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Conceptual divergence—canons and taboos—and critique: reflections on explanatory categories

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Peter Damerow und Wolf Wucherpfennig gewidmet, auf Anlaß deren sechzigsten Geburtstage

Abstract

Since the late 19th century it has been regularly discussed whether, e.g., the ancient Egyptian way to deal with fractions or the Greek exclusion of fractions and unity from the realm of numbers was a mere matter of imperfect notations or due to genuine "conceptual divergence," that is, to a mathematical mode of thought that differed from ours. After a discussion of how the notion of a "mode of thought" can be made operational through the linking of concepts to mathematical operations and practices it is argued (1) that cases of conceptual divergence exist, but (2) that the discussion of notational imperfection versus conceptual divergence is none the less too simplistic, since differences may also be due to deliberate choices and exclusions on the part of the authors of the ancient texts—for instance because such a choice helps to fence off a profession, because it expresses appurtenance to a real or imagined tradition, or as a result of a *critique* in the Kantian sense, an elimination of expressions and forms of reasoning that are found theoretically incoherent. The argument is based throughout on historical examples.

1. Prolegomenon

In *Physics* IV.4, 212ᵃ5–6 [Hussey, 1993, 28], Aristotle states that *place* [τόπος] is "the limit of the surrounding body, at which it is in contact with that [body] which is surrounded"—in 212ᵃ20 modified into "unchangeable limit of what surrounds." In IV.11, 219ᵃ8–9 *time* is explained to be "either change [κίνησις] or some aspect of change" [Hussey, 1993, 43]—a point made in almost the same words in *Metaphysics* Λ, Chapter 6, 1071ᵇ10. The *Physics* passage goes on to argue that since (as just shown) it "is not change, it must be some aspect of change," and concludes in 220ᵃ3 that it is "the number of the motion" [ὁ τῆς φορᾶς ἀριθμός].

Nobody who has gone beyond an introductory course in the history of philosophy would get the idea that Aristotle thought so because he was unable to grasp space and time in (our) quasi-Newtonian way, as receptacles within which bodies are located but which themselves do not depend on the presence of such bodies. Regarding space he even explains (*Physics* IV.1, 208ᵇ33–35) that Hesiod thought "as most

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people do" that place "is prior to all things, since that, without which no other thing is, but which itself is without the others, must be first. (For place does not perish when the things in it cease to be)"—and in *Categories* 6, $5^{a}6$–13 [Ackrill, 1963, 13] he himself comes disturbingly close to these "vulgar" opinions[1] about space and time:[2]

> Time also and place are of this kind [continuous quantities]. For present time joins on to both past time and future time. Place, again, is one of the continuous quantities. For the parts of a body occupy some place, and they join together at a common boundary. So the parts of the place occupied by the various parts of the body, themselves join together at the same boundary at which the parts of the body do.

Aristotle did not reject the seemingly modern views of space and time because they could not be thought in his epoch; after many pages of arguments he rejects them in the *Physics* because he cannot make philosophical sense of them: how, indeed, can "place" be established unless it be with regard to something known, and how can time be understood if we do not observe or think of something changing with time? Aristotle's solutions certainly do not coincide with those offered by Berkeley, Kant, Mach, and Einstein, but on a general level the problems he tries to solve belong to the same family as theirs.[3]

The preceding remarks had to do with the history of natural philosophy. They may serve to introduce a similar problem for the historiography of mathematics. In this domain, the problem of past conceptual structures differing from ours has certainly been discussed for well over a century,[4] but the discussion has turned around a different pivot: were historical concepts really different and the historical actors unable to think or express themselves in our terms, or is everything just a question of terminology and notation? In the following I shall argue that this debate is unduly simplistic, and that more attentive reading of pre-modern sources reveals that early mathematical writers, and not only Aristotle, might have other reasons than failing conceptual capacity or inadequate terminology to think or express themselves in ways that differ from ours. However, since mathematical writers tend to use their concepts or at most to define them rather than analyzing them or explaining their *raison-d'être*, we rarely have anything similar to

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1 This is how such opinions were characterized by the schoolmen, who were no less familiar with them; see [Grant, 1981, 9].

2 I leave aside as immaterial for the present discussion the question of whether Aristotle's thought had developed over time or a different agenda made him speak differently.

3 Cf. Einstein's introduction of the problem of time and contemporaneity in Section 1 of his treatise "Zur Elektrodynamik bewegter Körper" [Einstein, 1905/1913, 28]:

> Wollen wir die *Bewegung* eines materiellen Punktes beschreiben, so geben wir die Werte seiner Koordinaten in Funktion der Zeit. Es ist nun wohl im Auge zu halten, daß eine derartige mathematische Beschreibung erst dann einen physikalischen Sinn hat, wenn man sich vorher darüber klar geworden ist, was hier unter "Zeit" verstanden wird. [...]. Wenn ich z. B. sage: "Jener Zug kommt hier um 7 Uhr an," so heißt dies etwa: "Das Zeigen des kleinen Zeigers meiner Uhr auf 7 und das Ankommen des Zuges sind gleichzeitige Ereignisse."

Up to this point, the main distinction between Aristotle's reference to motion and Einstein's reflections on the meaning of time consists in the latter's specification of the kind of moving object he refers to (viz. the pointer of a clock). Serious divergence between the two only starts five lines later, when the finite speed of light is taken into acount.

4 In the later 19th century, we have Rodet's attack on Eisenlohr's and Cantor's use of modern algebraic symbolism in their interpretation of the Rhind Mathematical Papyrus (see below) and the Zeuthen–Cantor debate [Lützen and Purkert, 1994] about the (il)legitimacy of the reading of the historical record as contemporary mathematics. In more recent decades, the still cited standard example is the Unguru [1975]–Weil [1978]–Freudenthal [1977] debate, with van der Waerden [1976] in an intermediate position not too different from Cantor's. More examples are referred to in the following.

Aristotle's many pages discussing the shortcomings of rival views to help us. We shall therefore start with some reflections on how to approach a "mathematical mode of thought."

Tools and mode of mathematical thought

A "mode of thought" is *prima facie* as intangible as a *Zeitgeist*, and claiming that the mathematics of an ancient culture was rooted in a distinct "mode of thought" therefore does not in itself assist us much in understanding whether, why, or in which respect this mathematics differed from ours—it amounts to little more than a reformulation of the same matter in more airy terms. Speaking of the mathematical "concepts" of the culture in question is somewhat less elusive, but concepts should not be identified with the mere words into which they are put. Disregarding general epistemological discussions we may start from the metaphor that a mathematical concept is *a tool*: a mental tool, but a tool only by being a tool for operations. The shared properties and conditions of the whole network of connected mathematical concepts with participating operations then characterize the corresponding mode of thought.

This statement remains pretty abstract, but may be elucidated by an example. If we want to know (or, perhaps better, *decide*) whether, for instance, late medieval *abbaco* treatises operate with a concept of "negative numbers" it is not enough to notice that they use the word *meno*; even the observation that they state the rule that *meno via meno fa più* does not suffice.[5] As it is made manifest by the general adequacy of the translation "less" for *meno*, the rule might simply refer to a notion of "subtractive" members of a polynomial.[6] We should rather observe whether "numbers *meno*" also occur as results, or the actual use is restricted to expressions "a and less b" where b is not (or cannot, if roots are involved, easily be seen to be) larger than a (a as well as b being "non-*meno*" numbers or roots); further, whether the rule is used not only in multiplications of polynomials but also when a polynomial involving members *meno* is subtracted from another polynomial. If one of these conditions is not fulfilled, the notion of "less" is so different from our conception of "negative numbers" that it is misleading rather than illuminating to identify the two.

We could be more restrictive and refuse to speak of "negative" numbers before we have replaced the idea of two categories of numbers—normal and *meno*—by a single category divided "in the middle" by 0; but we may also decide that the idea of two separate categories is nothing but another version of the concept. In any case, the two concepts or two versions of the concept are linked to different practices with appurtenant tools: *the two categories* to the practices of accounting and rhetorical equation algebra,[7] *the*

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[5] An early published appearance is in the *Trattato dell'alcibra amuchabile* from c. 1365 [Simi, 1994, 17]. The unpublished occurrence in the *Aliabraa argibra* (ms. Chigiana M VIII 170, fol. 5^{v}) is linked to the example $10-2$ times $10-2$ and may go back to c. 1340.

[6] A "subtractive" number or member of a polynomial is an "ordinary," that is, nonnegative, member of an arithmetical expression or a polynomial whose role it is to be subtracted—as 3 has to be subtracted in the expression $5-3$. As a practical category, subtractive numbers appear (together with the corresponding sign rule) in ordinary elementary school mathematics and often in introductory algebra long before negative numbers are introduced. Widespread lore notwithstanding, negative numbers are absent from Babylonian mathematics, but subtractive numbers are spoken of explicitly, see [Høyrup, 1993].

[7] We may of course remember Leonardo Fibonacci's observation in the *Flos* [Boncompagni, 1862, 238] that a certain problem "is insolvable, unless it is conceded that the first man has a debt," and the similar passage in his *Liber abbaci* [Boncompagni, 1857, 256] (here, as everywhere in the following where no other translator is identified, the translation is

single category to the new practices evolving around symbolic algebra, analytical geometry, and *analysis infinitorum*.[8]

We may further observe that the two-category version is not simply an incomplete version of the single number line but in itself a mental tool which enabled Cardano and Bombelli to accept the only slightly more "false" categories of imaginary and complex numbers,[9] in a way which would have been barred once the single-category understanding was established, and which was only opened again by the invention of the geometric representation and the formal operation with pairs of real numbers.

A similar example is offered by the discussion whether the ancient Greeks possessed a notion of "general fractions" or merely one of repeated aliquot parts. Does $\overset{5}{16}$ (Diophantos, *Arithmetica* II.8 [Tannery, 1893–1895, I, 93]),[10] mean $\frac{16}{5}$? Obviously yes, in the sense that this is the correct solution; but if we read the way such correct solutions are expressed in I.23 we see that $\frac{50}{23}$ appears as $\bar{\nu}\ \kappa\gamma^{\omega\nu}$, "50 of 23rds," and $\frac{150}{23}$ slightly later as "150 of the said part." Obviously, what *we* would express as $\frac{p}{q}$ is thought of as p times the qth part; this is different from the Pharaonic Egyptian canon, according to which each aliquot part should appear only once in a number, but it is not for that reason by necessity identical with our concept. In III.11, however, we also find that $30\frac{1}{4}$ times $\frac{41}{77}$ is $\overset{77}{1240\frac{1}{4}}$, and that the "part denominated by" $\frac{41}{77}$ is $\frac{77}{41}$.[11] It seems reasonable to assume that Diophantos's concept was richer in operational links than a mere heaping of identical aliquot parts would suggest, and that he presupposes a similar richer concept on the part of his reader; but since he does not tell how he operates, we still

mine); but the real argument for the link is and remains the correspondence between the sets of operations on possessions and debts or incomes and expenses, booked in separate columns, and the treatment of numbers simply and *meno*.

[8] Actually, the first explicit reference to the single category which I know about (though for integers only) goes back to 1544, thus antedating symbolic algebra proper as well as analytical geometry and *analysis infinitorum*. In the *Arithmetica integra*, Stifel [1544, 249^{r}] explains that the "fictitious numbers" are "below 0, that is, below nothing." But it is also explained that this fiction is introduced because of its "supreme utility in things mathematical," a claim that is illustrated by the transformation of the subtraction $(8+5)-(10+2)$ into $(8-2)-(10-5)$ (both expressed in schemes, not by means of the parentheses invented by Bombelli—but anyhow in an early form of symbolization if this is understood as a representation that allows operation directly at the level of nonverbal representatives); on the verso of the same folio, furthermore, we find explicitly the sequence $-3, -2, -1, 0, 1, 2, 3, 4, 5, 6$ counting the exponents of the geometrical progression $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, 32, 64$. This first beginning of the one-category view, like its later consolidation, is thus linked to considerations of consistency derived from intramathematical practice.

[9] The link is betrayed by the vocabularies they invent—for instance Bombelli's *più di meno* for $+\sqrt{-a}$ and *meno di meno* for $-\sqrt{-a}$. We might speak of "Papageno tolerance" ("Es gibt ja schwarze Vögel. Warum soll es nicht auch schwarze Menschen geben?"): the existence of one irregular category, once habit has made it acceptable, opens the mind to the acceptability of other aberrations of a similar kind even if they remain aberrations.

[10] Reviel Netz [2002] has observed that Tannery uses the stenographic symbols much more consistently in his edition than the manuscripts do, and that these latter do not use them in the same place. No manuscript, moreover, goes back to Diophantos's own epoch. Though complicated fractions and reciprocals are less liable to variation in this respect than the symbols ς and ↑ ("number" and "less," respectively), we should not feel too confident that the expressions appearing as shorthand symbols in the present argument were all written in that way by Diophantos himself, though some similar expressions certainly were. But the argument does not really hinge on the stenographic writings.

[11] In more detail: $\frac{1}{4}$ is written δ^{x} in agreement with the explanation given in the introduction (p. 6); the *arithmós* is found to be $\frac{41}{77}$, and the number which was posited as *arithmós*$^{\times}$ is then $\frac{77}{41}$.

cannot decide exactly how similar his "practical concept" was to ours.[12] Only in the late medieval *abbaco* treatises, where cross-multiplication and other arithmetical operations are explained in detail, and where polynomial denominators are involved, can we be sure that the concept is really close to ours.[13]

Structures of mathematical operations grow out of operations with tools in the proper sense: the manipulation of bamboo sticks on a counting board, geometrical construction on a dust abacus or paper, routines for accounting or for solving equations, etc. But they are never *identical* with the more or less structured set of operations with these tools but always contain both (*qua* abstractions) *less* and (*qua* intellectual elaborations) *more*: for instance, a diminished expense results in an increased possession, which corresponds to the rule that *meno*(*meno* α) is simply α—but no straightforward accounting operation corresponds to the rule that *meno via meno* is *più*. Therefore it cannot be excluded that mathematical conceptual structures that are fairly congruent with something *we* know grow out of manipulations of tools that are quite different from those from which we are now accustomed to see them evolve. Identifying underlying tools that differ from ours does not prove that the corresponding concepts were also fundamentally different.

This is exemplified by the Cardano–Bombelli and the post-Gauss notions of imaginary and complex numbers. Another example (which goes both ways) is this: A couple of years ago a "historically interested" mathematician (I shall leave him anonymous and hence for copyright reasons not quote his words) claimed in a discussion on a Web site that the Babylonians could hardly have failed to recognize the particular character of irrational square roots because they will have seen that the equally nonfinishing sexagesimal reciprocals of irregular numbers are periodical, and have to be so because of the finite number of possible remainders. He forgot that the structure on which he himself was first taught about irrational numbers—probably decimal fractions and operations with rational numbers—was not the one on which the Greeks developed their notion of magnitudes that "could not be spoken" or "were not in ratio." He failed to notice that only the existence of a distinction between rational and irrational magnitudes once developed in relation to a different set of operations makes the distinction between periodic and nonperiodical decimal fractions interesting. He presupposed (without knowing to presuppose anything) that the Babylonians divided by irregular numbers in a way that leaves successive remainders (he may have been right, but that is a different question and so far undecided); and he overlooked that the sources that elucidate the question—the few listings of the reciprocals of irregular numbers—all stop before getting to the point where periodicity shows up, with the sole exceptions of the reciprocals of 59, told to be 1 1 1, and of 1 1 (i.e., 61), told to be 59 59 (meaning $0;59,0,59$)—hardly cases that invite one to consider the total set of possible remainders. All in all, the partial agreement between the ancient and the modern concepts of irrationals (and between ancient and modern place value notations) has veiled the fact that the underlying sets of operations are different, and therefore invite different further extensions. Leaving the anonymous mathematician aside we may also note that the aim of *Elements* X is very hard to understand if one's concept of irrationals is based on decimal fractions, whereas—conversely—the Greek concept does not allow the formulation of the distinction between

[12] A similar conclusion is reached by Jean Christianidis [forthcoming], from analysis of the more complex calculations of IV.36, in which fractions denominated by binomials are added and multiplied. In this case, as pointed out by Christianidis, Diophantos does refer to a general rule for the addition of fractions (μόρια).

[13] Being unable to read the Indian texts I prefer nor to include the indubitably earlier Indian fractions in this discussion.

algebraic and transcendental irrationals (not to speak of the theorems about the different decimal-fraction convergence patterns of the two classes).

These hints and sketched arguments should suffice to illustrate, both the fertility of the claim that mathematical concepts and conceptual structures are formed in interaction with tools within a practice and the dilemma presented by the lack of clear one-to-one correspondences between practices and mathematical conceptual structures. If we leave out the epithet "mathematical," this is of course a well-established Hegelo-Marxist point of view.[14] They should also suffice to show that "conceptual divergence"—differences between concepts that cannot be reduced to more or less full development of the same ideal concept—is something that must be taken into account.

The latter point can hardly be considered a historiographic revolution. As mentioned above, attempts to trace the differences between foreign and familiar conceptual worlds are certainly not new within the history of mathematics. The linking of the divergence to different practices is less hackneyed[15]—the prevailing tendency has been to find the inner logic of a certain conceptual world and explain its character or limits from there (a somewhat circular argument). Moreover, claims that the mathematical concepts of other cultures differ from ours were mostly challenged by proponents of the view that mathematics is only plural from the grammatico-etymological point of view, and that differences are to be found at the level of notations, not of thought (apart from that increasing scope and sophistication of mathematical thought which nobody could or would deny). We shall encounter more examples of both views below.

However, the fruitfulness of the notion of conceptual divergence is no proof that it explains *all* differences between the ways ancient and more recent texts speak about what from a Zeuthen–Weil point of view is basically *the same* mathematics. Some examples will show that other factors have sometimes been in play.

Egyptian discussions

The historiography of Egyptian mathematics is a classical ground for fighting the battle about dissimilar concepts. The first skirmish, mentioned in passing above, was between Eisenlohr [1877] and Cantor [1880] on one side and Rodet [1881] on the other—the former two explaining the procedures of the *ʿḥʿ*-problems in the Rhind Mathematical Papyrus by means of symbolic first-degree algebra, the latter claiming that this betrayed the underlying thought of the Egyptian calculator and proposing (with ample references to premodern counterparts) the use of a single false position. This discussion went on for long—I shall only mention Peet's identification of Rodet's reference magnitude or *bloc extractif* with a common denominator [Peet, 1923, 18] and Neugebauer's arguments that this modernizing

14 However, insight into the connection between practices and conceptual structures certainly precedes both Marx and Hegel—in Goethe's *Wahlverwandschaften* (II.8; [Goethe, n.d., VIII, 149]), the romanticist *Gehilfe* describes the relation between the changing view of nature and general material conditions and corresponding economic practice as follows:

> Menschen, die ihren Grund und Boden zu nutzen genötigt sind, führen schon wieder Mauern um ihre Gärten auf, damit sie ihrer Erzeugnisse sicher seien. Daraus entsteht nach und nach eine neue Ansicht der Dinge. Das nützliche erhält wieder der Oberhand, und selbst der Vielbesitzende meint zuletzt auch, das alles nutzen zu müssen.

15 But see the articles in [Damerow and Lefèvre, 1981].

view "misunderstands the inner unity of Egyptian computation completely" [Neugebauer, 1934, 138*ff*, quotation 145].

Slightly later came discussions about the particular Egyptian way of expressing fractional quantities—strikingly different from ours yet coherently developed and hence apparently the best candidate for a way to think about numbers that disagrees with our ways without being merely incomplete. As a typical representative of the attempts to be loyal to the Egyptian pattern of thought we may quote Gardiner's *Egyptian Grammar* [1957, 196]:

> For the Egyptian the number following the word r had ordinal meaning [...]. As being the part which completed the row into one series of the number indicated, the Egyptian r-fraction was necessarily a fraction with, as we should say, unity as the numerator. To the Egyptian mind it would have seemed nonsense and self-contradictory to write r-7 4 or the like for 4/7; in any series of seven, only one part could be the seventh, namely that part which occupied the seventh place in the row of seven equal parts laid out for inspection. Nor would it have helped matters from the Egyptian point of view to have written  r-7 (+) r-7 (+) r-7 (+) r-7, a writing that would have likewise assumed that there could be more than one actual "seventh." Consequently, the Egyptian was reduced to expressing (e.g.) $\frac{4}{7}$ by $\frac{1}{2}$ (+) $\frac{1}{14}$.

Already Hultsch [1895, 9] had pointed out that what we regard as fractions with a numerator greater than 1 was "nach Ägyptischer Anschauung Vielheitstheilungen oder noch nicht zu Ende geführte Divisionen." Evidently the Egyptians knew how to express 4 as measured by 7, namely as a sum of aliquot parts; paraphrasing Hultsch one might say that $\frac{4}{7}$ could well be thought, but as *a problem*, whose *solution* was $\dot{2}\,\dot{14}$.[16]

Peet and others objected that this was a question of notation, not one of *conception*; in Peet's words [1923, 16], "the argument from what the [Egyptian] was capable of expressing in symbols to what he was capable of conceiving is a *non sequitur*, and the suggestion that his notation must surely have kept pace with his conception will fall on deaf ears in the case of those acquainted with the amazing conservatism of the Egyptian mind in every branch of life."

More compelling than such general appeals to the supposedly familiar character of the Egyptian mind (though scarcely ever noticed) were perhaps Vogel's objections. He pointed [1929, 43] at two telltale slips in RMP #81 (noted by Peet [1923, 123] in the translation but not commented upon by him): in one place the scribe writes $\dot{5}$ instead of $\dot{2}\,\dot{8}$, thus betraying that something like $\frac{5}{8}$ (probably 5 times $\dot{8}$) was on his mind; in the following line, $\dot{4}\,\dot{8}$ is replaced by 3, with the implication that he was thinking of $\frac{3}{8}$. Vogel and others also pointed out that the unhesitating doublings of aliquot parts with denominator $2n$ as $\dot{n}$ implies knowledge that $2\dot{n}(+)2\dot{n} = \dot{n}$, and thus presupposes some concept of $\frac{2}{p}$ that entails $\frac{2}{2n} = \dot{n}$.

In [1975], Silberman pointed out that a late Old Kingdom use of the aliquot part notation (in fact the *only* Old Kingdom instance of fractions beyond the "natural fractions" $\ddot{3}, \dot{3}, \dot{4}$, "$\frac{3}{4}$" and $\dot{6}$, as far as I am informed) registers a damage to a cup as being large $\dot{5}\,\dot{5}$ finger.[17] All in all it seems legitimate to conclude, not only that the Egyptians knew p:q as a problem, but also that they were able, so to speak, to manipulate this *problem* (presumably thought of as $\dot{q}\,\dot{q}\ldots\dot{q}$) as a *representative of the solution*, that is, as a number. But this observation does not change the fact that Middle Kingdom scribes refused to use

[16] In the manner of hieratic Egyptian I use a dot over a number to indicate the corresponding "weak" number or aliquot part.

[17] Silberman explains this as an instance of scribal ignorance, but in the context of Middle Kingdom mathematics the point is so fundamental that it would correspond to a modern accountant ignorant of the place value system (as I have pointed out at an earlier occasion). The text in question is published in [Posener-Kriéger and de Cenival, 1968], with fractions $\dot{4}$, $\dot{6}$, and $\dot{5}\,\dot{5}$ on plates 23–25; translations can be found in [Posener-Kriéger, 1976].

this kind of number when writing down a result. We shall return to the conclusions that may be drawn from this fact.

Babylonian mysteries

Discussions similar to those concerning the Egyptian "equations" and "fractions" are almost nonexistent in connection with Babylonian mathematics. Discussions there certainly are—but they have concerned the question whether Babylonian "algebra" was really algebra or not, and if not, whether it was a collection of empirical recipes or based on arguments hidden from view.[18] I shall not pursue these topics, as they are not very relevant for the present discussion.

However, statements about the particular mathematical mode of thought of the Babylonians have certainly been made.[19] One example is Vajman's explanation [1961, 100] of the custom of subtracting an entity before it is added elsewhere, which he saw as an expression of a primal "concrete" organization of thought not yet ready for abstraction: we cannot add something before it has been made available.

In [Høyrup, 1990, 264] I cited Vajman's observation and explained (away) the only exception I had noticed by then. More recently, however, a fuller survey of texts made me discover that the exceptions are numerous, and that there is a pattern in their distribution.

Let me first present an example where the rule is followed: the problem solved in the text YBC 6967 [Neugebauer and Sachs, 1945 [MCT], 129]. It deals with two numbers *igûm* and *igibûm* that belong together in the table of reciprocals (the names mean "the reciprocal" and "its reciprocal"; for short in the following, n and $\tilde{n}$), and whose product is hence 1 or (in the actual case) 60; their difference is told to be 7. The product is spoken of as a "surface," which allows the interpretation of the procedure which is shown in the diagram in Fig. 1. First, the excess of $\tilde{n}$ over n is bisected and moved around. This transforms the rectangle into a gnomon, which can be completed as a square by being joined to the smaller square $\square(3\frac{1}{2})$ which it encloses. The area of the completed square is $72\frac{1}{4}$ and its sides, both vertical and horizontal, hence $8\frac{1}{2}$; from the vertical side we now remove that part which was moved around, leaving $8\frac{1}{2} - 3\frac{1}{2} = 5$ as n; putting *the same piece* back into its original place and joining it to the horizontal side of the completed square gives us $\tilde{n} = 8\frac{1}{2} + 3\frac{1}{2} = 12$.

Rectangle problems where the sum of the two sides and not their difference is given together with the area do not require that *the same* piece be removed and joined. Here, as in all cases where there is no inner constraint (not least when independent variations of problems are listed in sequence), the Old Babylonian texts let addition precede subtraction exactly as we do. On the other hand, rectangle problems are not the only ones where concrete meaningfulness requires subtraction to precede addition. As an example I shall mention the first-degree problem VAT 8389 #1 [Neugebauer, 1935, 1935, 1937 [MKT], I, 317*f*], in which a field is divided into two partial fields. The rent per area unit for each partial field is given together with

[18] Until recently (and even today in much of the general literature), "Babylonian mathematics" was conventionally understood as an undifferentiated whole. The following regards only the mathematics of the Old Babylonian period (2000–1600 B.C.E.), during which the overwhelming majority of known texts were produced.

[19] Outside the domain of mathematical thought, similar statements have also called forth discussion. I shall restrict myself to mentioning [von Soden, 1936; Larsen, 1987].

the complete area and the difference between the total rents paid from the partial fields.[20] At first the two total rents are found under the assumption that the partial fields are equally large. The amount by which these hypothetical rents differ is too small, and the next step is to find how much must be transferred from one partial field to the other in order to give the required difference; this piece is then really transferred, first removed, then joined.

Other texts, also texts treating the same type of rectangle problem as YBC 6967, do not respect the principle. Often, after having found the side of the completed square, they have the abbreviated formula "join and remove"—at times expanded into "join to one, remove from the other"—and then state the two resulting values. Moreover, it turns out that several of the texts that use the short version of the formula belong to the earliest phase of Old Babylonian "second-degree algebra," being thus close to the adoption into the scribe school of a set of mathematical riddles treating of the sides and areas of squares and rectangles originally belonging to an environment of nonscribal surveyors; in contrast, all those texts that respect concrete meaningfulness are younger.[21]

The use of the concretely absurd ellipsis thus cannot be explained as a result of a century's school routine in which once concretely meaningful operations were worn down to their arithmetical essentials. Quite the reverse, it turns out to be the scribe school that *invented* concreteness, or made it a canonical rule—or rather, certain scribe schools that did so: texts from other late text groups do not respect the canon.

I just referred to Old Babylonian "algebra," claiming in the same breath that it deals with geometric problems. The ideas of a Babylonian "algebra" and of geometry did not originally go together. Neugebauer claimed already in 1935 [MKT II: 63*f*] that the "nonsensical" inhomogeneous additions of sides and areas found in many texts prove that the problems are numerical and the geometric appearance an external dress; the same argument was advanced for instance by van der Waerden [1962, 71*f*]. Neither drew any consequences from the fact that the texts in question regularly use two different words for addition and distinguish between the situations where one or the other should be used (though Neugebauer appears to have been fully aware of it). One (*wasābum*, "joining") is meant to be concretely meaningful; the other (*kamārum*, in general interpretation "to heap," "to accumulate") can be used to add together the measuring *numbers* of discordant entities—lengths and areas, areas and volumes, or men, working days, and bricks produced. As a rule, problem statements formulate the addition of such discordant magnitudes as a "heaping."

When *solving* problems in which, for instance, the "heap" of a square area and the corresponding side is given, the texts may then employ various devices in order to make the sum concretely and not only numerically meaningful and permit a geometrical procedure analogous to the one that was shown in Fig. 1. One major text (BM 13901, in MKT III, 1–5) refers to an entity called "*wāṣītum* 1," derived from a verb meaning "to protrude," "to go out." The side s is represented by a rectangle $\sqsubset\!\sqsupset(1, s)$, the other side of which is a line of length 1 protruding from the square—heavily drawn in Fig. 2.

[20] Vajman's primary example is a problem from the closely related text VAT 8391. As a second example he refers to YBC 6967.

[21] For brevity and in agreement with established tradition I use the term "algebra" about the solution of problems about square or rectangular areas and sides. It is immaterial for the present discussion whether and in which sense this usage is justified; if "algebra" began with Emmy Noether, as maintained by some mathematicians, then of course there was no "Babylonian algebra." For the relative dating of the Old Babylonian texts I refer to [Høyrup, 2000a], for the derivation of the "algebra" from a set of nonscribal mathematical riddles, for instance, to [Høyrup, 2001]. Both issues are also treated in [Høyrup, 2002].

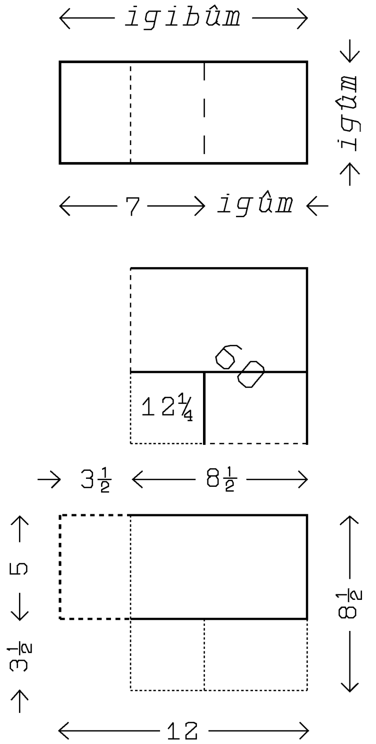

Fig. 1. The procedure of YBC 6967.

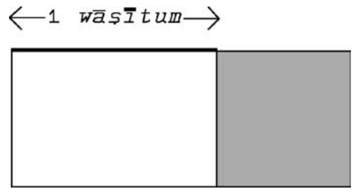

Fig. 2. The square area and the side provided with a *wasītum*.

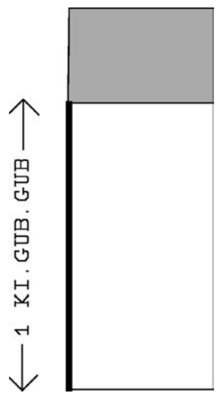

Fig. 3. The rectangle provided with a "base."

A similar trick but a different word is used in TMS IX #1,[22] in which the length is added to a rectangular area, and it is explained that this corresponds to the "joining" of a "base 1" (KI.GUB.GUB 1) to the width, see Fig. 3. A third trick, used in the text YBC 4714 #30–39, consists in introducing a "second" width of 25 in order give concrete meaning to the statement that the difference between the squares on the two sides of a rectangle is equal to 25 times the smaller of these sides.

However, the texts that avoid "joining" sides to areas and "heap" them instead are relatively late; those belonging to the earliest text groups express no scruple when "joining" sides to areas, and thus make implicit use of a notion of "broad lines," lines which on their own possess a virtual breadth of one length unit (an inherent "*wāsītum* 1"). Broad lines turn out to be wide spread in practical geometries, where the use of a fixed basic unit of length can be presupposed (we still sell cloth according to the same system).[23] They have always tended to disappear from more scholarly mathematics—as expressed by Plato (*Laws* VII, 819D–820B [Bury, 1926, II, 104–10]), the Greeks should be ashamed for being ignorant, not only of the problem of incommensurability of magnitudes of the same kind but also of the fact that lengths, surfaces, and solids are neither exactly nor "moderately" [ἠρέμα] commensurate—that is, for example, that a surface of "3 feet" (viz. in length, tacitly 1 foot broad) has no common measure with a line of 5 feet.

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[22] Text edition in [Bruins and Rutten, 1961, 63]—but see the corrections and the reinterpretation in [Høyrup, 1990, 320*f*; 2002, 89–95].

[23] See the discussion in [Høyrup, 1995].

The use of "heaping" of measuring numbers as a way to make sense of what the Old Babylonian school masters no less than Neugebauer would consider "nonsensical" joinings is thus another secondary development, a creation of the school; the nonscribal environment from which the problems were first taken over had no use for it. The various devices by which "broad lines" are transformed into rectangles whose lengths are the corresponding "Euclidean lines" (i.e., "lengths without breadth," in the words of a definition which is already found in Aristotle's *Topica* 143^{b}11 [Tredennick and Forster, 1960, 591]) were also inventions of the scribe school. According to the vacillating verbal expression of the same basic idea they were probably later than the elimination of broad lines through the distinction between "heaping" and "joining"—"heaping" is in general use throughout the later corpus and always spoken of in the same term.[24]

"Heaping" is certainly not the only term to be used without variation throughout the Old Babylonian corpus or most of it (actually, even the text groups and often those very texts that "join" lines to areas employ "heaping" for certain other additions); most of the essential terminology is shared, which is the main reason that the only consistent attempt made until recently to distinguish between separate text groups [Goetze, 1945] had to be based on orthographic criteria. But closer inspection reveals a number of subtle differences, of which I shall list some of the more significant:[25]

- In some text groups, the fact that (e.g.) 3 is the side of a square with area 9 is expressed in the Sumerian phrase "9.e 3 íb.si$_8$", "alongside 9, 3 is equal"; others, probably influenced by the use of tables of square roots, employ the Sumerian verb íb.si$_8$ as a noun (we may translate it "the equalside"), and state that "3 is the equalside of 9".
- In two (early) text groups, íb.si$_8$ is replaced occasionally or consistently by another conjugated form ba.si$_8$ of the same verb; in one of them this term is used as a verb, in the other as a noun. All other text groups use ba.si$_8$ only when sides of a cube or a rectangular prism are referred to and in more generalized functions.
- Some text groups (early as well as late) refer to "each" of the sides of a square or to "all four" sides—according to various criteria groups that remain close to that nonscribal geometrical practice that had once supplied the riddles; others avoid this usage consistently.
- Text groups from the periphery of the ancient Sumerian area—regions that had long been under the cultural influence of Sumer but had been subjected only briefly to the "Neo-Sumerian" empire in the 21st century B.C.E.—announce the appearance of a numerical result by saying that "you see" the number. All groups from what had once been the Sumerian core area avoid the phrase, with the exception of one very early group from Ur that sometimes uses the Sumerian equivalent pàd; they do so not in ignorance of existence of the expression—for instance, a question what to do "in order to see" the value of a magnitude[26] shows that the idea was familiar—but apparently as a consequence of deliberate choice. Some of the core groups state that a result "comes up," one group that the calculation "gives" it. In all groups but this one, "giving" occurs exclusively in connection with numerical calculations within the sexagesimal system. Nine (early) texts from Tell Harmal in Eshnunna in the periphery, all of them found in the same room, use "seeing" in problems linked to

24 Counting the Akkadian verb *kamārum* in syllabic writing and the two logographic writings of the term (g̃ar.g̃ar and UL.GAR) as one.

25 I refer again to [Høyrup, 2000a] and to the more matured discussion in [Høyrup, 2002, 317–361].

26 YBC 4608, obv. 22, 28 [MCT, 49*ff*].

the riddle tradition and "coming up" in problems belonging with traditional scribal computation, and couples the two terms consistently to different ways to ask for values; texts from other localities, and even one from a neighboring room, confirm the historical affiliations of "coming up" and "seeing" but reveal that the linking between a problem's "home tradition" and its way to ask questions is mistaken, and should have been turned upside down.

– The nine texts that couple the way to announce results, the way to ask questions, and the "home tradition" of problems start all problems with the formula "If somebody asks you thus, ..." This obvious borrowing from the riddle tradition (used, however, also for problems with a different historical affiliation) is present in a few texts from the same region, but never appears elsewhere, *except* in one text that uses it in abbreviated form in a single problem which also on several other counts can be demonstrated to be a folkloristic citation of nonschool usage. Once again the formula (which survives within the practical-geometrical tradition until the late Middle Ages together with "each" or "all four" sides of a square) is seen to have been known, and its absence from the texts thus to be a result of filtering.
– Some text groups invariably start the prescription by a formula "You, by your proceeding"; others restrict themselves to a terse "You"; still others omit the opening formula altogether.
– Early texts often employ two terms for removal, one (*nasāḫum*) meaning "to tear out," the other (*ḫarāsum*) "to cut off," or use the latter only; if making use of both, they tend to "cut" from lines and "tear" from areas; the first of these verbs possesses a Sumerian logographic equivalent (zi), the second not (which implies that it will have belonged to the nonscribal tradition). Later groups eliminate "cutting."

These and a number of similar observations demonstrate that the Babylonian school masters were actively engaged in the creation of canons for *how mathematics should present itself*, tabooing alternatives; they also show that different schools—even schools located within the same town and active during the same decades, as revealed by comparison of different texts from Tell Harmal in Eshnunna—did not agree fully on what was canonical. Some of the choices may have aimed at fencing off the school from nonschool practice—thus the avoidance of "seeing" in texts from the core, the elimination of "cutting" and of "each side"/"all four sides," and the ousting of the riddle formula "If somebody ... ." Others—for instance the deliberate linking of the ways to ask questions and announce results—seem to reflect a wish to keep traditions alive in memory (mathematicians of our days are not the first to be "historically interested," nor are they the first to reinvent history). Still others—the use or nonuse of introductory formulae in prescriptions, the generalization of the riddle formula to all problems types in one group and of "giving" to all kinds of resulting in another—can hardly be seen as anything but *stylistic* choices.

All of these possibilities—fencing off, *Traditionspflege*, style—belong to the category with which postmodernist history of science has been much concerned, those characteristics which are "shared by science and organized crime," in a locution borrowed from the Popper–Kuhn exchange in [Lakatos and Musgrave, 1974]. The "rule of concreteness" and the elimination/justification of the "broad lines" represent a different category, that cardinal virtue on account of which, in Benjamin Farrington's words, science cannot be ethically neutral but "must be true."[27] More precisely, they are instances of *critique* in a Kantian sense, asking for the *extent to which* and the *conditions under which* what is usually

[27] [Farrington, 1938, 437]. Even this norm is, of course, *qua* institutional imperative, essential to the long-term "career strategy" of the scientific institution as such; but since postmodern science studies committed academic parricide on Robert

done or believed is justified—“Untersuchung der Möglichkeit und Grenzen derselben,” “examination of its possibility and limits,” in Kant’s words (*Critik der Urtheilskraft*, B III [1956–1964, V, 237]). Explaining the solution of the *igûm–igibûm* problem as done above,[28] the teachers can be supposed to have discovered that the moving back of the piece that had been displaced contradicted the precise wording of the traditional formula “join and remove,” and that they had to turn the phrase around; that they also expanded it is likely to reflect a will to emphasize the concrete meaningfulness of the operation. Similarly, the school environment will have made the notion of the broad line implausible or outright inconceivable.[29] At an early moment, this will have induced the schoolmasters to separate the statement of a problem, made in terms of the “heaping” of measurable numbers, from the geometric procedure by which it was solved; in the latter, the sides could be represented by rectangles. Later, various schools invented (each in its own words) ways to justify the trick of the procedure.

The geometry of *Elements* II.1–10 can be understood as a critique of the cut-and-paste procedures of the surveying tradition. These propositions presuppose the *definition* of what a right angle is (likely never to have been discussed by the practical geometers of earlier times, who will have had no difficulty in distinguishing a good from a skew corner) as well as the *postulate* that was necessitated by this definition (since it turned out not to be self-evident from the definition that all right angles are equal). On this basis, the proof of II.6 (which we may take as our prototype) constructs the rectangles and squares of Fig. 1 meticulously and shows the necessary equalities; in this way the text shows that what had “always” been done is indeed justifiable on the best theoretical foundations. This corresponds to a general characteristic of Greek philosophy, and vindicates the view that the “Greek miracle” consisted to a large extent in this kind of critical questioning. As we see, however, critique was no Greek privilege but also undertaken by the Old Babylonian schoolmasters. They did not make a critique for all times to come, and Euclid had his role to play. But after Hilbert’s *Grundlagen* the fleeting character of every critique should no longer come as a surprise: even Euclid’s critique turned out to be in need of re-critique[30]—and Kant’s *Critik der Urtheilskraft* is in a way a critique of the preceding critiques of pure and practical reason.

Merton as part of their own career strategy, this has largely escaped their attention. Moreover, the fact that even Old Babylonian scribe school teachers felt obliged toward it shows that this norm, just as any other moral norm, was a generalization that went beyond what immediate utility and self-interest seemed to ask for (in their case, a generalization from the institutional obligations to find the correct result and to be able to teach efficiently how to find it).

[28] Most of the Babylonian mathematical texts are parsimonious in giving such explanations (though less so than was believed as long as the whole terminology was interpreted in a purely arithmetical key), but several texts from Susa contain didactical expositions (for instance, expounding the use of the KI.GUB.GUB); a few texts from other regions contain rudiments of a similar pedagogy, confirming the hunch of Neugebauer and others that the texts went together with oral instruction explicating the meaning and purpose of steps). See [Høyrup, 2002, 85 and *passim*].

[29] This may have been a consequence of the teaching of the topic, of the need to have a particular notion of the line of *no* breadth; in similar torment when adding “roots” to “squares,” Pedro Nuñez [1567, fols. 6^r, 232^r] had to explain that roots are to be understood as rectangles whose width is “la unidad lineal.” It may, however, also be correlated to the cognitive organization of the Mesopotamian school since its fourth-millennium beginning around what Luria [1976, 48*ff*] calls “categorical classification,” in contradistinction to his “situational thinking,” see [Høyrup, 2000b, 16]. “Situational thinking,” mental organization of the world in terms of customary and invariable situations, is indeed a generalized correlate of the presupposition of the “broad line”: that everybody knows and agrees what *the* standard breadth has to be.

[30] For this Hilbert was of course only needed in view of the ever-recurrent returns of didactics to Euclid as the supreme model; much of the medieval commentary tradition, Islamic as well as Latin, had already submitted the holy text to critical desacralization.

Summing up the observations made here on the Old Babylonian material we may conclude that much of what the texts do not say or do not do must be explained, not from what their authors *could not think* but instead in terms, either of *what they did not find it professionally fitting to say*, or of *what they found it incoherent to say*.[31] Thus, in both cases, of what they *refused to say*. Better, perhaps, in terms of what they *refused to write down*—some of the slips suggest that they may have used the tabooed language in their oral expositions.

Egyptian flashback

With this in mind we may return to the question of the Egyptian canon. If the Egyptians knew to treat the problem $p{:}q$ as "a *representative of the solution*, that is, as a number" but "refused to use this kind of number when stating a result," then we are again confronted not with a case of what the calculators were *unable to think* but of what they *refused to write down*.

Even in this case, the canon is likely to have been produced by the school. Firstly, there is the argument *post hoc, ergo propter hoc*: the scribe school only replaced master–apprenticeship teaching at the onset of the Middle Kingdom [Brunner, 1957, 11–15], that is, at the dividing point between the Old Kingdom irregular $\dot{5}\,\dot{5}$ and the canonical expression of fractional quantities in Middle Kingdom mathematical and administrative papyri. Second, third millennium computation had made use of subunits instead of fractions, which is indeed much more convenient for practical purposes; but subunits presuppose rounding and thus preclude the teacher's unambiguous decision whether "you have found it correctly" (the recurrent phrase from the teacher's annotations to the Moscow Mathematical Papyrus [Struve, 1930]). Since the full and systematic unfolding of the unit fraction *system* in the Middle Kingdom thus corresponded to *a need* that only came into being through the emergence of the school, it is likely to have been *brought about* by the school—and with this system, in which denominators might go into the hundreds or even further, repetitive writings of p times $\dot{q}$, in the vein of $\dot{5}\,\dot{5}$, were certainly neither practical nor practicable.

But this can hardly be the *raison d'être* of the canon. For the higher numerals, the Egyptians made use of multiplicative writings much in the manner of Diophantos, first eliminating in this way the unit 1,000,000 and next also 100,000 [Sethe, 1916, 9].[32] The Egyptians clearly *could* think in this way if they wanted to. Gardiner knew so—this kind of multiplicative writing is precisely what is meant by his "r-7 4 or the like." The hieratic slips $\dot{5}$ and 3 in RMP #81 show that they actually *did* think like this on occasion.

Why then? It is not to be excluded that Gardiner got a point, and that the Egyptian school masters when figuring out what could be meant by an aliquot part q explained it in a way that precluded that more than

[31] My impression from the texts that were used in school to inculcate professional attitudes and self-importance ("examination texts" and proverbs dealing with scribes) is that intellectual coherence was no part of the explicit norms regarding what was professionally fitting. But not all norms are in need of being made explicit: few of us ever had to be told that it is unfitting to eat your soup with your feet on the table—it is as self-defeating as teaching mathematics through incoherent explanations.

[32] For instance, 27,000,000 could be written as 270 below the sign for 100,000, and 40,000 as 4 below the sign for 10,000; as in Diophantos, we see, the unit which is counted ("the denominator") is written above the number counting it ("the numerator").

Even in Jacopo of Florence's *Tractatus algorismi* from 1307 [Høyrup, 1999, 6], the same notation is used when the meaning of the Hindu–Arabic numerals is explained, "700" being for instance explained as $\genfrac{}{}{0pt}{}{c}{vii}$ and "400000" as $\genfrac{}{}{0pt}{}{m}{cccc}$. I shall leave aside as undecidable the question whether this constitutes a case of borrowing (through channels unknown to us) or of independent invention and thus evidence that the notation falls "naturally."

one copy could legitimately be present. It may also have to do with the computational technique and its use of repeated doublings, as proposed by van der Waerden [1937–1938, 361] and accepted by Clagett [1999, 25]. We cannot know, nor can we exclude the possibility that both explanations are wrong and that a third motive has to be looked for. In any case, the canon was the outcome of *deliberate choice*, not of mental divergence.

Greek "numbers"

Nobody suspects that the ancient Greeks made their geometry in the Euclidean manner because they were intellectually incapable of thinking in more heuristic ways. For this, the testimonials of Greek heuristic thinking are too copious. The only account where mental inability has been imputed to the Greek geometers is Sabetai Unguru's rejection (1975) of the idea that the real reasoning of *Elements* II, *Elements* X, and Apollonios's *Conics* is algebraic. I see no reason to challenge Unguru's arguments.

When it comes to Greek theoretical *arithmetic*, however, claims about the limits or distinctiveness of Greek thought abound. As is known, the *arithmói* of Greek arithmetic, translated "numbers," are supposed to be the integers 2, 3, 4, ...—1 being the "root of number" but no number itself. This is born out by numerous passages in Aristotle's *Metaphysics*, at times as a plain and obvious fact, at times as something which "is said" or "said by some"; it is stated less clearly in *Elements* VII, deff. 1–2; and it was repeated countless times until Boethius (and, in the wake of the latter, another set of countless times until the Renaissance). Fractions, of course, are no *arithmói*.

Here, it is often claimed (names and exact quotations are omitted for reasons of charity) that the Greeks *could not* think otherwise. Since they understood number as a "collection of units," they "failed to understand" that 1 is a number.

Several fallacies are involved. Firstly, endemic preaching against sin is evidence of the existence of endemic sin, not of virtue; no ancient Greek writer ever asserted that "nothing" is not a number, because this was not an idea that would ever come to him. If it was necessary to explain so often that unity was no number, then the temptation must have been great to see it as one. That unity and "numbers" were treated together and on a par in practical reckoning is obvious and may already suffice to explain from where temptation might come. But we do not need to leave the domain of the theoreticians. Reading one definition further in *Elements* VII we find a definition of "being a part" which presupposes that the part is a number;[33] accordingly, 2 is a part of 12 (the 6th part), but 1 is not a part of 6 if the definitions are taken seriously.

Discussions about the legitimacy of definitions always tend to become futile, and we might well allow Euclid this quirk. But definitions have consequences, and this one has the consequence that the parts of 6 are only 2 and 3, for which reason proposition IX.36 about perfect numbers becomes false. Obviously Euclid did not mean exactly what he said, but rather that "unity or a smaller number is a part of a

[33] Μέρος έστὶν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος, ὅταν καταμετρῇ τὸν μείζονα—"A number is a part of a number, the smaller of the larger, if it measures the larger" [Heiberg, 1883–1888, II, 184]. It is not said explicitly that only numbers can be parts, but no other definition states which other kinds of parts exist. In consequence, 1 cannot be meant to be a part of any number if not, at least for practical purposes, a number itself.

larger number if it measures it"; the slip is not serious *unless* we believe that there was a fundamental difference—which we may conclude that there was not.[34]

The case of fractions is no different. Again, accountants would certainly divide unity "in many parts." This, indeed, is Socrates's complaint in the *Republic* (525D–526A [Shorey, 1930, II, 162–164]). But so did Diophantos—his $\frac{77}{1240\frac{1}{4}}$ is the answer to a request for *a number*.

We are forced to conclude once again that the conceptual otherness that is reflected in the sermons about the nature of number is not caused by any inability to think otherwise; the sermons censure an ever-recurrent tendency to neglect in mathematical practice taboos resulting from philosophical critique. This critique (maybe Pythagorean, maybe not) had once asked what number *really is* (a question which practical reckoners may never have asked, knowing number too well as the stuff they were always dealing with); it had been found that the only justifiable answer was that number was a *collection of units*. But learning thus what number really was entailed learning also what it *could not possibly*, and therefore *should not* be.[35] The Greek mathematicians has some difficulties in taking to heart the latter part of the lesson, as we have seen.

This was certainly not the last time in history that a mathematician cared more about conquering new mathematical ground (whether insights or results) than about consolidating philosophically what he already possessed. Accumulating a treasure in Popper's Third world often involves paying a price in the first, and even the philosopher may sometimes be in doubt whether the gain outweighs the loss: Kierkegaard, as is known, tried to reestablish a bond with his former fiancée Regine once she had found a husband who did not write monumental books about why he should forsake her. Within mathematics, the price to be paid for respecting a critical taboo is the blocking of such further conceptual and operational developments as might spring from active use of the tabooed concepts or operations. The prevailing Greek proscription of fractions affected the development of arithmetic until the 17th century and beyond (even after being rediscovered Diophantos remained peripheral on this account, modern "Diophantine" analysis treating of integers), just as the Egyptian outlawing of those repeated aliquot fractions which Greek calculators accepted prevented them from preparing the further steps we find with Diophantos; in other words, tabooing at one level may produce genuine conceptual divergence at the following. On the other hand, forgetting that *one* should be excluded from the realm of number along with fractions enabled the Greeks to do things that otherwise might have been forbiddingly cumbersome.

These observations suggest a further twist to the preceding argument. As argued in the bulk of the article, the absence of such conceptualizations from ancient sources as a modern mathematical reader might expect to find there does not prove that the ancient authors *were not able* to think more or less in our patterns—it may also be due to an explicit rejection of this way of thinking, either because of the existence of some canon or because they deemed it conceptually incoherent. Only close analysis of

[34] Van der Waerden [1962, 108] was thus in excellent company when treating as mere "quibbles" the distinction between unity and numbers—quibbles with which there was no reason to burden an introduction to Greek arithmetic (as distinct from Greek philosophy of mathematics).

[35] We may remember Vogel's demonstration [1936] that the whole terminology for ratios—claimed not to be *numbers* but *relations between pairs of numbers* (tacitly including unity)—was derived from the terminology for fractions, in a way that shows ratios to be a way to save fractions in a philosophically acceptable way once they had been outlawed as numbers. It was a fortunate accident that this concept could later be extended beyond the scope of fractions, once the "ineffable" ratios turned up—those ratios that had no possible name within a language forged after the practice of fractions. In the modern epoch the notion of decimal fractions (and later the critiques of Dedekind and others) reopened the gates to the realm of numbers for both.

the sources at large will, in the best of cases, allow us to distinguish between cognitive divergence and cognitive proscription. Conversely, however, the absence of critique where we would like to find it does not necessarily imply that the ancient authors were unable to realize the inconsistencies in what they did; they may have decided that they did not care about critical hairsplitting as long as things worked as they should—anticipating thus d'Alembert's famous recommendation to carry on work in infinitesimal calculus and his conviction that faith would result from success or habit.

Note added in proof

Since an already published reference to the above paper (Reviel Netz, "It's not that they couldn't," *Revue d'Histoire des Mathématiques* 8 (2002), 263–289, here p. 284) shows a possible misreading, I would like to add that taboos in ancient mathematical writings are no more "external to the mathematical thinking itself" than the quest for rigor is to 19th-century mathematics; first, they characterize it just as much as the theorems or rules it knows. Second, in pre-Modern as well as recent times, the taboos and other "second-level" choices characterizing one period may determine the questions that arise and the way they are approached, and thus co-determine the development of "first-level" mathematics in ensuing periods.

References

Ackrill, J.L., 1963. Aristotle, *Categories* and *De interpretatione*. Clarendon, Oxford.

Boncompagni, Baldassare (Ed.), 1857. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. I. Il *Liber abaci* di Leonardo Pisano. Tipografia delle Scienze Matematiche e Fisiche, Rome.

Boncompagni, Baldassare (Ed.), 1862. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. II. *Practica geometriae* et *Opusculi*. Tipografia delle Scienze Matematiche e Fisiche, Rome.

Bruins, E.M., Rutten, M., 1961. Textes mathématiques de Suse, Mémoires de la Mission Archéologique en Iran, XXXIV. Paul Geuthner, Paris.

Brunner, Hellmut, 1957. Altägyptische Erziehung. Otto Harrassowitz, Wiesbaden.

Bury, R.G. (Ed., Transl.), 1926. Plato, *Laws*. In Two Volumes. In: Loeb Classical Library. Harvard Univ. Press, London/Heinemann/Cambridge, MA.

Cantor, Moritz, 1880. Vorlesungen über Geschichte der Mathematik, Erster Band. Teubner, Leipzig.

Christianidis, Jean, forthcoming. Did the Greeks have the notion of common fraction and did they use it? In: Christianidis, Jean (Ed.), A Century of Greek Mathematics. Classics in the Twentieth Century Historiography, In: Boston Studies in the Philosophy of Science. Kluwer Academic, Dordrecht, Introduction to Chapter V.

Clagett, Marshall, 1999. Ancient Egyptian Science. A Source Book. Volume III. Ancient Egyptian Mathematics. In: Mem. Am. Philos. Soc. No. 232. Am. Philos. Society, Philadelphia.

Damerow, Peter, Lefèvre, Wolfgang (Eds.), 1981. Rechenstein, Experiment, Sprache. Historische Fallstudien zur Entstehung der exakten Wissenschaften. Klett-Cotta, Stuttgart.

Einstein, Albert, 1905/1913. Zur Elektrodynamik bewegter Körper. In: Lorentz, H.A., Einstein, A., Minkowski, H. (Eds.), Das Relativitätsprinzip, Eine Sammlung von Abhandlungen. In: Fortschritte der mathematischen Wissenschaften, vol. 2. Teubner, Leipzig/Berlin. Annalen der Physik 17 (1905).

Eisenlohr, A., 1877. Ein mathematisches Handbuch der alten Ägypter (papyrus Rhind des British Museum) übersetzt und erklärt. Leipzig.

Farrington, Benjamin, 1937, 1938. Prometheus bound: government and science in classical antiquity. Science and Society 2, 435–447.

Freudenthal, Hans, 1977. What is algebra and what has it been in history? Arch. Hist. Exact Sci. 16, 189–200.

Gardiner, Alan, 1957. Egyptian Grammar. Being an Introduction to the Study of Hieroglyphs, third ed., revised. Griffith Institute, Ashmolean Museum, Oxford.

Goethe, Johann Wolfgang von, n.d. Werke. Auswahl in zehn Teilen. Auf Grund der Hempelschen Ausgabe neu herausgegeben von Karl Alt. Deutsches Verlagshaus Bong and Co, Berlin, etc.

Goetze, Albrecht, 1945. The Akkadian dialects of the Old Babylonian mathematical texts. In: Neugebauer, O., Sachs, A. (Eds.), Mathematical Cuneiform Texts. In: American Oriental Series, vol. 29. American Oriental Society, New Haven, CT, pp. 146–151.

Grant, Edward, 1981. Much Ado About Nothing: Theories of Space and Vacuum from the Middle Ages to the Scientific Revolution. Cambridge Univ. Press, Cambridge, UK.

Heiberg, J.L. (Ed., Transl.), 1883. Euclidis *Elementa* (Euclidis Opera omnia, vols. I–V). Teubner, Leipzig.

Høyrup, Jens, 1990. Algebra and naive geometry: an investigation of some basic aspects of Old Babylonian mathematical thought. Altorientalische Forschungen 17, 27–69, 262–354.

Høyrup, Jens, 1993. On subtractive operations, subtractive numbers, and purportedly negative numbers in Old Babylonian mathematics. Zeitschrift für Assyriologie und Vorderasiatische Archäologie 83, 42–60.

Høyrup, Jens, 1995. Linee larghe. Un'ambiguità geometrica dimenticata. Boll. Storia Sci. Mat. 15, 3–14.

Høyrup, Jens, 1999. VAT. LAT. 4826: Jacopo da Firenze, *Tractatus algorismi*. Preliminary transcription of the manuscript, with occasional commentaries. Filosofi og Videnskabsteori på Roskilde Universitetscenter. 3. Række: Preprints og Reprints 1999, No. 3.

Høyrup, Jens, 2000a. The finer structure of the Old Babylonian mathematical corpus: elements of classification, with some resultats. In: Marzahn, Joachim, Neumann, Hans (Eds.), Assyriologica et Semitica, Festschrift für Joachim Oelsner anläßlich seines 65. Geburtstages am 18. Februar 1997, Münster 1999. In: Altes Orient und Altes Testament, vol. 252. Ugarit, Münster, pp. 117–177.

Høyrup, Jens, 2000b. Human Sciences: Reappraising the Humanities through History and Philosophy. State Univ. of New York Press, Albany, NY.

Høyrup, Jens, 2001. On a collection of geometrical riddles and their role in the shaping of four to six 'algebras'. Science in Context 14, 85–131.

Høyrup, Jens, 2002. Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin, Studies and Sources in the History of Mathematics and Physical Sciences. Springer, New York.

Hultsch, Friedrich, 1895. Die Elemente der ägyptischen Theilungsrechnung: Erste Anhandlung, Abhandlungen der philologisch-historischen Classe der Königlich-Sächsischen Gesellschaft der Wissenschaften, 17,1. Leipzig.

Hussey, Edward (Ed., Transl.), 1993. Aristotle's *Physics* Books III and IV. Clarendon, Oxford.

Kant, Immanuel, 1956–1964. Werke. Insel Verlag, Wiesbaden.

Lakatos, Imre, Musgrave, Alan (Eds.), 1974. Criticism and the Growth of Knowledge. Proc. Internat. Colloq. Philos. Sci., London, 1965, vol. 4. Corrected reprint. Cambridge Univ. Press, Cambridge, UK.

Larsen, Mogens Trolle, 1987. The Mesopotamian lukewarm mind: reflections on science, divination and literacy. In: Rochberg-Halton, F. (Ed.), Language, Literature, and History: Philological and Historical Studies presented to Erica Reiner. In: American Oriental Series, vol. 67. American Oriental Society, New Haven, CT, pp. 203–225.

Luria, Aleksandr R., 1976. Cognitive Development. Its Cultural and Social Foundations, Edited by Michael Cole. Harvard Univ. Press/Nauka, Cambridge, MA/London/Moscow, 1974.

Lützen, Jesper, Purkert, Walter, 1994. Conflicting tendencies in the historiography of mathematics: M. Cantor and H.G. Zeuthen. In: Knobloch, Eberhard, Rowe, David (Eds.), The History of Modern Mathematics, vol. III, Images, Ideas, and Communities. Academic Press, Boston, pp. 1–42.

Netz, Reviel, 2002. Paper presented at the Workshop Histoire et historiographie de la démonstration mathématique dans les traditions anciennes, Columbia University Institute for Scholars at Reid Hall, Paris, 17.–19.5.2002.

Neugebauer, Otto, 1934. Vorlesungen über Geschichte der antiken mathematischen Wissenschaften. I. Vorgriechische Mathematik. In: Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. XLIII. Julius Springer, Berlin.

Neugebauer, O., 1935, 1935, 1937. Mathematische Keilschrift-Texte, I–III. Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3. Band, erster–dritter Teil. Julius Springer, Berlin [Reprint Berlin, etc.: Springer, 1973].

Neugebauer, O., Sachs, A., 1945. Mathematical Cuneiform Texts. In: American Oriental Series, vol. 29. American Oriental Society, New Haven, CT.

Nuñez, Pedro, 1567. Libro de Algebra en Arithmetica y Geometria. Anvers: En casa de los herederos d'Arnaldo Birckman.

Peet, T. Eric, 1923. The Rhind Mathematical Papyrus, British Museum 10057 and 10058. Introduction, Transcription, Translation and Commentary. Univ. Press of Liverpool, London.

Posener-Kriéger, Paule, de Cenival, Jean Louis, 1968. Hieratic Papyri in the British Museum. Fifth Series. The Abu Sir Papyri. Edited, together with Complementary Texts in Other Collections. The Trustees of the British Museum, London.

Posener-Kriéger, Paule, 1976. Les Archives du temple funéraire de Néferirkare-kakaï (les papyrus d'Abousir). Traduction et commentaire. Institut Français d'Archéologie Orientale du Caire, Paris.

Rodet, Léon, 1881. Les prétendus problèmes d'algèbre du manuel du calculateur égyptien (Papyrus Rhind). Journal asiatique, septième série 18, 184–232, 390–559.

Sethe, Kurt, 1916. Von Zahlen und Zahlworten bei den Alten Ägyptern, und was für andere Völker und Sprachen daraus zu lernen ist. In: Schriften der Wissenschaftlichen Gesellschaft in Straßburg, 25. Heft. Trübner, Straßburg.

Shorey, Paul (Ed., Transl.), 1930. Plato, *The Republic*. With an English Translation. In: Loeb Classical Library, Nos. 237, 276. Harvard Univ. Press, London/Heinemann/Cambridge, MA.

Silberman, David P., 1975. Fractions in the Abu Sir Papyri. Journal of Egyptian Archaeology 61, 248–249.

Simi, Annalisa (Ed.), 1994. Anonimo (Sect. XIV), *Trattato dell'alcibra amuchabile* dal Codice Ricc. 2263 della Biblioteca Riccardiana di Firenze. In: Quaderni del Centro Studi della Matematica Medioevale, 22. Servizio Editoriale dell' Università di Siena, Siena.

Stifel, Michael, 1544. Arithmetica integra. Petreius, Nürnberg.

Struve, W.W., 1930. Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau, Herausgegeben und Kommentiert, Quellen und Studien zur Geschichte der Mathematik, Abteilung A: Quellen, 1. Band. Julius Springer, Berlin.

Tannery, Paul (Ed., Transl.), 1893–1895. Diophanti Alexandrini *Opera omnia* cum graecis commentariis. Teubner, Leipzig.

Tredennick, Hugh, Forster, E.S. (Ed., Transl.), 1960. Aristotle, *Posterior Analytics* and *Topica*. In: Loeb Classical Library, No. 391. Harvard Univ. Press, London/Heinemann/Cambridge, MA.

Unguru, Sabetai, 1975. On the need to rewrite the history of Greek mathematics. Arch. Hist. Exact Sci. 15, 67–114.

Vajman, A.A., 1961. Šumero-vavilonskaja matematika. III–I Tysjačeletija do n.e. Izdatel'stvo Vostočnoj Literatury, Moscow.

van der Waerden, B.L., 1937–1938. Die Entstehungsgeschichte der ägyptischen Bruchrechnung. Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung B: Studien 4, 359–382.

van der Waerden, B.L., 1962. Science Awakening. Noordhoff, Groningen.

van der Waerden, B.L., 1976. Defence of a 'shocking' point of view. Arch. Hist. Exact Sci. 15, 199–210.

Vogel, Kurt, 1929. Die Grundlagen der ägyptischen Arithmetik, Dissertation, Munich.

Vogel, Kurt, 1936. Beiträge zur griechischen Logistik. Erster Theil. Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayerischen Akademie der Wissenschaften zu München. pp. 357–472.

von Soden, Wolfram, 1936. Leistung und Grenze sumerischer und babylonischer Wissenschaft. Die Welt als Geschichte 2, 411–464, 507–557.

Weil, André, 1978. Who betrayed Euclid? Arch. Hist. Exact Sci. 19, 91–93.