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Mathematical justification as non-conceptualized practice: the Babylonian example*

Jens Høyrup

Speaking about and doing – doing without speaking about it

Greek philosophy, at least its Platonic and Aristotelian branches, spoke much about demonstrated knowledge as something fundamentally different from opinion; often, it took mathematical knowledge as the archetype for demonstrated and hence certain knowledge – in its scepticist period, the Academy went so far as to regard mathematical knowledge as *the only* kind of knowledge which could really be based on demonstrated certainty.[1]

Not least in quarters close to Neopythagoreanism, the notion of mathematical demonstration may seem not to correspond to our understanding of the matter; applying our own standards we may judge the homage to demonstration to be little more than lip service.

Aristotle, however, discusses the problem of finding principles and proving mathematical propositions from these in a way that comes fairly close to the actual practice of Euclid and his kin. Even though Euclid himself only practises demonstration and does not discuss it we can therefore be sure that he was not not only making demonstrations but also explicitly aware of doing so in agreement with established standards. The preface to Archimedes's *Method* is direct evidence that its author knew demonstration according to established norms to be a cardinal virtue – the alleged or real heterodoxy consisting solely in his claim that discovery without strict proof was also valuable. Philosophical commentators like Proclos, finally, show beyond doubt that they too saw the mathematicians' demonstrations in the perspective of the philosophers' discussions.

As to Diophantos and Hero we may find that their actual practice is not quite

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[1] See. e.g., Cicero, *Academica* II.116-117 [ed. Rackham 1933].

in agreement with the philosophical prescriptions, but there is no doubt that even *their* presentation of mathematical matters was meant to agree with such norms as are reflected in the philosophical prescriptions.

Justification unproclaimed – or absent?

But is it not likely that mathematical demonstration has developed as a practice in the same process as created the norms, and thus before such norms crystallized and were hypostasized by philosophers? And is it not possible that mathematical demonstration – or, to use a word which is less loaded by our reading of Aristotle and Euclid, *justification* – developed in other mathematical cultures without being hypostasized?

A good starting point for the search for a mathematical culture of this kind might be that of the Babylonian scribes – if only for the polemical reason that "hellenophile" historians of mathematics tend to deny the existence of mathematical demonstration in this area. In Morris Kline's (relatively moderate) words [1972: 3, 14], written at a moment when non-specialists tended to rely on selective or not too attentive reading of popularizations like Neugebauer's *Science in Antiquity* [1957] and *Vorgriechische Mathematik* [1934] or van der Waerden's *Erwachende Wissenschaft* [1956]:

> Mathematics as an organized, independent, and reasoned discipline did not exist before the classical Greeks of the period from 600 to 300 B.C. entered upon the scene. There were, however, prior civilizations in which the beginnings or rudiments of mathematics were created.
>
> ...
>
> The question arises as to what extent the Babylonians employed mathematical proof. They did solve by correct systematic procedures rather complicated equations involving unknowns. However, they gave verbal instructions only on the steps to be made and offered no justification of the steps. Almost surely, the arithmetic and algebraic processes and the geometrical rules were the end result of physical evidence, trial and error, and insight.

The only opening toward any kind of demonstration beyond the observation that a sequence of operations gives the right result is the word "insight", which is not discussed any further. Given the vicinity of "physical evidence" and "trial and error" we may suppose that Kline refers to the kind of insight which makes us understand in a glimpse that the area of a right triangle must be the half of that of the corresponding rectangle.

Evident validity

In order to see how much must be put into the notion of “insight” if Kline’s characterization is to be defended we may look at some texts.[2] I shall start by problem 1 from the Old Babylonian tablet VAT 8390 (as also in following examples, an explanatory commentary follows the translation):[3]

Obv. I

1. [Length and width] I have made hold:[4] 10`[5] the surface.
2. [The length t]o itself I have made hold:
3. [a surface] I have built.
4. [So] much as the length over the width went beyond[6]

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2 I use the translations from [Høyrup 2002], leaving out the interlinear transliterated text and explaining key operations and concepts in notes at their first occurrence – drawing for this latter purpose on the results described in the same book. In order to facilitate checks I have not straightened the very literal (“conformal”) translations.

The first text (VAT 8390 #1) is translated and discussed on pp. 61–64.

3 The Old Babylonian period covers the centuries from 2000 BCE to 1600 BCE (according to the “middle chronology”). The mathematical texts belong to the second half of the period.

4 To make the lines *a* and *b* “hold” or “hold each other” (with further variations of the phrase in the present text) means to construct (“build”) the rectangular surface ⊏⊐(*a*,*b*) which they contain. If only one line *s* is involved, the square □(*s*) is built.

5 I follow Thureau-Dangin’s system for the transliteration of sexagesimal place value numbers, where `, ``, ... indicate increasing and ´, ˝, ... decreasing sexagesimal order of magnitude, and where “order zero” when needed is marked ° (I omit it when a number of “order zero” stands alone, thus writing 7 instead of 7°). 5`2°10´ thus stands for $5\cdot60^1+2\cdot60^0+10\cdot60^{-1}$. It should be kept in mind that absolute order of magnitude is not indicated in the text, and that `, ´ and ° correspond to the merely mental awareness of order of magnitude without which the calculators could not have made as few errors as actually found in the texts.

The present problem is homogeneous, and therefore does not enforce a particular order of magnitude. I have chosen the one which allows us to distinguish the area of the surface (10‘) from the number $^1/_6$ (10´).

6 The text makes use of two different “subtractive” operations. One, “by excess”, observes how much one quantity *A* goes beyond another quantity *B*; the other,

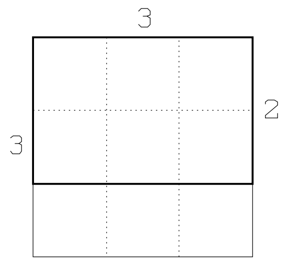

Figure 1. The configuration of VAT 8390 #1.

5. I have made hold, to 9 I have repeated:[7]
6. as much as that surface which the length by itself
7. was [ma]de hold.
8. The length and the width what?
9. 10\` the surface posit,[8]
10. and 9 (to) which he has repeated posit:
11. The equalside[9] of 9 (to) which he has repeated what? 3.
12. 3 to the length posit
13. 3 t[o the w]idth posit.
14. Since “so [much as the length] over the width went beyond
15. I have made hold”, he has said
16. 1 from ⌞3 which t]o the width you have posited
17. tea[r out:] 2 you leave.
18. 2 which yo[u have l]eft to the width posit.
19. 3 which to the length you have posited
20. to 2 which ⟨to⟩ the width you have posited raise,[10] 6.

“by removal”, finds how much remains when a quantity *a* is “torn out” (in other texts sometimes “cut off”, etc.) from a quantity *A*. As suggested by the terminology, the latter operation can only be used if *a* is part of *A*.

[7] “Repetition to / until *n*” is concrete, and produces *n* copies of the object of the operation. *n* is always small enough to make the process transparent, $1<n<10$.

[8] “Positing” a number means to take note of it by some material means, perhaps in isolation on a clay pad, perhaps in the adequate place in a diagram made outside the tablet. “Positing *n* to” a line (obv. I 12, etc.) is likely to correspond to the latter possibility.

[9] The “equalside” *s* of an area *Q* is the side of this area when it is laid out as a square (the “squaring side” of Greek mathematics). Other texts tell that *s* “is equal along” *Q*.

[10] “Raising” is a multiplication that corresponds to a consideration of proportionality; its etymological origin is in volume determination, where a prismatic volume with height *h* cubits is found by “raising” the base from the implicit “default thickness” of 1 cubit to the real height *h*. It also serves to determine the areas of rectangles which were constructed previously (lines İ 20 and II 7), in which case, e.g., the “default breadth” (1 “rod”, c. 6 m) of the length is “raised” to the real width.

In the case where a rectangular area is constructed (“made hold”), the arithmetical determination of the area is normally regarded as implicit in the operation, and the value is stated immediately without any intervening “raising” (thus lines II 7 and 10).

21. Igi 6[11] detach: 10´.
22. 10´ to 10` the surface raise, 1`40.
23. The equalside of 1`40 what? 10.

Obv. II

1. 10 to 3 wh[ich to the length you have posited]
2. raise, 30 the length.
3. 10 to 2 which to the width you have po[sited]
4. raise, 20 the width.
5. If 30 the length, 20 the width,
6. the surface what?
7. 30 the length to 20 the width raise, 10` the surface.
8. 30 the length together with 30 make hold: 15`.
9. 30 the length over 20 the width what goes beyond? 10 it goes beyond.
10. 10 together with [10 ma]ke hold: 1`40.
11. 1`40 to 9 repeat: 15` the surface.
12. 15` the surface, as much as 15` the surface which the length
13. by itself was made hold.

This problem about a rectangle exemplifies a characteristic of numerous Old Babylonian mathematical texts, namely that the description of the procedure already makes its adequacy evident. In Obv. I 4–5 we are told to construct the square on the excess of the length of the rectangle over its width and to take 9 copies of it, in lines I 6–7 that these can fill out the square on the length. Therefore, these small squares must be arranged in square, as in Figure 1, in a 3×3-pattern (lines I 11–13). But since the side of the small square was defined in the statement to be the excess of length over width (I 14–15, an explicit quotation), removal of one of three rows will leave the original rectangle, whose width will be 2 small squares.[12] In this unit, the area of the rectangle is $2{\cdot}3 = 6$ (I 18–20); since the rectangle is already there, there is no need for a "holding" operation. Because the area measured in standard units (square "rods") was 10`, each small square must be $^1/_6 \cdot$10` = 1`40 and its side √1`40 = √100 = 10 (I 21–23).

[11] "Igi n" designates the reciprocal of n. To "detach igi n", that is, to find it, probably refers to the splitting out of one of n parts of unity. "Raising a to igi n" means finding $a{\cdot}{}^1/_n$, that is, to divide a by n.

[12] In our understanding, 2 times the side of the small square. However, the Babylonian term for a square configuration (*mitḫartum*, literally "[situation characterized by a] confrontation [between equals]", was numerically identified by and hence with its side – a Babylonian square (primarily though of as a square frame) "was" its side and "had" an area, whereas ours (primarily thought of as a square-shaped area) "has" a side and "is" an area.

From this follows that the length must be $3 \cdot 10 = 30$ and the width $2 \cdot 10 = 20$ (II 1–3).

The one who follows the procedure on the diagram and keeps the exact (geometrical meaning and use of all terms in mind will feel no more need for an explicit demonstration than when confronted with a modern step-by-step solution of an algebraic equation,[13] in particular because numbers are always concretely identified by their role ("3 which to the length you have posited", etc.). The only place where doubts might arise is why 1 has to be subtracted in I 16–17, but the meaning of this step is then duly explained by a quotation from the statement (a routine device). There should be no doubt that the solution *must be* correct.

None the less a check follows, showing that the solution is valid (II 5 onwards). This check is very detailed, no mere numerical control but an appeal to the same kind of understanding as the preceding procedure: as we see, the rectangle is supposed to be already present, its area being found by "raising"; the large and small squares, however, are derived entities and therefore have to be constructed (the tablet contains a strictly parallel problem that follows the same pattern, for which reason we may be confident that the choice of operations is not accidental).

A similar instance of evident validity is offered by problem 1 of the text BM 13901,[14] the simplest of all mixed second-degree problems (and by numerous other texts, which however present us with the inconvenience that they are longer):

Obv. I

1. The surfa[ce] and my confrontation[15] I have accu[mulated]:[16] 45´ is it.

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[13] For instance,

$$\begin{aligned} & 3x+2 = 17 \\ \Rightarrow\ & 3x = 17–2 = 15 \\ \Rightarrow\ & x = {}^{1}\!/_{3} \cdot 15 = 5 \end{aligned}$$

[14] Translation and discussion [Høyrup 2002: 50–52].

[15] The *mitḫartum* or "[situation characterized by the] confrontation [of equals]", as we remember from note 11, is the square configuration parametrized by its side.

[16] "To accumulate" is an additive operation which concerns or may concern the measuring numbers of the quantities to be added. It thus allows the addition of lengths and areas, as here, in line 1, and of areas and volumes or of bricks,

1, the projection,[17]

2. you posit. The moiety[18] of 1 you break, [3]0´ and 30´ you make hold.

3. 15´ to 45´ you append: ⌊by] 1, 1 is equalside. 30´ which you have made hold

4. in the inside of 1 you tear out: 30´ the confrontation.

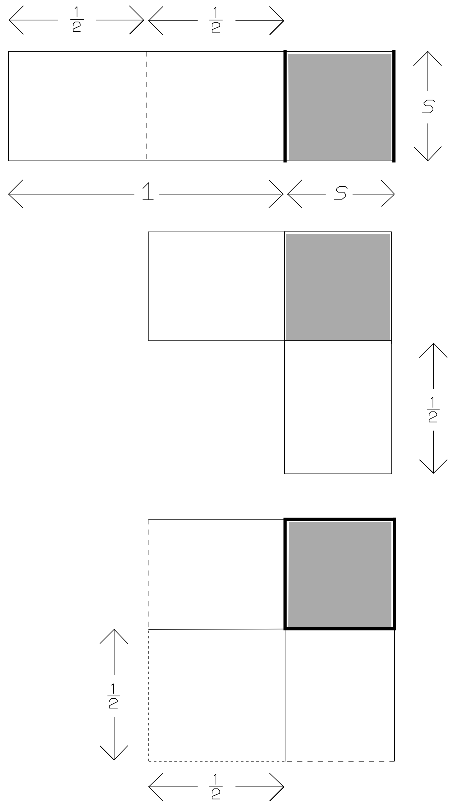

Figure 2. The procedure of BM 13901 #1, in slightly distorted proportions.

The problem deals with a “confrontation”, a square configuration identified by its side *s* and possessing an area. The sum of (the measures of) these is told to be 45´. The procedure can be followed in Figure 2: The left side *s* of the shaded square is provided with a “projection” (I 1). Thereby a rectangle ⊏⊐(*s*,1) is produced, whose area equals the length of the side *s*; this rectangle, together with the shaded square area, must therefore also equal 45´. “Breaking” the “projection 1” (together with the adjacent rectangle) and moving the outer “moiety” so as to make the two parts “hold” a small square □(30´) does not change the area (I 2), but completing the resulting gnomon by “appending” the small square results in a large square, whose area must be 45´+15´ = 1 (I 3). Therefore, the side of the large square must also be 1 (I 3). “Tearing out” that part of the rectangle which was moved so as to make it “hold” leaves 1–30´ for the “confrontation”, [the side of] the square

men and working days in other texts.

Another addition (“appending”) is concrete. It serves when a quantity *a* is joined to another quantity *A*, augmenting thereby the measure of the latter without changing its identity (as when interest, Babylonian “the appended”, is joined to *my* bank account while leaving it as mine).

[17] The “projection” (*wāṣītum*, literally something which protrudes or sticks out) designates a line of length 1 which, when applied orthogonally to another line *L* as width, transforms it into a rectangle ⊏⊐(*L*,1) without changing its measure.

[18] The “moiety” of an entity is its “necessary” or “natural” half, a half that could be no other fraction – as the circular radius is by necessity the exact half of the diameter, and the area of a triangle is found by raising exactly the half of the base to the height. It is found by “breaking”, a term which is used in no other function in the mathematical texts.

configuration.

As in the previous case, once the meaning of the terms and the nature of the operations is understood, no explanation beyond the description of the steps seems to be needed.

In order to understand *why* we may compare to the analogous solution of a second-degree equation:

$$
\begin{aligned}
& x^2 + 1\cdot x = {}^3\!/_4 \\
\Leftrightarrow\quad & x^2 + 1\cdot x + ({}^1\!/_2)^2 = {}^3\!/_4 + ({}^1\!/_2)^2 \\
\Leftrightarrow\quad & x^2 + 1\cdot x + ({}^1\!/_2)^2 = {}^3\!/_4 + {}^1\!/_4 = 1 \\
\Leftrightarrow\quad & (x + {}^1\!/_2)^2 = 1 \\
\Leftrightarrow\quad & x + {}^1\!/_2 = \sqrt{1} = 1 \\
\Leftrightarrow\quad & x = 1 - {}^1\!/_2 = {}^1\!/_2
\end{aligned}
$$

We notice that the numerical steps are the same as those of the Babylonian text, and this kind of correspondence was indeed what led to the discovery that the Babylonians possessed an "algebra". At the same time, the terminology was interpreted from the numbers – for instance, since "making $^1/_2$ and $^1/_2$ hold" produces $^1/_4$, this operation was identified with a numerical multiplication; since "raising" and "repeating" were interpreted in the same way, it was impossible to distinguish them.[19] Similarly, the two additive operations were conflated, etc. All in all, the text was thus interpreted as a numerical algorithm:

Halve 1: $^1/_2$.
Multiply $^1/_2$ and $^1/_2$: $^1/_4$.
Add $^1/_4$ to $^3/_4$: 1.
Take the square root of 1: 1.
Subtract $^1/_2$ from 1: $^1/_2$.

A similar interpretation as a mere algorithm results from a reading of the symbolic solution if the left-hand side of all equations is eliminated. It is indeed this left-hand side which establishes the identity of the numbers appearing to the right, and thereby makes it obvious that the operations are justified and lead to the solution. In the same way, the geometric reference of the operational terms

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[19] Actually, both Neugebauer and Thureau-Dangin knew that this was not the whole truth: none of them ever uses a wrong operation when reconstucting a damaged text. On one occasion Neugebauer [1935–37: I, 180] even observes that the scribe uses a wrong multiplication. However, they never made this insight explicit, for which reason less brilliant successors did not get the point. For instance, [Bruins & Rutten 1961] abounds in wrong choices (even when Sumerian word signs are translated into Akkadian).

in the Babylonian text is what establishes the meaning of the numbers and thereby the pertinence of the steps.

Didactical explanations

Kline wrote at a moment when the meaning of the terms and the nature of the operations was *not* yet understood and where the text was therefore usually read as a mere prescription of a numerical algorithm; his opinion is therefore explainable (we shall return to the fact that this opinion of his also reflects deeply rooted post-Renaissance scientific ideology). How this understanding developed concerns the history of modern historical scholarship.[20] But how did Old Babylonian students come to understand these matters? (Even we needed some explanations and some training before we came to consider algebraic transformations as self-explanatory.)

Neugebauer, fully aware that the complexity of many of the problems solved in the Old Babylonian texts presupposes deep understanding and not mere glimpses of insight, supposed that the explanations were given in oral teaching. In general this will certainly have been the case, but after Neugebauer's work on Babylonian mathematics (which stopped in the late 1940s) a few texts have been published which turn out to contain exactly the kind of explanations we are looking for.

Most explicit are some texts from late Old Babylonian Susa: TMS VII, TMS IX, TMS XVI.[21] Since TMS IX is closely related to the problem we have just dealt with, whereas TMS VII investigates non-determinate linear problems and TMS XVI the transformation of linear equations, we shall begin by discussing the former. It falls in three sections, of which the first two run as follows:

#1

[20] See [Høyrup 1996] for what evidently cannot avoid being a partisan view.

[21] All were first published by E. M. Bruins and M. Rutten [1961] who, however, did not understand their character. Revised transliterations and translations as well as analyses can be found in [Høyrup 2002], on pp. 181–188, 89–95 and 85–89 (only part 1), respectively. A full treatment of TMS XVI is found in [Høyrup 1990: 299–302].

1. The surface and 1 length accumulated, 4[0´. ¿30, the length,? 20´ the width.][22]
2. As 1 length to 10´ ⌈the surface, has been appended,]
3. or 1 (as) base to 20´, [the width, has been appended,]
4. or 1°20´ [¿is posited?] to the width which together ⌈with the length ¿holds?] 40´
5. or 1°20´ toge⟨ther⟩ with 30´ the length hol[ds], 40´ (is) [its] name.
6. Since so, to 20´ the width, which is said to you,
7. 1 is appended: 1°20´ you see. Out from here
8. you ask. 40´ the surface, 1°20´ the width, the length what?
9. [30´ the length. T]hus the procedure.

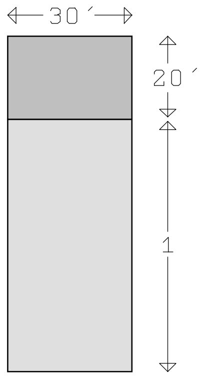

Figure 3. The configuration discussed in TMS IX #1.

#2

10. [Surface, length, and width accu]mulated, 1. By the Akkadian (method).
11. [1 to the length append.] 1 to the width append. Since 1 to the length is appended,
12. [1 to the width is app]ended, 1 and 1 make hold, 1 you see.
13. [1 to the accumulation of length,] width and surface append, 2 you see.
14. [To 20´ the width, 1 appe]nd, 1°20´. To 30´ the length, 1 append, 1°30´.[23]
15. [¿Since? a surf]ace, that of 1°20´ the width, that of 1°30´ the length,
16. [¿the length together with? the wi]dth, are made hold, what is its name?
17. 2 the surface.
18. Thus the Akkadian (method).

Section 1 explains how to deal with an equation stating that the sum of a rectangular area $\sqsubset\!\sqsupset(\ell,w)$ and the length ℓ is given, referring to the situation that the length is 30´ and the width 20´. These numbers are used as identifiers, fulfilling thus the same role as our letters ℓ and w. Line 2 repeats the statement but identifying the area as 10´. In line 3, this is told to be equivalent to adding "a base" 1 to the width, as shown in Figure 3 – in symbols, $\sqsubset\!\sqsupset(\ell,w)+\ell = \sqsubset\!\sqsupset(\ell,w)+\sqsubset\!\sqsupset(\ell,1) = \sqsubset\!\sqsupset(\ell,w+1)$; the "base" evidently fulfils the same role as the

[22] As elsewhere, passages in plain square brackets are reconstructions of damaged passages that can be considered certain; high or low writing of the square brackets indicate that only the lower respectively upper part of the signs close to that bracket is missing. Passages within ¿...? are reasonable reconstructions which however may not correspond to the exact formulation that was once on the tablet.

[23] My restitutions of lines 14–16 are somewhat tentative, even though the mathematical substance is fairly well established by a parallel passage in lines 28–31.

“projection” of BM 13901. Line 4 tells us that this means that we get a (new) width 1°20´, and line 5 checks that the rectangle contained by this new width and the original length 30´ is indeed 40´, as it should be. Lines 6–9 sum up.

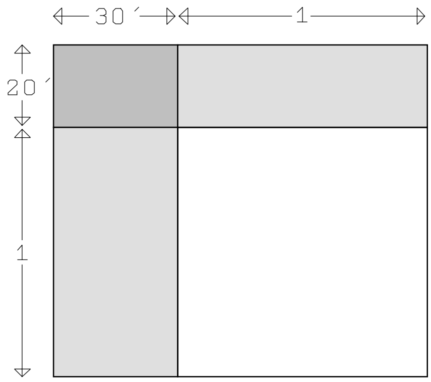

Figure 4. The configuration of TMS IX #2.

Section 2 again refers to a rectangle with known dimensions – once more $\ell = 30'$, $w = 20'$. This time the situation is that both sides are added to the area, the sum being 1. The trick to be applied in the transformation is identified as the “Akkadian method”. This time, both length and width are augmented by 1 (line 11); however, the resulting rectangle $\sqsubset\!\sqsupset(\ell+1,w+1)$ contains more than it should (cf. Figure 4), namely beyond a quasi-gnomon representing the given sum (consisting of the original area $\sqsubset\!\sqsupset(\ell,w)$, a rectangle $\sqsubset\!\sqsupset(\ell,1)$ whose measure is the same as that of ℓ, and a rectangle $\sqsubset\!\sqsupset(1,w) = w$), also a quadratic completion $\sqsubset\!\sqsupset(1,1) = 1$ (line 12). Therefore, the area of the new rectangle should be 1+1 = 2 (line 13). And so it is: the new length will be 1°30´, the new width will be 1°20´, and the area which they contain will be 1°30´·1°20´ = 2 (lines 15–17).

Since extension also occurs in section 1, the “Akkadian method” is likely to refer to the quadratic completion (this conclusion is supported by further arguments which do not belong within the present context).

After these two didactical explanations follows a problem in the proper sense. In symbolic form it can be expressed as follows:

$$\sqsubset\!\sqsupset(\ell,w)+\ell+w = 1 \ , \quad {}^{1}\!/_{17}(3\ell+4w)+w = 30' \ .$$

The first equation is the one whose transformation into

$$\sqsubset\!\sqsupset(\lambda,\omega) = 2$$

($\lambda = \ell+1$, $\omega = w+1$) was just explained in section 2. The second is multiplied by 17, thus becoming,

$$3\ell+21w = 8°30' \ .$$

and further transformed into

$$3\lambda+21\omega = 32°30 \ ,$$

whereas the area equation is transformed into

$$\sqsubset\!\sqsupset(3\lambda,21\omega) = 2\text{`}6 \ .$$

Thereby, the problem has been reduced to a standard rectangle problem (known area and sum of sides), and it is solved accordingly (by a method similar to that of BM 13901 #1).

The present text does not explain the transformation of the equation $^{1}/_{17}(3\ell+4w)+w = 30'$, but a similar transformation is the object of section 1 of TMS XVI:

1. [The 4th of the width, from] the length and the width to tear out, 45´. You, 45´
2. [to 4 raise, 3 you] see. 3, what is that? 4 and 1 posit,
3. [50´ and] 5´, to tear out, ˹posit˺. 5´ to 4 raise, 1 width. 20´ to 4 raise,
4. 1°20´ you ⟨see⟩, 4 widths. 30´ to 4 raise, 2 you ⟨see⟩, 4 lengths. 20´, 1 width, to tear out,
5. from 1°20´, 4 widths, tear out, 1 you see. 2, the lengths, and 1, 3 widths, accumulate, 3 you see.
6. Igi 4 de[ta]ch, 15´ you see. 15´ to 2, the lengths, raise, [3]0´ you ⟨see⟩, 30´ the length.
7. 15´ to 1 raise, [1]5´ the contribution of the width. 30´ and 15´ hold.[24]
8. Since “The 4th of the width, to tear out”, it is said to you, from 4, 1 tear out, 3 you see.
9. Igi 4 de⟨tach⟩, 15´ you see, 15´ to 3 raise, 45´ you ⟨see⟩, 45´ as much as (there is) of [widths].
10. 1 as much as (there is) of lengths posit. 20, the true width take, 20 to 1´ raise, 20´ you see.
11. 20´ to 45´ raise, 15´ you see. 15´ from $30_{15'}$ [tear out],
12. 30´ you see, 30´ the length.

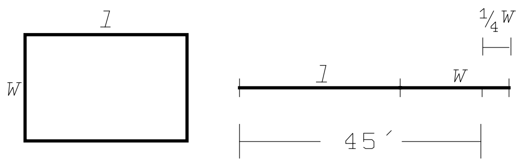

Figure 5. The situation of TMS XVI #1.

Even this explanation deals formally with the sides ℓ and w of a rectangle, although the rectangle itself is wholly immaterial to the discussion. In symbolic translation we are told that

$$(\ell+w)-{}^{1}/_{4}w = 45' \ .$$

The dimensions of the rectangle are not stated directly, but from the numbers in line 3 we see that they are presupposed to be known and to be the same as before, 50´ being the value of $\ell+w$, 5´ that of $^{1}/_{4}w$ – cf. Figure 5.

The first operation to perform is a multiplication by 4. 4 times 45´ gives 3, and the text then asks for an explanation of this number (line 2). The subsequent explanation can be followed on Figure 6, which certainly is a modern reconstruction but which is likely to correspond in some way to what is meant by the

[24] This “hold” is an ellipsis for “make your head hold”, the standard phrase for retaining in memory.

explanations. The proportionals 1 and 4 are taken note of ("posited"), 1 corresponding of course to the original equation, 4 to the outcome of the multiplication. Next 50´ (the total of length plus width) and 5´ (the fourth of the width that is to be "torn out") are taken note of (line 3), and the multiplied counterparts of the components of the original equation (the part to be torn out, the width, and the length) are calculated and described in terms of lengths and widths (lines 3–4); finally it is shown that the outcome (consisting of the components 1 = $4w$–$1w$ and 2 = 4ℓ) explain the number 3 that resulted from the original multiplication (lines 4–5).

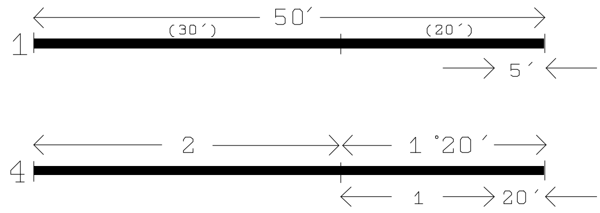

Figure 6. The transformations of TMS XVI #1.

Now the text reverses the move and multiplies the multiplied equation that was just analyzed by $^1/_4$. Multiplication of 2 (= 4ℓ) gives 30´, the length; multiplication of 1 gives 15´, which is explained to be the "contribution of the width"; both contributions are to be retained in memory (lines 6–7). Next the contributions are to be explained; using an argument of false position ("if one fourth of 4 was torn out, 3 would remain; now, since it is torn out of 1, the remainder is $3\cdot{}^1/_4$"), the coefficient of the width ("as much as (there is) of widths") is found to be 45´. The coefficient of the length is seen immediately to be 1 (lines 1–10).

Next (line 10) follows a step whose meaning is not certain; the text distinguishes between the "true length" and the "length" *simpliciter*, writing however the value of both in identical ways. One possible explanation (in my opinion quite plausible, and hence used in the translation) is that the "true width" is the width of an imagined "real" field, which could be 20 rods (120 m), whereas the width *simpliciter* is that of a model field that can be drawn in the school yard (2 m); indeed, the normal dimensions of the fields dealt with in second-degree problems (which are school problems without any practical use) are 30´ and 20´ rods, 3 and 2 m, much too small for real fields but quite convenient in school. In any case, multiplication of the value of the width by its coefficient gives us the corresponding contribution once more (line 11), which indeed has the value that was assigned to memory. Subtracting it from the total (which is written in an unconventional way that already shows the splitting) leaves the length, as indeed it should (lines 11–12).

Detailed didactical explanations as these have only been found in Susa; once they have been understood, however, we may recognize in other texts rudiments of similar explanations, which must have been given in their full form orally,[25] as once supposed by Neugebauer.

These explanations are certainly meant to impart *understanding*, and in this sense they are demonstrations. But their character differs fundamentally from that of Euclidean demonstrations (which, indeed, were often reproached their opacity during the centuries where the *Elements* were used as a school book). Euclidean demonstrations proceed in a linear way, and end up with a conclusion which readers must acknowledge to be unavoidable (unless they find an error) but which may leave them wondering where the rabbit came from. The Old Babylonian didactical texts, in contrast, aim at building up a tightly knit conceptual network in the mind of the student.

However, conceptual connections can be of different kinds. Pierre de la Ramée when rewriting Euclid replaced the "superfluous" demonstrations by explanations of the practical uses of the propositions. Numerology (in a general sense including also analogous approaches to geometry) links mathematical concepts to non-mathematical notions and doctrines; to this genre belong not only writings like the ps-Nicomachean *Theologoumena arithmetica* but also for some of their aspects, according to [Chemla 1997], Liu Hui's commentaries to the *Nine Chapters on Arithmetic*, which cannot be understood in isolation from the *Book of Changes*. Within this spectrum, the Old Babylonian expositions belong in the vicinity of Euclid, far away from Ramism as well as numerology: the connections which they establish all belong strictly within the same mathematical domain as the object they discuss.

Justifiability and critique

Whoever has tried regularly to give didactical explanations of mathematical procedures is likely to have encountered the situation where a first explanation turns out on second thoughts – maybe provoked by questions or lacking success of the explanation – not to be justifiable without adjustment. While didactical explanation is no doubt one of the sources of mathematical demonstration, the scrutiny of the *conditions under which* and the *reasons for which* the explanations given hold true is certainly another source. The latter undertaking is what Kant

[25] Worth mentioning are the unpublished text IM 43993, which I know about through Jöran Friberg and Farouk al-Rawi (personal communication), and YBC 8633, analyzed from this perspective in [Høyrup 2002: 254–257].

termed *critique*, and its central role in Greek mathematical demonstration is obvious.

In Old Babylonian mathematics, critique is less important. If read as demonstrations, explanations oriented toward the establishment of conceptual networks tend to produce circular reasoning, in the likeness of those persons referred to by Aristotle "who [...] think that they are drawing parallel lines; for they do not realize that they are making assumptions which cannot be proved unless the parallel lines exist" (*Prior Analytics* II, 64b34–65a9 [trans. Tredennick 1938: 485–487]). In their case, Aristotle told the way out – namely to "take as an axiom" (αξιόω) that which is proposed. This is indeed what is done in the *Elements*, whose fifth postulate can thus be seen to answer metatheoretical critique.

However, though less important than in Greek geometry, critique is not absent from Babylonian mathematics. One instance is illustrated by the text YBC 6967,[26] a problem dealing with two numbers *igûm* and *igibûm*, "the reciprocal and its reciprocal", the product of which, however, is supposed to be 1` (that is, 60), not 1:

Obv.

1. [The *igib*]*ûm* over the *igûm*, 7 it goes beyond
2. [*igûm*] and *igibûm* what?
3. Yo[u], 7 which the *igibûm*
4. over the *igûm* goes beyond
5. to two break: 3°30´;
6. 3°30´ together with 3°30´
7. make hold: 12°15´.
8. To 12°15´ which comes up for you
9. [1` the surf]ace append: 1`12°15´.
10. [The equalside of 1`]12°15´ what? 8°30´.
11. [8°30´ and] 8°30´, its counterpart,[27] lay down.[28]

Rev.

1. 3°30´, the made-hold,
2. from one tear out,
3. to one append.
4. The first is 12, the second is 5.

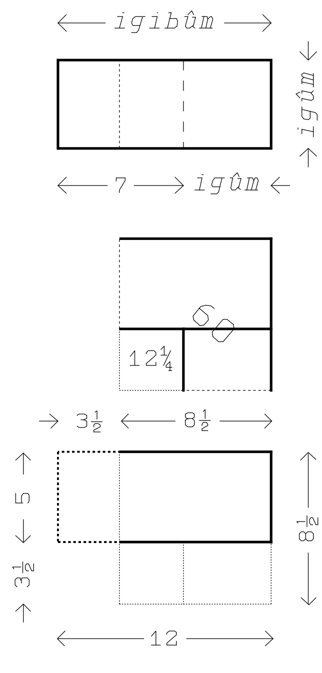

Figure 7. The procedure of YBC 6967.

[26] Transliterated, translated and analyzed in [Høyrup 2002: 55–58].

[27] The "counterpart" of an equalside is "the other side" meeting it in a common corner.

[28] Namely, lay down in writing or drawing.

5. 12 is the *igibûm*, 5 is the *igûm*.

The procedure can be followed in Figure 7; the text is another instance of self-evident validity, and only differs from those discussed under this perspective in having the sides and the area of the rectangle *represent* numbers and not just themselves. The interesting point is found in Rev. 2–3. In cases where there is no constraint on the order, the Babylonians always speak of addition before subtraction. Here, however, the 3°30´ that is to be added to the left of the gnomon (that is, to be put back in its original position) must first be at disposition, that is, it must already have been torn out below.

This compliance with a request of concrete meaningfulness should not be read as evidence of some “primitive mode of thought still bound to the concrete and unfit for abstraction”; this is clear from the way early Old Babylonian texts present the same step in analogous problems, often in a shortened phrase “append and tear out” and indicating the two resulting numbers immediately afterwards, in any case never respecting the norm of concreteness. This norm thus appears to have been introduced precisely in order to make the procedure justifiable – corresponding to the introduction in Greek theoretical arithmetic of the norm that fractions and unity could be no numbers in consequence of the explanation of number as a “collection of units”.[29]

But the norm of concreteness is not the only evidence of Old Babylonian mathematical critique. Above, we have encountered the “projection” and the “base”, devices that allow the addition of lines and surfaces in a way that does not violate homogeneity, and the related distinction between “accumulation” and “appending”. Even these stratagems turn out to be secondary developments. A text like AO 8862 (probably from the early phase of Old Babylonian mathematics, in any case reflecting early usages) does not make use of them. Its first problem starts thus:

1. Length, width.[30] Length and width I have made hold:
2. A surface have I built.
3. I turned around (it). As much as length over width
4. went beyond,
5. to inside the surface I have appended:

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[29] See [Høyrup 2004: 148*f*].

[30] That is, the object of problem is told to be the simplest configuration determined solely by a length and a width – namely, according to Babylonian habits, a rectangle.

6. 3`3. I turned back. Length and width

7. I have accumulated: 27. Length, width, and surface w[h]at?

As we see, a line (the excess of length over width) is “appended” to the area; “accumulation” also occurs, but the reason for this is that “appending” for example the length to the width would produce an irrelevant increased width and no symmetrical sum (cf. the beginning of TMS XVI, above, which first creates a symmetrical sum and next removes part of it).

This “appending” of a line to an area does not mean that the text is absurd. In order to see that we must understand that it operates with a notion of “broad lines”, lines that carry an inherent virtual breadth. Though not made explicit, this notion underlies the determination of areas by “raising” (cf. note 9); it is widespread in pre-Modern practical mensuration, in which “everybody” (locally) would measure in the same unit, for which reason it could be presupposed tacitly[31] – land being bought and sold in consequence just as we are used to buying and selling cloth, by the yard and not the square yard. However, once didactical explanation in school has taken its beginning (and once it is no longer obvious which of several metrological units should serve as standard breadth), a line which at the same time is “with breadth” and “without breadth” becomes awkward. In consequence, critique appears to have outlawed the “appending” of lines to areas and to have introduced devices like the “projection” – the latter in close parallel to the way Viète established homogeneity and circumvented the use of broad lines of Renaissance algebra.[32]

All in all, mathematical demonstration was thus not absent from Old Babylonian mathematics. Procedures were described in a way which, once the terminology and its use have been decoded, turns out to be as transparent as the self-evident transformations of modern equation algebra and in no need of further explicit arguing in order to convince; teaching involved didactical explanations which aimed at providing students with a corresponding understanding of the terminology and the operations; and mathematical concepts and procedures were transformed critically so as to allow coherent explanation of points that may initially have seemed problematic or paradoxical. No surviving texts suggests, however, that all this was ever part of an explicitly formulated

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[31] See [Høyrup 1995].

[32] Namely the “roots”, explained by Nuñez [1567: fols 6r, 232r] to be rectangles whose breadth is “la unidad lineal”.

programme, nor do the texts we know point to any thinking about *demonstration as a particular activity*. All seems to have come as naturally as speaking in prose to Molière's monsieur Jourdain, as consequences of the situations and environments in which mathematics was practised.

Mathematical Taylorism: practically dubious but an effective ideology

Teachers, in the Bronze Age just as in modern times, may have gone beyond what was really needed in the "real" practice of their future students, blinded by the fact that the practice they themselves knew best was that of their own trade, the teaching of mathematics. None the less, the social *raison d'être* of Old Babylonian mathematics was the training of future scribes in practical computation, and not deeper insight into the principles and metaphysics of mathematics. Why should this involve demonstration? Would it not be enough to teach precisely those *rules* or algorithms which earlier workers have found in the texts and which (in the shape of paradigmatic cases) also constitute the bulk of so many other pre-Modern mathematical handbooks? And would it not be better to teach them precisely as rules to be obeyed without distracting reflection on problems of validity?

That "the hand" should be governed in the interest of efficiency by a "brain" located in a different person but should in itself behave like a mindless machine is the central idea of Frederick Taylor's "scientific management" – "hand" and "brain" being, respectively, the worker and the planning engineer. In the pre-Modern world, where craft knowledge tended to constitute an autonomous body, and where (with rare exceptions) practice was not derived from theory, Taylorist ideas could never flourish.[33] In many though not all fields, autonomous practical knowledge survived well into the nineteenth, sometimes the twentieth century; however, the *idea* that practice should be governed by theory (and the ideology that practice is derived from the insights of theory) can be traced back to the early Modern epoch. Already before its appearance in Francis Bacon's *New Atlantis* we find something very similar forcefully expressed in Vesalius's *De humani corporis fabrica,* according to which the art of healing had suffered immensely from being split into three independent practices: that of the theoretically schooled physicians, that of the pharmacists, and that of vulgar

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[33] Aristotle certainly thought that master artisans had insight in "principles" and common workers not (*Metaphysics* I, 981^{b}1–5), and that slaves were living instruments (*Politics* I.4); but reading of the context of these famous passages will reveal that they do not add up to anything like Taylorism.

barbers supposed to possess no instruction at all; instead, Vesalius argues, all three bodies of knowledge should be carried by the same person, and that person should be the theoretically schooled physician.

In many fields, the suggestion that material practice should be the task of the theoretically schooled would seem inane; even in surveying, a field which was totally reshaped by theoreticians in the eighteenth century, the scholars of the *Académie des Sciences* (and later Wessel and Gauß), even when working in the field, would mostly instruct others in how to perform the actual work and control they did well. Such circumstances favoured the development of views close to those of Taylorism – why should those who merely made the single observations or straightened the chains be bothered by explanations of the reasons for what they were asked to do? If the rules used by practitioners were regarded in this perspective, it also lay close at hand to view these as "merely empirical" if not recognizably derived from the insights of theoreticians.

Such opinions, and their failing in situations where practitioners have to work on their own, are discussed in Christian Wolff's *Mathematisches Lexikon* [1716: 867, trans. JH]:

> It is true that performing mathematics can be learned without reasoning mathematics; but then one remains blind in all affairs, achieves nothing with suitable precision and in the best way, at times it may occur that one does not find one's way at all. Not to mention that it is easy to forget what one has learned, and that that which one has forgotten is not so easily retrieved, because everything depends only on memory.

Wolff certainly identified "reasoning mathematics" (also called "*Mathesis theorica* or *speculativa*") with established theoretical mathematics, but none the less he probably hit the point not only in his own context but also if we look at the conditions of pre-Modern mathematical practitioners: without insight in the reasons why their procedures worked they were likely to err except in the execution of tasks that recurred so often that their details could not be forgotten.[34] Even the teaching of practitioners' mathematics through paradigmatic

[34] The "rule of three", with its intermediate product deprived of concrete meaning, only turns up in environments where the problems to which it applies were really the routine of every working day – notwithstanding the obvious computational advantage of letting multiplication precede division. Its extensions into "rule of five" and "rule of seven" never gained similar currency.

A more recent example, directly inspired by Adam Smith's theory of the division of labour, is Prony's use of "several hundred men who knew only the cases exemplifying rules that were or were not stated explicitly will always have involved some level of explanation and thus of demonstration – and certainly, as in the Babylonian case, internal mathematical rather than philosophical or otherwise "numerological" explanation. Whether critique would also be involved probably depended on the level of professionalization of the teaching institution itself.

But those mathematicians and historians who were not themselves involved in the teaching of practitioners were not forced to discover such subtleties. For them, it was all too convenient to accept Taylorist ideologies (whether *ante litteram* or *post*) and to magnify their own intellectual standing by identifying the appearance of explicit or implicit rules with mindless rote learning (if derived from supposedly *real* mathematics) or blind experimentation (if not to be linked to recognizable theory). Such ideologies did not make opinions such as Kline's necessary and did not engender them directly, but they shaped the intellectual climate within which he and his mental kin grew up as mathematicians and as historians.

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