \mathscr{C}LR_\ell(X)\}$, $a \le s \le b$ is a continuous family and ∂N is compact, there exists an $\varepsilon > 0$ such that for all $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$, $\sigma(A_s) \cap N = \emptyset$. Then $\{P_N(A_s)\}$, $|s - s_0| < \varepsilon$ is a well-defined continuous family of projections on X and $\dim \operatorname{im} P_N(A_s) \le \dim \operatorname{im} P_N(A) < +\infty$. We also have $\sigma(A_s) \cap \ell \subset \mathfrak{m} \subset N$. So there exists an open subset N_s of N such that $\sigma(A_s) \cap \ell = \sigma(A_s) \cap N_s$, and $\dim \operatorname{im} P_{N_s}(A_s) \le \dim \operatorname{im} P_N(A_s) < +\infty$. (a) is proved.

b. Since $A \in \mathscr{A}_\ell(X)$, there exists a triple $(N; N^+, N^-) \in \mathscr{A}_{\ell,A}$. As in the proof of (a), we can choose N such that $N \cap \ell = \mathfrak{m}$ and $\sigma(A) \cap N = \sigma(A) \cap \ell$. Let ε be as in the proof of (a). If $|s - s_0| < \varepsilon$, we have $\sigma(A_s) \cap \ell \subset \mathfrak{m} \subset N$, $\sigma(A_s) \cap \partial N = \emptyset$ and $\dim \operatorname{im} P_N(A_s) < +\infty$. So $(N; N^+, N^-) \in \mathscr{A}_{\ell,A_s}$ for $|s - s_0| < \varepsilon$.

Given a triple $(N'; N'^+, N'^-) \in \mathscr{A}_{\ell,A}$ with $N' \subset N$ and $N'^\pm \subset N^\pm$, we have $\overline{N' \cap \ell} = \overline{N \cap \ell} \subset \overline{\mathfrak{m}} \subset \ell$ and $\sigma(A) \cap \partial N' = \emptyset$. Since $\partial N'$ is compact, for $|s - s_0| \ll 1$, we have $\sigma(A_s) \cap \partial N' = \emptyset$. Then $\dim \operatorname{im} P_{N'}(A_s)$ does not depend on s and is finite. Since $\sigma(A) \cap N = \sigma(A) \cap \ell \subset \sigma(A) \cap N'$, we have $\sigma(A) \cap N = \sigma(A) \cap N'$. By

$$\dim \operatorname{im} P_{N'}(A_s) \;=\; \dim \operatorname{im} P_{N'}(A) \;=\; \dim \operatorname{im} P_N(A) \;=\; \dim \operatorname{im} P_N(A_s),$$

we get $P_{N'}(A_s) = P_N(A_s)$. So we have $\sigma(A_s) \cap \ell \subset \sigma(A_s) \cap N = \sigma(A_s) \cap N' \subset N'$ and $(N'; N'^+, N'^-) \in \mathscr{A}_{\ell,A_s}$.

Given an interval $K \subset (s_0 - \varepsilon(s_0), s_0 + \varepsilon(s_0))$ and a triple $(N'; N'^+, N'^-) \in \mathscr{A}_{\ell,A_s}$ with $N' \subset N$, $N'^\pm \subset N^\pm$ and $s \in K$, we have $\sigma(A_s) \cap \partial N = \sigma(A_s) \cap \partial N' = \emptyset$ for all $s \in K$. Since $\sigma(A_s) \cap \ell \subset N$ and $\sigma(A_s) \cap \ell \subset N'$, we have $\sigma(A_s) \cap (\ell \setminus N') = \emptyset$ and $\sigma(A_s) \cap (\mathfrak{m} \setminus N') = \emptyset$. Define the closed curve

$$C^\pm \;:=\; (\partial N \cap N^\pm) \cup (\mathfrak{m} \setminus N') \cup (\partial N \cap N'^\pm)$$

with the orientation of $\partial N \cap N^\pm$ and opposite orientation of $\partial N \cap N'^\pm$. Then we have

$$P_{N^\pm \setminus N'^\pm}(A_s) \;=\; -\frac{1}{2\pi i} \int_{C^\pm} (A_s - \zeta I)^{-1} \zeta.$$

The families of projections $\{P_{N'}(A_s)\}$ and $\{P_{N^\pm \setminus N'^\pm}(A_s)\}$, $s \in K$ are continuous. So we have $\dim \operatorname{im} P_{N'}(A_s)$ and $\dim \operatorname{im} P_{N^\pm \setminus N'^\pm}(A_s)$ do not depend on $s \in K$. (b) is proved.

We close the appendix by discussing the invariance of the spectral flow under embeddings in larger spaces, assuming a simple regularity condition.

Lemma 19 *Let $\{Y_s; s \in [a,b]\}$ and $\{X_s; s \in [a,b]\}$ be two families of (complex) Banach spaces with $X_s \subset Y_s$ (no density or continuity of the embeddings are assumed). Let $\{A_s \in \mathscr{C}LR(Y_s); s \in [a,b]\}$ be a spectral-continuous curve near a fixed co-oriented curve $\ell \subset \mathbb{C}$. We assume that $A_s(X_s) \subset X_s$ for all s and that the curve*

$$\{A_s|_{X_s} \in \mathscr{C}LR(X_s); s \in [a,b]\}$$

is also spectral-continuous near ℓ. Then we have

$$\operatorname{sf}_\ell\{A_s; s \in [a,b]\} \;=\; \operatorname{sf}_\ell\{A_s|_{X_s}; s \in [a,b]\}$$

if the difference $\dim \nu_\ell(A_s) - \dim \nu_\ell(A_s|_{X_s})$, $s \in [a,b]$, is a constant m. In this case, $m \ge 0$.

Proof We go back to the local definition of sf_ℓ and reduce to the finite-dimensional case. Let $s_0 \in [a,b]$. Choose a triple

$$(N_1; N_1^+, N_1^-) \in \mathscr{A}_{\ell, A_{s_0}}$$

such that N_1 satisfies (16) for A_{s_0} and $A_{s_0}|_{X_{s_0}}$. By (16) we have

$$P_{N_1}(A_{s_0}) \;=\; \nu_\ell(A_{s_0}) \quad \text{and} \quad P_{N_1}(A_{s_0}|_{X_{s_0}}) \;=\; \nu_\ell(A_{s_0}|_{X_{s_0}}).$$

Then by spectral continuity, there exists a triple $(N; N^+, N^-)$ with $\overline{N} \subset N_1$ with

$$(N; N^+, N^-) \in \mathscr{A}_{\ell, A_s} \cap \mathscr{A}_{\ell, A_s|_{X_s}} \quad \text{for } |s - s_0| \ll 1.$$

Then we have

$$P_{N_1}(A_{s_0}) \;=\; \nu_\ell(A_{s_0}) \;=\; P_N(A_{s_0}) \quad \text{and} \quad P_{N_1}(A_{s_0}|_{X_{s_0}}) \;=\; \nu_\ell(A_{s_0}|_{X_{s_0}}) = P_N(A_{s_0}|_{X_{s_0}}),$$

and for $|s-s_0| \ll 1$

$$\dim \operatorname{im} P_N(A_s) \;=\; \nu_\ell(A_{s_0}) \;=\; \nu_\ell(A_{s_0}|_{X_{s_0}}) + m \;=\; \dim \operatorname{im} P_N(A_s|_{X_s}) + m \tag{21}$$

by spectral-continuity and our assumption. Now we consider for each $\lambda \in \mathbb{C} \cap N$ the algebraic multiplicities and find

$$\dim\ker(A_s|_{X_s} - \lambda I|_{X_s})^k \le \dim\ker(A_s - \lambda I)^k \tag{22}$$

for each $k \in \mathbb{N}$. By our assumption, we have $\nu_\ell(A_s) = \nu_\ell(A_s|_{X_s}) + m$. Comparing

$$\begin{aligned}
\dim \operatorname{im} P_N(A_s) \;&=\; \sum_{\lambda \in \sigma(A_s) \cap N} \sum_{k \in \mathbb{N}} \dim\ker(A_s - \lambda I)^k\,,\\
\dim \operatorname{im} P_N(A_s|_{X_s}) \;&=\; \sum_{\lambda \in \sigma(A_s|_{X_s}) \cap N} \sum_{k \in \mathbb{N}} \dim\ker(A_s|_{X_s} - \lambda I|_{X_s})^k\,,\\
\nu_\ell(A_s) \;&=\; \sum_{\lambda \in \sigma(A_s) \cap \ell} \sum_{k \in \mathbb{N}} \dim\ker(A_s - \lambda I)^k\,, \text{ and}\\
\nu_\ell(A_s|_{X_s}) \;&=\; \sum_{\lambda \in \sigma(A_s|_{X_s}) \cap \ell} \sum_{k \in \mathbb{N}} \dim\ker(A_s|_{X_s} - \lambda I|_{X_s})^k\,,
\end{aligned}$$

we obtain from equation (21) and the inequalities (22) that $m \ge 0$ and

$$\dim\ker(A_s|_{X_s} - \lambda I|_{X_s})^k \;=\; \dim\ker(A_s - \lambda I)^k$$

for each $\lambda \in N \setminus \ell$ and $k \in \mathbb{N}$. So $\sigma(A_s) \cap (N \setminus \ell) \;=\; \sigma(A_s|_{X_s}) \cap (N \setminus \ell)$; and the algebraic multiplicities with respect to A_s and $A_s|_{X_s}$ coincide in each point. By the definition of the spectral flow, the two spectral flows must coincide.

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