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Old Babylonian "Algebra", and What It Teaches Us about Possible Kinds of Mathematics

JENS HØYRUP

ABSTRACT

John Britton in memoriam

Until recently, Old Babylonian algebra (mostly identified simply as Babylonian) either looked very much like recent equation algebra in presentations of the history of mathematics, or it was characterized as empirical, a collection of rules found by trial and error or other (unidentified) methods not based on reasoning. In the former case, the implicit message was a confirmation of the status of our present type of mathematics as mathematics itself. The message inherent in the second portrait is not very different: if mathematics is not of the type we know, and whose roots we customarily trace to the Greeks, it is just a collection of mindless recipes (a type we also know, indeed, from teaching of those social classes that are not supposed to possess or exercise reason)—*tertium non datur*!

More precise analysis of Old Babylonian mathematical texts—primarily the so-called algebraic texts, the only ones extensive enough to allow such analysis— shows that both traditional views are wrong. The prescriptions turn out to be neither renderings of algebraic computations as we know them nor mindless rules to be followed blindly; they describe a particular type of geometric manipulation, which like modern equation algebra is analytical in character, and which displays the correctness of its procedures without being explicitly demonstrative.

Key words: Old Babylonian algebra; mathematical justification.

This is a strongly reduced version—better, perhaps, an extended abstract—of the much longer paper which I presented at the conference. The full paper, containing more textual examples and more thorough discussion than possible here, can be found at the address

http://rudar.ruc.dk/handle/1800/5878

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The paper explains this, adding substance, shades and qualifications to the picture, and then takes up the implications for our global understanding of the possible types of mathematics.

1. THREE READERS OF "BABYLONIAN ALGEBRA"

In many general expositions of the history of mathematics one finds a treatment of "Babylonian mathematics", mostly presented without chronological distinction between the "Old Babylonian" epoch (2000 to 1600 BCE according to the "middle chronology") and the Seleucid period (third and second century BCE)—the only periods from which these expositions know about mathematical texts.

As a matter of fact, the texts from the two periods are rather different in character, and in what follows I shall only speak about Old Babylonian mathematics. The so-called "algebra" is indeed best known from texts written between 1800 and 1600 BCE. Being the earliest example of what can be considered "advanced mathematics", this mathematical genre is treated sometimes briefly, sometimes extensively in many historical presentations. The authors of these depend on what they have found in the commentaries in the text editions of Otto Neugebauer, Françis Thureau-Dangin and (after 1961) Evert M. Bruins, read selectively through their own understanding of mathematics. We may look at three examples.

In 1953, J. E. Hofmann published the first volume of his *Geschichte der Mathematik* [5]. Misreading Neugebauer he believes (p. 11) that[1]

> the particular ideographic writing causes the prescriptions for the solution of practical problems to be almost untranslatable into language, showing thus a certain kinship with the representation through algebraic formulae.

More significant (and still wrong even on Hofmann's own premises) is perhaps what follows on p. 14:

> On other occasions, linear equations with several unknowns are solved diligently, and further equations of the form $ax + by = \mathrm{C}$, $xy = D$ and those that can be reduced to it, where the transformation $(x+y)^2 = (x-y)^2+4xy$ is constantly made use of.

[1]My translation, as all translations in the following when nothing else is stated.

Maybe Hofmann is aware that his "linear equations with several unknowns" are word problems whose translation into algebraic equations presupposes choices and therefore is not unambiguous; but he gives his readers no possibility to discover—what he speaks about is exclusively the result of blunt mapping onto the conceptual grid of present-day mathematics.

Carl Boyer's *A History of Mathematics* [1] from 1968 is not very different on this account. He states (p. 33) that

> in an Old Babylonian text we find two simultaneous linear equations in two unknown quantities, called respectively the "first silver ring" and the "second silver ring".

and goes on to explain that "if we call these x and y in our notation, the equations are $x/7 + y/11 = 1$ and $6x/7 = 10y/11$"; again, the problem seems to be born as an equation, not as a description of a situation whose translation into an equation already involves choices.

In one respect, even Morris Kline's 1972 book *Mathematical Thought from Ancient to Modern Times* [9] is similar. However, in a summary "evaluation of Babylonian mathematics" Kline observes (p. 14) that the Babylonians

> did solve by correct procedures rather complicated equations involving unknowns. However, they gave verbal instructions only on the steps to be made and offered no justification of the steps. Almost surely, the arithmetic and algebraic processes and the geometrical rules were the end result of physical evidence, trial and error, and insight. That the methods worked was sufficient justification to the Babylonians for their continued use.

The final period removes any substance one might believe to be implied by the term "insight". Since

> the concept of proof, the notion of logical structure based on principles warranting acceptance on one ground or another, and the consideration of such questions as under what conditions solutions to problems can exist, are not found in Babylonian mathematics,

Babylonian calculators—thus Kline—did not base their mathematics on understanding. To his eyes, Babylonian mathematics was not really mathematics at all—in complete agreement with his claim (p. 3) that

> mathematics as an organized, independent and reasoned discipline did not exist before the classical Greeks of the period from 600 to 300 B.C. entered upon the scene.

So, while Hofmann and Boyer find our kind of mathematics in the Babylonian texts, Kline finds something different—but so different that it does not really count as mathematics. The implied message (probably resulting because it is the implicit starting point for all three) is the same: there is only one kind of mathematics: *ours*.

2. INTERPRETATIONS

All of this depended on an interpretation of the Babylonian terminology made before 1935. By then, the mathematical meaning of only a few terms could be guessed from their general interpretation; for the others, the only possibility was to start from the numbers. An operation which produces 30 from 5 and 6 was thus supposed to be a multiplication. The translations thereby came to presuppose a purely arithmetical reading.

In the early 1980s I discovered that two distinct (not synonymous) operations had been interpreted in this way as addition; similarly, there were two different "subtractions" (only one of which is the inverse of one of the "additions"), two different "halves", and four distinct "multiplications". This did not fit the arithmetical interpretation, within which there is space for only one of each class.

In what follows, I shall use a "conformal translation", in which different terms are always translated differently, and the same term always translated in the same way.[2] The translations are chosen so as to correspond to the general meaning of the original terms; word order is conserved in as far as possible, since it structures the architecture of the argument.

The two additions are:

- to *append,* the asymmetric concrete joining of one entity to another one, which conserves its identity;
- to *accumulate*, a symmetric operation, collecting into one sum two magnitudes *or their measuring numbers*. It allows the addition of (the

[2]The corresponding Babylonian and Sumerian terms are given in the full version of the paper, cf. note (*).

measuring numbers of) magnitudes of different kinds (lengths and areas, etc.).

The two subtractions are:

- *to tear out*, the inverse of *appending*, a concrete removal of an entity from another quantity of which it is a part;
- *comparison*, the observation that *one quantity goes so and so much beyond another one*. Even this operation is concrete.

The inverse of *accumulating* is no subtraction but a splitting into constituent parts, occurring in only a couple of texts.

The four operations originally interpreted as multiplications are:

- *steps of*, the multiplication of pure number by pure number used in tables of multiplication, "5 steps of 6" being 30;
- *to raise*, designating the determination of a concrete magnitude by multiplication, always implying some kind of proportionality;
- *to make* [two segments] *hold*, namely "hold" a rectangle—thus no genuine multiplication but a construction, often however implying the determination of the resulting area;
- to *repeat* or *repeat until n*, a concrete doubling (e.g., of a right triangle into a rectangle) or *n*-doubling.

The construction of a square can be described as "making the side hold". Alternatively, one may *make it confront itself*. The square configuration itself is called a *confrontation* (namely of equals). It refers to the square frame rather than to the area it contains: whereas our square *is* (say) 4 m^2 and *has* a side 2 m, a "confrontation" *is* 2 m and *has* an area 4 m^2. Finding the side s of an area Q laid out as a square is expressed in a Sumerian phrase meaning (*close*) by Q, *s is equal*. In certain text groups, the phrase is used as a noun, which I shall translate "the equalside". When one side is found, the other side which it meets in a corner may be spoken of as its *counterpart*.

It is an oft-repeated claim that the Babylonians did not know division. This is only partly true. They knew the division *problem* "What shall I raise to b which gives me A"—many mathematical texts formulate that question. However, division was no *operation.* Here, when possible, they multiplied by the reciprocal of the divisor[3]—by its IGI, the reciprocal as listed in a table. Finding the IGI was spoken of as "detaching" it.

In practical calculation, it was always possible to find the IGI: technical constants were chosen so as to have one. In the mathematical school texts, on the other hand, division by irregular numbers turns up time and again. Here, the text asks exactly the division question "What shall I ...", and states the answer immediately. Since the problems where it happens were always constructed backwards from known results, the quotient would always exist—and always be known by the author of the problem.

Halves, as stated, were two. One is the "normal" half, a fraction belonging to the same family as 1/3, 1/4 , etc. It can be a number ($30'$) or the *half of something*, found then via multiplication by $30'$. But a half (then only the *half of something*) can also be a "natural" or "necessary" half, as the radius of a circle is the necessary half of the diameter: it has a role quite different from that of, say, 1/3 of the diameter. This natural half I shall designate a *moiety*; the operation producing it is called to *break*.

Old Babylonian "algebra" deals with squares and rectangles and their sides. These were taken in the first interpretation to be mere names for *numbers* and their products. Actually, this was a mistake, as we shall see. In any case, the essential terminology is geometrical, and comprises the following:

- the *length* of a rectangle;
- the *width* of a rectangle;
- the *confrontation,* the square configuration numerically parametrized by the side, cf. above;
- and the *surface*, the area of a rectangle or square (or any other figure), designated by a word which in general usage means "field".

[3]The mathematical texts use a floating-point place value system with base 60. Multiplication by the reciprocal was hence only possible when the reciprocal was expressible as a finite sexagesimal fraction. Numbers possessing this kind of reciprocal are called ''regular''. In the translations below, I have fixed a possible order of magnitude, which the Babylonians only kept track of mentally, by writing a number $2 \cdot 60 + 24 + 46 \cdot 60^{-1} + 46 \cdot 60^{-2}$ as $2`24°46'40''$ (the original text only writes 2 24 46 40).

As already follows from this list of essential terms, the geometry in question is of a particular kind: it is a geometry of measurable segments and the areas they contain, so to say within a square grid.

3. SOME TEXTS

We are now prepared to look at a few texts. We may start by considering a problem (VAT 8390 #1)[4] which Kline [9, 9] sees as "a fourth-degree equation in x [that] was solved as a quadratic in x^2".

VAT 8390 #1[5]

Obv. I

1. Length and width I have made hold: 10` the surface.
2. The length to itself I have made hold:
3. a surface I have built.
4. So much as the length over the width went beyond
5. I have made hold, to 9 I have repeated:
6. as much as that surface which the length by itself
7. was made hold.
8. The length and the width what?
9. 10` the surface posit,
10. and 9 (to) which he[6] has repeated posit:
11. The equalside of 9 (to) which he has repeated what? 3.

[4]Cuneiform tablets are written in lines, mostly on the obverse as well as the reverse, and often in columns (indicated in editions by Arabic respectively Roman numerals). Mostly, they are identified by their museum numbers (in the present case, Berlin, *Vorderasiatische Texte*, tablet no. 8390). Because the texts are heavily repetitive, damaged passages can often be reconstructed. I have done so tacitly in what follows.

[5]First published in [11, I, 335f]. Here following [7, 61–63].

[6]This "he" in the reference to the statement shows that the voice which explains the procedure is supposed to differ from the one which states the problem.

12. 3 to the length posit
13. 3 to the width posit.
14. Since “so much as the length over the width went beyond
15. I have made hold”, he has said
16. 1 from 3 which to the width you have posited
17. tear out: 2 you leave.
18. 2 which you have left to the width posit.
19. 3 which to the length you have posited
20. to 2 which to the width you have posited raise, 6.
21. IGI 6 detach: 10′.
22. 10′ to 10` the surface raise, 1`40.
23. The equalside of 1`40 what? 10.

Obv. II

1. 10 to 3 which to the length you have posited
2. raise, 30 the length.
3. 10 to 2 which to the width you have posited
4. raise, 20 the width.
5. If 30 the length, 20 the width,
6. the surface what?
7. 30 the length to 20 the width raise, 10` the surface.
8. 30 the length together with 30 make hold: 15`:
9. 30 the length over 20 the width what goes beyond? 10 it goes beyond.
10. 10 together with 10 make hold: 1`40.
11. 1`40 to 9 repeat: 15` the surface.
12. 15` the surface, as much as 15` the surface which the length
13. by itself was made hold.

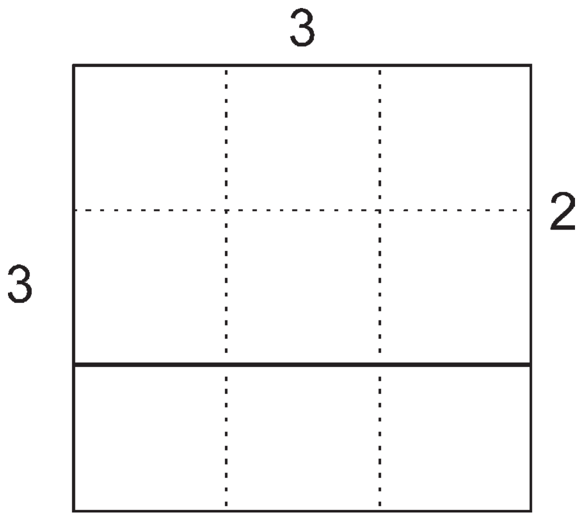

Figure 1: The configuration dealt with in VAT 8390 # 1.

The text starts by constructing a rectangle. The sides are unknown, but the area is given. In the next step, first the square on the length, next the square on the excess of the length over the width are constructed; the latter is "repeated until 9", which is "as much as" the square on the length. The length and width are then asked for.

This already shows us what a Babylonian "equation" is: a statement that (the measure) of a (possibly composite) entity is so and so much, or that (the measure of) one entity is "as much as" (the measure of) another entity. This is no different from the equations of any applied algebra; the difference is that the Babylonians did not *operate* on their equations.

In lines I.9−10 the given numbers are "posited", that is, taken note of materially. Then (lines 12−13), since the square on the excess is "repeated until" 9 and thereby becomes a square, the "equalside" of 9 is found to be 3, a number which is posited to length as well as width (see Figure 1). The following step is argued with a quotation from the statement "since . . . he has said" (a standard phrase for this): the small square, of which the large square contains 3 × 3 copies, is the square on the excess of length over width. When removing ("tearing out") 1 of these from the width, 2 are left, which must correspond to the width of the rectangle. Therefore, the number 2 is posited to the width. In total, the rectangle therefore contains 2 × 3 = 6 small squares (found by "raising", the rectangle being already there; line 20).

Lines 21–22 now perform an IGI-division of 10` (identified as the rectangle surface) by 6, finding the area of the single small square to be 1`40; line 23 then finds that its side is 10. Finally (lines II.1–4) the sides are found by raising the numbers "posited to" the length and the width to this side. Lines II.5–13 contain a proof, that is, a control of the correctness of the result.

BM 13901 #1[7]

This is the first and the simplest problem from a tablet containing in total 24 "algebraic" problems about one or several squares; translated into modern algebraic symbols, it also becomes the simplest of all mixed second-degree problems, $x^2+x = \alpha$ ($\alpha = 3/4$).

Obv. I

1. The surface and my confrontation I have accumulated: 45′ is it. 1, the projection,
2. you posit. The moiety of 1 you break, 30′ and 30′ you make hold.
3. 15′ to 45′ you append: by 1, 1 is equal. 30′ which you have made hold
4. in the inside of 1 you tear out: 30′ the confrontation.

The sum of (the measures of) a square area and the side (the "confrontation") is thus 45′ (= 3/4). In order to make this sum concretely meaningful, the side is provided with a "projection"—in Babylonian *waṣītum*, meaning something which sticks out or projects, e.g., (in architecture) from a building. Thereby the side is transformed into a rectangle, which can meaningfully be glued onto the area—see Figure 2.[8] This "projection" is bisected and the outer part (together with the appurtenant part of the rectangle) is moved so as to form a gnomon. The two halves are now caused to "hold" a supplementary square, which is "appended" to

[7]First published by Thureau-Dangin [12, 31]. Here following [7, 50–52].

[8]Some early texts "append" lines directly to surfaces, operating with a notion of "broad lines", lines provided with a standard width equal to the length unit. This way of thinking of lines is fairly widespread in pre-Modern mensuration [6], and also likely to have characterized the surveyors' environment from which the Old Babylonian school borrowed the first elements of its "algebra" (see [7, 362–387]). The "projection" (and the equivalent "base" which we shall encounter below) appear to be secondary creations of the school.

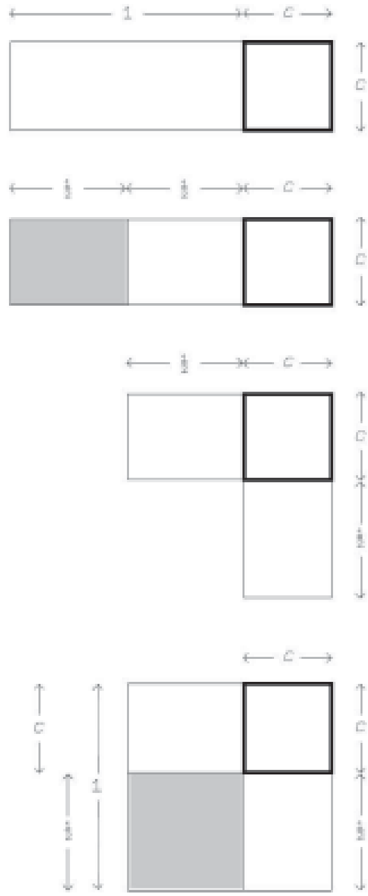

Figure 2: The procedure of BM 13901 # 1, in slightly distorted proportions.

the gnomon. The surface of the resulting square is 1, close by which "1 is equal"– that is, its side is 1. Removing from "the inside" of this side that half of the projection which was moved, we are left with the original side, which must hence be $1 - 30' = 30'$.

We may compare with the procedure by which we solve the corresponding modern equation (disregarding negative numbers, which the Babylonians did not have):

$$x^2 + 1 \cdot x = \frac{3}{4} \qquad \Leftrightarrow \qquad x^2 + 1 \cdot x + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \left(\frac{1}{2}\right)^2$$

$$\Leftrightarrow \qquad x^2 + 1 \cdot x + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$$

$$\Leftrightarrow \qquad \left(x+\frac{1}{2}\right)^2 = 1$$

$$\Leftrightarrow \qquad x+\frac{1}{2} = \sqrt{1} = 1$$

$$\Leftrightarrow \qquad x = 1-\frac{1}{2} = \frac{1}{2}.$$

We observe that the sequence of numbers occurring in the Babylonian text coincides with what we find here. Moreover, the Babylonian method is *analytical* in the same (classical) sense as the solution by equation: we suppose that the object sought after (the "confrontation" respectively the number represented by x) exists; we take note of what we know about it, treat it as a normal entity, thus transforming what we know about it until we have disentangled it.

Finally, both methods are "naive": we do not argue explicitly for their correctness, but we "see" immediately that they are correct. We could undertake a Kantian critique, investigating in which sense and to which extent our procedure is justified—but even we mostly do not feel the need for that.

There are thus much better reasons to consider this an algebraic text than the mere possibility to translate its presumed "mathematical substance" into equation transformations. Whether the reasons are sufficient is a different matter.

TMS XVI #1[9]

Our next examples show that the Babylonians *did* argue in some way for the correctness of their procedures—not deductively, as did Euclid and as does modern theoretical mathematics, but by imparting conceptual understanding of what goes on. At first we shall look at the transformation of a first-degree equation.

1. The 4th of the width, from the length and the width to tear out, 45′. You, 45′
2. to 4 raise, 3 you see. 3, what is that? 4 and 1 posit,
3. 50′ and 5′, to tear out, posit. 5′ to 4 raise, 1 width. 20′ to 4 raise,
4. 1°20′ you see, 4 widths. 30′ to 4 raise, 2 you see, 4 lengths. 20′, 1 width, to tear out,
5. from 1°20′, 4 widths, tear out, 1 you see. 2, the lengths, and 1, 3 widths, accumulate, 3 you see.

[9]First published in [2, 91f], where the commentary is unfortunately almost as mistaken as can be. I follow the presentation in [7, 85–89].

6. IGI 4 detach, $15'$ you see. $15'$ to 2, lengths, raise, $30'$ you see, $30'$ the length.
7. $15'$ to 1 raise, $15'$ the contribution of the width. $30'$ and $15'$ hold.
8. Since "The 4th of the width, to tear out", it is said to you, from 4, 1 tear out, 3 you see.
9. IGI 4 detach, $15'$ you see, $15'$ to 3 raise, $45'$ you see, $45'$ as much as (there is) of widths.
10. 1 as much as (there is) of lengths posit. 20, the true width take, 20 to $1'$ raise, $20'$ you see.
11. $20'$ to $45'$ raise, $15'$ you see. $15'$ from ${}^{30'}{}_{15'}$ tear out,
12. $30'$ you see, $30'$ the length.

As we see, the equation deals with the length and width of a rectangle; in line 1 these are added with a mere "and", an ellipsis for "accumulation". In symbolic translation:

$$(l + w) - \frac{1}{4}w \quad = 45'$$

The text starts by raising $45'$—the right-hand side of the equation—to 4, from which results 3, and then asks for an explanation of this number. This explanation is given in lines 2–5. In line 3 we see that the two "unknowns" are already known; the transformation is thus explained on the basis of a figure with known dimensions. Each of the contributions $5'$, $20'$ and $30'$ is "raised" to 4, and the resulting numbers $20'$, $1°20'$ and 2 are identified, respectively, with w, $4w$ and $4l$. Tearing out $20' = w$ from $1°20' = 4w$ yields $1 = 3w$—in total thus 3, as was to be explained:

$$4l + 3w = 3.$$

Next, the text goes back, dividing this new equation by 4, that is, raising to IGI $4 = 15'$. This gives the contributions of length and width in the original equation, $30'$ respectively $15'$ ("held" in memory in line 7), and "how much there is" of each, that is, the coefficients, respectively 1 and $45'$. A final control shows that multiplication of length and width with these *coefficients*, and removal of $45'w$ from the sum (already written in a way that corresponds to the numbers memorized in line 7) leaves the (contribution of the) length.

The occurrence of a "true width" that is apparently indistinguishable from the width without epithet (line 10) is remarkable. Probably, the "true" length and width were meant to be 30 NINDAN and 20 NINDAN, the NINDAN or "rod" (the standard unit of horizontal distance) being c. 6 m. Evidently, a rectangle of 120 m × 180 m would not fit in the school yard; so, the standard "school extensions" were reduced by an order of magnitude, resulting in a rectangle 2 m × 3 m, which could be drawn in the yard of a typical Babylonian house.

In line 8 we observe once more a quotation from the statement used as argument for a particular step.

This is certainly no deductive proof of anything. It is a nice didactical explanation of concepts relevant to the understanding of the equations and of what goes on in their transformation.

TMS IX[10]

The next text we shall look at starts by two similar didactical explanations:

#1

1. The surface and 1 length accumulated, 40′. ¿30′ the length,? 20′ the width.
2. As 1 length to 10′ the surface, has been appended,
3. or 1 (as) base to 20′, the width, has been appended,
4. or 1°20′ ¿is posited? to the width which 40′ together with the length ¿holds?
5. or 1°20′ together with 30′ the length holds, 40′ (is) its name.
6. Since so, to 20′ the width, which is said to you,
7. 1 is appended: 1°20′ you see. Out from here
8. you ask. 40′ the surface, 1°20′ the width, the length what?
9. 30′ the length. Thus the procedure.

#2

10. Surface, length, and width accumulated, 1. By the Akkadian (method).

[10]First published in [2, 63f], once again with a misleading commentary (which has been largely accepted in the literature). Here after [7, 89–95]. The tablet is damaged and the text sometimes without parallel, for which reasons the wording of some restitutions (marked ¿. . . ?) is uncertain.

11. 1 to the length append. 1 to the width append. Since 1 to the length is appended,
12. 1 to the width is appended, 1 and 1 make hold, 1 you see.
13. 1 to the accumulation of length, width and surface append, 2 you see.
14. To 20′ the width, 1 append, 1°20′. To 30′ the length, 1 append, 1°30′.
15. ¿Since? a surface, that of 1°20′ the width, that of 1°30′ the length,
16. ¿the length together with? the width, are made hold, what is its name?
17. 2 the surface.
18. Thus the Akkadian (method).

Both sections are based on the usual rectangle ▬ (20′, 30′). #1 has the task to explain the geometric interpretation of the equation that the accumulation of the area and 1 length is 40′. It explains that *appending* 1 length to the area is

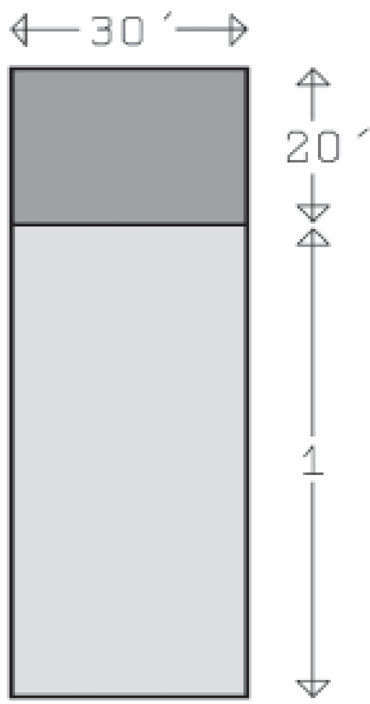

Figure 3: The configuration described in TMS IX # 1.

equivalent to appending a “base” of 1 to the width[11]—cf. Figure 3. Together with the length, this extended width holds a rectangle ▬ (1°20′, 30′) with surface 40′. In the end it is shown how one may find the length from the surface and the extended width.

As we see, the “base” of the present text is equivalent to the “projection” of BM 13901 #1.

#2 accumulates both sides and the surface. It applies the same stratagem in order to make this addition concretely meaningful, but in the first instance this only brings about a quasi-gnomon, a rectangle from which a square ■(1) is lacking in a corner—cf. Figure 4. In order to get a useful configuration this square has to be appended, which gives a total surface 1 + 1 = 2 (line 13). And indeed, the rectangle ▬(1°20′, 1°30′) has the surface 2 (line 17).

The trick to be used is announced in line 10 to be “the Akkadian [method]”, and referred to as such again in line 18. Since the only innovation with respect to #1 is the quadratic completion (though an aberrant variant), this trick must then be what is known as “Akkadian”.

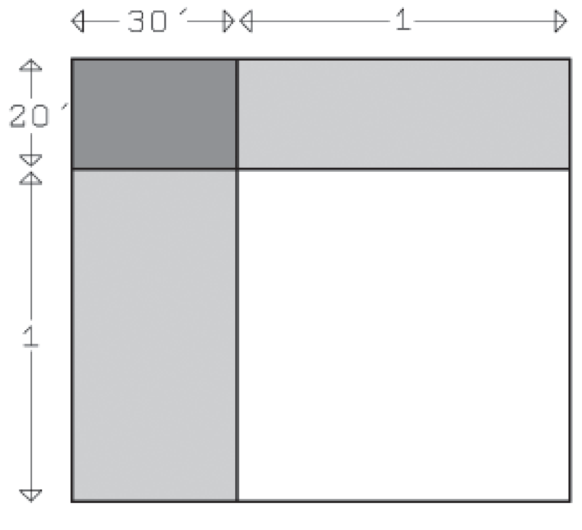

Figure 4: The configuration of TMS IX # 2.

[11]The logogram KI.GUB.GUB is not known from elsewhere, but seems to indicate something which stands erected permanently on the ground. ‘‘Base” seems an adequate translation.

A third section (omitted from the translation) combines the equation that was examined in #2 with an equation of the type that was dealt with in TMS XVI #1—in short:

$$\blacksquare(l, w) + l + w = 1, \qquad \frac{1}{17}(3l + 4w) + w = 30'.$$

The second equation is multiplied by 17:

$$3l + (17 + 4)w = 8°30'.$$

Next, the trick from #2 is repeated (lines 28–33). The length and width "of 2 the surface" are introduced—we may call them $\lambda = l + 1$ and $\omega = w + 1$, where thus $\blacksquare\ (\lambda, \omega) = 2$. Moreover (the damages prevent us from knowing the exact formulation) it is found that

$$3\lambda + 21\omega = 32°30'.$$

If Λ designates 3λ and Ω stands for 21ω (no particular name for these entities occur in the text, we should notice), we thus have (lines 34–39)[12] that

$$\Lambda + \Omega = 32°30, \qquad \blacksquare\ (\Lambda, \Omega) = (3 \cdot 21) \cdot 2 = 2`6.$$

This is a standard problem, to find the sides of a rectangle from its area and the sum of its sides. It is solved by a trick different from but similar to the one used in BM 13901—see Figure 5.

A complex text like this one should bury any belief that Old Babylonian mathematics was merely "empirical" and based on trial-and-error.

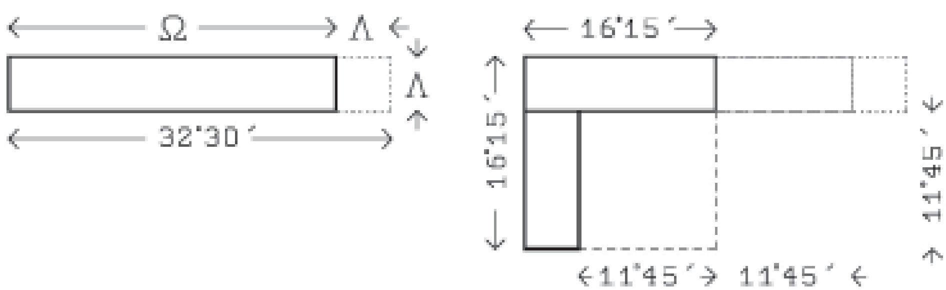

Figure 5: The transformed system of TMS IX # 3.

[12]A simpler variant of the same trick is used to solve non-normalized problems about squares and sides, transforming a problem $\alpha \blacksquare\ (\mathrm{c}) + \beta\mathrm{c} = \delta$ into $\blacksquare(\alpha\mathrm{c}) + \beta(\alpha\mathrm{c}) = \alpha\delta$.

YBC 6967[13]

In a way—but only in a way—this simple text presents us with the rectangle version of BM 13901 #1:

Obv.

1. The *igibûm* over the *igûm*, 7 it goes beyond
2. *igûm* and *igibûm* what?
3. You, 7 which the *igibûm*
4. over the *igûm* goes beyond
5. to two break: 3°30′;
6. 3°30′ together with 3°30′
7. make hold: 12°15′.
8. To 12°15′ which comes up for you
9. 1\` the surface append: 1\`12°15′.
10. The equalside of 1\`12°15′ what? 8°30′.
11. 8°30′ and 8°30′, its counterpart, lay down.

Rev.

1. 3°30′, the made-hold,
2. from one tear out,
3. to one append.
4. The first is 12, the second is 5.
5. 12 is the *igibûm*, 5 is the *igûm*.

Indeed, the topic is not geometrical at all. The problem deals with a couple of numbers from the IGI-table, *igûm* and *igibûm*, Akkadianized versions of IGI and IGI.BI, "the IGI" and "its IGI". However, the reference to the product of the two as a "surface" in obv. 9 shows that these numbers are *represented* by the sides of a rectangle (just as we *represent* geometrical magnitudes by pure numbers when treating geometrical objects and relations algebraically).

[13]First published in [10, 129]. Here following [7, 55–58].

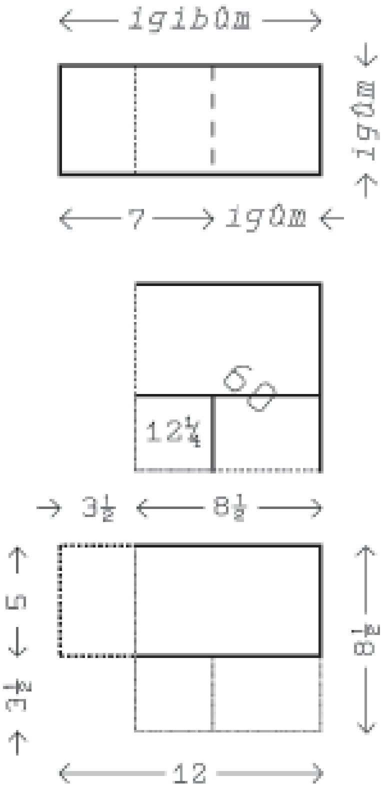

Figure 6: The procedure of YBC 6967.

Algebraic representation was used widely. Lines may represent areas of squares; cubic volumes; inverse prices ("so much oil for one shekel of silver"); in one text, one line represents a number of workers, another one the days they work, the number of bricks they produce being thus proportional to the rectangle they "hold".

The product of *igûm* and *igibûm* is not 1°, as we might expect, but 1\`; on the tablet, 1\` and 1° are of course written in the same way.

With this in mind, we can easily follow the text and see that it describes the transformations of Figure 6. (The "made-hold"/*takīltum* of Rev. 1 is a verbal noun, equivalent to the relative clause "which you have made hold" of BM 13901, line 3.) This time it is the gnomon which is "appended" to the completing square, not vice versa. Since both remain in place, this is possible; in contrast, no text ever "appends" an entity which stays in place in this geometric reading to something which is moved around.

Lines Rev. 2–3 are interesting. Normally, the Babylonians would let addition precede subtraction, just as we do. Here, however, the subtractive process comes first. The explanation is that it is *the same* piece which is "torn out" and "appended", and before it can be appended it must of course be at disposal.

This "norm of concreteness" is not respected in all texts. In the texts IM 53965, IM 54559, and Db_2-146, all three quite early (initial eighteenth century BCE), we just find the prescription "to 1 append, from 1 tear out". Like the "projection" allowing the elimination of the notion of "broad lines", this norm appears to be a secondary development, introduced by school teachers insisting that operations should be meaningful.

4. A GENERAL CHARACTERIZATION

The few examples discussed so far do not show all aspects of Old Babylonian "algebra", and even less all aspects of Old Babylonian mathematics. However, they already allow us to make some generalizations.

It is obvious that anything like deductive chains is absent from all texts. Moreover, the absence of all traces even from such didactic texts as TMS XVI and TMS IX and the sometimes rather circular character of their exposition—a feature which goes against the very gist of deductivity—allows us to conclude that their authors had no familiarity with the possibility of deductive presentation. In so far Kline is right that

> The concept of proof, the notion of logical structure based on principles warranting acceptance on one ground or another [...] are not found in Babylonian mathematics.

He is mistaken, however, when he concludes from this that Babylonian mathematics was not reasoned and thus not really mathematics—unless he would also exile the whole history of analysis before Cauchy (or before Weierstrass?) to the pre-history of mathematics.

Old Babylonian "algebra" was certainly "naive"—but so was infinitesimal analysis during the epoch where faith was promised by d'Alembert to come from practising. So was even European algebra until the early nineteenth century—the attempts to make it rigorous by basing it on Euclidean geometry failed as soon as theoretical developments went beyond the second degree (and already when it operated with negative numbers).

This naive character of essential branches of Early and not so early Modern mathematics did not prevent the appearance of criticism—criticism is a project and hardly ever a totally stabilized outcome. But even Old Babylonian "algebra" presents us with such attempts at criticism. The "norm of concreteness" can be considered such an attempt; the abolition of the "broad lines" represents another one. Both were apparently introduced as the ''algebraic'' discipline became the object of institutionalized teaching in or in the vicinity of the scribe school.[14] Both did indeed give a guarantee of consistency and possible existence by linking the "algebraic" technique to a domain of which one could feel sure, eliminating thus the risk that the entities dealt with had "no existence, if not that on paper", in Georg Cantor's vicious words [3, 501] about Veronese's infinite numbers. Much in the same vein, modern metamathematics ascertains the consistency of a mathematical domain by linking it to the integers, about whose consistency we do feel confident.

Induction based on only a couple of examples is a daring leap. None the less, the Old Babylonian example taken together with "our" mathematics and its Greek, Arabic and Early Modern European ancestor types suggests that mathematics going beyond a very elementary level cannot avoid being reasoned.[15]

5. AN ALGEBRA?

Much ink, and much ire, have been disbursed in order to disprove that various pre-Cartesian mathematical theories and techniques can justly be considered "algebraic".

[14]It should perhaps be stressed that we have no external, independent evidence for the existence of such an institutionalized teaching of sophisticated mathematics. However, the standardized format of the texts leaves no doubt that institutionalization had taken place. What we cannot know for certain is the precise relation of this kind of teaching to the normal scribe school (similarly known from the evidence presented by a highly standardized syllabus, not from independent external testimonials).

[15]Other mathematical traditions could be certainly included in the induction, strengthening its validity; I abstain from mentioning them, not having worked enough on them myself.

Taking these arguments to their full consequence we easily end up in the position once formulated by my friend and colleague Bernhelm Booss-Bavnbek: there was no algebra before Emmy Noether! Leo Corry [4, 397], finding the question about the essence of algebra "ill-posed", suggests instead to "ask 'What is the algebra of Fermat, Descartes and Viète?' or 'What is van der Waerden's algebra?', or even, 'What was the algebra of the Greeks?"' and then discuss whether "the Greeks were, or were not, doing algebra like it was later done in the seventeenth century, or like it is done in the twentieth century". Similarly, let us summarize the characteristics of Old Babylonian "algebra":

Firstly, what we know about was a technique, no mathematical theory. Insights of a quasi-theoretical kind may have been necessary in order to see that certain very complicated problems *could be solved*,[16] but they have left no written trace, and we can only guess at their nature.

Its "fundamental representation" was a geometry of measurable segments and squares and rectangles in a square grid. However, it could be applied to entities of other sorts whose mutual relations were analogous to those of the fundamental representation—numbers, workers and their working days, areas instead of segments, etc.[17]

It was formulated in words, in a very standardized but not always unambiguous language. However, in contrast to rhetorical algebra of al-Khwarizmi's type, its operations were not made *within* language. Even though it seems justified to speak of the verbal statements as "equations", there is thus the fundamental difference that the Babylonian calculators did not *operate* on their equations, as did al-Khwarizmi and as does modern symbolic equation algebra. The Babylonian prescriptions *describe* what is done in the geometric representation, just as we may describe in words what we are doing to the equation—"then we halve the coefficient of x and square it and add it to both sides of the equation", etc., in the above example.

[16]For instance, TMS XIX #2 [7, 197−200], where the area of a rectangle is given together with the area of the rectangle contained by the diagonal of the first rectangle and the cube on its length. This leads to a bi-biquadratic equation, which is solved correctly.

[17]It is noteworthy, but says more about the kind of real-world problems encountered by Babylonian calculators than about their mathematical technique that all problems of the second or higher degrees which we find in the texts are artificial. Not a single one of them corresponds to a task a scribe might encounter in his working practice—unless his work was to teach mathematics!

As already discussed, Old Babylonian "algebra" was "naive", though with attempts at criticism. So, as a last characteristic we shall recall that it was analytical, as is modern equation algebra: it presupposed the existence and the properties of the objects it was looking for.

Whether all of this is sufficient to include the Old Babylonian technique in an extended family of "algebras" is a matter of taste and epistemological convenience. However, whether we include it or we exclude it we should remain aware of the precise criteria used to delimit this family and which Old Babylonian "algebra" fulfils or fails to fulfil. To assert that it *was* an algebra merely because its procedures can be described in modern equations, or to declare that *it was not* because it did not itself write such equations, is perhaps a bit superficial.

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