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What did the abbacus teachers really do when they (sometimes) ended up doing mathematics?

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Abstract

Italian fourteenth- and fifteenth-century abbacus algebra presents us with a number of deviations from what we would consider normal (or proper) mathematical behaviour: the invention of completely false algebraic rules for the solution of cubic and quartic equations, and of rules that pretend to be generally valid but in fact only hold in very special cases; and (in modern terms) an attempt to expand the multiplicative semi-group of non-negative algebraic powers into a complete group by treating roots as negative powers. In both cases, the authors of the fallacies must have known they were cheating. Certain abbacus writers seem to have discovered, however, that something was wrong, and devised alternative approaches to the cubics and quartics, and developed safeguards against the latter misconception.

The paper analyses both phenomena, and correlates them with the general norm system of abbacus mathematics as this can be extracted from the more elementary level of the abbacus treatises.

Contents

Kreuger, Enron and abbacus algebra: three scandals

There may still be Swedes who consider Ivar Kreuger a businessman of genius (at least when I was young there were). After his suicide in 1932 and the opening of his books the rest of the world, in so far as it remembers him and his attempt to create a world monopoly of matches, tends to agree that he was a crook blown up into heroic wide-screen format. That he succeeded as a star for so long – and that the Enron directors did so seven decades later[1] – depended on the construction of a scheme so complex that nobody was able to look through it.

The history of abbacus mathematics presents us with a similar episode, and some members of the tribe of historians of mathematics wave a patriotism that recalls that of certain Swedes – a phenomenon which illuminates particular features of the mathematical endeavour, just as Kreuger and Enron illuminate particular aspects of the market economy. But before that story can be told, the notion of "abbacus mathematics" should itself be explained.

Abbacus mathematics (Italian *abbaco*) is known from Italy (primarily from the region between the Genova-Milan-Venice arc to the north and Umbria to the south) from the late thirteenth to the mid-sixteenth century (but with an aftermath which makes much of its contents familiar to anybody who learned arithmetic in junior secondary school in the 1950s, as I did). Its social base was the "abbacus school", a school frequented by merchant and artisan youth (but also sons of the aristocracy) for two years around age 11–13. In smaller cities, the abbacus masters were often employees of the city, in large cities like Florence and Venice they were run on a completely private basis.[2]

It has been commonly assumed that the abbacus school and its mathematics descended, at most with minor secondary contributions, from Leonardo Fibonacci, his *Liber abbaci* and his *Pratica de geometria*. Thus, according to Elisabetta Ulivi [2002b: 10], the *libri d'abbaco* "were written in the vernaculars of the various regions, often in Tuscan vernacular, taking as their models the two important

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[1] I abstain from referring to corresponding Danish affairs, not because they do not exist (they do, and mostly have as protagonists leading members of the major, "liberal" government party, reduced to ex-members only after they have been discovered or convicted) but because readers may not know about them. Those who are curious and read Danish may find information on specific cases at http://da.wikipedia.org/wiki/Klaus_Risk%C3%A6r_Pedersen and http://da.wikipedia.org/wiki/Peter_Brixtofte.

[2] A convenient survey of the topic is [Ulivi 2002a].

works of Leonardo Pisano, the *Liber abaci* and the *Practica geometriae*" – while, as Warren Van Egmond's sees it [1980: 7], all abbacus writings "can be regarded as [...] direct descendants of Leonardo's book".

On close analysis of the texts involved – early Italian abbacus books, texts of a similar kind from the Ibero-Provençal area, and the *Liber abbaci* – this turns out to be a mistake, due to what at another occasion I called "the syndrom of the Great Book": the "conviction that every intellectual current has to descend from a *Great Book* that is *known to us*" [Høyrup 2003: 11]. Instead, as argued in [Høyrup 2005b], the beginning of abbacus mathematics must be traced to an environment which precedes the *Liber abbaci*; which was known to Fibonacci; which (if it had not fully reached Italy in his days) he may have encountered in Provençal area; but which is likely to have spanned both sides of the maritime and the religious divide of the Mediterranean world. The beginning of abbacus *algebra*, taking place in the early fourteenth century, seems to be inspired by borrowings from an environment located in the Provençal-Catalan area, with a Catalan rather than a Provençal barycentre. This is argued in [Høyrup 2006], on which I draw for the following outline of the events. The precise location of the area is unimportant for what follows; it is more important that the inspiration did *not* come directly from Arabic "scientific" algebra as represented for instance by the treatises of al-Khwārizmī, Abū Kāmil and al-Karajī.

The "scandal" belongs precisely within the field of algebra. The earliest extant treatment of the subject (and plausibly the earliest treatment at all in Italian vernacular) is found in Jacopo da Firenze's *Tractatus algorismi*, written in Montpellier in 1307.[3] In what in my transcription of the manuscript is labelled chapters 16–17 – the algebra section proper – rules are given for the following cases

(1) $\alpha t = n$	(3) $\alpha C = \beta t$	(5) $\beta t = \alpha C+n$
(2) $\alpha C = n$	(4) $\alpha C+\beta t = n$	(6) $\alpha C = \beta t+n$
(7) $\alpha K = n$	(12) $\alpha K = \beta C+\gamma t$	(17) $\alpha CC+\beta K = \gamma C$
(8) $\alpha K = \beta t$	(13) $\alpha CC = n$	(18) $\beta K = \alpha CC+\gamma C$

[3] [Høyrup 2000a] is an edition of the algebraic chapter with mathematical commentary, [Høyrup 1999] is a preliminary transcription of the complete Vatican manuscript (Vat. lat. 4826); the other two extant manuscripts of the treatise (Milan, Trivulziana MS 90, Florence, Riccardiana MS 2236), of which [Høyrup 2007a] is a semi-critical edition, represent a redaction from which the algebra chapter is eliminated. Both are also included in my forthcoming *Jacopo da Firenze's 'Tractatus Algorismi´ and Early Italian Abbacus Culture* [2007b].

(9) $\alpha K = \beta C$	(14) $\alpha CC = \beta t$	(19) $\alpha CC = \beta K+\gamma C$
(10) $\alpha K+\beta C = \gamma t$	(15) $\alpha CC = \beta C$	(20) $\alpha CC+\beta C = n$
(11) $\beta C = \alpha K+\gamma t$	(16) $\alpha CC = \beta K$	

Here, t stands for *thing* (*cosa*), C for *censo*, n for *number* (*numero*), K for *cube* (*cubo*), CC for *censo di censo*. *Censo* is the product of *thing* with *thing*, *cubo* the product of *censo* with *thing*, and *censo di censo* the product of *censo* with *censo*.[4]

For the first six cases, one or more illustrating examples are given, for the rest only rules. All twenty rules are valid, since all the cubic and quartic cases (7)–(20) are either homogeneous, biquadratic or reducible to one of the cases (1)–(6) through division. No mathematical scandal so far.

But scandal was not far away, neither in time nor in space. In 1328, and still in Montpellier, a certain Paolo Gherardi wrote a *Libro di ragioni*, another abbacus book containing an algebra section.[5] Gherardi repeats most of Jacopo's rules and examples – dropping however those of the fourth degree, offering only one example for each case, changing the numerical parameters in some cases, and replacing two of the examples by entirely different ones.

The important innovations are two. Firstly, Gherardi introduces four new cases, one of which (G1) is rather trivial and the other three (G2–G4) not resolvable by means of techniques known at the time:

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[4] These terms come from Arabic algebra, which is the evident basis for all abbacus algebra. *Cosa* translates *šay'*, *censo* comes from Latin *census*, a translation of *māl*, "possession" or "amount of money". Originally, Arabic *al-jabr* was centred around riddles dealing with a possession and its (square) root, for instance "a possession and ten of its roots equal 39 dinars". Al-Khwārizmī, in his presentation of the topic (which may be the earliest presentation at all in a systematic written treatise) still remembers this: when he has found the root, he multiplies it by itself in order to find also the possession. But already in *his* treatise these riddles with their solutions serve as representation of second-degree problems, in which the fundamental unknown is a *šay'*, whose second power is identified with a *māl* (whence the *šay'* becomes its root).

Almost all abbacus algebras do as Jacopo: the "roots" are replaced by "things" in the formulation of the rules, and the number is a number, not (as in the Latin translations of al-Khwārizmī) a quantity of dragmas. This is one of several reasons that abbacus algebra (in particular Jacopo's algebra) can be seen not to descend from the "learned" level of Arabic algebra but from a type which has disappeared from the sources – probably from a practice which was integrated with the teaching of commercial arithmetic, just as abbacus algebra itself.

[5] An edition of this chapter, with translation and mathematical commentary, is [Van Egmond 1978]. The whole treatise is found (without translation and mathematical commentary) in [Arrighi 1987].

$$\text{(G1) } \alpha K = \sqrt{n} \qquad \text{(G3) } \alpha K = \beta C+n$$
$$\text{(G2) } \alpha K = \beta t+n \qquad \text{(G4) } \alpha K = \gamma t+\beta C+n$$

For the latter three, Gherardi gives rules modelled after those for the second degree – for (G2) and (G3) those which hold if K is replaced by C, for (G4) the rule for the equation

$$\mathrm{a}C = \beta t+(\gamma+n)$$

Finally, Gherardi offers illustrative examples for those higher-degree cases where Jacopo had given none – all of a kind that is easily constructed, whereas some of those proposed by Jacopo are so intricate that a modern reader does not immediately see that they lead to second-degree equations. For instance, Jacopo's illustration of case (6) runs as follows (the translation is mine, as all translations in the following where no translator is identified):

> Somebody has 40 *fiorini* of gold and changed them to *venetiani*. And then from those *venetiani* he grasped 60 and changed them back into *fiorini* at one *venetiano* more per *fiorino* than he changed them at first for me. And when he has changed thus, he found that the *venetiani* which remained with him when he detracted 60, and the *fiorini* he got for the 60 *venetiani*, joined together made 100. I want to know how much was worth the *fiorino* in *venetiani*.

Gherardi's examples for the third-degree cases all follow the model used in Jacopo's illustration for case (3):

> Find me 2 numbers that are in the same proportion as is 4 of 9. And when one is multiplied against the other, it makes as much as when they are joined together. I want to know which are these numbers.

Such illustrations are of course easily constructed for any given polynomial equation and look more complex than the equation itself without really being so – for instance Gherardi's illustration [ed. Arrighi 1987: 106] of the case (G4) "cubes are equal to things and *censi* and number":

> Find me three numbers which are in proportion as 2 to 3 and as 3 to 4, and that the first multiplied by itself and then by the [same] number makes as much as when the second is multiplied by itself and the third number is added above, and then 12 are added above.

As we shall see, Gherardi did not invent all of this, he copied it from an earlier source. We may ask why he did not discover that he was filling his treatise with nonsense. The answer is that all the wrong solutions contain irreducible radicals, and that Gherardi made no attempt to find the approximate value of the solutions. This was no idiosyncrasy. Even Jacopo, when finding a correct irrational solution to one of his examples, leaves things there. Being satisfied with exactly expressed but irrational solutions remained the habit of abbacus algebra. In contrast, abbacus *geometry* always approximated the square roots that turned up in applications of the "Pythagorean rule" (as it must be called in a context where it was always presented as a rule without proof).[6] This difference already tells us that algebra and geometry served different purposes: geometry (as a whole, not necessarily each single problem) had to lead to results that were applicable in practice, and which could thus be compared to the reading of a yard stick. Abbacus algebra, at least beyond the first degree, must in some sense (which we shall get closer at below) have been a purely theoretical discipline without intended practical application.

Almost honest business

If Gherardi does not represent the beginning of false solutions, nor is he their end. In 1344, Master Dardi da Pisa (as unknown as Jacopo and Gherardi) wrote a treatise *Aliabraa argibra*, the earliest extant European-vernacular treatise dedicated to algebra alone.[7] After presenting the arithmetic of roots and binomials and giving geometric demonstrations for the correctness of the rules (the latter are very rare in abbacus algebra, and in particular not present in any of the earlier treatises) Dardi deals with 194 "regular" cases and 4 whose rules are told only to hold under special conditions (which are not analyzed).[8] The huge number of regular cases (all with the exception of two lapses solved correctly) is reached because of ample use of radicals – for instance in these ways:

$$\alpha t+\beta\sqrt{K} = \gamma C \qquad\qquad \alpha t+\beta\sqrt{C} = \gamma C$$

$$\alpha CC = n+\sqrt[3]{m} \qquad\qquad \alpha CC+n+\sqrt[3]{m} = \beta C$$

For a generation which has come to see no difference between rational and

6 This role of square roots and their approximation was so important for geometry that the topic was mostly taught in the geometry chapter of abbacus treatises (when these were ordered in separate chapters and geometry was actually covered). In the Latin algorisms, in contrast, root extraction (not approximation) was one of the arithmetical "species"; they contain no geometry.

7 The extant complete manuscripts are younger. One is from c. 1395 (Vatican, Chigi M.VIII 170), one from 1429 (Arizona State University Library, Tempe), and one from c. 1470 (Siena, Biblioteca Comunale, I.VII.17). Apart from lost sheets and some reordering of the material in the last manuscript, there are no major differences between the three. Of a fourth manuscript from c. 1495 (Florence, Biblioteca Mediceo-Laurenziana, Ash. 1199) I have only seen the extract in [Libri 1838: III, 349–356], but to judge from this it appears to be very close to the Siena manuscript.

8 [Van Egmond 1983] lists all the cases in symbolic transcription.

irrational numbers, to see all the "cossic numbers" (as *thing*, *censo* etc. were to be called when abbacus algebra reached Germany under the name of *Coss*) as powers of the same unknown and to express everything in symbols and not in words, these are trivial extensions – and by reducing many of the cases to other cases that are dealt with previously, Dardi shows that he understood things in the same way without having access to our tools (tools without which the extensions are often *not* trivial).

All of these cases are illustrated by one or more examples. All are pure-number problems, with a few exceptions either about a single number, about two numbers with sum 10, or about numbers in given proportion.

Then there are the four "irregular" cases, cases governed by non-general rules. It is clear from Dardi's words that he knows these rules to be valid only when the equations to which they correspond have particular properties – but he states that "by some accident the said rules may appear in some computation". The cases in question are these:

$$\gamma t+\beta C+\alpha K = n \qquad\qquad \alpha t+\gamma C+\alpha CC = n+\beta K$$
$$\delta t+\gamma C+\beta K+\alpha \mathrm{CC} = n \qquad\qquad \delta t+\alpha CC = n+\gamma C+\beta K$$

All four are provided with examples, the former two of which reveal how the rules have been found. We may look at the first example – a capital grows in three years with composite interest from 100 £[9] to 150 £ (Jacopo has the same problem, only with two years; it illustrates his case (4)). If the value of the capital after 1 year – or, even simpler, the value of 1 £ after one year – had been taken as the *thing*, we would have been led to a homogeneous equation,

$$t^3 = 1500000 \quad \text{respectively} \quad t^3 = 1^1/_2 \ .$$

Instead, Dardi takes the monthly interest of 1 ß expressed in δ as his *thing*. The yearly interest of 1 £ is therefore $^1/_{20}$ £ *thing*. The same choice is made by Jacopo, and in the present case it leads to the equation

$$100+15t+{}^3/_4C+{}^1/_{80}K = 150 \ .$$

The rule used to solve it is

$$t = \sqrt[3]{\left(\frac{\gamma/\alpha}{\beta/\alpha}\right)^3+\frac{n}{\alpha}}-\frac{\gamma/\alpha}{\beta/\alpha} \ ,$$

– or rather, since the rule first tells to divide by [the coefficient of] the cubes and afterwards speaks only of the resulting coefficients,

9 £ stands for *lira*/*lira*. 1 £ = 20 ß (*soldi*), 1 ß = 12 δ (*denari*). Whoever is familiar with the traditional British pound-shilling-penny system will recognize it.

$$t = \sqrt[3]{\left(\frac{\gamma'}{\beta'}\right)^3 + n'} - \frac{\gamma'}{\beta'} \ ,$$

where $\beta' = \beta/\alpha$, etc. At first view, this may seem an astonishingly good guess (since it works), but it requires nothing beyond some training in the arithmetic of polynomials and awareness that a different position for the *thing* leads to a homogeneous equation:

For simplicity, let us consider the homogeneous equation

$$(t+\phi)^3 = \mu$$

(in the actual problem, $\phi = 20$, $\mu = 12000$). Performing the multiplication we get

$$\phi^3+3\phi^2t+3\phi C+K = \mu \quad \text{or} \quad 3\phi^2t+3\phi C+K = \mu-\phi^3 \ ,$$

which should correspond to

$$\gamma' t+\beta' C+K = n' \ .$$

Therefore, $\phi = \gamma'/\beta'$, $n' = \mu-\phi^3$, $\mu = \phi^3+n' = (\gamma'/\beta')^3+n'$. Now, the solution obtained from the homogeneous equation is

$$t = \sqrt[3]{\mu}-\phi \ ,$$

that is,

$$t = \sqrt[3]{\left(\frac{\gamma'}{\beta'}\right)^3+n'} - \frac{\gamma'}{\beta'} \ ,$$

exactly Dardi's rule. Whoever invented the rule must have done so from a numerical example, but following the numerical steps precisely and seeing from which operations the coefficients arise it would not be too difficult to see that the 20 of our example results, in the words of the rule, "when [the coefficient of] the *things* [is]/are divided by [the coefficient of] the *censi*"; similarly for the rest of the rule – and similarly for the remaining three irregular rules.

The inventor of Gherardi's rules may have been a pure bluffer – for imitating the rules for the second case it was not even necessary to know how these were derived, all that was needed was to know the rules themselves. In contrast, the rules for Dardi's irregular cases, guesses though they are in a certain sense, can only have been guessed by someone who understood polynomial operations quite well.

The irregular rules turn up in many later manuscripts, mostly without the warning about their restricted validity. One of these, an anonymous *Libro di conti e mercatanzie* [ed. Gregori & Grugnetti 1998] from c. 1395, is related to Gherardi's *Libro di ragioni* in a way which shows them to build on common sources (also shared with an equally anonymous *Trattato dell'Alcibra amuchabile* from c. 1365 [ed. Simi 1994].[10] Quite apart from the internal evidence (the use of a business dress when all other examples are in pure numbers, and the reservations expressed by Dardi himself), this is strong evidence that these rules were borrowed by Dardi and thus that they antedate 1344, just as the false rules in Gherardi's *Libro di ragioni* must have been borrowed by Gherardi from a source shared with the *Trattato dell'Alcibra amuchabile*. We may conclude that the presence of regular higher-degree cases in Jacopo's algebra created a fashion or a need to do even better – a need which was then fulfilled, first by the invention of false rules that could not be controlled,[11] and then by the construction of irregular rules that worked if tested on the proposed example.[12] We shall discuss this process below, but for the moment only observe that false solutions survived for long. Luca Pacioli, after having made the check proposed in note 11, pointed out in his *Summa de Arithmetica* [1494: 150^r] that so far no rule had been found for the solution of cases where, as he says, the three algebraic powers that are present are not "equidistant". On that background, del Ferro's genuine solution of the cubic equation and Cardano's publication of a corresponding proof can be seen not only to be mathematically impressing but to deliver what others were known by then to have promised in vain for two centuries. But Pacioli's book did not kill off the fraud completely – in 1555, the Portuguese Bento Fernandes still included them in his *Tratado da arte de arismetica* [Silva 2006: 16, 30–33].

Aiming high – and failing honestly

If we are to learn from the abbacus masters about what mathematics *is* it does not serve to consider solely such aspects of their activity as correspond to what we routinely expect from a mathematician. So, we shall go on with another anomaly.

It is found in yet another Vatican manuscript, Vat. Lat. 10488 (fol. 29^v–30^v):[13]

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[10] For this, see [Høyrup 2006: 18–25].

[11] That is, unless one constructed alternative examples with (most conveniently *from*) a known integer solution; and that seems not to have been a widespread idea.

[12] This test was easy: in the example that was analyzed above, $t = \sqrt[3]{12000} - 20$. Since this is the yearly interest, the yearly growth factor of the capital is $1+{}^1/_{20}t = \sqrt[3]{3/2}$. After three years the capital is thus multiplied by ${}^3/_a$, just as required.

[13] I use the most recent of the two discordant foliations.

¶ These are some computations collected from a book made by the hand of Giovanni di Davizzo dell'abbaco from Florence written the 15th of September 1339, and this is 1424.

¶[1] Know that to multiply number by cube makes cube
and number by censo makes censo
and number by thing makes thing

¶[2] And plus times plus makes plus
and less times less makes plus
and plus times less makes less
and less times plus makes less.

¶[3] And know that a thing times a thing makes 1 censo
and censo times censo makes censo of censo
and thing times censo makes cube
and cube times cube makes cube of cube
and censo times cube makes censo of cube

¶[4] And know that dividing number by thing gives number
and dividing number by censo gives root
and dividing thing by censo gives number
and dividing number by cube gives cube root
and dividing thing by cube gives root
and dividing censo by cube gives number
and dividing number by censo of censo gives root of root
and dividing thing by censo of censo gives cube root
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and dividing cube by cube of cube gives cube root
and dividing censo of censo by cube of cube gives root
and dividing censo of cube by cube gives ~~number~~ censo[14]
and dividing number by censo of censo of censo of censo gives root of root of root of root
and dividing number by cube of cube of cube of cube gives cube root of cube root of cube root of cube root.

[14] From later versions it can be seen that this line was originally

"and dividing censo of cube by cube *of cube* gives number"

Somewhere in the process, this had become

"and dividing censo of cube by cube gives number"

Noticing the error, somebody – presumably the writer of the manuscript, since the correction is made there – discovered that this was wrong, and stated a correct result (but of a division Giovanni had not intended).

¶[5] If you want to multiply root by root, multiply root of 9 times root of 9, say, 9 times 9 makes 81, and it will make the root of 81, and it is done.

To divide root of 40 by root of 8, divide 40 by 8, it gives 5, and root of 5 let it be.

To divide root of 25 by root of 9, divide 25 by 9, it gives root of $2^{7}/_{9}$, done.

If you want to multiply 7 less root of 6 by itself, do 7 times 7, it makes 49, join 6 with ⟨49, it makes⟩ 55, and 7 times 6 makes 42, then multiply 7 times 42, it makes 294, and multiply then 4 times 294, it makes 1176, I say that 55 less root of 1176 will it make when 7 less root of 6 is multiplied by itself.

¶[6] If you want to detract root of 8 from root of 18, do 8 times 18, it makes 144, its root is 12, and say, 8 and 18 makes 26, detract 24 from 26, and root of 2 will remain, done.

It you want to join root of 8 with root of 18, do 8 times 18, it makes 144, its root is 12, and say, 12 and 12 makes 24, and say, 8 and 18 makes 26, and join 24 and 26, it makes 50, and root of 50 will the number be.

If you want to multiply 5 and root of 4 times 5 less root of 4, do thus and say, 5 times 5 makes 25, and say, 5 times root of 4, do thus, bring 5 to root, it makes 25, and do root of 25 times root of 4, it makes root of 100, and make 5 times less root of 4, it makes less root of 100, 25 still remains, now detract 4 from 25, 21 remains, and 21 they make.

If you want to multiply 7 and root of 9 times 7 and root of 9, do 7 times 7, it makes 49, put (above) this 9, you have 58, and 9 times 49 makes 441, multiply by 4, it makes 1764, you have that it will make 58 and root of 1764, which is 42, done.

If you want to divide 35 by root of 4 and by root of 9, do thus, from 4 to 9 there is 5, multiply 5 times 5, it makes 25, and say, bring 35 to root, it makes 1225, now say, 4 times 1225 makes 4900, divide by 25, it makes 196, and do 9 times 1225, it makes 11025, divide by 25, it gives 441. We have that dividing 35 by root of 4 and by root of 9 gives root of 441 less root of 196, and it is done.

This is followed by 19 rules for solving reduced equations of the first, second, third and fourth degree: Jacopo's 20 cases, with two omissions, and a new false case which cannot be read because somebody discovered that it did not work and glued a paper slip over it (this slip has been removed or fallen off, but the glue has made the paper as dark as the ink).

First of all we should know that Giovanni's composition of the "cossic numbers" is multiplicative and not made by nesting: *cube of cube* stands for $t^3 \cdot t^3$, not for $(t^3)^3$. This corresponds to what we find with Diophantos and in Arabic algebra.[15] Once we know this we see that the first and third paragraphs

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15 In the present case κυβόκυβος respectively *kāb kāb*. None of these involve the genitive – in the case of the Arabic because a possibly spoken genitive ending *-in* was not written (but presumably a genitive would also ask for the article, *kāb al-kāb*). At least in writing none of them therefore suggests nesting, as does the genitive used in the Italian and Latin translations. In the short run this caused a problem to the abbacus writers; for instance, it probably caused Dardi's two lapses, cf. [Van Egmond 1983: 417]. In the longer run,

present what we might call the multiplicative semi-group of non-negative algebraic powers through examples; the interrupting second paragraph gives the "sign rules". So far, everything goes well; from the correction made in the Vatican manuscript at a later point (see note 14) it is also clear that the author of this manuscript understood it well, and was able to perform divisions within the semi-group to the extent they can be performed.

But Giovanni does not stop here. Skipping the divisions that correspond to multiplications within the semi-group (which he may have considered unproblematic) he jumps to those that have no such solution. Obviously what he does is wrong, and he should have discovered that if he had been a bit careful. Indeed, if "dividing number by thing gives number", then, since the quotient multiplied by the divisor gives the dividend (any abbacus algebraist would know that, it is often told explicitly in the texts), number multiplied by thing should give number. But "number by thing makes thing", Giovanni knows it well and states it in paragraph [2].

However, the nonsense conceals a system. If, *in this paragraph*, we read "root" as t^{-2}, "cube root" as t^{-3}, if we compose these "roots" multiplicatively, and if we finally interpret "number" *when occurring as a result* as t^{-1} – then everything is perfect, and the semi-group is extended into a group.

We shall return to the implications of Giovanni's undertaking as a whole. At this point we may try to trace how he thought. The background appears to be an intuitive and only implicit arithmetization of the series of algebraic powers. Multiplying by *censo*, so more or less he may have reasoned, we take two steps "upwards"; multiplying by cube we take three steps. Multiplying cube by *censo* we get *censo* of cube (this is stated). Dividing *censo* of cube by *censo* we therefore get cube, two steps "downwards". Dividing instead by cube we have to take three step downwards. Now multiplying the thing by itself we get a *censo*, and taking the root of the *censo* we return to the thing; similarly, the cube root of a cube is a thing. Therefore "root" must be some kind of opposite of the *censo*, and cube root some kind of opposite of cube. Taking two steps upwards from number (number by *censo*) gives us *censo*, taking two steps downwards (number divided by *censo*) therefore root; taking three steps downward must give us cube

however, the linguistic trouble was probably what drove the trend toward an interpretation through nesting (common in the later fifteenth century, and practised for instance by Pacioli). Since the creation of new names for the fifth and seventh power (etc.) then caused new confusion, this may have been one of the driving forces behind the introduction of numerical exponents (first in Chuquet and Bombelli).

root.

This explains everything except those rules where the result is “number” – for instance “dividing *censo* by cube”. Here the idea must more or less have been that “root” is a “second root”, just as *censo* is a second (being thing times thing, and corresponding to two steps in multiplication and division); correspondingly, the cube root is a “third root”. Therefore, the result of *censo* divided by cube must be a “first root”, which Giovanni then identifies with number (probably because it seemed to him that “thing” was an impossible choice, being the result of the division of cube by *censo*). If we take care that this is only the meaning of “number” when it results from a division, everything becomes correct – but like Hogarth’s false-famous perspective drawing only locally correct, and absurd as soon as one tries to move back and forth through the whole network of possible operations.

Even Giovanni’s fallacies were borrowed faithfully. As we have seen, his text was copied in 1424 by somebody who understood it well enough to repair a copying error correctly. Later Giovanni’s system turns up in Piero della Francesca’s *Trattato d’abaco* (earlier than c. 1480) [ed. Arrighi 1970: 84*f*] with some change of the order and without the mistake discussed in note 14, and almost identically in Giovanni Guiducci’s *Libro d’arismetricha* from c. 1465 – see [Giusti 1993: 205]. Finally, Giovanni’s first 15 rules turn up in exactly the same order in Bento Fernandes’ *Tratado da arte de arismetica* from 1555 [Silva 2006: 14] (which thus stops just before the corrupted line, which may be no accident). Piero, like Fernandes, also repeats the false algebraic rules, apparently without suspecting that something is rotten. Evidently Piero, claiming to write about “certain abbacus things that are necessary for merchants” [ed. Arrighi 1970: 39], could do so because neither he nor any merchant had the least operatory need for it.

A better intuition

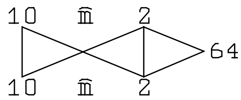
Intuitions like those which can be read out of Giovanni’s system can be found in other abbacus writings, and they often worked better. One which is also made much more explicit gives a proof for the sign rule “less times less makes plus”. The earliest known occurrence is in Dardi’s *Aliabraa argibra*:[16]

> Now I want to demonstrate by number how less times less makes plus, so that every times you have in a construction to multiply less times less you see with certainty that it makes plus, of which I shall give you an obvious example. 8 times 8 makes 64, and this 8 is 2 less than 10, and to multiply by the other 8, which is still 2 less

[16] I translate from the Vatican manuscript, Chigi M.VIII.170, fol. 5^v.

> than 10, it should similarly make 64. This is the proof. Multiply 10 by 10, it makes 100, and 10 times 2 less makes 20 less, and the other 10 times 2 less makes 40 less, which 40 less detract from 100, and there remains 60. Now it is left for the completion of the multiplication to multiply 2 less times 2 less, it amounts to 4 plus, which 4 plus join above 60, it amounts to 64. And if 2 less times two less had been 4 less, this 4 less should have been detracted from 60, and 56 would remain, and thus it would appear that 10 less 2 times 10 less two had been 56, which is not true. And so also if 2 less times 2 less had been nothing, then the multiplication of 10 less 2 times 10 less 2 would come to be 60, which is still false. Hence less times less by necessity comes to be plus.

The passage is followed by a diagram:



The reason this must be characterized at least in part as an intuition and not as a genuine piece of analysis is the final part: instead of finding that the contribution of less 2 by less 2 must be the lacking 64–60 = 4, Dardi expects (from similarity) that it must be either an additive or a subtractive contribution of 4, or possible nothing at all, and then eliminates the second and the last possibility, leaving only the first one.

Luca Pacioli repeats the argument in his *Summa* [1494: 113^{r}], now with the diagram in the margin, and with an explicit reference to the cross-multiplication; he finds the very concept to be *absurda* and an abuse but none the less necessary – Pacioli, indeed, thinks in terms of *negative numbers*, not merely subtractive contributions to an equation as does Dardi.[17] Apart from that the only innovation is that the alternative to the alternative is now $(-2)\cdot(-2) = -2$, not $(-2)\cdot(-2) = 0$.

An alternative to the false solutions

Some abbacus authors thus had better intuitions than others. Similarly, some of them understood better than others that the false solutions to the higher-degree equations *were* false and even devised alternatives.

One such alternative is described in yet another anonymous manuscript from the outgoing fourteenth century (Florence, Biblioteca Nazionale, Fond. princ. II.V.152). After the presentation of the "22 rules of algebra" (Jacopo's 20 rules,

[17] Pacioli indeed explains that a number of this kind is "less than zero and in consequence a debt".

and the two biquadratics that are absent from his list), the author goes on to explain [ed. Franci & Pancanti 1988: 98] that other rules can be made for certain other cases. He continues:[18]

> Wanting to treat of this it is needed first to show how there are other roots than those one normally speaks about, that is, there are other roots than square roots and cube roots, and among these there is one which is called cube root with addition of some number, and about this I intend to show something.

The concept is then explained through several examples, starting with "the cube root of 44 with addition of 5". This root is 4, because $4^3 = 44+5\cdot4$; in general, expressed in symbols, the cube root of n with addition α – say, $\sqrt{}^{c}(\alpha,n)$ – is t if

$$K = n+at \ .$$

(We recognize the normalized version of equation (G2). Evidently, this allows us to give a name to the solution of the above equation; but if we follow Pascal's advice about how one should understand definitions, this name is just an abbreviation of "the solution to the equation $K = n+\alpha t$", which makes the whole thing rather circular.

However, several further observations must be added to this. Firstly, as long as irrational square and cube roots were not approximated in abbacus algebra, expressing the solution to the equation $C = 3$ as "root of 3" was just as circular. Secondly, the trick is also used in much more recent mathematics – elliptic functions could be said to suffer from the same defect. What makes square roots and elliptic functions mathematically interesting (beyond the possibility of numerical approximation) is the network of relations they allow us to establish.

What can we say about our author and his "cube root with addition" in this respect? Firstly, that he must have been aware of the objection just discussed. He does not find it worthwhile to discuss a single problem of the type which is immediately solved by his particular root; instead he explains that it is of limited use, since for many numbers this root cannot be expressed. What he does beyond that is to establish a (limited) network of relations: he gives (correct) rules for reducing equations of the types $K+\beta C = m$, $K = \beta C+m$ and $\beta C = K+m$ to the form $K = n+\alpha t$, and in the ensuing example he then makes use of the cube root with addition.[19] He also shows in the examples that solutions may exist

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[18] See also [Franci 1985].

[19] He does *not* show that α can be eliminated and thus that a single table of $\sqrt{}^{c}(1,n)$ is all that is needed. The reason could be that tables did not enter his mind, but it could also be that the transformation was too difficult. It asks indeed for a substitution $z = t/\sqrt{\alpha}$, which gives the equation

even if the number term turns out to be "a debt", that is, negative. In order to find this reduction rule, the author[20] must have performed manipulations similar to those behind Dardi's first irregular rule.[21] The author must have been an adroit mathematician.[22]

Luca Pacioli may have heard about the solutions of particular higher cases by means of these specious roots, but in that case he does not seem to have appreciated them. In any case he goes on, after the statement that cases where the three algebraic powers that are present are not "equidistant" had not been solved so far, to admit that certain particular cases can be solved *a tastoni*, "feeling one's way". There is another trace in Pacioli's text of these solutions by special roots, which he may not have recognized as such. Our anonymous author, as we remember, refers to "other roots than those one normally speaks about" in the plural, but only mentions one. In particular he does not speak about the *radice pronica* which is referred to by several other authors. Pacioli [1494: 155$^{\mathrm{v}}$] explains

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$$z^3 = \frac{n}{\alpha\sqrt{\alpha}} + z \ .$$

This is more difficult to find and explain without symbolic algebra than the additive substitutions needed for the transformation which *is* explained (finding the transformation factor to be $\sqrt{\alpha}$ asks for manipulation of several powers of two variables at a time, something which was so far beyond the horizon of abbacus algebra that even Bombelli when creating his new formalism happened to exclude it (cf. below, p. 23). *Vive* Descartes!

[20] Or the one from whom he borrows – a reservation which must always be made for the abbacus authors when they seem to be original; I shall not repeat it but ask the reader to keep it in mind.

[21] This is not fully explicit, but obvious from the detailed appearance of the rule. If, for convenience, we reformulate the first equation as

$$t^3+3at^2 = n$$

completion gives

$$(t+a)^3 = n+a^3+3a^2t = n+a^3+3a^2(t+a)-3a^2\cdot a \ ,$$

which is exactly what the rule tells, in this order and without contraction of any kind of the expression $n+a^3+3a^2(t+a)-3a^2\cdot a$. Similarly for the other two cases.

[22] Indeed more than that, he was also more honest than many colleagues. He not only avoids the false rules, when dealing with the problem type to which Dardi applies his second irregular rule the present author [ed. Franci & Pancanti 1985: 76] takes the *thing* to be the value of the capital after one year, thus showing that the problem is fundamentally homogeneous. Further, when presenting [ed. Franci & Pancanti 1985: 3–6] the arithmetic of the algebraic powers he accompanies the rules by numerical examples that show how things really work. If Giovanni di Davizzo had done that, his marvellous construction would have collapsed immediately.

that by “pronic root”

> one normally understands a number multiplied by itself and above its square add the root of the said number; of this sum that number is called the pronic root. As 9 multiplied by itself makes 81, and above 81 add the root of 9, which is 3, makes 84, the pronic root is said by practitioners to be 9.

This does not seem very useful, and does not seem even loosely related to the notion of “pronic numbers”, numbers of the type $n \cdot (n+1)$ (also known as oblong numbers). However, in Pierpaolo Muscharello’s *Algorismus* from 1478 [ed. Chiarini et al 1972: 163] we read that

> Pronic root is as if you say, 9 times 9 makes 81. And now take the root of 9, which is 3, and this 3 is added above 81: it makes 84, so that the pronic root of 84 is said to be 3.

This makes better sense – according to Muscharello, n is the pronic root of $n^4+n = n^3 \cdot (n+1)$. Moreover, as we see, this pronic root can be used to “solve” equations of the type $CC+\alpha t = n$. It therefore seems plausible that the cube root with addition was not the only attack at higher-degree equations made by abbacus authors before Pacioli’s time.[23]

Some general characteristics of abbacus mathematics

Before we discuss the implications of the material presented so far – which after all represents only a small although prestigious corner of abbacus mathematics,[24] far too difficult to be taught to the young students of the standard two-

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23 Benedetto da Firenze [ed. Pieraccini 1983: 26] also mentions the pronic root in his discussion of Biagio il Vecchio’s solution of the problem CC+t = 18, which he points out to be valid only for this particular parameter. It is not clear, however, whether the pronic root to which he refers is 4 (as Pacioli would have it), 2 (in agreement with Muscharello), or perhaps Biagio’s solution $\sqrt{18+(½)^4}-(½)^2$, which is 4 (not 2, as claimed by Pieraccini in her preface [1983: vi]). The coincidence of Biagio’s formula with Pacioli’s interpretation depends on the specific parameter 18, it should be noted.

24 An expression of the prestige of algebra is found in Pacioli’s words when he comes to the presentation of the rules for the algebraic cases [1494: 144r],

> “having come with the help of God to the much desired place, that is, to the mother of all the cases called popularly “the rule of the thing”, or the Great art, that is, a theoretical practice also called Algebra and almucabala in the Arabic tongue ...”.

The words “theoretical practice” (*pratica speculativa*) confirm what was derived from internal evidence on p. 5, that algebra was “a purely theoretical discipline without intended practical application”. We shall still need to ascribe a more precise meaning year course.

Some of the orderly abbacus treatises start by presenting the Hindu-Arabic numerals and their use (in multiplication tables, in the algorithms for numerical computation, and/or in particular divisions); others start directly by the rule of three.[25] In both cases they show how things *are* or *are to be done*, without giving arguments for this. In particular in the case of the rule of three, this is noteworthy. We may look at the way the presentation is done in Jacopo's *Tractatus algorismi* (I translate from the Vatican manuscript, fol. 17r):

> If some computation should be given to us in which three things were proposed, then we should always multiply the thing that we want to know against that which is not similar, and divide in the other thing, that is, in the other that remains.

Then follows the first example (*tornesi* and *parigini* are coins minted in Tours and Paris, respectively):

> VII *tornesi* are worth VIIII *parigini*. Say me, how much will 20 *tornesi* be worth. Do thus, the thing that you want to know is that which 20 *tornesi* will be worth. And the not similar (thing) is that which VII *tornesi* are worth, that is, they are worth 9 *parigini*. And therefore we should multiply 9 *parigini* times 20, they make 180 *parigini*, and divide in 7, which is the third thing. Divide 180, from which results 25 and $^5/_7$. And 25 *parigini* and $^5/_7$ will 20 *tornesi* be worth.

We notice that the intermediate product has no concrete interpretation (apart from the awkward and intuitively unattractive "as many times *p parigini* as there are *tornesi* in 20 *tornesi*). If instead the division had been performed first, it would have been easy to explain 9/7 to be the worth of 1 *tornese* in *parigini*, for which reason 20 *tornesi* must be worth $20\cdot(9/7)$ *parigini*. Alternatively, one might have explained that 20 *tornesi* must be worth 20/7 times as much as 7 *tornesi*, and hence $(20/7)\cdot 9$. These methods are not totally absent from the abbacus record, but they are uncommon.[26]

In a somewhat similar vein, the Pythagorean rule is always presented as a naked rule, and the perimeter of the circle is simply stated to be $3^1/_7$ times the diameter.

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to this.

25 Still other treatises are less ordered problem collections. Finally, some of the "abbacus books" are not treatises at all in any proper sense but private notebooks.

26 In contrast, the latter method is so common in Arabic treatises belonging to the same genre that it has a specific name, namely "by *nisba*" ("relation", specifically "ratio"); thus, for instance, al-Karajī in the *Kāfī* [ed., trans. Hochheim 1878: II, 17], who states that he prefers it over the (rule-of-three) solution by "multiplication and division".

This does not mean that the abbacus treatises contain nothing but isolated rules. Firstly, dressed problems have to be analyzed in such a way that they can be reduced to the application of a standard rule, which means that abbacus mathematics is argued though not as thoroughly so as philosophers or modern mathematicians might prefer; secondly, the rules may also serve in more theoretical contexts. For instance, when Dardi wants to show how to divide 8 by $3+\sqrt{4}$,[27] he first makes the calculation $(3+\sqrt{4})\cdot(3-\sqrt{4}) = 5$ and concludes that *5* divided by $3+\sqrt{4}$ gives *3–√4*. *What*, he next asks, will result if *8* is divided similarly, finding the answer by means of the rule of three (5, $3-\sqrt{4}$ and 8 being the three numbers involved).[28]

Even though abbacus mathematics does not in any way attempt to construct an axiomatic structure, these rules offered without proof but serving to justify other procedures function as axioms or postulates.[29] But this is not the sole expression of the norm that mathematics should be not only argued but also consistent.[30] Firstly, when different ways to solve a problem are presented, the identity of the outcomes may be followed by an explanation like the one Jacopo gives after having found the circular area first according to the normal "Arabic" formula $(1-\frac{1}{2}\cdot\frac{1}{7})$diameter^2 and next as (diameter×perimeter)/4 [ed. Høyrup 2007b: 352*f*]:

[27] Vatican manuscript, Chigi M.VIII.170, fol. 12^v.

[28] Similarly, the *Istratti di ragioni* [ed. Arrighi 1964: 26] from c. 1440, plausibly extracts from Paolo dell'Abbaco who wrote a century before, teaches how to divide $^4/_5$ by $^1/_3$ by means of the rule of three.

[29] A rather explicit and very simple instance of this function is found in Jacopo's *Tractatus algorismi* 15.2 [ed. Høyrup 2007b: 285] when the circle is treated:

> Always do, that when you know its circumference around, that is, its measure, and you want to know how much is its straight in middle, then divide its circumference by 3 and $^1/_7$. And that which results from it, so much will its diameter be, that is, the straight in middle. And similarly when you know the straight in middle of a circumference and you want to know in how much it goes around, then multiply the straight in middle by 3 and $^1/_7$, and as much as it makes, in so much does the said round go around. And if you should want to know for which cause you divide and multiply by 3 and $^1/_7$, then I say to you that the reason is that every round of whatever measure it might be is around {...} 3 times and $^1/_7$ as much as is its diameter, that is, the straight in middle. And for this cause you have to multiply and divide as I have said to you above.

[30] There are indeed good reasons to maintain that being reasoned is an "institutional imperative" [Merton 1942] for any institutionalized and cognitively autonomous teaching of mathematics – see [Høyrup 2005a: 109–112].

> And you see that it becomes as the one above, which we make without knowing the circulation around, which is also *braccia* 44, and they become the same. And therefore I have made this beside that, so that you understand well one as well as the other, and that one as well as the other is a valid rule. And they go well.

Secondly, solutions to problems are regularly followed by numerical proofs (in the sense of "verifications"). At times these control directly that the application of the rule actually gives what is asked for; at times, however, only a more indirect control is possible, which means that the proof shows the compatibility of two approaches. In one such case, the Milan-Florence redaction of Jacopo's *Tractatus* observes [ed. Høyrup 2007b: 454] that "Thus we have made the alloy well, since we have found again the said 700 δ. It would have been a pity if we had found more or less".[31]

At times, the computations lead to approximate results – not only when the roots of non-square numbers are found but also, for instance, when discounting of a debt is computed by means of an iterative procedure or in application of *welsche Praktik* (a kind of combined division-cum-multiplication by stepwise emptying used in practical trade). In such cases I do not remember ever to have seen a proof. Consistency was apparently meant to be exact, and once approximations were made exactness could no longer be expected[32] – approximation, so to speak, was a one-way street leading away from the world of consistency toward measurement and business.

All in all (and many more arguments could be given from the texts), the following norms or expectations[33] can be seen to have regulated abbacus

[31] In modern elementary arithmetic we are accustomed to the need for rounding called forth by the use of decimal fractions, for which reason checks of many practical calculations will not be exact however correct the calculations. Since abbacus mathematics operated with genuine fractions it did not encounter that problem, and exactness was therefore possible.

[32] In particular, I have never seen an analogue of the reversal of the approximate determination of a diagonal in the Old Babylonian text BM 96957+VAT 6598 # xxv [ed. Robson 1999: 259], made by reversal of the approximation formula.

[33] "Norms or expectations": indeed, expectations concerning the *object* of the activity of abbacus masters ("mathematics") are involved along with norms for the way these masters *should act*. It might be better to speak of an "ideology", since an ideology is exactly to be characterized as an inextricable unity of descriptive and prescriptive (supposed) knowledge. Cf. [Høyrup 2000b: 342].

mathematics:[34]

– it should, *in so far as authors and users could do it respectively follow it*, be argued;
– it should be consistent;
– and it should be exact, unless some real-world application asked for approximation.

False rules revisited

How do these norms agree with the invention of the false algebraic rules? At the surface of things, not at all. Those of Gherardi can never have been argued in a pertinent way. Dardi's irregular rules were certainly derived from arguments, but arguments which could never be told publicly because they would show *how* restricted their validity was, while only Dardi and those who copied them from him reveal at all that their validity was restricted. They were never tested by the inventors (or if they were the inventors did not betray themselves by telling the negative outcome), so the consistency they fulfil is merely that of superficial similarity.

The display of wrong results is thus not to be understood (as is the sweeping unacknowledged copying from the writings of predecessors) as "what was generally done and accepted at the time / within the environment". Instead it should be understood as a parallel of scientific fraud nowadays, which also exists, *in spite of* its conflict with what is expected from its perpetrators, and in the likeness of the economic fraud of Kreuger and the managers of Enron and Parmalat.

The background is also the same. Abbacus masters were in liberal profession, and had to impress municipal authorities or the fathers of prospective students if they wanted to earn their living. That could at best be done by solving problems that were too difficult for competitors; the prestige of algebraic problem solving (see note 24) made it an adequate instrument in that fight for prestige, and the inability of the judges to distinguish gilt lead from gold made it profitable to choose the easy way of fraud.

However, the fraud could only succeed *because of* the existence of those very norms which it violated. The general predilection for exactness barred control

[34] It may seem somewhat circular to read norms from a text corpus and then (as we shall do) apply them in the understanding of the same corpus. However, the norms are read out of one part of the corpus (the scattered casual remarks, the basic level), and we shall discuss their impact on other parts of the corpus, in particular the algebra.

of the *approximate* validity of the false solutions, and faith was instilled by the expectation that abbacus authors had arguments for their mathematical claims even if their public – whether municipal councillors or fathers, perhaps even less brash competitors – felt themselves to be unable to follow these.[35] In the same way, Enron could generate faith by being the client of prestigious accountant firms like Arthur Andersen and PriceWaterhouseCooper,[36] and by being apparently successful operators on a market supposed to be transparent by nature even though common citizens cannot look through it.[37]

Norm systems, indeed, are double-edged. They keep together a social system and regulate the behaviour of most members of the system; but they also allow those who hide behind them without complying with them to be far more successful than they could have been without the trust of others in the system and its effectiveness – beyond regulation, norm systems provide expectations, namely regarding the behaviour of others. No Tartuffe without religion and reverence for it!

Understanding Giovanni, and understanding more through Giovanni

Giovanni di Davizzo apparently did non know that his marvellous complete group had "no existence, if not that on the paper", in Georg Cantor's vicious words [1895: 501] about Veronese's transfinite numbers. In so far he may have profited from the cover of the norm system without actually knowing that he disobeyed it. This is not very illuminating, scientific mistakes are made, and if nobody discovers them to *be* mistakes their authors may earn degrees, positions and prestige from them in good faith.

But there is something more to say about what Giovanni did. His expansion of the semi-group may be seen as a search for consistency – but then not only for consistency as a condition that had to be obeyed but as something which should be actively created. Since his scheme was taken over by others, a fair

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35 More or less in the same vein, readers of the present pages probably suppose that I have really consulted the unpublished manuscripts I quote and to which they have no access (I promise I have!).

36 See, for instance, [McNamee 2002].

37 This is one aspect of what Robert Merton [1973: 439–459] baptized the "Matthew effect". Similarly, because Cyril Burt was already famous when he started making his glaring statistical fabrications, for decades nobody noticed their character (admittedly, it also played a role that his "conclusions" – the intellectual superiority of the better classes – were politically convenient). Cf. [Kamin 1977, *passim*].

number of abbacus writers seem to have shared the norm that mathematical knowledge ought to expand – since the scheme was completely useless for practical as well as mathematically-theoretical purposes, they can have adopted it for no other reason. This norm agrees well with a passage in the introduction to Jacopo's *Tractatus* [ed. Høyrup 2007b: 195] (copied more often than any other introduction by other abbacus authors and thus likely to correspond to prevailing moods):

> ... by mind and good and subtle intelligence men make many investigations and compose many treatises which were not made by other people, and know to make many artifices and written arguments which for us bring to greater perfection things that were made by the first men.

This wish for expansion of the art throws further light on the creation of the false solutions: whereas being able to solve (or give pretended solutions) to complicated algebraic problems gave prestige, prestige (probably more prestige) was specifically conferred to those who expanded the reach of existing algebraic knowledge. This is also the reason that many historians of mathematics tend spontaneously to see the fraud as praiseworthy because of the cognitive ambition it reveals, as pointing toward the breakthroughs of del Ferro, Tartaglia and Cardano. However, we should rather reverse this verdict. Those who committed the fraud consciously had no ambition to expand knowledge, just as modern scientific swindlers they were parasites on the cognitive ambition of others. They gained their prestige because of an existing norm system but in fact, in so far as they succeeded in having their fraud accepted as good knowledge (and the abbacus frauds went undetected much longer than the Piltdown fabrication) they undermined the creation of genuine new knowledge.

The power of the norm system

Some palaeontologists doubted the Piltdown man from the very beginning, and in the end this notorious potpourri of man and ape was exposed. The invention of the "cube root with addition" shows us that not all abbacus authors believed in the Gherardi solution to equation (G2). Similarly, the reductions of other equation types in the anonymous treatise in which we find this peculiar "root" explained shows that the norm for expanding the art consistently could lead to genuinely extensions of mathematical insight – extensions which, when combined with the breakthrough of del Ferro etc., led to the solution of all cubics and quartics in the sixteenth century.

A similar argument could be made (now in contrast to Giovanni's "group") around the way the same text (like Pacioli in his *Summa*) correlates the algebraic powers with powers of a number (see note 22). This led directly toward the arithmetization of the sequence of such powers – for instance, Bombelli's arithmetical notation for powers, in which $\overset{n}{\smile}$ corresponds to our x^n.

Not to be contrasted with any fraud or fallacy is the use of purely formal algebraic operations – another consequence of the faith in the consistency and expandability of mathematics.

In the above-mentioned *Trattato dell'Alcibra amuchabile* from c. 1365 it is stated in direct words [ed. Simi 1994: 41*f*] that the addition $\frac{100}{a\ thing}+\frac{100}{a\ thing\ plus\ 5}$ is to be performed "in the mode of a fraction", explained with the parallel $\frac{24}{4}+\frac{24}{6}$. It is thus taken for granted that operations with algebraic expressions could be handled exactly as numbers, and thus that for instance the notation for fractions was a mere form that could be filled out by any contents, numerical as well as algebraic.[38] This formal use of the fraction notation could not be used by Dardi, since he had already chosen to use the same notation for multiples of *ç* (*censo*) and *c* (*cosa* / "thing"), writing the "denominator" below the "numerator" with a stroke in between – for instance, $\frac{10}{c}$ for "10 things".[39] Nor was the usage broadly accepted at first (nor understood by all those who copied material where it was used[40]). In the longer run, however, mathematical writers got accustomed to it, and when Viète makes use of it in his *In artem analyticen isagoge*

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[38] We take note that formal operations could be made without abbreviations, even though the introduction of standard abbreviations was a prerequisite for maturation of the technique. Even though letter abbreviations had been used by both scholastic philosophers and Jordanus de Nemore, the true precursors of later mathematical symbols are the standard abbreviations of abbacus algebra.

[39] This notation had to remain unproductive because Dardi did not see the fraction as a symbol for an arithmetical operation but linked it instead to something like the medieval *denominatio* for ratios, or (more likely) saw $/_3$ simply as an abbreviation for the (ordinal form of the) number 3. But it lived on for at least a century and a half alongside the formal operations, being still used in a German algebra from 1481 [Vogel 1981: 10].

[40] In the *Libro di conti e mercatanzie* from c. 1395 (see p. 7), 100 divided by a *thing* plus 5 is thus stated [ed. Gregori & Grugnetti 1998: 103] to be "$\frac{1}{1\ thing}$ and 5", but afterwards the operations – copied from elsewhere – are performed correctly. The same treatise, it should be noted, solves the problem $C = \alpha+\sqrt{\beta}$ by taking the root of α and $\sqrt{\beta}$ separately, claiming the solution to be $t = \sqrt{\alpha}+\sqrt{\sqrt{\beta}}$ [ed. Gregori & Grugnetti 1998: 115*f*].

[ed. van Schooten 1646: 7*f*], all he feels the need to explain is his geometrical interpretation – for instance, that $\frac{B\ cubus}{A\ plano}$ is “the latitude which B cube makes when applied to A plane”. When coming to the arithmetic of such fractions he just prescribes the customary operations for numerical fractions without mentioning this parallel as an argument – he appears simply to see no difference.

The norm system which governed the practice of abbacus mathematics was not identical with that of Greek-inspired Humanist and university mathematics, and could not be already because the practices they governed were different in spite of similarities. For instance, a request for exactness could not mean the same in numerical computation and in geometry made exclusively by ruler and compass.[41] But the two systems were sufficiently similar to one another to allow a merger, not only of the two types of mathematical knowledge but also of the two norm sets. We may remember that both Maurolico and Clavius in their voluminous production also wrote on abbacus matters although from the Humanist perspective, and that Clavius’s stance on the matter of exactness was more tolerant than that of, for example, Viète and the classicist Kepler, at least for a while.[42] Without the partial merger of norms for what constituted legitimate practice, seventeenth-century scientific mathematics would hardly have been able to integrate the tools created by abbacus algebra – and without the heritage from abbacus algebra, it would have remained restricted to the possibility of finding something more (perhaps brilliant, but not very much more) of the same kind with respect to the Greek heritage, just as had been the case for medieval Islamic theoretical geometry. The total transformation of the mathematical enterprise taking place from Descartes to (say) Bernoulli would not have been possible.

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[41] On the conflicts around the concept of exactness in the latter context, see [Bos 1993]. The conflict can also be seen when Viète – as much a Humanist mathematician as there ever was – insists on a meaningful geometric interpretation of the algebraic powers.

[42] See [Bos 1993: 33–35] for a convenient confrontation, and [Bos 2001: 159–166] for the details of Clavius’s “idealizations of practical methods”.

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