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Jacopo da Firenze and the beginning of Italian vernacular algebra

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To Mohamed Souissi and *in memoriam* Khalil Jaouiche

Abstract

In 1307, a certain Jacopo da Firenze wrote in Montpellier a *Tractatus algorismi* that contains the earliest extant algebra in a European vernacular and probably, as is argued, the first algebra in vernacular Italian. Analysis of the text shows that it cannot descend from any of the algebras written in Latin, nor from any published Arabic treatise, for which reason it presents us with evidence for a so far unexplored level of Arabic algebra. Further, since it contains no Arabisms, it must build on an already existing Romance-speaking environment engaged in algebra. Comparison with other Italian algebras written during the next 40 years show that all are linked to Jacopo or to this environment (perhaps Catalan) and disconnected from Leonardo Fibonacci's *Liber abbaci*.

Sommario

Nel 1307, un certo Jacopo da Firenze scrisse a Montpellier un *Tractatus algorismi* che contiene la prima presentazione sopravvissuta dell'algebra in un volgare europeo – probabilmente la prima presentazione in volgare italiano in assoluto. L'analisi del testo dimostra che l'algebra di Jacopo non è basata su nessuno dagli scritti algebrici latini, e neanche su un trattato arabo pubblicato; è dunque una testimonianza di un livello finora inesplorato dell'algebra araba. D'altra parte, Jacopo non utilizza un solo arabismo, e deve dunque aver preso la sua ispirazione da un ambiente di lingua romanza. Un'ispezione attenta di altri scritti algebrici italiani risalenti alla prima metà del Trecento svela che tutti sono legati a Jacopo o a questo ambiente (possibilmente catalano) e che nessuno ha legami con il *Liber abbaci* di Leonardo Fibonacci.

Keywords: Jacopo de Firenze; Dardi da Pisa; Giovanni di Davizzo; Algebra, Arabic; Algebra, Italian medieval; Montpellier; Catalonia

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First discovery of a central source—and neglect

In [1929], Louis Karpinski published a short description of "The Italian Arithmetic and Algebra of Master Jacob of Florence, 1307"—the "Jacopo da Firenze" of the present paper. Among other things he pointed out that the algebra chapter of the treatise in question—written according to its incipit in Montpellier in September 1307—presents the algebraic "cases" (the fundamental first and second-degree equations) in a different order than al-Khwārizmī, Abū Kāmil, and Leonardo Fibonacci, and that the examples that follow the rules are also different than those of the same predecessors. Karpinski did not state explicitly that Jacopo offers no geometric proofs of the rules, nor that the examples that illustrate these rules differ from those of the other authors already in general style, not only in detailed contents; but attentive reading of Karpinski's text and excerpts from the manuscript leave little doubt on either account.

In retrospect, these discrepancies should have made historians of late medieval algebra aware that the reception of Arabic *al-jabr* may have been more complex than assumed so far. However, I have not been able to discover any echo whatsoever of Karpinski's publication. Actually, everybody interested in the development of Christian-European algebra before the late 16th century conserved for decades the undisturbed conviction that a single line of development led from the Latin presentations of the subject (the translations of al-Khwārizmī and the last part of Fibonacci's *Liber abbaci*) to Luca Pacioli, Cardano, and Tartaglia.

This conviction still prevailed in 1997 when I inspected the algebra section of the Vatican manuscript of Jacopo's treatise (the manuscript used by Karpinski) in order to verify my hunch that it might be very different from the just-mentioned Latin presentations of the subject. Since this inspection showed Jacopo's algebra to be even more different from the Latin precursors than I had suspected, I set on to prepare an edition of it, which appeared as [Høyrup, 2000]. In order to be sure that the algebra in question was really due to Jacopo and not a later interpolation in a manuscript copy from c. 1450, I further undertook a detailed comparison of the Vatican manuscript with another manuscript that also claims to be Jacopo's *Tractatus*, and from which the algebra chapter is absent. The results of this comparison were presented in preliminary form at the meeting "Commerce et mathématiques du moyen âge à la renaissance, autour de la Méditerranée," Beaumont de Lomagne, 13–16 May 1999, and published in the proceedings of this meeting [Høyrup, 2001]. Since this volume appears to have reached nobody but the contributors,[1] I integrate a partial recapitulation of what I reported on that occasion in the present complete presentation.

The initial neglect of Karpinski's article may be due to at least three interacting causes.

First, 1929 fell in a period where the interest in European medieval mathematics was at a low ebb—probably the lowest since the Middle Ages, lowest at least since 1840. From 1920 to c. 1948 (from the death of Moritz Cantor to the beginning of Marshall Clagett's work in the field), the total number of scholarly publications dealing with Latin and European-vernacular mathematics does not go much beyond a dozen.

Second, the existence of the distinct *abbaco* mathematical tradition was not recognized, although Karpinski had already described another *abbaco* treatise in [1910]. As early as 1900, it is true, Cantor [1900, 166] had spoken of the existence throughout the 14th century of two coexisting "schools" of

[1] I was unable to locate it in the online catalogs of the Bibliothèque Nationale, the British Library, the Library of Congress, and the Deutsche Bibliothek. One French university library (Lille) knew about the volume but had been unable to get hold of it.

mathematics, one "geistlich" ("clerical," i.e., universitarian), the other "weltlich oder kaufmännisch" ("secular or commercial" and supposedly derived from Leonardo Fibonacci's work); part of Cantor's basis for this (but only a modest part) was Libri's edition [1838–1841, III, 302–349] of a major section of what has now been recognized as Piero della Francesca's *Trattato d'abaco*[2] (which Cantor, accepting Libri's wrong dating, had located in the fourteenth century). Eneström [1906] had done what he could to ridicule Cantor's claim about the existence of a separate school of commercial mathematics by twisting his words.[3] Sensitive reading would easily have exposed Eneström's arrogant fraud; but the kind of knowledge that would have been required for such a reading had come to be deemed irrelevant for historians of mathematics and hence forgotten, and Sarton [1931, 612*f*] not only cites Eneström's article but embraces the whole thesis uncritically.

Third, like Cantor, Karpinski took the continuity from Fibonacci onward for granted, and concluded on p. 177 that the

> treatise by Jacob of Florence, like the similar arithmetic of Calandri, marks little advance on the arithmetic and algebra of Leonard of Pisa. The work indicates the type of problems which continued current in Italy during the thirteenth to the fifteenth and even sixteenth centuries, stimulating abler students than this Jacob to researches which bore fruit in the sixteenth century in the achievements of Scipione del Ferro, Ferrari, Tartaglia, Cardan and Bombelli.

Only those interested in manifestations of mathematical stagnation—thus Karpinski invited readers to conclude—would gain anything from looking deeper into Jacopo's treatise.

The manuscripts of Jacopo's *Tractatus*

Whatever the reason, nobody seems to have taken an interest in the treatise before Warren Van Egmond inspected it in the mid-seventies during the preparation of his global survey of Italian Renaissance manuscripts concerned with practical mathematics [1976; 1980]. By then, the autonomous existence of the *abbaco* tradition in the 14th and 15th centuries was well established; but Van Egmond noticed that the manuscript that Karpinski had examined (Vatican ms. Vat. Lat. 4826, henceforth **V**) could be dated by watermarks to the mid-15th century, and that the algebra chapter (and certain other matters) was missing from two other manuscripts containing Jacopo's *Tractatus algorismi* (Florence, Riccardiana Ms. 2236, henceforth **F**; and Milan, Trivulziana Ms. 90, henceforth **M**).[4] Because **M** can be dated by watermarks

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2 On the identification of Libri's manuscript with the very manuscript from which Arrighi made his edition [1970], see [Davis, 1977, 22*f*].

3 Arguing from his own blunt ignorance of the institution within which university mathematicians moved, Eneström rejected the epithet "clerical" as absurd ("Sacrobosco und Dominicus Clavasio waren meines Wissens nicht Geistliche"; actually, all university scholars were at least in lower holy orders, as evident from the familiar fact that they were submitted to canonical jurisdiction). Because Fibonacci is supposed to be spoken of as a merchant only in late and unreliable sources (it was no part of Cantor's argument that he was one, although Cantor does refer to him elsewhere in pseudo-poetical allusions as the "learned merchant"—pp. 85*f*, 154; yet in the very preface to the *Liber abbaci* Fibonacci speaks of his commercial traveling), and because merchants' mathematics teaching was supposed never to treat fanciful problems such as the "100 fowls," no "commercial" school could have been inspired by Fibonacci and teach such useless problems.

4 An edition of **F** was prepared by Annalisa Simi [1995]. A critical edition of **F** and **M** by the late Jean Cassinet and Annalisa Simi has not yet appeared; for the moment, **M** is inaccessible, but the description in [Van Egmond, 1980, 166] confirms what I was told by Jean Cassinet in 1999, namely that the differences between **F** and **M** are minor.

to c. 1410, some 40 years before **V**, and since **V** contains rules for the fourth degree not present in the algebra of Paolo Gherardi's *Libro di ragioni* from 1328, Van Egmond decided [personal communication] "that the algebra section of Vat. Lat. 4826 [was] a late 14th-century algebra text that [had] been inserted into a copy of Jacopo's early 14th-century algorism by a mid-15th-century copyist."

Close textual examination of **V** shows that this manuscript is very coherent not only in style but also regarding the presence of various characteristic features in the chapters that are shared with **F** as well as in those that are not; **F**, on the other hand, is less coherent.[5] Van Egmond's explanation of the differences between the two versions must therefore be turned around: **V** is a quite faithful descendant of Jacopo's original (or at least of the common archetype for **F** and **V**), whereas **F** (and its cousin **M**) is the outcome of a process of rewriting and abridgement, an adapted version apparently meant to correspond to the curriculum of the abbacus school as described in a document from Pisa from c. 1430 [ed. Arrighi, 1967] and in a Florentine contract from 1519 [ed. Goldthwaite, 1972, 421–425].

Internal evidence shows that **V** is a meticulously made (but not a blameless) library copy made from another meticulous copy[6]; seeming setoffs from Provençal orthography suggest that preceding steps in the copying process (if any there are) can have been no less meticulous.[7] All in all it is thus legitimate to treat **V** as identical with Jacopo's treatise from 1307 apart from minor errors and a few omissions.

Jacopo's algebra

The algebra section proper of **V** runs from fol. 36^{v} to fol. 43^{r}. It is followed by an alligation problem about grain which is solved without algebra, and four problems which *we* would consider algebraic but whose solutions do not make use of *cosa*, *censo* (the terms representing the first and second power of the algebraic unknown; cf. Appendix A), etc. Like the algebra section proper, these problems are absent

[5] See [Høyrup, 2001]. Repetition of the details of the extensive argument would lead too far; but let me list a few points that on the whole speak for themselves:

- In one place **F** refers to a diagram that is only present in **V**.
- In another problem, the illustrating diagram in **F** is so fancifully different from what is needed that Simi inserts a "(sic!)"; the diagram in **V** corresponds to the description of the situation in both texts.
- One problem in **F** starts "egli è uno terreno lo qual è ampio 12 braccia, cioè uno muro, et è alto braccia 7 ed è grosso braccia 1 et 1/4"; the counterpart in **V** starts "egli è uno muro, el quale è lungho 12 braccia e alto sette. Et grosso uno et 1/4." The solution in both speaks of the wall presented in **V**.
- **V** states regularly that the first-order approximation to an irrational square root is approximate, and regularly also gives a (mistaken but easily explainable) second-order approximation. Occasionally, **F** also mentions the approximate character of the first-order formula; but in one place it believes it may be exact, while in another it mixes up the wrong second-order formula found in **V** with a correct formula, which makes the whole thing quite nonsensical.
- In **V**, the commercial partnership serves (both in sections that have a counterpart in **F** and in those that have none, for instance, in the algebra) as a general model for proportional partition; in **F**, this trick is mostly avoided—but in one place it is not. As we shall see, descendant treatises show that the algebra section in **V** must antedate 1328 by so much that 1307 seems a quite reasonable date.

[6] On fol. 46^{v} we find what according to its contents is a marginal note indicating that the list of silver coins has been forgotten by mistake and comes later. But the note is *not* in the margin but within the normal text frame, which shows it to have been copied.

[7] In one place, moreover, the text of **V** should transform $4\sqrt{54}$ into a pure square root; instead we find a blank, and in the margin the words "così stava nel'originale spatii." Obviously, the author did not want to compute 16×54 mentally but postponed—and forgot; and all intermediate copyists have conserved the blank.

from **F** and **M**. One of the latter four problems is of the third degree and is solved correctly even though it cannot be reduced to a second-degree or a homogeneous problem. We shall return to this group below.

The rules

The algebra section proper gives rules for the following cases—C stands for *censo*, t for *thing* (*cosa*), n for *number* (*numero*), K for *cube* (*cubo*), CC for *censo di censo*, i.e., the fourth power of t[8]:

(1) $\alpha t = n$	(3) $\alpha C = \beta t$	(5) $\beta t = \alpha C + n$
(2) $\alpha C = n$	(4) $\alpha C + \beta t = n$	(6) $\alpha C = \beta t + n$
(7) $\alpha K = n$	(12) $\alpha K = \beta C + \gamma t$	(17) $\alpha CC + \beta K = \gamma C$
(8) $\alpha K = \beta t$	(13) $\alpha CC = n$	(18) $\beta K = \alpha CC + \gamma C$
(9) $\alpha K = \beta C$	(14) $\alpha CC = \beta t$	(19) $\alpha CC = \beta K + \gamma C$
(10) $\alpha K + \beta C = \gamma t$	(15) $\alpha CC = \beta C$	(20) $\alpha CC + \beta C = n$
(11) $\beta C = \alpha K + \gamma t$	(16) $\alpha CC = \beta K$	

The first six cases are the traditional first- and second-degree cases, familiar since al-Khwārizmī's *Kitāb al-jabr wa'l-muqābala*. The remaining ones are all reducible to homogeneous problems or second-degree problems, and thus nothing new compared to what had been done in the Arabic world since centuries. As already mentioned, the order of the six fundamental cases differs, both from that of al-Khwārizmī (extant Arabic text as well as Latin translations) and Abū Kāmil (both have 3-2-1-4-5-6) and from that of Fibonacci (who has 3-2-1-4-6-5). Jacopo's higher cases, as we see, are ordered group-wise according to the same principles as the groups (2)–(3) and (4)–(5)–(6).[9]

Another noteworthy characteristic is that all cases are defined as non-normalized problems (that is, the coefficient of the highest power is not supposed to be 1), the first step of each rule thus being a normalization.[10] In the Latin treatises, all cases except "roots equal number" (where the normalized equation *is* the solution[11]) are defined as normalized problems, and the rules are formulated correspondingly.[12] (All

[8] The Latin translations of al-Khwārizmī (but not the *Liber abbaci*) would refer to the numbers as *dragmas*, but this idiom is absent from Jacopo's formulation of the rules. Similarly, Jacopo refers to the first power of the unknown as *thing* (*cosa*), never as *root* (*radix*), as do the Latin treatises (including the *Liber abbaci*) when stating the rules.

Appendix A contains translations of select passages from al-Khwārizmī's and Jacopo's algebras that exemplify these differences. The selections may also introduce readers who only know Arabic and medieval algebra from symbolic translations to the original style and the basic terminology.

[9] That is, $\alpha x^n = \beta x^p$, n fixed (either 3 or 4), p increasing from 0 to $n - 1$, and equation groups obtained from the group (4)–(6) by multiplication by x or x^2. According to this principle, (20), the biquadratic obtained from (4), should obviously be followed by two other biquadratic equations, (21*) $\beta C = \alpha CC + n$ and (22*) $\alpha CC = \beta C + n$. Jacopo must have intended to include one of them, since the text announces "15 rules which [...] lead back to the six rules from before." Whether he forgot himself or the omission was due to an early copyist cannot be decided (a late copyist can be excluded; see footnote 35).

[10] The rule for the third case thus says that "when the *censi* are equal to the number, one shall divide the number by the *censi*. And the root of that which results from it is the thing."

[11] Fibonacci actually defines even this case in normalized form—but gives no example and thus escapes the absurdity.

[12] Here and elsewhere I disregard the brief excerpts "de libro qui dicitur gleba mutabilia" in *Liber Alchorizmi de pratica arismetice* [ed. Boncompagni, 1857b, 112*f*]. They are not in Allard's partial edition of the *Liber Alchorizmi* [1992], but they are present in manuscripts that are as distant from each other in the stemma as possible—see [Høyrup, 1998b, 16, n.7]; there is thus no doubt that they were present in the original and have not been interpolated. But the few paragraphs in question can hardly count as a presentation of the field and appear to have had no impact whatsoever.

also teach how to proceed when a non-normalized problem is encountered, but this is done outside the regime of rules—see the example in Appendix A.)

The examples

For each of the first six cases, Jacopo gives at least one, sometimes two or three examples after setting forth the rule in abstract form. For the remaining cases, only the rules and no examples are given. In translation[13] the statements of these problems run as follows:

1a. Make two parts of 10 for me, so that when the larger is divided into[14] the smaller, 100 results from it.
1b. There are three partners, who have gained 30 libre. The first partner put in 10 *libre*. The second put in 20 *libre*.[15] The third put in so much that 15 *libre* of this gain was due to him. I want to know how much the third partner put in, and how much gain is due to (each) one of those two other partners.
2. Find me two numbers that are in the same proportion as is 2 of 3: and when each (of them) is multiplied by itself, and one multiplication is subtracted from the other, 20 remains. I want to know which are these numbers.
3. Find me 2 numbers that are in the same proportion as is 4 of 9. And when one is multiplied against the other, it makes as much as when they are joined together. I want to know which are these numbers.
4a. Someone lent to another 100 *libre* at the term of 2 years, to make (up at) the end of year.[16] And when it came to the end of the two years, then that one gave back to him *libre* 150. I want to know at which rate the *libra* was lent a month.
4b. There are two men that have *denari*. The first says to the second, if you gave me 14 of your *denari*, and I threw them together with mine, I should have 4 times as much as you. The second says to the first: if you gave me the root of your *denari*, I should have 30 *denari*. I want to know how much each man had.
5a. Make two parts of 10 for me, so that when the larger is multiplied against the smaller, it shall make 20. I ask how much each part will be.
5b. Somebody makes two voyages, and in the first voyage he gains 12. And in the second voyage he gains at that same rate as he did in the first. And when his voyages were completed, he found himself with 54, gains and capital together. I want to know with how much he set out.[17]
5c. Make two parts of 10 for me, so that when one is multiplied against the other and above the said multiplication is joined the difference which there is from one part to the other, it makes 22.[18]
6. Somebody has 40 *fiorini* of gold and changed them to *venetiani*. And then from those *venetiani* he grasped 60 and changed them back into *fiorini* at one *venetiano* more per *fiorino* than he changed them at first for me. And when he has changed thus, that one found that the *venetiani* which remained with him when he detracted 60, and the *fiorini* he got for the 60 *venetiani*, joined together made 100. I want to know how much was worth the *fiorino* in *venetiani*.

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13 Here as everywhere in the following, translations into English are mine if nothing else is indicated. For the present list (and everywhere below where it is adequate) I use the very literal translation from [Høyrup, 2000] with minor emendations.
14 Cf. below, footnote 31, about "division into."
15 The *libra* (*lira* in many contemporary and in later Italian texts) is a monetary unit. It is divided into 20 *soldi*, each being worth 12 *denari*—cf. the recent British pound–shilling–penny system.
16 That is, at compound interest, computed yearly.
17 Both solutions are shown to be valid.
18 This example serves to demonstrate that one of the two solutions may be false (unless, as we would say, the difference between the two numbers can be counted as negative).

The first observation to make is that none of Jacopo's problems are stated in terms of numbers, *things*, and *censi* (afterward, of course, a "position" is made identifying some magnitude with the *thing*; without this position, no reduction to the corresponding case could result). In the Latin treatises, in contrast, the first examples are always stated directly in the same number–roots–*census* terms as the rules.

Second, we notice that three of Jacopo's pure-number examples (*viz* 1a, 5a, and 6a) follow the pattern of the "divided ten," familiar since al-Khwārizmī's treatise and abundantly represented in the *Liber abbaci*. Others, however, are of a type with no such precedent: those where the ratio between two unknown numbers is given.[19] For any given polynomial equation with a single unknown it is of course easy to create an example of this kind, thereby adding cheap seeming complexity.

Further, we should be struck by the abundant presence of problems (5 out of 10) that pretend to deal with commercial questions—mu'āmalāt-problems ("problems dealing with social life"), in the classification of Arabic mathematics. The only problem belonging to this category that we find in the Latin algebra translations is the one where a given sum of money is distributed evenly first among an unknown number x of people, next among $x+1$ [ed. Hughes, 1986, 255], with a given difference between the shares in the two situations. Among the problems treated in the algebra section of the *Liber abbaci* at most some 8% belong to the *mu'āmalāt* category: four variants of the problem type just mentioned, one problem treating of the purchase of unspecified goods, and one referring to interest and commercial profit.

Finally, we should take notice of the presence of the square root of an amount of real money in (4b); this is without parallel even in the nonalgebraic chapters of the *Liber abbaci*, where *mu'āmalāt* problems abound.[20]

Peculiar methods

In the main, the methods used by Jacopo of course coincide with what we know from the Latin works. But some differences can be observed here and there. We may look at the solution to (1b)—a paradigmatic example of how to break a butterfly on the algebraic wheel—in which several idiosyncrasies are represented (fols 36^{v}–37^{r}):

> Do thus, if we want to know how much the third partner put in, posit that the third put in a thing. Next one shall aggregate that which the first and the second put in, that is, *libre* 10 and *libre* 20, which are 30. And you will get that there are three partners, and that the first puts in the partnership 10 *libre*. The second puts in 20 *libre*. The third puts in a thing. So that the principal of the partnership is 30 *libre* and a thing. And they have gained 30 *libre*. Now if we want to know how much of this gain is due to the third partner, when we have posited that he put in a thing, then you ought to multiply a thing times that which they have gained,

[19] There is an analogue of Jacopo's superficially similar problem (1a) in al-Khwārizmī's treatise [ed. Hughes, 1986, 248], repeated by Abū Kāmil [ed. Sesiano, 1993, 360]; but like Jacopo's (1a) these problems speak of division, not of "proportion," and like Jacopo's they are primarily divided-ten problems.

[20] The problems in *Liber abbaci*, Chapter 12, Part 3 ("Questions of trees and similar things"), that involve square roots all treat of *numbers*: "On finding a certain number, of which 1/6 1/5 1/4 1/3 of the same is the root of the same number" [ed. Boncompagni, 1857a, 175], etc.

Elsewhere in the medieval world, problems involving the square roots of real entities may go together with problems that consider their product—thus in Mahāvīra's *Gaṇita-sāra-sangraha* [ed. Raṅgācārya, 1912, 75–85]. Of this type, several specimens are present in the *Liber abbaci*, namely a number of problems about three or five men finding *bizanti* respectively having *denari*, where relations between the *products* of their possessions taken pairwise are given [ed. Boncompagni, 1857a, 204–206, 281].

> and divide into the total principal of the partnership. And therefore we have to multiply 30 times a thing. It makes 30 things, which you ought to divide into the principal of the partnership, that is, 30 and a thing, and that which results from it, as much is due to the third partner. And this we do not need to divide, because we know that 15 *libre* of it is due to him. And therefore multiply 15 times 30 and a thing. It makes 450 and 15 things. Hence 450 numbers and 15 things equal 30 things. Restore each part, that is, you shall remove from each part 15 things. And you will get that 15 things equal 450 numbers. And therefore you shall divide the numbers in the things, that is, 450 into 15, from which results 30. And as much is the thing. And we posited that the third partner put in a thing, so that he comes to have put in 30 *libre*. The second 20 *libre*. The first 10 *libre*. And if you should want to know how much of it is due to the first and to the second, then remove from 30 *libre* 15 of them which are due to the third. 15 *libre* are left. And you will say that there are 2 partners who have gained 15 *libre*. And the first put in 10 *libre*. And the second put in 20 *libre*. How much of it is due to (each) one. Do thus, and say, 20 *libre* and 10 *libre* are 30 *libre*, and this is the principal of the partnership. Now multiply for the first, who put in 10 *libre*, 10 times 15 which they have gained. It makes 150. Divide into 30, from which results 5 *libre*. And as much is due to the first. And then for the second, multiply 20 times 15, which makes 300 *libre*. Divide into 30, from which results 10 *libre*, and as much is due to the second partner. And it is done, and it goes well. And thus the similar computations are done.

Let us first concentrate on the start of the procedure, the one that leads to the determination of what the third partner put in. It makes use of the "partnership rule," a special case of the rule of three: the share of each partner in the profit is found by first multiplying his share of the capital by the total profit, and next dividing the outcome by the total capital of the partnership,

$$p_i = \frac{c_i \cdot P}{C}.$$

We notice that division by a binomial is treated as a matter of course, as is the cancellation of this division by a corresponding multiplication. Such operations are also found in Ibn Badr's *Ikhtiṣār al-jabr wa'l-muqābala* [ed., trans. Sánchez Pérez, 1916, 43 and *passim*] and in al-Karajī's *Fakhrī* [Woepcke, 1853, 88, 91*f*, and *passim*]—but in al-Khwārizmī's algebra [ed. Hughes, 1986, 248] we only encounter the division by a simple *thing* (in the illustration of the case "things made equal to number").

The second part of the procedure, the one determining the shares of the first two partners by means of a fictitious new partnership, illustrates a feature of Jacopo's text that was already mentioned above: his recurrent use of the commercial partnership as a general model or functionally abstract representation within which all kinds of proportional distributions can be made.

Other idiosyncrasies

Al-Khwārizmī's "algebra" was entitled "Book of *al-jabr* and *al-muqābala*," *al-jabr* being derived from the verb *jabara* (mostly translated "to restore") and *al-muqābala* from the verb *qabila* ("to accept," etc., the nontechnical meaning of *muqābala* being "encounter," "comparison," etc.). *Al-jabr* and *al-muqābala* must hence be central operations for the discipline—and it must be significant that Jacopo does not use the terms in the same way as the Latin algebra writings.

In these, *restaurare* (the translation of *jabara*) designates the cancellation of a subtractive term by addition. Jacopo uses the corresponding term *ristorare* both in this function and for the cancellation of an additive term (an instance of the latter use is quoted above). In Abū Bakr's *Liber mensurationum* as translated by Gherardo da Cremona [ed. Busard, 1968] *restaurare* is also used a couple of times (#7,

#55) in the function of multiplicative completion (changing 2/5 and 1/4 into 1 through multiplication by $2\frac{1}{2}$ and 4, respectively). Cancellation of an additive term, on the other hand, is nowhere spoken of in this way in any of the Latin treatises but instead as *opponere*, the Latin equivalent of *qabila* (from which *muqābala* is derived).

Opporre, the vernacular counterpart of *opponere*, is absent from Jacopo's text, but that probably does not mean that it contains no equivalent of *qabila/muqābala*. Indeed, in Raffaello Canacci's *Ragionamenti d'algebra* [ed. Procissi, 1954, 302] we read, in a passage ascribed to Guglielmo de Lunis, that *elmelchel* (the neighbor of *geber* (i.e., *jabr*) in Canacci/Guglielmo's text and thus certainly a transcription of *al-muqābala*[21]) means "exempio hovvero aghuaglamento," "exemple or equation." This term (in the form *raoguaglamento*) is indeed used in the end of Jacopo's example (5b), precisely in the sense of "equation."

A final characteristic by which Jacopo's treatise differs from all Latin algebra writings is the complete absence of geometric proofs for the correctness of the rules by means of which the cases 4–6 are solved.

The fondaco problems

As mentioned above, Jacopo's treatise contains four problems that *we* would consider algebraic but that do not make use of the technique of *thing* and *censo* (fols 43^v–45^v). All deal with the yearly wages of the manager of a *fondaco* or warehouse. Their statements run as follows:

a. Somebody stays in a warehouse 3 years, and in the first and third year together he gets in salary 20 *fiorini*. The second year he gets 8 *fiorini*. I want to know accurately what he received the first year and the third year, each one by itself.
b. Somebody stays in a warehouse 4 years, and in the first year he got 15 *fiorini* of gold. The fourth he got 60 *fiorini*. I want to know how much he got the second year and the third at that same rate.
c. Somebody stays in a warehouse 4 years. And in the first year and the fourth together he got 90 *fiorini* of gold. And in the second year and the third together he got 60 *fiorini* of gold. I want to know what resulted for him, each one by itself.
d. Somebody stays in a warehouse 4 years. And in the first year and the third together he got *fiorini* 20 of gold. And in the second and the fourth year he got *fiorini* 30 of gold. I want to know what was due to him the first year and the second and the third and the fourth. And that the first be such part of the second as the third is of the fourth.

Obviously, we are missing some information which Jacopo takes for granted. The solution to (a) shows what:

> Do thus, and let this always be in your mind, that the second year multiplied by itself will make as much as the first in the third. And do thus, multiply the second by itself, in which you say that he got 8 *fiorini*. Multiply 8 times 8, it makes 64 *fiorini*. Now you ought to make of 20 *fiorini*, which you say he got in the first and third year together, two parts which when multiplied one against the other make 64 *fiorini*. And you will do thus, that is that you always halve that which he got in the two years. That is, halve 20, 10

[21] As pointed out to me by Ulrich Rebstock [personal communication], the transcription (with assimilated *b*) appears to render a mozarabic pronunciation.

Canacci's explanation is similar to but not directly copied from a passage in Benedetto da Firenze's exposition of "La reghola de algebra amuchabale" [ed. Salomone, 1982, 1*f*], which lends credibility to their common reference to Guglielmo de Lunis. Nothing similar is found in the Latin version of al-Khwārizmī's algebra contained in MS Lyell 52 (Bodleian Library Oxford) [ed. Kaunzner, 1986]; this version is therefore *not* likely to represent Guglielmo's translation, as sometimes claimed—cf. also [Kaunzner, 1985, 11*f*].

> result. Multiply the one against the other, it makes 100. Remove from it the multiplication made from the second year which is 64, 36 is left. And of this find its root, and you will say that one part, that is, the first year, will be 10 less root of 36. And the other part, that is, the second year, will be 8 *fiorini*. And the third will be from 10 less root of 36 until 20 *fiorini,* which are *fiorini* 10 and added root of 36. And if you want to verify it, do thus and say: the first year he gets 10 *fiorini* less root of 36, which is 6. Detract 6 from 10, 4 *fiorini* is left. And 4 *fiorini* he got the first year. And the second year he got 8 *fiorini*. And the third he got *fiorini* 10 and added root of 36, which is 6. Now put 6 *fiorini* above 10 *fiorini,* you will get 16 *fiorini*. And so much did he get the third year. And it goes well. And the first multiplied against the third makes as much as the second by itself. And such a part is the second of the third as the first of the second. And it is done.

The beginning of this solution provides the clue: the yearly wages are tacitly assumed to increase in geometric progression. When this is taken into account, all four problems possess unique solutions, which are found correctly in the text. In (a), it is used that the wages of the three years fulfill the condition

$$S_1 \cdot S_3 = S_2 \cdot S_2 = 64.$$

At the same time, $S_1 + S_3 = 20$. This problem *could* be solved by means of algebra (of the *al-jabr* kind)—it is of exactly the same type as (5a) above. But the text offers an alternative, a purely numerical algorithm—which coincides with the solution to the corresponding rectangle problem given by Abū Bakr (that is, with the solution known from the tradition of geometric rectangle riddles since this tradition is first attested in the Old Babylonian clay tablets).

Problem (b) first finds the quotient p of yearly increase (without giving it any name) as $\sqrt[3]{S_4/S_1}$, and then finds S_2 and S_3 as $p \cdot S_1$ and $p^2 \cdot S_1$, respectively. (d) finds p as $(S_2 + S_4)/(S_1 + S_3)$ (again without telling *what* is found) and next S_1 as $(S_1 + S_3)/(1 + p^2)$. Both solutions are straightforward for anybody who possesses a fair understanding of the nature of the ascending algebraic powers as a geometric series, but less straightforward for the one who knows his algebra through al-Khwārizmī or Fibonacci alone.

Problem (c) is more complex. The solution makes use of the identity

$$S_1 \cdot S_4 = S_2 \cdot S_3 = \frac{(S_2 + S_3)^3}{3(S_2 + S_3) + (S_1 + S_4)},$$

which can be explained by the transformations ($S_1 = a$)

$$\frac{(S_2 + S_3)^3}{3(S_2 + S_3) + (S_1 + S_4)} = \frac{a^3 p^3 (1 + p)^3}{a(3p + 3p^2 + 1 + p^3)} = a^2 p^3 = a \cdot ap^3 = ap \cdot ap^2$$

—transformations which certainly require more than a merely "fair" understanding of the nature of the ascending algebraic powers as a geometric series. Who understood this (no explanation in the text suggests that Jacopo himself understood, fond though he elsewhere is of giving pedagogical explanations) will have had no difficulty in seeing how the cases (7) through (20) in the algebra proper could be solved either directly or by reduction to appropriate second-degree cases.

Abbreviations and notation

It is a general and noteworthy characteristic of Jacopo's algebra (or at least of manuscript **V**, but there are good reasons to believe the manuscript to be true to the original in this regard) that it avoids all abbreviations in the technical algebraic terminology, as if the author was conscious of introducing a new field of knowledge where readers would be unfamiliar with the terminology and therefore unable to expand abbreviations correctly.[22] A fortiori, nothing in his algebra even vaguely recalls algebraic symbolism or syncopation. Early in the treatise, however, we find an unusual variant of the Roman numerals—for instance in the explanation of 400,000 as $\overset{\text{m}}{\text{cccc}}$. This way to put the "denominator" above the number being denominated coincides exactly with the algebraic notation found in Maghreb writings from the 12th century onward [Abdeljaouad, 2002, 11*f*; Souissi, 1969, 92 n. 2]—but since the same system was also used by Diophantos and other ancient Greeks to write multiples of aliquot parts, and by Middle Kingdom Egyptian scribes for the writing of large numbers, the similarity remains suggestive and nothing more for the time being.

Jacopo's possible sources: Arabic writings on algebra

Jacopo's algebra is derived neither from Fibonacci nor from the Latin translations of al-Khwārizmī (or Abū Kāmil)—that much should already be clear. That it is ultimately derived from Arabic *al-jabr* is no less certain. In consequence, Jacopo's algebra confronts us with a hitherto unknown channel from the Arabic world and its mathematics.

This conclusion raises two difficult questions. First, Jacopo's algebra, if fundamentally different from the Latin translations of al-Khwārizmī and Abū Kāmil, must also be fundamentally different from their Arabic originals, and his Arabic inspiration must therefore be of a different kind; second, his treatise contains no single Arabism, and direct use of Arabic sources on his part can thus be safely excluded. We must therefore ask, first, which kind of Arabic material provided his ultimate inspiration?

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22 In a table listing the fineness of coins, *meno* is abbreviated m (as was the standard); in the rest of the text, this abbreviation will be looked for in vain. Abbreviations for *radice*, *cosa*, and *censo* are equally absent, even though they were current when **V** was written. In contrast, terms that are not part of the algebraic technical vocabulary (*moltiplicare*, *libra*, *compagnia*, etc.) are regularly abbreviated.

It may then seem strange that no explanation is given in the beginning of the algebra chapter of what *cosa* and *censo* mean. The reason could be that an introduction to the chapter has disappeared during transmission. Other chapters indeed start by announcing what comes next—for instance,

– Abiamo dicto dele multiplicationi et dele divisioni et de tucto quello che intorno a ciò è di necessità. Ora lasciamo questo, et dirremo per propria et legitima forma et regola sopre tucti manere de numeri rocti [...].
– Abiamo dicto de rotti abastanza, però che dele simili ragioni de rotti tucte se fanno a uno modo e per una regola. E però non ne diremo più al punte. Et incominciaremo ad fare et ad mostrare alcune ragioni secondo che appresso diremo. Se ci fosse data alcuna ragione nela quale se proponesse tre cose [...].
– In nomine Domini amen. Qui appresso incominciaremo, et dirremo de tucte maniere de mesure. Et primamente dirremo del tundo ad conpasso [...].
– In Christi nomine amen. Qui sonno sotto scripte tucte maniere de leghe de monete. Et similmente tucti allegamenti de oro, argento et ramo [...].

The algebra chapter, in contrast, simply begins by stating the first rule (fol. 36^{v}), "Quando le cose sonno eguali al numero, si vole partire el numero nelle cose, et quello che ne vene si è numero. Et cotanto vale la cosa." However, *after* the example to the sixth rule we read (fol. 42^{r}). "Qui finischo le sey regole conposte con alquanti assempri. Et incomincia l'altre regole che sequitano le sopradicte sey como vederete." Seemingly, the text presupposes that these six rules have already been spoken of as a set. Cf. also footnote 37.

And second, where in the Romance-speaking world did he find an environment using this material actively?

The two questions must be addressed one by one. In the present section we shall therefore look at a larger range of Arabic algebraic writings in relation to the parameters where Jacopo's algebra differs from the Latin treatises. Appendix B contains a list of the Arabic works that are taken into consideration. The following section examines Italian writings.

The order of the six cases

As already mentioned, al-Khwārizmī as well as Abū Kāmil presents the six fundamental cases in the order 3-2-1-4-5-6 (Jacopo's order being 1-2-3-4-5-6). This "classical order" recurs in Ibn al-Bannā's presentation of the cases in the *Talkhīṣ* [ed., trans. Souissi, 1969], in al-Qalaṣādī's *Kashf* [ed., trans. Souissi, 1988], in Ibn Badr's *Ikhtiṣār al-jabr wa'l-muqābala* [ed., trans. Sánchez Pérez, 1916], and in Ibn al-Yāsamīn's *Urjūza fī'l-jabr wa'l-muqābala* [ed., trans. Abdeljaouad, 2005, 4*f*].

Al-Karajī arranges things differently. In the *Kāfī* [ed., trans. Hochheim, 1878] as well as the *Fakhrī* [Woepcke, 1853], his order is 1-3-2-4-5-6. The same pattern is found in al-Samaw'al and al-Kāshī [Djebbar, 1981, 60*f*] and in Bahā' al-Dīn al-'Āmilī's *Khulāṣat al-ḥisāb* [ed., trans. Nesselmann, 1843] from c. 1600. In his solution of the equations, Ibn al-Bannā' follows the pattern 3-2-1-4-6-5 (that of the *Liber abbaci*).

Jacopo's order is referred to around 1500 by al-Māridīnī in his commentary to Ibn al-Yāsamīn's *Urjūza* as the one that is used "in the Orient," and it is indeed that of al-Miṣṣīṣī, al-Bīrūnī, al-Khayyāmī, and Sharaf al-Dīn al-Ṭūsī [Djebbar, 1981, 60]. Not only there, however; al-Qurashī, born in al-Andalus in the 13th century and active in Bejaïa, has the same order [Djebbar, 1988, 107].

Normalization

Al-Khwārizmī's original text, like the Latin translations, defines all cases except "things made equal to number" in normalized form and gives corresponding rules.[23] This also applies to Ibn Turk's [ed., trans. Sayılı, 1962] and Thābit's [ed., trans. Luckey, 1941] demonstrations of the correctness of the rules, and to al-Khayyāmī's algebra [ed., trans. Rashed and Djebbar, 1981]. Al-Karajī's *Kāfī* confronts us with a mixed situation: the three simple cases (1)–(3) are nonnormalized (definitions as well as rules); case (4) is defined as nonnormalized, but its rule presupposes normalization; the two remaining composite cases are presented only through normalized paradigmatic examples, and the formulation of the rules presupposes this normalization. The *Talkhīṣ* and the *Kashf* treat the simple cases like the *Kāfī*; they give no explicit definitions of the composite cases, but give rules that presuppose normalization. Ibn Badr gives nonnormalized definitions for all cases, and corresponding rules for the simple cases; his rules for the composite cases apply to the normalized equation; as far as can be judged from the very concise versified

[23] The Arabic manuscript published first by Rosen [1831] and later by Musharrafa and Ahmad [1939] defines the cases in nonnormalized form, even though its rules presuppose normalized equations. However, Gherardo's extreme grammatical faithfulness in other respects attests to his reliability even on this account. The different pattern of the Arabic text is thus an innovation—an adaptation of the original to changing customs within the field (a partial adaptation only, the rules being unchanged and the resulting totality thus incoherent). Indeed, comparison of the published Arabic version with Gherardo's and Robert of Chester's Latin translations shows that it must have been submitted to at least three successive revisions, two of which have also affected Robert's Arabic text—see [Høyrup, 1998a, 172*f*].

text, this is also Ibn al-Yāsamīn's intention (only cases 1–3 and 6 are quite explicit).[24] Only Bahā' al-Dīn states all definitions as well all as rules in nonnormalized form, as does Jacopo.

Examples

Basic examples formulated in the same terms as the rules, i.e., dealing with a *māl* ("possession," the equivalent of Jacopo's *censo*) and its *jidhr* ("[square] root"), are found in almost all the Arabic works I have looked at—in al-Khwārizmī's, Abū Kāmil's, and al-Khayyāmī's treatises, in al-Karajī's *Kāfī* and *Fakhrī*, in al-Qalaṣādī's *Kashf* and in Ibn Badr's *Ikhtiṣār*. Only Ibn al-Yāsamīn's *Urjūza*, Ibn al-Bannā's *Talkhīṣ* and Bahā' al-Dīn's *Khulāṣa* contain no examples of this kind—but the *Urjūza* and the *Talkhīṣ* because they give no examples at all.[25]

The divided 10 turns up everywhere (except where no examples are given), from al-Khwārizmī and Abū Kāmil to Bahā' al-Dīn. Problems where two unknown numbers are given in proportion are as absent from the Arabic treatises I have inspected as from the Latin ones.

Abū Kāmil, like al-Khwārizmī, deals with the division of a given amount of money between first x, then $x + p$ men, but apart from that neither of the two treat of *mu'āmalāt* problems in the properly algebraic parts of their treatises. Most other treatises keep *mu'āmalāt* matters wholly apart from their algebra. The only exceptions among the works I have inspected are the *Fakhrī* and Ibn Badr's and Bahā' al-Dīn's treatises. Ibn Badr, after a large number of divided-10 and *māl-jidhr* problems, has others dealing with the remuneration of a principal, dowries,[26] the mixing of grain, the distribution of booty among soldiers, travels of couriers, and reciprocal gifts (three or four of each type). Of Bahā' al-Dīn's illustrations of the six fundamental cases, two deal with pure numbers and four with feigned *mu'āmalāt* (that is, with "recreational") problems. In a later chapter listing nine problems that can be resolved by more than one method, the share of recreational problems is the same.

Square roots of real money

One of Jacopo's problems—(4b), the only one of his *mu'āmalāt* problems that belongs to a familiar recreational type—refers to the square root of an amount of real money. From a purely formal point of view this is highly traditional, the basic *al-jabr* cases being defined as problems dealing with a *māl* or "possession" and its square root, and treating the known number as a number of dirhams. But already in al-Khwārizmī's time this had become a formality. It is true that he states not only the root when it has been found but also the *māl*, remembering thus that once this had been the real unknown quantity of the problem. But stating the case "*māl* made equal to number" in normalized form (and defining first the *root* as one of the number types and next the *māl* as the product of this number by itself [ed. Hughes, 1986, 233*f*]) he clearly shows that he considers the *root* as the unknown proper—in perfect agreement indeed with his later identification of the root with the *shay'* or *thing*. From al-Khwārizmī onward we may thus claim that the *root* was a square root of formal, not real money.

24 However, ibn al-Yāsamīn has a very explicit discussion of how to treat nonnormalized mixed problems, either through division by the coefficient of the possession or by multiplication (the Babylonian–Diophantine method).

25 The *Khulāṣa* does contain a first-degree problem about a *māl*, but apparently meant to stand for real money.

26 Principal as well as dowry is designated *māl*, but the problem texts show that real invested money and real dowries are meant.

Roots of real money are absent from almost all of the Arabic algebra writings I have examined—the only exceptions being al-Karajī, who in the *Fakhrī* once takes the root of an unknown price and twice of unknown wages, and Ibn Badr, who twice takes the root of a dowry. However, the *Liber mahamaleth*, a Latin composition made in Spain during the 12th century, contains at least two algebraic problems of the kind: in one, the square roots of a capital and a profit are taken, in another the square root of a wage [Sesiano, 1988, 80, 83].

In order to find copious square roots of real entities (not only money but also, for instance, a swarm of bees, the arrows fired by Arjuna, or a horde of elephants) we have to go to India.

Commercial calculation within algebra

Jacopo employs the rule of three as a tool for algebraic computation; further, he uses the commercial partnership as a functionally abstract representation for proportional distributions. I have never noticed anything similar in an Arabic treatise—al-Khwārizmī *presents* the rule of three in a separate chapter (said to deal with *mu'āmalāt*) after the algebra proper and before the geometry, but this is a different matter.

Jabr and muqābala

Jacopo's use of the equivalent of *jabr* (*ristorare*) and of the likely equivalent of *muqābala* (*raoguaglamento*) differs from al-Khwārizmī's use of the original terms (which is also the main usage of Abū Kāmil, and that of Ibn al-Bannā', al-Qalaṣādī and Bahā' al-Dīn). However, the Arabic usage is far from uniform.

First, Abū Bakr's *Liber mensurationum*, whose multiplicative use of *restaurare* was mentioned above, uses the phrase *restaura et oppone* repeatedly in situations where no subtraction is to be made. The meaning of "opposition" is clearly in concordance with Canacci's explanation, namely *to form a (simplified) equation*—and thus with Jacopo's usage. Even in Abū Kāmil's *Algebra* the same phrase turns up time and again with the same sense (see the index in [Sesiano, 1993]). Similar ambiguities are found in Ibn Badr [Sánchez Pérez, 1916, 24, n. 1].

In the *Fakhrī* [Woepcke, 1853, 64], *jabr* refers to the elimination of additive as well as subtractive terms, just as in Jacopo's treatise. *Muqābala*, on its part, is explained to be the formation of a simplified equation where two terms are equal to one (or vice versa)—that is, the formation of one of the equations that define the basic cases. In the *Kāfī* [ed., trans. Hochheim, 1878, III, 10], *jabr* is also said to include multiplicative completion (as it does in the *Liber mensurationum*). For the rest, this text seems to be ambiguous (as far as can be judged from the translation). Perhaps it means to leave the elimination of an additive term unnamed and uses *muqābala* as the *Fakhrī*; perhaps this latter term is meant instead to designate the removal that leads to the formation of the simplified equation.[27]

Geometric proofs

Geometric proofs for the correctness of the rules for the three composite cases are found in al-Khwārizmī and Ibn Turk, and (with new ones added) in Abū Kāmil and in the *Fakhrī*. They are absent

[27] As Saliba [1972] has argued, the *Fakhrī* usage appears to be the original one; the ambiguity in the *Kāfī* illustrates the way in which the new interpretation as the subtractive counterpart of *jabr* can have come about.

Raffaello Canacci, in the passage where he explains *elmelchel* to stand for "exemple or equation" [ed. Procissi, 1954, 302], states that *elchel* (*al-qābila*, according to the parallel) stands for "opposition," explained to be the simplified equation.

from the *Kāfī*, from the treatises belonging to the Maghreb school (Ibn al-Yāsamīn, Ibn al-Bannā', al-Qalaṣādī), and from those of Ibn Badr and Bahā' al-Dīn.

Polynomial algebra and geometric progressions

I have seen nothing similar to Jacopo's four *fondaco* problems in Arabic works, and never received a positive answer when asking others who might know better. But the basic underlying theory—that which also allows one to see that Jacopo's cases (7) through (20) can be solved—was known at least since al-Karajī and al-Samaw'al,[28] and part of it was inherent in all writings that presented the sequence of algebraic powers as a geometric progression and also stated the rules for multiplying binomials—thus in the *Urjūza*, the *Talkhīṣ*, and the *Kashf*.[29]

Summing up

Almost every seeming idiosyncrasy we find in Jacopo can be found in Arabic writings (the exceptions being the use of the rule of three and the partnership structure as tools for algebra, the examples asking for numbers in given proportion, and the idea that wages increase by default in geometric progression). But they never occur together in treatises I have inspected. Those that are furthest removed from Jacopo are al-Khwārizmī and Abū Kāmil. The exponents of the Maghreb school are somewhat closer (in their omission of geometric proofs and, hypothetically, in the similarity between their algebraic notation and Jacopo's multiplicative writing of Roman numerals). But Jacopo's order of cases, his use of the *jabr*- and *muqābala*-equivalents, his square roots of real money, and his ample use of *mu'āmalāt*-problems within the algebra links him to (some middle ground between) al-Karajī's writings, Ibn Badr's possibly Iberian *Compendium of Algebra*, the certainly Iberian *Liber mahamaleth*, and Bahā' al-Dīn's *Essence of the Art of Calculation*; his consistent presentation of nonnormalized cases is only shared with the latter much younger work. In other, more explicit words: We do not know the kind of Arabic algebra that provided him with his ultimate inspiration, but it was certainly different from those (scholarly or "high") currents that have so far been investigated by historians of mathematics; we may also conclude with fair certainty that it was linked to an institution that taught algebra as integrated in *mu'amalāt*-mathematics.

Jacopo's possible sources: a look at the next Italian generation

We should now concentrate on the second aspect of the "source" question: where in the Romance-speaking world did Jacopo find an environment actively engaged in algebra?

However, an answer to this question (indirect and partially negative as it will be) can only be given if we look closely at the still extant Italian expositions of algebra written during the decades that followed immediately after Jacopo.

[28] In the *Fakhrī*, al-Karajī makes use of the formula for the third power of a binomial [Woepcke, 1853, 58]. At first he exemplifies it by $(2+3)^3$, next he uses it to show that $\sqrt[3]{2}+\sqrt[3]{54}=\sqrt[3]{128}$.

[29] With hindsight, not only "part" but all that is required to resolve all of Jacopo's *fondaco* problems was implied. But hindsight may amount to historiographical blindness: Cardano's solution to the third-degree equation is "implied" in Old Babylonian "algebra," in the sense that he combines tricks that were in use in that discipline; but it took more than three millennia to discover that it could be done.

One of these (**G**) is contained in Paolo Gherardi's *Libro di ragioni*, written in Montpellier in 1328.[30] Two others are contained in an *abbaco* manuscript from Lucca from c. 1330 [ed. Arrighi, 1973], a conglomerate written by several hands. Its fols 80^v–81^v (pp. 194–197) contain a section on "le reghole dell'aligibra amichabile" (henceforth **L**); another section on "le reghole della chosa con asenpri" is found on fols 50^r–52^r (pp. 108–114; henceforth **C**).

Somewhat later but so closely related to one or more members of the first generation that they can inform us about it are two other items: **A**, a *Trattato dell'Alcibra amuchabile* from c. 1365 [ed. Simi, 1994]; and **P**, an anonymous *Libro di conti e mercatanzie* [ed. Gregori and Grugnetti, 1998] kept today in the Biblioteca Palatina of Parma and probably compiled in the Tuscan-Emilian area—according to problems dealing with interest in the years immediately after [13]89–95.

All of these depend to some extent on what we know from **V**, that is, on Jacopo. The first extant vernacular algebra that does not depend on him—and the earliest vernacular work dedicated exclusively to algebra—is the *Aliabraa argibra*, which according to one manuscript was written by an otherwise unidentified Master Dardi from Pisa in 1344 (henceforth **D**; on a *possible* identification of its author, see footnote 48). Dardi's work is analyzed in some depth in the next section. Slightly earlier and also independent of Jacopo is a treatise written by Giovanni di Davizzo, from which however nothing but a fragment (**Z**) survives, whose importance only becomes clear when we compare it with **V** as well as **D**.

Table 1 summarizes some important features of these presentations of algebra. If a work has a rule for a particular case, it is marked R if the rule is true; X if it is false and constructed merely as an illegitimate imitation of the solution to a similar-looking second-degree problem; and S if the rule is valid only in a special case modeled after Jacopo's example (4a), from which the rule has been guessed (S_n if stated for the normalized case). The presence of examples is indicated by E, marked by subscript digits (E_{12} thus indicates that two examples are given; E_1 and E_2 in the same row but different columns indicate that the examples are different, E_1 and E_{1*} that they are identical apart from the choice of numerical parameters). The letters "p" and "n" indicate whether the division by which the equation is normalized is expressed as "partire per" or "partire in"; we shall see that this "neutral mutation" is an interesting parameter.[31] K stands for *cubo*, C for *censo*, CC for *censo di censo*, t for *cosa*, n for *numero* (in whatever spellings the manuscripts may use), and Greek letters for coefficients (implied by the plurals *cubi*, *censi*, and *cose*). We notice immediately that all works have the six fundamental cases in the same characteristic "non-Latin" order as Jacopo.

Paolo Gherardi

Let us first concentrate on the column for **G**, Gherardi's algebra from 1328, composed in that very town where Jacopo had written 21 years before him. Gherardi, as we see, follows Jacopo fairly closely in the six fundamental cases. The differences are the following:

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30 Published by Gino Arrighi in [1987]—the chapter on algebra separately with translation and mathematical commentary by Van Egmond in [1978]; mentioned above.

31 Etymologically, "partire a in b" refers to the division of the quantity a into b equal parts, and "partire a per b" to the numerical computation; but I have never remarked any reference to the "parts" in question in any Italian *abbaco* writing which divides "in"—the etymology must already have been forgotten. Any systematic choice of one or the other formulation (for instance, Jacopo dividing always the product of circular diameter and perimeter *in* 4 in order to find the area, and the perimeter invariably *per* $3\frac{1}{7}$ in order to find the diameter) therefore points to a source in time or space where the distinction was still semantically alive.

Table 1

Case	**V**	**G**	**L**	**C**	**A**	**P**	**D**	**Z**
$\alpha t = n$	1.R,E_{12},n	1.R,E_{1*},n	1.R,E_1,n	1.R,E_{1*},**p**	1.R,E_{12},n	1.R,E_{1*},**p**	1.R,E_{1**},**p**	1.R.**p**
$\alpha C = n$	2.R,E_1,**p**	2.R,E_2,n	2.R,E_2,n	2.R,E_{2*},n	2.R,E_1, **p**	2.R,E_2[a],**p**	2.R,E_3,**p**	2.R.**p**
$\alpha C = \beta t$	3.R,E_1,**p**	3.R,E_{1*},n	3.R,E_{1*},**p**	3.R,E_2,**p**	3.R,E_1,**p**	3.R,E_1,**p**	3.R,E_{2*},**p**	3.R.**p**
$\alpha C + \beta t = n$	4.R,E_{12},n	4.R,E_{1*},n	4.R,E_{1*},n	4.R,E_{1**},n	4.R,E_{12},n	4.R,E_{1*},**p**	4.R,E_3**p**	4.R.n
$\beta t = \alpha C + n$	5.R,E_{123},n	5.R,E_{2*},n	5.R,E_{2**},**p**	5.R,E_{2***}[b],n	5.R,E_{123},n	5.R,E_{2*},**p**	5.R,E_{1*45},**p**	5.R.n
$\alpha C = \beta t + n$	6.R,E_1,n	6.R,E_2,n	6.Omitted[c]	6.R,E_3,n	6.R,E_1,n	6.R,E_2,**p**	6.R,E_4[d],**p**	6.R.n
$\alpha K = n$	7.R,**p**	7.R,E_1,**p**	7.R,n	7.R,**p**	7.R,E_1,**p**	7.R,E_2,**p**	7.R,E_3,**p**	7.R.n
$\alpha K = \beta t$	8.R,**p**	9.R,E_1,**p**	8.R,n	8.R,**p**	8.R,E_1,**p**	9.R,E_1[e],**p**	8.R,E_2,**p**	8.R.**p**
$\alpha K = \beta C$	9.R,**p**	10.R,E_1,**p**	9.R,**p**	9.R,**p**	9.R,E_1,**p**	10.R,E_1,**p**	9.R,E_2,**p**	9.R.**p**
$\alpha K + \beta C = \gamma t$	10.R,n	15.R,E_1,n	10.R[f],**p**	14.R,n	15.R,n	15.R,E_1,**p**	14.R,E_{1*},**p**	
$\beta C = \alpha K + \gamma t$	11.R,n		11.R,n	15.R,n	16.R,n			
$\alpha K + \gamma t = \beta C$					14.R,E_1,n	16.R,E_1,**p**	15.R,E_{234},**p**	10.R.n
$\alpha K = \beta C + \gamma t$	12.R,n	11.R,E_1,n	12.R[g],n	16.R,**p**	10.R,E_1,n	11.R,E_1,**p**	16.[h]R,E_2,**p**	11.R.n
$\alpha K = \sqrt{n}$		8.R,E_1,**p**			11.R,E_1,**p**	8.R,E_1,n	21.R,E_2,**p**	
$\alpha K = \beta t + n$		12.X,E_1,n			12.X,E_1[i],n	12.X,E_1,**p**		
$\alpha K = \beta C + n$		13.X,E_1,n			13.X,E_1,n	13.X,E_1,**p**		
$\alpha K = \gamma t + \beta C + n$		14.X,E_1,n				14.X,E_1,n		
$\alpha CC = n$	13.R,n		13.R,**p**	11.R,**p**	17.R,n	17.R,E_1,**p**	11.R,E_2,**p**	12.R.**p**
$\alpha CC = \beta t$	14.R,**p**			12.R,**p**	18.R,**p**	18.R,E_1,p	12.R,E_2,**p**	13.R.**p**
$\alpha CC = \beta C$	15.R,**p**			13.R,**p**	19.R,**p**	19.R,E_1,**p**	13.R,E_2,**p**	14.R.**p**
$\alpha CC = \beta K$	16.R,**p**			10.R,**p**	20.R,**p**	22.E_1,**p**	10.[j]R,E_1,**p**	
$\alpha CC + \beta K = \gamma C$	17.R,n				21.R,n			15.R.n
$\beta K = \alpha CC + \gamma C$	18.R,n				22.R,n			16.R.n
$\alpha CC = \beta K + \gamma C$	19.R,n				23.R,n			17.R.n
$\alpha CC + \beta C = n$	20.R,n				24.R,n			18.R.n
$\alpha CC + n = \gamma C$						20.R[k],E_1,n		
$\alpha C = \sqrt{n}$						21.R,E_1,n		
$\alpha C = n + \sqrt{v}$						23.X,E_1,**p**		
$\alpha K + \beta C + \gamma t = n$						24.S_n,E_1	A1.S,E_1,**p**	
$\alpha CC + \beta K + \gamma C + \delta t = n$						25.S,E_1,n	A2.E_1,**p**	
$\gamma t + \alpha CC = \beta K$?								19.X.?

[a] With the difference that $1/3 + 1/4$ has been replaced by $7/12$.
[b] In the end of the solution, the compiler of **C** tinkers with the double solution which was present in his original. In the short collection of further illustrative examples, **C** also has the problem E_1 of **V**.
[c] Absent; but since the ensuing text refers to "6 reghole," this is clearly by involuntary omission.
[d] E_4 in this line is closely related to E_3.
[e] With a copying error in the statement which might look like being inspired by E_2.
[f] The rule should read "Quando li chubi ⟨e li censi⟩ sono egualj alle cose [...]."
[g] The rule should read "Quando li chubi sono egualj ⟨a' censi⟩ e alle chose [...]."
[h] Formulated $\beta C + \gamma t = \alpha K$.
[i] Correcting a lacuna in the statement, which should read "Trouami 2 numeri che tale parte sia l'uno dell'altro come 2 di 3 e, multiprichato il primo per se medesimo et poi ⟨per⟩ quello numero faccia tanto quanto e più 12."
[j] Formulated $\beta K = \alpha CC$.
[k] With a copying error, "traendone" instead of "più."

- Gherardi never gives more than one example.
- He replaces Jacopo's pure-number example for case (2) with a different pure-number example.
- In example (4), he divides the amount borrowed by 5.
- In Jacopo's example (5b), he changes the given numbers in such a way that the result becomes irrational, and omits the second solution even though his rule mentions it.
- He replaces Jacopo's example (6) by a pure-number version of the problem of dividing a given quantity (here 100), first among x, then among $x + p$ (here $x + 5$) persons and adding the two results: $\frac{100}{t} + \frac{100}{t+5} = 20$. The description of the procedure refers to a number diagram[32]

```
100       1  cosa

100       1  cosa  piu  5
```

  in a way (with "cross-multiplication" and all the other operations needed to add fractions) that implies underlying operations with the "formal" fractions $\frac{100}{1\ cosa}$ and $\frac{100}{1\ cosa\ piu\ 5}$.

Further on, major differences turn up:

- Gherardi leaves out all fourth-degree cases.
- He introduces $\alpha K = \sqrt{n}$ as a case on its own.
- He introduces three irreducible third-degree cases, giving false rules fashioned after those for the second degree—solving for instance the case $\alpha K = \beta t + n$ as if it had been $\alpha C = \beta t + n$.
- The higher-degree rules are illustrated by examples, all of which are pure-number problems of the kind that could easily be constructed ad hoc ("to find two or three numbers in given proportion so that . . .").

The illustrations to the false rules all lead to solutions containing irrational roots. This allowed the fraud to go undetected, since no approximate value of these solutions was computed—approximation was not the custom; even Jacopo, when finding correctly a monthly interest of $\sqrt{600} - 20$ *denari* in his example (4a), left it there.

The Lucca manuscript

The two algebraic components of this conglomerate (**L** and **C**) are closer to **V**, and largely to be described as somewhat free abridgments of Jacopo's algebra.[33] The changes they introduce in the numerical parameters of certain examples do not change the character of these. Two of the examples where Gherardi differs from Jacopo are shared with **L**, but both are too simple to prove particular affinity.

Trattato dell'Alcibra amuchabile

While sharing the title with **L**, this *Trattato* (**A**) is much closer to **V** in those cases and problems that have a counterpart in that treatise than are **L** and **C**; it has all of Jacopo's examples with identical

[32] The diagram is actually missing from the manuscript, but it can be reconstructed from the verbal description and coincides with what is known from later manuscripts—see [Van Egmond, 1978, 169, n. 11].

[33] Evidently, it cannot be excluded that they descend from a source very close to Jacopo and not from Jacopo's own manuscript. However, the close agreement in the distribution of divisions *in* and divisions *per* excludes less direct relationships.

parameters, deviating mainly at the level of orthography; however, where Jacopo left spaces open in example (4b) in order to insert later the result of $4\sqrt{54}$ (cf. footnote 7), **A** has the correct result "radicie di 864." As we see, it even agrees strictly with **V** in the decision whether to divide *in* or *per*; both must hence descend by careful copying from a common archetype (which can hardly be anything but Jacopo's original manuscript or an early copy).

With a single exception, however—viz Gherardi's only four-term case[34]—**A** has all the examples for the higher-degree cases that we find in Gherardi, including his false rules for irreducible cases; but the agreement is not verbatim as with Jacopo. **A** also contains a rule and an example for the reducible case $\alpha K + \gamma t = \beta C$, which **A** distinguishes from its mirror image $\beta C = \alpha K + \gamma t$; only the latter and not the former shape is present in **V**. Those higher-degree rules that are found in **V** but not in **G** (including the just-mentioned $\beta C = \alpha K + \gamma t$) follow **V** and are equally devoid of examples. The two biquadratic cases that are missing in **V** are also absent from **A**.[35]

So far, only the middle part of the tripartite *Trattato dell'alcibra amuchabile* has been spoken of. The first part starts by presenting the sign rules ("più via più fa più e meno via meno fa più ...," "plus times plus makes plus, and minus times minus makes plus ...") and then goes on to teach operations with roots—number times root, root times root, products of binomials containing roots, and the division of a number or one such binomial by another binomial. For the product of binomial by binomial, a diagram is introduced to illustrate the procedure—for instance, for $(5 + \sqrt{20}) \cdot (5 - \sqrt{20})$,

```
5 e   piu  ℞ di 20
via
5 e   meno ℞ di 20.
```

As was usual in algebraic manuscripts from the Maghreb [Abdeljaouad, 2002], the diagram stands outside the running text and recapitulates what is done by rhetorical means in the text. For the division of a number by a binomial, for instance 100 by $10 + \sqrt{20}$, we find the similar diagram

```
10   piu  ℞ di 20
10   meno ℞ di 20,
```

which serves to illustrate that both dividend and divisor are to be multiplied by $1 - \sqrt{20}$. Whether the writer thinks in terms of formal fractions is not clear at this point.

However, in the third part [ed. Simi, 1994, 41*f*], we find Gherardi's example for the sixth case; in **A** it is stated in direct words that the addition $\frac{100}{t} + \frac{100}{t+5}$ is to be performed "in the mode of a fraction," explained with the parallel $\frac{24}{4} + \frac{24}{6}$.

34 $\alpha K = \gamma t + \beta C + n$, solved as if it had been $\alpha K = (n + \gamma)t + \beta C$, $t = \sqrt{\frac{\gamma+n}{a} + \left(\frac{b}{2a}\right)^2} + \frac{b}{2a}$.

35 Since **A** has no reference to "15 cases which [...] lead back" to the basic rules, this observation excludes **V** descending from a model also inspiring **A**: the error committed by Jacopo or an early copyist of his is repeated in **A**. Given **G**'s dependence on an intermediary between Jacopo's original and **A**, Gherardi must therefore also depend on Jacopo and not (or not exclusively) on a common archetype. Cf. also the section below on Giovanni di Davizzo, according to which the distribution *in*/*per* found in **V** errs in two cases from the canon prevailing in the environment where Jacopo found his inspiration; if **A** were inspired directly from here, it is very unlikely that exactly the same errors would be committed.

The Parma manuscript

The algebra section of the Parma manuscript *Libro di conti e mercatanzie* (**P**) is closer to **G** than **A**, also in the treatment of those cases that had been dealt with by Jacopo. But in the illustration of the case $\alpha C = \beta t + n$ (still the problem $\frac{100}{t} + \frac{100}{t+5} = 20$) it has the explicit formal fractions of **A** (distorted in the beginning in a way that suggests that the writer did not understand) and not Gherardi's diagram. It also has the case $\alpha K + \gamma t = \beta C$ that was absent from **G** but present in **A**, with the same example as **A**—but the mirror case $\beta C = \alpha K + \gamma t$ is absent from **P** though present in **A**. Gherardi's only four-term problem ($\alpha K = \gamma t + \beta C + n$), absent from **A**, is present in **P**.

P also provides examples to four of those fourth-degree rules which had none in **A**; three of these are of the usual facile pure-number type, but one ($\alpha CC = n$) is illustrated by a geometric question—to find the side of an equilateral triangle with given area. Further we find a biquadratic that was omitted in **V** (and **A**), and more examples involving roots of numbers ($\alpha C = n + \sqrt{v}$ being solved by taking the roots of the right-hand terms separately!). The four-term problem and the three problems involving roots of numbers are all normalized by division *in*, where all other normalizations are *per*.

The two cases $\alpha K + \beta C + \gamma t = n$ and $\alpha CC + \beta K + \gamma C + \delta t = n$ are of a new kind. The rules are still false, but they are not copied from rules for second-degree cases—and they work for the examples that are given. The former example coincides with Jacopo's example (4a), with the difference that the 100 *libre* are lent for three, not two years—but the capital still grows to 150 *libre*, which speaks in favor of inspiration from Jacopo's or some related text (starting with 100 *libre*, on the other hand, seems to have been the standard, and thus does not tell much). In the latter example, 100 *libre* are lent for four years and grow to 160 *libre*. The rules (complicated as they look because the *thing* is put equal to the interest in *denari* per month of one *libra*) appear to be constructed from the solutions that may be found from $\sqrt[3]{150/100}$ and $\sqrt[4]{160/100}$. The fraud is certainly more intelligent than that behind Gherardi's formulae—but it remains a fraud, and was probably recognized as such by its inventor (who was certainly not the compiler of **P**).[36]

Lines of ancestry and descent

We have now come to the point where it is possible to construct an approximate stemma (Scheme 1) showing the connections between the various Italian treatises discussed so far (the vertical axis corresponds to time, Jacopo writing in 1307, **G** being from 1328, and **V** from c. 1450). On top, we have Jacopo's original writing. **V**′ is the hypothetical archetype for all the actual manuscripts—perhaps identical with Jacopo's original work.[37] **V**″ is the faithful copy from which **V** is made (cf. footnote 6 and

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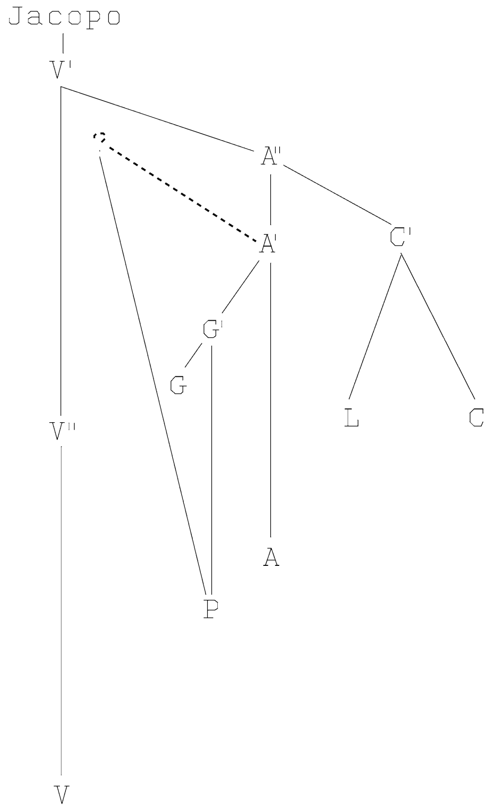
[36] One should not wonder that mathematicians would invent and publicize wrong formulae. As a rule, the authors of the *abbaco* texts were not "mathematicians" but teachers advertising and selling their abilities in a free market, where cheating the customers (parents of potential students or communal councils) successfully was just as efficient as convincing them honestly. The condition for successful fraud was not mathematical truth but the inability of competitors to unmask the deceit (whence the usefulness of solutions containing roots). Tartaglia's fortunes and misfortunes illustrate the point well.

Compilers of texts like **P** were probably quite unaware of the fraud; they merely repeated what they believed to be good algebra.

[37] But probably not if the hypothesis formulated in footnote 22 is correct, and the beginning of the algebra chapter has disappeared in transmission: **A** starts the chapter in question exactly as **V**. Possibly, Jacopo's autograph could have become **V**′ by losing a sheet.



Scheme 1.

preceding text). **A**″ is the common archetype for **A**, **L**, and **C**, which must still have been very faithful to **V**′ and can have contained none of the false rules, nor examples for the higher-degree rules. **C**′ is the common ancestor of **L** and **C** (since everything that is in **C** is also in **L** they are likely to have a common ancestor not very different from **C** but already free with respect to **A**″). **A**′ is a common ancestor to **A** and **G**, faithful to **V**′ in the parts coming from Jacopo but already provided with examples for some of the higher-degree cases and false rules for some irreducible cases. **G**′ is an ancestor to **G** from which **P** descends (the agreement of **P** and **A** in the case $\alpha K + \gamma t = \beta C$ appears to exclude direct descent of **P** from **G**). The extra cases in **P**, for instance involving square roots of numbers (and the prevalence of division *in* in the cases not shared with **A**, where division *per* is its standard choice in the shared cases) suggests that these have been borrowed from an unidentified source or area (labeled "**?**") and not created between **G**′ and **P** as generalizations of the case $\alpha K = \sqrt{n}$.[38] It is likely that the latter problem (shared by **G** and **A**) has been adopted into **A**′ from the same area.

Crosswise contamination is not to be totally excluded, but the distribution of shared versus particular features in the various treatises makes substantial importance of such influences unlikely. The stemma suggested here should hence be close to the truth.

[38] Perhaps with the exception of #23, the one which finds the root of $n + \sqrt{v}$ as $\sqrt{n} + \sqrt{\sqrt{v}}$, and which divides *per*. This could be an independent misshaped addition.

This means, first, that everything written on algebra in the Italian vernacular in the first generation after Jacopo depended on his work, with only a marginal influence from the “area ?.” This excludes the existence of an Italian environment practicing algebra before Jacopo’s times. Jacopo must have gone abroad in order to find the discipline—and his whole treatise indeed suggests that he was very conscious of presenting knowledge that was *new* to his public. Second, since **A**, **L**, and **C** are all written in Tuscan with no traces of non-Tuscan orthography, even **A**″ and **A**′ are likely to have been written in the Tuscan area; if this is so, then Paolo Gherardi must have sought his inspiration in Italian writings[39] and found little of algebraic interest in Montpellier.[40] But if there was no strong environment practicing algebra in Montpellier in 1328, there can hardly have been any in 1307.

This gives us no direct answer to the question concerning the localization of that Romance-speaking area from which Jacopo drew his knowledge of algebra. Indirectly, however, things begin to narrow down: if Italy and Provence are excluded, little beyond Catalonia remains—easily reached from Montpellier, and at the time involved in intense trading relations with the Arabic world as far as Egypt, and also an obvious channel for Ibero-Islamic influences.[41] Alternatively, the Iberian peninsula at large may be thought of[42]—a recently published Castilian *Libro de arismética que es dicho alguarismo* from 1393 [ed. Caunedo del Potro and Córdoba de la Llave, 2000], astonishingly close to Jacopo in many formulations, is even closer to the various extant 15th-century Provençal–Catalan algorisms (and closer to these than to the Italian counterparts); it contains no algebra, which prevents us from drawing too definite conclusions.

Maestro Dardi da Pisa

Dardi’s *Aliabraa argibra*, apparently from 1344, is the first full-scale vernacular algebra that does *not* depend on Jacopo (as will be argued below); it thus represents a different strand in the “beginning of Italian vernacular algebra.” It is the earliest extant vernacular work devoted solely to algebra—and it is more than four times as long as the *Trattato dell’Alcibra amuchabile* from c. 1365, also solely algebraic.[43] Like Jacopo’s treatise, it contains no single Arabism (unless we count the word “algebra” of the title as one). As it turns out, its independence of Jacopo does not preclude its being informative about some of

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39 In the introductory passage [ed. Arrighi, 1987, 15] he also presents himself as being from Florence.

40 Pure veneration for Jacopo can be excluded, since his name does not appear in Gherardi’s treatise. Since Jacopo knows none of the false rules (according to the style of his work he would have mentioned it if he knew about them and understood them to be false), even they are not likely to come from Montpellier.

41 It is worth observing in this connection that the semantic distinction between “partire in” and “partire per” (see footnote 31) is still fairly present in Francesc Santcliment’s Catalan *Summa de l’art d’aritmètica* from 1482 [ed. Malet, 1998]. Thus, fol. 27^{v}, “digues: que partisses 589 en 6 parts,” “say, you divide 589 into 6 parts,” versus fol. 32^{r}, “no es nenguna altra cosa partir per 25, ho per 35 ho 57 ho 77 [. . .] sino partir per 12 ho per 19,” “it is no different to divide by 25, or by 35 or by 57 or by 77 [. . .] than to divide 12 by 19.”

42 Sicily seems less likely but is perhaps not to be totally excluded—Fibonacci [ed. Boncompagni, 1857a, 1] lists it along with Egypt, Syria, Greece (i.e., Byzantium), and Provence as one of the places where he had pursued the study of the “nine Indian figures” and what belonged together with them after having been introduced to the topic in Bejaïa.

43 I used the Vatican manuscript Chigi M.VIII.170 from c. 1395 ($\mathbf{D_1}$); Raffaella Franci’s edition [2001] of the Siena manuscript I.VII.17 from c. 1470 ($\mathbf{D_2}$); and Warren Van Egmond’s personal transcription of the Arizona manuscript, written in Mantova in 1429 ($\mathbf{D_3}$). The datings of $\mathbf{D_1}$ and $\mathbf{D_2}$ are based on watermarks and according to [Van Egmond, 1980]; that of $\mathbf{D_3}$ is stated in the manuscript. $\mathbf{D_1}$ is in Venetian, $\mathbf{D_2}$ in Tuscan, and $\mathbf{D_3}$ as far as I can judge in a northern dialect not too different from Venetian. For further information, see [Van Egmond, 1983] and [Hughes, 1987]. I thank Raffaella Franci for supplementary information on $\mathbf{D_2}$ and Van Egmond for giving me access to his transcription of $\mathbf{D_3}$.

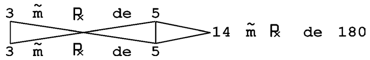
the questions left open in the preceding section. The following concentrates on the aspects of the work that are relevant in this respect, but can only do so on the condition of presenting the treatise in more general terms.[44]

Its basic structure is fairly similar to that of the first two sections of **A**. However, first come an introduction and an index listing all $194 + 4$ cases to be dealt with.[45] The sign rules of **A** are missing—but Dardi proves[46] when arriving to the point where it is first needed that "meno via meno fa più" (using the example $(10 - 2) \cdot (10 - 2)$). The index is thus followed directly by a "Treatise on the rules which belong to the multiplications, the divisions, the summations, and the subtractions of roots."[47] Then comes a presentation of the six fundamental cases, with geometric demonstrations (**A** has nothing similar), and finally a presentation of 194 "regular" and 4 "irregular" cases, all with rules and one or more examples. The distinction regular/irregular is made in the introduction; a note to the index uses different words, distinguishing between cases governed by general and by nongeneral rules.

In $\mathbf{D_1}$, the following abbreviations are made use of consistently: *ç* for *censo*, *c* for *cosa*, *nũo* for *numero*, ℞ for *radice*, *m̃* for *meno*; the notation for multiples of *ç* and *c* emulates that for fractions, writing the "denominator" below the "numerator" with a stroke in between—for instance, $\frac{10}{c}$ for "10 things". *ç*, *c*, and ℞ are also used in the later manuscripts $\mathbf{D_2}$ and $\mathbf{D_3}$. The fraction-like notation does not occur in $\mathbf{D_2}$ but often (not always) in $\mathbf{D_3}$; it therefore seems plausible that it was used in Dardi's original.[48]

Chapter 1: calculating with roots

In the chapter on roots, we find diagrams illustrating the multiplication of binomials similar to those in **A**—for instance, for $(3 - \sqrt{5}) \cdot (3 - \sqrt{5})$,[49]



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We notice that Dardi's diagram is fuller than that of **A**, which makes it implausible that **A** could have simply borrowed from him.

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44 For other aspects of the treatise, see [Van Egmond, 1983; Hughes, 1987; Franci, 2001, 1–33].

45 The index is absent from $\mathbf{D_2}$, but the introduction promises to provide it and leaves three empty pages—the obvious intention being to insert it once the equally promised corresponding folio numbers were known. In $\mathbf{D_1}$, the introduction and the first page of the index are missing, and the first folio number is 2.

46 $\mathbf{D_2}$ p. 44; $\mathbf{D_1}$ fol. 5^{v}; $\mathbf{D_3}$ fol. 11^{v}.

47 $\mathbf{D_1}$ fol. 3^{v}; **D2** p. 38. $\mathbf{D_3}$ does not have this general caption but has separate captions for the single sections.

48 In general, $\mathbf{D_1}$ is not only earlier in time than $\mathbf{D_2}$ and $\mathbf{D_3}$ but also textually closer to their common archetype in various respects. One example is the reference to the rule of three in the passage of $\mathbf{D_1}$ quoted below (footnote 50) and the absence of the reference in $\mathbf{D_2}$; since $\mathbf{D_2}$ cites it when referring backward to the passage (p. 62, corresponding to $\mathbf{D_1}$ fol. 14^{r}), it must have been present in the common archetype (it is indeed also found in $\mathbf{D_3}$). Another example is the use of the term *adequation* in $\mathbf{D_1}$, corresponding to *dequazione* in $\mathbf{D_2}$; they are indistinguishable in the definite form *ladequation*/*ladequazione*, which explains that one of the manuscripts has misunderstood the intended term of the original, but in one place (p. 77) $\mathbf{D_2}$ has an unexpected and indubitable *adequazione*, which can therefore be assumed to be the original form (indeed, $\mathbf{D_3}$ also uses *adequation*).

In single readings, $\mathbf{D_3}$ often seems better than $\mathbf{D_1}$, but at the level of overall structure (captions, etc.), $\mathbf{D_1}$ is apparently to be preferred. Since Dardi *could* be identical with one Ziio Dardi present in Venice in 1346 [Hughes, 1987, 170], even the language of $\mathbf{D_1}$ might be closest to the original.

Globally, the differences between the three manuscripts are fairly modest.

49 $\mathbf{D_2}$ p. 45; $\mathbf{D_1}$ fol. 6^{r}; $\mathbf{D_3}$ fol. 12^{r}.

When looking at the explanation of how to divide a number by a binomial we find greater differences. In order to divide 8 by $3+\sqrt{4}$, Dardi first makes the calculation $(3+\sqrt{4})\cdot(3-\sqrt{4})=5$ and concludes that 5 divided by $3+\sqrt{4}$ gives $3-\sqrt{4}$. *What*, he next asks, will result if 8 is divided similarly, finding the answer by means of the rule of three (5, $3-\sqrt{4}$ and 8 being the three numbers involved).[50]

Chapter 2: the six fundamental cases

The chapter proving the correctness of the second-degree rules has no counterpart in **A**, nor in any of the other Italian treatises discussed so far. The demonstrations descend from those found in al-Khwārizmī's algebra, but their style is as different as it would be if somebody not versed in the received conventions governing the use of letters in geometric diagrams were to relate al-Khwārizmī's proofs from memory to somebody not too well versed in geometry. As an illustration (which should speak for itself as soon as it is confronted with any version of al-Khwārizmī's text—cf. also Appendix A) I translate the beginning of the first proof verbatim (repeating the grammatical inconsistencies of the text),[51] reproducing also the first diagram (Fig. 1):

> How 1 *ç* and 10 *c* are proved to be equal to 39. Since the *c*, which is said to be ℞ of the *ç*, the *ç* now comes to be a quadrangular and equilateral surface, that is, with 4 corners and four equal and straight sides. Now we shall make a square with equal sides and right corners, and we shall say that the *ç* is its surface, which is *ab*, and since the *c* is the ℞ of the *ç*, it comes to be the sides of the said square, and since to the *ç* $\frac{10}{c}$ are added, we divide this $\frac{10}{c}$ into 4 parts, which comes to be $\frac{2}{c}\frac{1}{2}$ each,[52] and since the *c* comes to be the sides of the *ç*, we shall place each of these four parts along *ç*, each along its own side of *ç*, the surface of each being *cd*, and outside each of the corners of *ç* falls an equilateral quadrangle with right corners, which as side will have the breadth of the *c*, that is, $2\frac{1}{2}$, which breadth, or length, multiplied by itself amounts to $6\frac{1}{4}$, that is, *ef*, [...].

A closer look at some textual details reveals that the chapter has been adopted from the same environment as Jacopo's algebra (which was not a priori to be expected, given that Jacopo presents no geometric

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50 I render the text of $\mathbf{D_1}$ (fol. 12^{v}; similarly $\mathbf{D_3}$ fol. 19^{v}); punctuation and diacritics have been adjusted/added; words in ⟨ ⟩ are corrections of copyist's omissions inserted between the lines in a different hand (as is evident from the presence of the same words in $\mathbf{D_2}$):

> Se tu volessi partir nũo in ℞ e nũo, serave a partir 8 in 3 e ℞ de 4, tu die moltiplicar 3 e ℞ de 4 per 3 m̃ ℞ de 4, che monterà 5. Adonqua a partir 5 in 3 e ℞ de 4 te ne vien 3 m̃ ℞ de 4 perché ogne nũo moltiplicado per un'altro nũo, la moltiplication che ne vien partida per quel nũo si ne vien l'altro nũo moltiplicado per quello. Adunqua partando 5 in 3 e ℞ de 4 si ne vien 3 m̃ ℞ de 4, e partando 5 in 3 m̃ ℞ de 4 si ne vien l'altra parte, zoè 3 e ℞ de 4, e inperzò diremo che questo 5 sia partidor, e metteremo questo partimento alla regla del 3, e diremo, se 5, a partir in 3 e ℞ de 4, ne ven 3 m̃ ℞ de 4, che ne vegnirà de 8, e moltiplica 3 m̃ ℞ de 4 via 8, che monta 24 m̃ ℞ de 256, la qual moltiplication parti in 5, che ne vien $4\frac{4}{5}$ per lo nũo. Ora resta a partir ℞ de 256 ⟨meno⟩ in 5, che ne vien ℞ de $10\frac{6}{25}$, che a partir ℞ in nũo el se die redur lo nũo a ℞, zoè lo 5 redutto in ℞ monta ℞ de 25. E così avemo che a partir 8 in 3 e ℞ de 4 si ne vien $4\frac{4}{5}$ men ℞ de $10\frac{6}{25}$.

$\mathbf{D_2}$ omits the explicit reference to the rule of three, but as observed in note 48 it must have been present in the common archetype.

51 $\mathbf{D_2}$ pp. 68*f*; $\mathbf{D_1}$ fols 16^{v}–17^{r}; $\mathbf{D_3}$ fols 24^{v}–25^{r}.

52 We notice that Dardi extends his fraction-like notation into an "ascending continued fraction"; indeed, $\frac{2}{c}\frac{1}{2}$ means $\frac{2}{c}$ plus $\frac{1}{2}$ of $\frac{1}{c}$.

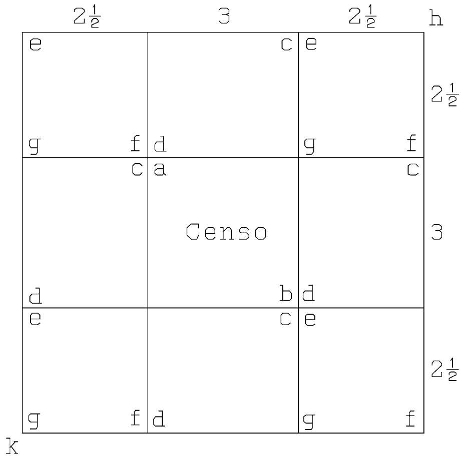


Fig. 1. Dardi's first proof for case (4).

proofs). Dardi's rule for the fifth case runs as follows in $\mathbf{D_1}$[53]:

> Quando li *ç* e'l numero è equali ale *c*, el se die partir tutta l'adequation per la quantità dei *ç*, e po partir le *c* in 2, e una de queste mità, zoè la quantità de una de queste parte, moltiplica in si medesima, e de quella moltiplication trazi lo numero e la ℞ de quello che roman zonzi all'altra mità dela quantità dele *c*, e tanto vegnirà a valer la *c*, e *sappi che in algune raxon te convegnirà responder esser la c per lo primo modo, zoè la mità dela quantità dele c più ℞ de quello che roman, e algun fiade per lo secondo modo, zoè la mità dela quantità dele c la ℞ de quello che roman, e algune se pò responder per tutte e 2 li modi, com'io te mostrerò*.

Jacopo's corresponding rule (fol. 39^{v}) is not very similar (except, by necessity, in mathematical substance):

> Quando le cose sonno oguali ali censi et al numero, se vole partire nelli censi, et poi dimezzare le cose et multiprichare per se medesimo et cavare el numero, et la radice de quello che romane, et poi el dimezzamento dele cose vale la cosa. Overo el dimezzamento dele chose meno la rad ice de quello che remane.

However, when Jacopo comes to present the double solution of example (5b), we find the following passage (fol. $40^{r–v}$, emphasis added):

> Siché tu vedi che all'uno modo et all'altro sta bene. Et però quella così facta regola è molto da lodare, che ce dà doi responsioni et così sta bene all'una come all'altro. Ma *abbi a mente che tucte le ragioni che reduchono a questa regola non si possono respondere per doi responsioni se non ad certe. Et tali sonno che te conviene pigliare l'una responsione, et tale l'altra. Cioè a dire che a tali ragioni te converà rispondere che vaglia la cosa el dimezzamento dele cose meno la radice de rimanente. Et a tale te converrà dire la radice de remanente e più el dimezzamento dele cose*. Onde ogni volta che te venisse questo co'tale

53 Fol. 16^{r}, emphasis added; similarly $\mathbf{D_2}$ p. 66 and $\mathbf{D_3}$ fol. 24^{r}.

raoguaglamento, trova in prima l'una responsione. Et se non te venisse vera, de certo si piglia l'altra senza dubio. Et averai la vera responsione.

The similarities between the two italicized passages are too particular to allow explanation merely from shared general vocabulary and style. However, several reasons speak against Dardi copying directly from Jacopo's text, not least the total absence of shared examples and of anything similar to Jacopo's *fondaco* problems from the *Aliabraa argibra*. Moreover, *if* Dardi had found the italicized passage in Jacopo interesting and moved it to the rule (because the examples he promises only come in the following chapter), he would not have changed its finer texture as seen in the excerpt[54]; nor would he have had any reason to invent the term *adequation* in replacement of *raoguaglamento* if using Jacopo's treatise. In consequence, Dardi must have drawn his inspiration for this chapter from the very environment which Jacopo had once drawn on. And he must have kept fairly close to his direct source: only too faithful copying explains the sudden appearance of "78 dramme, zoè numeri" in the example illustrating the fourth case ($\mathbf{D_1}$ fol. 16^r, similarly $\mathbf{D_2}$ p. 65)—up to this point, all numbers have been nothing but *numeri*.

Chapter 3: 194+4 regular and irregular cases

As mentioned, the final chapter presents 194 "regular" cases with rules, only a small selection of which are listed in Table 1. A very large part of them involve radicals, not only roots of numbers but also of *things*, *censi*, *cubi*, and *censi di censo*—thus, for instance, no. 59, $\alpha t = \sqrt[3]{\beta C}$, and no. 123, $\sqrt{\beta t} + \sqrt[3]{n} = \alpha t$ (notation as in Table 1). All are solved correctly (apart from two slips, convincingly explained in [Van Egmond, 1983, 417]), and all provided with an illustrative example (at times two or, with rules allowing a double solution, three examples[55]). All are pure-number problems, almost half of them are of the fraudulently complicated type asking for two or three numbers in a given proportion; a good fourth ask for a single number fulfilling conditions fashioned in agreement with the equation type; some 15 percent deal with a divided 10. The order of the six fundamental cases is the same as in the other treatises we have looked at, which corroborates the conclusion that Jacopo and Dardi were inspired from the same area. Even the order of the next three cases coincides with that of Jacopo—but since these are just the simplest higher-degree cases (cubes equal to number/*things*/*censo*), this agreement is hardly significant. After that, Dardi's order is wholly his own.

In $\mathbf{D_1}$ and $\mathbf{D_2}$, the four "irregular" cases are inserted between regular cases 182 and 183, after the observation that all equations up to this point contain no more than three terms.[56] In contrast, the regular cases from 183 onward all correspond to four-term equations. The rules for the irregular cases are presented at this point as "adapted solely to their problems, and with the properties these possess,"[57] but included all

54 When we are able to compare Dardi's text with another one deriving from the same source, such as Dardi's first irregular case with the corresponding case in **P** (see presently), Dardi can be seen to change at most the wording of the single phrases while conserving their order and mutual relation (but since **P** is later and hence more likely than **D** to have changed with regard to the original source, Dardi may well be even more faithful).

55 Even this, we notice, corresponds to Jacopo's treatment of the six fundamental cases, three examples showing that case (5) is sometimes solved by one solution, sometimes by the other, sometimes by both.

56 In $\mathbf{D_3}$, the irregular cases come after the last regular case, but the observation that all preceding cases involve at most three terms is found on fol. 113^r.

57 "[...] reghulati solamente alle loro ragione, e di quelle proprietà delle quale elle sono ordinate" ($\mathbf{D_2}$ p. 269; similarly $\mathbf{D_1}$ fol. 102^r). The wording in $\mathbf{D_3}$ (fol. 121^r) is slightly different but equivalent.

the same because they may turn up in certain problems. This, and their separate numbering, suggests that Dardi has adopted the group wholesale and inserted it into the main body of his treatise. The character of the examples supports this inference. Two of them (no. 1 and no. 2) are strictly identical with examples (24) and (25) from **P**, which means that they are the only problems in Dardi's treatise that do not treat of pure numbers (but of lending with interest, as we remember), and that they are directly inspired by Jacopo's example (4a). The other two, $\alpha t + \beta C + \gamma CC = n + \delta K$ and $\alpha t + \gamma CC = n + \beta C + \delta K$, are based on the divided ten; had it not been for their constituting a closed group together with the former two, they could have been Dardi's invention; as things actually stand, this is unlikely.[58]

Dependency or independence

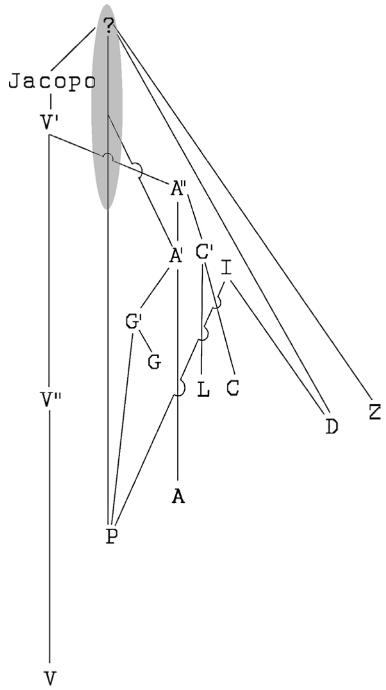
Dardi's many rules involving radicals and roots of numbers show him to share in the inspiration coming from "area ?." They do not tell whether he only received general inspiration and used that as a starting point for something going far beyond what his source tradition had done, or he borrowed in large scale. Some details in the chapter on roots suggest dependency on a model,[59] and the importance of a model for several features of the presentation of the six fundamental cases was already discussed. But the main body of the last chapter, the regular cases 1–194, may still have been structured by Dardi. Of the single cases, quite a few had been dealt with before, as we have seen, and Dardi may plausibly have known about that, just as he knew about the way to construct pseudo-complex examples by asking for numbers in given proportion (while copying no examples directly from predecessors known to us, neither from Jacopo nor from Gherardi); yet no evidence contradicts the conjecture that most were devised by Dardi.

The principle of creating new algebraic cases involving roots, as argued, was inspired from the unidentified "area ?." For the use of diagrams in the multiplication of binomials, Dardi seems to have shared the inspiration with **A**; **A** and **G** (and hence their shared archetype **A**′) make use of the related calculations with formal fractions. Finally, the order of the fundamental cases, the discussion of the double solution to the fifth case and the use of the rule of three as an algebraic tool shows affinity with Jacopo, while, as we have seen, the details of Dardi's text speak against direct borrowing; even Jacopo and Dardi hence share a source of inspiration.

Occam's razor is a dangerous weapon—wielding it was what led to the assumption that *abbaco* algebra had to come from Fibonacci. But ad hoc multiplication of explanatory entities beyond what is needed remains gratuitous, and a reasonable working hypothesis is that all these unidentifiable sources

58 Raffaella Franci [2002, 96–98] supposes that **P** and the very similar treatment of algebra in ms. 2Qq E13 (1398, d), Biblioteca Comunale di Palermo, which I have not seen, represent a synthesis combining material borrowed from Jacopo, Gherardi and Dardi. If this is meant to imply that the four irregular cases were Dardi's own invention and the borrowing made from his treatise, it is implausible. Quite apart from the above considerations speaking against Dardi's authorship, it would be strange that only these four wrong rules were borrowed and nothing else.

59 Thus, a number of procedures are illustrated by polynomials containing rational roots (e.g., $36/(\sqrt{4}+\sqrt{9}+\sqrt{16})$, treating them *as if* they were surds ("intendando de queste ℞ discrete como s'elle fosse indiscrete"—$\mathbf{D_1}$ fol. 3$^{\mathrm{v}}$, similarly $\mathbf{D_2}$ p. 62), the obvious point being that this allows control of the correctness of the result; however, no proof is ever made, nor is any other advantage taken of the choice of rational roots, except an unproven statement that the result coming from the calculation (in the example $\sqrt{40\frac{24}{25}}+\sqrt{92\frac{4}{25}}+\sqrt{5\frac{19}{25}}-\sqrt{163\frac{21}{25}}-\sqrt{10\frac{6}{25}}$) *can* be reduced. Omitting a proof when copying or missing the opportunity to make it when borrowing a style that prepares for it (or when using a model where such a thing has already happened at an earlier stage of transmission) may easily happen; but that the author prepares it repeatedly on his own initiative and then himself omits it each time is not very likely. Cf. also below.



Scheme 2.

of shared inspiration belong to the same area—that is, our "area ?" (in which case this area can hardly be Montpellier itself). The only extra entity we may be forced to accept could be the one that, in the wake of the success of Jacopo's higher-degree cases, invented **P**'s and Dardi's irregular cases—which we may designate **I**. These various observations cause the addition of new elements and links to our stemma (Scheme 2) without changing anything (except the age ascribed to "area ?"[60]) in what was already drawn up.

An instructive fragment: Giovanni di Davizzo

The manuscript Vat. Lat. 10488 of the Vatican Library, itself written in 1424, contains six pages with the heading "Algebra" (possibly more, see presently), said to be copied from a book written by Giovanni di Davizzo de l'abacho da Firenze on September 15th, 1339. Giovanni must have given this information

[60] As suggested in the diagram, however, an "area" or environment from which inspiration is drawn may well function for decades or even longer, unlike a particular treatise, and there is no reason it should appear as a single point in the stemma. In particular there is no reason that everybody who received inspiration from what was done in "area ?" had to have been inspired at the same moment. Nor is there evidently any reason to assume that the algebraic practice *within* this area underwent no development. **I**, the place where the new false rules of **D** and **P** originated, *could* thus be located within the "area ?"; below we shall encounter evidence suggesting that it was.

in his incipit in the same way as Jacopo—this is indeed the place where such information, when at all given, appears in the *abbaco* treatises. We can therefore safely assume it to be reliable.[61]

The first three pages (fols 28^{v}–29^{v}, original foliation) contain sign rules and rules for operations with monomials and binomials. Next follow rules for 19 algebraic cases (fols 29^{v}–31^{r}). After that comes a sequence of examples (fols 31^{r}–32^{r}) which are not likely to be from Giovanni's hand; it is improbable but not impossible that the heading "Algebra" was intended to cover even these.[62]

We shall first concentrate on the rules for the algebraic cases. According to [Franci, 2002, 87], the list contains all of Jacopo's 20 rules plus the 2 that are missing. This is mistaken, as can be seen in the scheme on p. 21, column **Z**. What we find is Jacopo's list of 20, with 2 omissions (no. 10, no. 16), and with no. 11 being replaced by the mirror case $\alpha K + \gamma t = \beta C$.[63] These 18 cases are numbered. The last, 19th case is unnumbered, and only partly legible—it equates $\gamma t + \alpha CC$ with a right-hand side that contains at least βK but perhaps more terms, and is thus neither to be found in **V** nor in any of the other treatises we have examined. The reason it is in part illegible is that a piece of paper has been glued over this rule, probably by someone who discovered that it was wrong; the paper has been removed, but the humid glue has made the paper almost as dark as the ink.[64]

The wording of the rules is mostly identical with that of Jacopo, but there are a fair number of deviations. Sometimes different expressions are used; sometimes, as mentioned, Jacopo's cases appear in mirror form. However, the decision whether to divide *per* or *in* is the same in all cases except three. If Giovanni had copied from Jacopo (whether directly or indirectly), there is no reason that agreement should be higher concerning this choice than in the rest of the wording. Instead, he must like Dardi be independent of Jacopo, but also (like Dardi, and with less independent initiative) draw on the same environment as Jacopo.

Confrontation of Jacopo's and Giovanni's lists of rules allows us to decipher the canon that governs the choice *in*/*per*. It is quite simple: two-term equations (those that can be reduced to homogeneous problems) divide *per*, while three-term equations (those that reduce to mixed second-degree equations) divide *in*. Jacopo, or a copyist between him and $\mathbf{V}'$, errs twice (no. 1 and no. 13); Giovanni, or his 15th-century copyist, errs once (no. 7). Once the canon is understood, we see that only one of the first-generation treatises errs in a way that might be independent of Jacopo, namely **C**. However, **G** and **A** both obey the canon in their new, falsely solved cases. Since they do not repair Jacopo's two errors, we may conclude that the canon was not known to the intermediate copyists ($\mathbf{A}''$, $\mathbf{A}'$, $\mathbf{G}'$), which means that the false rules

[61] Giovanni di Davizzo (fl. 1339–1344) belonged to a Florentine abbacist family, whose activity spanned almost the whole 14th century—his father, his brother, and two nephews were also *abbaco* masters, see [Ulivi, 2002, 39, 197, 200].

[62] The presentation of algebra is located within a long sequence of problems about finding numbers but just before the ones that make use of *cosa* and *censo*. It is therefore likely that the author discovered the need to present the tool for solving problems of this kind (and the conceptual framework within which they belong), and found an appropriate exposition in Giovanni's treatise.

[63] So is Jacopo's no. 18, Giovanni's case no. 16 being $\alpha CC + \gamma C = \beta K$. Like no. 11 of both, this case reduces to case no. 5. Since the mirror image of Jacopo's no. 18 does not appear separately in other treatises, I have not given it a separate line in the scheme.

[64] The headline "Algebra" stands outside the normal text frame, and must hence be a later addition. It is written in the same bright red ink as the numbering of the cases and the indication of paragraphs in the introduction—an ink type that is found nowhere else in the manuscript, although paragraphs are also indicated in other places in what perhaps was once red. Even the numbering of cases must therefore be a secondary addition, almost certainly made after the discovery that rule no. 19 was wrong. It is thus by mere accident that **Z** has a distinction between numbered and unnumbered cases that looks like Dardi's certainly genuine distinction between numbered rules of general validity and unnumbered rules that only hold in special cases.

are drawn from some other source where it was known and respected—that is, probably, the very area or environment where Jacopo and Giovanni (or somebody from whom he borrows) found their inspiration. Since Jacopo appears not to have known about these false rules, they could represent an invention made in that area after Jacopo's time.

The formulation of the sign rules coincides verbatim with that of **A**, which need not tell very much—the order "$++$, $--$, $+-$, $-+$" could be considered "natural," and the phrases themselves leave little room for variation. The rules for multiplying monomials are no more informative, beginning with the products $n \cdot K$, $n \cdot C$, and $n \cdot t$, then (after the insertion of the sign rules) going on in a rather disorderly way with $t \cdot t$, $C \cdot C$, $t \cdot C$, etc. Divisions are more interesting. Giovanni starts by stating that number divided by thing becomes number, number divided by *censo* becomes root, thing divided by *censo* becomes number, number divided by cube becomes cubic root ... and ends, after another 13 calculations of the kind,[65] by asserting that number divided by cube of cube of cube of cube becomes cubic root of cubic root of cubic root of cubic root. Close scrutiny reveals that the mathematical mistakes constitute a system—a rather ingenious but unfortunately incoherent experiment aiming, in modern terms, at extending the semigroup of nonnegative powers of the algebraic *thing* into a complete group. Not possessing negative exponents, Giovanni expresses t^{-p} as the pth "root," composing such "roots" additively in the way the positive powers are composed ("cube of cube" meaning $t^3 \cdot t^3$, not $(t^3)^3$); the "first root" is identified with number. The invention is likely to be Giovanni's own—it is difficult to see how it could be adopted for any algebraic purpose; but it may none the less reflect inspiration from an environment very interested in "roots" and experimenting with the power series of algebraic unknowns.

The rules for adding, subtracting, multiplying, and dividing square roots and for multiplying or dividing binomials are correct, and in so far uninformative. It is noteworthy, however, that five out of nine examples[66] operate with the roots of square numbers, *without taking advantage of this particular choice*, exactly as Dardi. As argued in the case of the latter, this must mean that the idea is borrowed from elsewhere (and borrowed badly). The idea is sufficiently unexpected to allow the conclusion that the two must have borrowed the inspiration from the same source tradition—though certainly not precisely the same source, given how different they are on almost all other accounts.

Giovanni is certainly much more similar to Jacopo than to Dardi, and the two appear to have used very similar sources though hardly precisely the same source. It is quite possible, indeed, that Giovanni's treatise contained examples which were omitted by the 15th-century compiler as not necessary for his purpose. The part of the Giovanni-excerpt which precedes the rules may therefore be similar to the introduction that can be presumed to have been lost from Jacopo's algebra (cf. footnotes 22 and 37). Indeed, Giovanni's introduction contains exactly the 66 lines normally found on two pages of **V**. Irrespective of the conscientious copying process leading to **V**, Jacopo's original need evidently not have had exactly the same number of lines to a page. Even with this proviso, however, the size of Giovanni's introduction fits the hypothesis that **V**′ is either is a copy of Jacopo's original having lost a sheet or identical with this mutilated original: the forgotten list of silver coins (see footnote 6), which fills out one page in **V**,

65 Only *censo* of cube (meaning $t^2 \cdot t^3$, not $(t^3)^2$) divided by cube, which leads to no negative exponent, is given correctly as *censo*.

66 Namely $\sqrt{9} \cdot \sqrt{9}$; $\sqrt{25}/\sqrt{9}$; $(5 + \sqrt{4}) \cdot (5 - \sqrt{9})$; $(7 + \sqrt{9}) \cdot (7 + \sqrt{9})$; $35/(\sqrt{4} + \sqrt{9})$.

must also have taken up one page in the manuscript from where it was forgotten (two steps back in the transmission chain, possibly back at **V**′).

Summing up

It should be firmly established by now that the algebra section of **V** belongs to the early 14th century, and thus that it is quite reasonable to trust both the ascription to Jacopo da Firenze and the date 1307; it should also be obvious that it does not draw even minimally on the preceding Latin treatises on algebra, neither on the translations from the Arabic nor on the *Liber abbaci*. In spite of his indubitable ultimate inspiration from the Arabic world it should also be incontrovertible that Jacopo has not drawn his material from the levels or types of Arabic algebra that have so far been examined by historians of mathematics; further, there can be no doubt that his access to the Arabic inspiration is indirect, mediated by a Romance-speaking (not Italian but most likely Catalan) environment already engaging in algebra.

Finally, it should be clear that for the next 30 years, all known Italian writers on algebraic matters drew on Jacopo's treatise, receiving only modest further inspiration from other sources. It was argued that the source for this supplementary inspiration (labeled "area ?") was also the area where Jacopo had found his inspiration, and that even Giovanni and Dardi, writing respectively 32 and 37 years after Jacopo, appear to have learned from this environment or area.

The existence of "area ?" followed from indirect arguments and, as far as its being a single area is concerned, from plying Occam's razor. However, several of the lines connecting "?" with known Italian writings in the revised Stemma 2 represent multiple inspirations: for instance, **V**′ and **D** having in common the order of the basic cases, the way the double solution to the fifth case is spoken of, and the use of the rule of three as an algebraic method. More decisively perhaps, Giovanni follows Jacopo's statement of the rules as precisely as can be done if no direct manuscript copying is involved while sharing with Dardi the futile predilection for taking roots of square instead of nonsquare numbers. Rejection of the assumption of one unitary area of inspiration would therefore force us to accept that each author belonging to the first generation of Italian vernacular algebra was inspired by several or all of a multiplicity of direct sources—a multiplicity of Romance-speaking sources, moreover, given the absence of Arabisms in the texts.

Since the only Romance-speaking area outside Italy where the next 150 years offer any evidence of algebraic interest is the Provençal–Catalan region (or perhaps the larger Iberian area), and since Montpellier itself appears not to have been a rich source, it seems reasonable to conclude that the "area ?" was indeed one area, identified with, located in, or encompassing the Catalan region (see also footnote 41 and preceding text).

Within this area, most of that by which the first generation of Italian algebra goes beyond al-Khwārizmī will already have been known either fully unfolded or in germ: polynomial algebra, the use of computational diagrams, the beginnings of formal computation. The easy way to create problems looking more complex than they are may have originated here, together with the interest in equations involving roots of numbers and perhaps other radicals. The carrying environment is likely to have been close to the teaching of commercial mathematics, given the generalized use of the rule of three and of the partnership structure and the preponderance of *mu'āmalāt* problems in **V**.

Quite independent of this we may notice that the points where the first generation of Italian vernacular algebra writers go beyond al-Khwārizmī were to become centrally important when, in Karpinski's words, two centuries of *abbaco* algebra "bore fruit in the 16th century in the achievements of Scipione del Ferro, Ferrari, Tartaglia, Cardan, and Bombelli": viz polynomial algebra, schematic number diagrams, the use of standard abbreviations in formal operations preparing the genuine symbolic operations of Descartes—and even the ambition to solve irreducible higher-degree problems, notwithstanding the fraud it had led to. The mathematical competence of a Jacopo and a Paolo Gherardi and even a Dardi is likely to have been well below that of Fibonacci, and many of the *abbaco* teachers may hardly deserve a characterization as "mathematicians"; but collectively *they* were the ones who prepared the algebraic takeoff of the 16th century and that whole transformation of the mathematical enterprise that it brought about in the 17th and 18th centuries.

Appendix A. Text excerpts from al-Khwārizmī's and Jacopo's algebra

The present Appendix serves two purposes. First, it contains representative text excerpts that illustrate the differences between al-Khwārizmī's and Jacopo's algebraic styles; second, it is meant to introduce those readers to the appearance of medieval rhetorical algebra who are not familiar with it.

Al-Khwārizmī's *Algebra* starts by introducing three kinds of numbers, "roots" (*jidhr*), "possessions" (*māl*, Latin translation *census*), and simple numbers belonging to neither of the preceding types.[67] "Possessions" are explained to be the products of roots with themselves (and the "roots" thus to be the square roots of possessions). If we choose the root to represent the unknown of a problem, the possession is thus the second power of this unknown (that this choice corresponds to al-Khwārizmī's thought is argued above).

This presentation of the basic entities is followed by six "cases," equation types combining two or three terms. The cases (1) to (3) are "simple":

(1) Possession is made equal to roots;
(2) Possession is made equal to number;
(3) Roots are made equal to number;

whereas the cases (4) to (6) are composite:

(4) Possession and roots are made equal to number;
(5) Possession and number are made equal to roots;
(6) Roots and number are made equal to possession.

As an example we may see how case (4) is dealt with:

> But possession and roots that are made equal to a number is as if you say, "A possession and ten roots are made equal to thirty-nine dragmas." The meaning of which is: from which possession, to which is added ten of its roots, is aggregated a total which is thirty-nine? The rule of which is that you halve the roots, which in this question are five. Then multiply them by themselves, and from them 25 are made. To which add thirty-nine, and they will be sixty-four. Whose roots you take, which is eight. Then subtract from it

67 I follow Gherardo da Cremona's translation [ed. Hughes, 1986, 233–249, *passim*], trying to be as literal as Gherardo himself with respect to the Arabic original. Gherardo's translation is, indeed, a better witness of the original than the published Arabic text, which is the outcome of several creative rewritings—see [Høyrup, 1998a].

> half of the roots, which is five. There thus remains three, which is the root of the possession. And the possession is nine. And if two possessions or three or more or fewer are mentioned, reduce them similarly to one possession. And what are with them of roots or numbers, reduce them similarly as you reduced the possession. Which is as if you say, “Two possessions and ten roots are made equal to forty-eight.” The meaning of which is that when to any two possessions are added the equal of ten roots of one of them, forty-eight are aggregated from it. Two possessions must hence be reduced to one possession. Now we know however that one possession is the half of two possessions. Reduce therefore everything that is in question to its half. And it is as if one said: “A possession and five roots are equal to twenty-four.” Which means that with any possession five of the roots of the same are added, from which twenty-four are aggregated. halve the roots, and they are two and a half. Multiply them then themselves, and they make six an a fourth. [. . .]

In the presentation of case (5), the possibility of a double solution is set forth in these words:

> But a possession and a number which are made equal to roots is as if you say: “A possession and twenty-one dragmas are made equal to ten roots.” The meaning of which is that when you add to any possession twenty-one, that which is aggregated will be equal to ten roots of that possession. Its rule is that you halve the roots, and they will be five, which you multiply in themselves, and twenty-five results. From this you then subtract the twenty-one which you mentioned together with the possession, and four will remain, of which you take the root, which is two. This you subtract from the half of the roots, which is five. Three thus remain, which is the root of the possession, which you wanted; and the possession is nine. And if you want it, add the same to the half of the roots, and it will be seven, which is the root of the possession; and the possession is forty-nine. Thus, when a question should happen to lead you to this case, try its truth with addition; and if it is not correct, then without doubt it will be with subtraction. And this is the only one of the three cases in which the halving of the roots must proceed with addition and subtraction. But know that when you halve the roots in this case and multiply them in themselves, and less results than the dragmas which are with the possession, then the question is impossible. And if it is equal to these same dragmas, then the root of the possession is equal to the half of the roots without addition or subtraction. And always when you get two possessions or more or fewer than one possession, reduce it to one possession, such as we showed in the first [mixed] case.

After the presentation of rules for the six cases, all followed by examples, al-Khwārizmī presents geometrical demonstrations for the composite cases. For case (4) he even offers two different proofs.[68]

The first proof is based on Fig. 2. It starts in this way:

> The cause [of the halving of the roots, characteristic of the mixed cases] is as follows. A possession and ten roots are made equal to thirty-nine dragmas. Make therefore for it a quadratic surface with unknown sides, which is the possession which we want to know together with its sides. Let the surface be *AB*. But each of its sides is its root. And each of its sides, when multiplied by a number, then the number which is aggregated from that is the number of roots of which each is as the root of this surface. Since it was thus said that there were ten roots with the possession, let us take a fourth of ten, which is two and a half. And let us make for each fourth a surface together with one of the sides of the surface. With the first surface, which is the surface *AB*, there will thus be four equal surfaces, the length of each of which is equal to the

[68] Linguistic and stylistic analysis suggests that they were written at different moments—the second proof being probably added in a process of rewriting; see [Høyrup, 1998a].

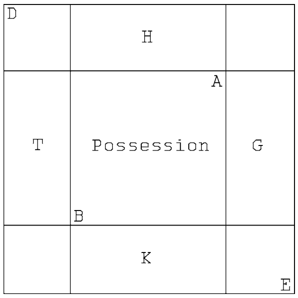


Fig. 2. Al-Khwārizmī's first proof for case (4).

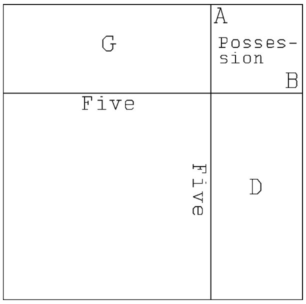


Fig. 3. Al-Khwārizmī's second proof for case (4).

> root of *AB* and the width two and a half. Which are the surfaces G, H, T, and K. From the root of a surface of equal and also unknown sides is lacking that which is diminished in the four corners, that is, from each of the corners is lacking the multiplication of two and a half by two and a half. What is needed in numbers for the quadratic surface to be completed is thus four times two and a half multiplied by itself. And from the sum of all this, twenty-five is aggregated.

Therefore, as the proof goes on, the completed square *DE* has the area $39 + 25 = 64$, and hence the side 8. Subtracting $2 \cdot 2\frac{1}{2} = 5 = 10/2$ we find the side of *AB* to be $8 - 5 = 3$.

The second proof corresponds better to the words of the rule. It is based on Fig. 3, where each of the rectangles D and G have an area $(10/2) \cdot r = 5r$, and the lower left completing square an area $(10/2)^2 = 5^2$. The total area of *AB* with rectangles D and G is thus $r^2 + 10r = 39$, and the area of the large square $39 + 25 = 64$.

After a section treating of the multiplication of binomials and other auxiliary matters come six problems, each illustrating one of the six cases. The illustration of the fourth case is somewhat atypical,[69] for which reason it is more illuminating to look at the illustration of the fifth case:

> "Divide ten into two parts, and multiply each of them with itself, and aggregate them. And it amounts to fifty-eight." Whose rule is that you multiply ten minus a thing by itself, and hundred and a possession minus

[69] "Multiply the third of a possession and a dragma with its fourth and a dragma, and let that which results be twenty." In order to resolve this problem, the possession is regarded as *a thing*, identified with the root, and the square of this root with the unknown possession that corresponds to the rule.

> twenty things results. Then multiply a thing with itself, and it will be a possession. Then aggregate them, and they will be one hundred, known, and two possessions minus twenty things, which are made equal to fifty-eight. Restore then one hundred and two possessions with the things that were taken away, and add them to fifty-eight. And you say: "One hundred, and two possessions, are made equal to fifty-eight and twenty things." Reduce it therefore to one possession. You therefore say: "Fifty and a possession are made equal to twenty-nine and ten things." Oppose hence by those, which means that you throw twenty-nine out from fifty. There thus remains twenty-one and a possession, which is made equal to ten things. Hence halve the roots, and five result [. . .].

Even though both solutions are valid, al-Khwārizmī only indicates the one obtained by subtraction.

Jacopo presents the fourth case, its rule and the corresponding example in **V**, fol. $38^{r–v}$. As we observe, the case is defined as nonnormalized and the rule formulated correspondingly; the "roots" are replaced by "things" even in the definition of the case; and no numerical example defined in terms of *censi* and things follows (*censo*, from Latin *census*, was the established translation of Arabic *māl* ("possession")):

> When the *censi* and the things are equal to the number, one shall divide by the *censi*, and then halve the things and multiply by itself and join above the number. And the root of the sum less the halving of the things is the thing.
>
> Example of the said rule. And I shall say thus: one lent to another 100 *libre* at the term of 2 years, to make (up at) the end of year.[70] And when it came to the end of the two years, then that one gave back to him *libre* 150. I want to know at which rate the *libra* was lent a month. Do thus: posit that it was lent at one thing in *denaro* a month, so that the *libra* turns out to be worth 12 things in *denaro* a year, which 12 things in *denaro* are the twentieth of a *libra*, so that the *libra* is worth 1/20 ⟨thing⟩ of a *libra* a year. And therefore say thus: if the *libra* is worth 1/20 of a *libra* a year, what will 100 *libre be* worth? Multiply 100 times 1/20. It makes 100/20, which are 5 things. Adjoin above 100 *libre*. They make 100 *libre* and 5 things for one year. Now if you want to know for the second year, multiply 100 *libre* and 5 things times 1/20 of thing. They make 5 things and 1/4 *censo*, which are to be adjoined to 100 *libre* and 5 things, which make 100 *libre* and 10 things and 1/4 *censo*. And as much are the 100 *libre* in 2 years, interest and capital together. And being lent the *libra* at one thing a month. And we know for sure that the 100 *libre* have gained 50 *libre* in 2 years. So that the 150 *libre* are the 100 *libre* and 10 things and 1/4 *censo*. So that the 100 *libre*, 10 things, and 1/4 *censo* are equal to 150 *libre*. Restore each part, that is, to remove 100 *libre* from each part, and you will get that 10 things and 1/4 *censo* are equal to 50. Now do so as our rule says, that is, to bring to one *censo*, that is, to divide by 1/4 *censo*, and you will get that i *censo* and 40 things are equal to 200 in numbers. Now halve the things. They are 20. Multiply by itself, it makes 400; adjoin above the numbers, they make 600. Find its root, which is surd, that is, as it is manifest, to have no precise root, and as much will we say that the thing is, that is the root of 600 less 20, that is the halving of the things. And we posited that the *libra* was lent at one thing of *denaro* a month, then we will say that the libra was lent at the root of 600 less 20 *denari* a month. And it goes well. And thus the similar computations are made.

Jacopo's presentation of the fifth case and its rule is much more concise than what al-Khwārizmī offers—the elaborate discussion of the double solution (quoted above) is brought within example (5b), omitting the question of solvability. Once again the problem is stated in nonnormalized form and the first step of the rule is a normalization, a division by the [coefficient of the] *censi*:

70 That is, at compound interest, computed yearly.

When the things are equal to the *censi* and to the number, one shall divide in the *censi*, and then halve the things and multiply by itself and remove the number, and the root of that which remains and then the halving of the things is the thing. Or indeed the halving of the things less the root of that which remains.

Appendix B. List of cited Arabic algebraic works

Abū Bakr, *Liber mensurationum* (*Kitāb al-misāḥa*?). Terminological considerations suggest an early date (c. 800?). [Ed. Busard, 1968, trans. Gherardo da Cremona].

Al-Khwārizmī, *Kitāb al-mukhtaṣar fī ḥisāb al-jabr wa'l-muqābala*. Written in Baghdad, earlier ninth century. [Ed. Hughes, 1986, trans. Gherardo da Cremona].

Ibn Turk, *Kitāb al-jabr wa'l-muqābala* (extant fragment containing geometrical proofs). Roughly contemporary with al-Khwārizmī. [Ed. Sayılı, 1962].

Thābit ibn Qurra, *Qawl fī taṣḥīḥ masā'il al-jabr bi'l-barāhīn al-handasīya*. Written in Baghdad, later ninth century. [Ed., trans. Luckey, 1941.]

Abū Kāmil, *Risāla fi'l-jabr wa'l-muqābala*. Late ninth or early tenth century. The surname al-Miṣrī means "the Egyptian," but does not prove that Abū Kāmil actually lived there. [Ed. Sesiano, 1993, trans. anon.]

Al-Karajī, *Kāfī fi'l-ḥisāb*. Written in Baghdad, c. 1011. [Ed., trans. Hochheim, 1878].

Al-Karajī, *Fakhrī fi'l-jabr wa'l-muqābala*. Written in Baghdad, c. 1011. Paraphrase [Woepcke, 1853].

Al-Khayyāmī, *Risāla fi'l-barāhīn 'alā masā'il al-jabr wa'l-muqābala*. Written in Samarkand, c. 1070. [Ed., trans. Rashed and Djebbar, 1981].

Ibn al-Yāsamīn, *Urjūza fi'l-jabr wa'l-muqābala*. Written in Morocco (or possibly Sevilla?) before 1190. [Ed., trans. Abdeljaouad, 2005.]

Ibn al-Bannā', *Talkhīṣ a'māl al-ḥisāb*. Written in Morocco in the later thirteenth century. [Ed., trans. Souissi, 1969.]

Ibn Badr, *Ikhtiṣār al-jabr wa'l-muqābala*. Written before 1343 (and after Abū Kāmil), perhaps in Muslim Spain. [Ed., trans. Sánchez Pérez, 1916.]

Al-Qalaṣādī, *Kashf al-asrār 'an 'ilm ḥurūf al-ghubār*. Written in Cairo in 1448, but the author had studied and taught in al-Andalus and the Maghreb. [Ed., trans. Souissi, 1988.]

Bahā' al-Dīn al-'Āmilī, *Khulāṣat al-ḥisāb*. Written in the late sixteenth or the early seventeenth century; the author was born in Syria and died in Iran. [Ed., trans. Nesselmann, 1843.]

Appendix C. Sigla

A: Florence, Riccardiana, Ms. 2263, fols 24^r–50^v. Anon., *Trattato dell'Alcibra amuchabile*. [Ed. Simi, 1994.]

C: Lucca, Biblioteca Statale, Ms. 1754, fols 50^r–52^r. Anon., "Le reghole della cosa". [Ed. Arrighi, 1973.]

D_1: Vatican Library, Chigi M.VIII.170, fols 2^r–114^r (original foliation). Dardi da Pisa, *Aliabraa argibra*.

D_2: Siena, Biblioteca Comunale, I.VII.17. Dardi da Pisa, *Aliabraa argibra*. [Ed. Franci, 2002.]

D_3: Manuscript in the Library of Arizona State University. Dardi da Pisa, *Aliabraa argibra*. I used Van Egmond's unpublished personal transcription.

F: Florence, Riccardiana, Ms. 2236. Jacobo da Firenze, *Tractatus algorismi* (abridged). [Ed. Simi, 1995.]

G: Florence, Magliabechiana, Cl. XI, 87, Paolo Gherardi, *Libro di ragioni*. [Ed. Arrighi, 1987; Van Egmond, 1978 (partial).]

L: Lucca, Biblioteca Statale, Ms. 1754, fols 81^r–82^v. Anon., "Le reghole dell'aligibra amichabile". [Ed. Arrighi, 1973.]

M: Milan, Trivulziana Ms. 90. Jacobo da Firenze, *Tractatus algorismi* (abridged).

P: Parma, Biblioteca Palatina, Ms. Pal. 312. Anon., *Libro di conti e mercatanzie*. [Ed. Gregori and Grugnetti, 1998.]

V: Vatican Library, Vat. Lat. 4826. Jacobo da Firenze, *Tractatus algorismi*. [Ed. Høyrup, 2000 (partial).]

Z: Vatican Library, Vat. Lat. 10488, The algebraic fragment from Giovanni di Davizzo.

References

Abdeljaouad, Mahdi, 2002. Le Manuscrit mathématique de Jerba: Une pratique des symboles algébriques maghrébins en pleine maturité. Présenté au Septième Colloque sur l'histoire des mathématiques arabes (Marrakech, 30–31 mai et 1er juin 2002).

Abdeljaouad, Mahdi, 2005. 12th century algebra in an Arabic poem: Ibn Al-Yāsamīn's *Urjūza fi'l-jabr wa'l-muqābala*. Manuscript, January 2005.

Allard, André (Ed.), 1992. Muḥammad ibn Mūsā al-Khwārizmī, Le Calcul indien (Algorismus). Histoire des textes, édition critique, traduction et commentaire des plus anciennes versions latines remaniées du XIIe siècle. Collection Science dans l'histoire: Collection d'Études classiques. Blanchard/Société des Études Classiques, Namur/Paris.

Arrighi, Gino, 1967. Un « programma » di didattica di matematica nella prima metà del Quattrocento (dal Codice 2186 della Biblioteca Riccardiana di Firenze). Atti e memorie dell'Accademia Petrarca di Lettere, Arti e Scienze di Arezzo, Nuova Serie 38, 117–128.

Arrighi, Gino (Ed.), 1970. Piero della Francesca, Trattato d'abaco. Dal codice ashburnhamiano 280 (359*–291*) della Biblioteca Medicea Laurenziana di Firenze. Testimonianze di storia della scienza, vol. 6. Domus Galileana, Pisa.

Arrighi, Gino (Ed.), 1973. Libro d'abaco. Dal Codice 1754 (sec. XIV) della Biblioteca Statale di Lucca. Cassa di Risparmio di Lucca, Lucca.

Arrighi, Gino (Ed.), 1987. Paolo Gherardi, Opera mathematica: Libro di ragioni – Liber habaci. Codici Magliabechiani Classe XI, nn. 87 e 88 (sec. XIV) della Biblioteca Nazionale di Firenze. Pacini-Fazzi, Lucca.

Boncompagni, Baldassare (Ed.), 1857a. Scritti di Leonardo Pisano matematico del secolo decimoterzo. I. Il Liber abaci di Leonardo Pisano. Tipografia delle Scienze Matematiche e Fisiche, Rome.

Boncompagni, Baldassare (Ed.), 1857b. Trattati d'aritmetica. I. Algoritmo de numero indorum. II. Ioannis Hispalensis Liber algorismi de pratica arismetrice. Tipografia delle Scienze Fisiche e Matematiche, Rome.

Busard, H.L.L., 1968. L'Algèbre au moyen âge : Le « Liber mensurationum » d'Abû Bekr. Journal des Savants (April–June), 65–125.

Cantor, Moritz, 1900. Vorlesungen über Geschichte der Mathematik. Zweiter Band, Von 1200–1668. Teubner, Leipzig.

Caunedo del Potro, Betsabé, Córdoba de la Llave, Ricardo (Eds.), 2000. El arte del alguarismo. Un libro castellano de aritmética comercial y de ensayo de moneda del siglo XIV. (Ms. 46 de la Real Colegiato de San Isidoro de León). Junta de Castilla y León, Consejeria de Educación y Cultura, Salamanca.

Davis, Margaret Daly, 1977. Piero della Francesca's Mathematical Treatises. The "Trattato d'abaco" and "Libellus de quinque corporibus regularibus'. Longo Editore, Ravenna.

Djebbar, A., 1981. Enseignement et recherche mathématiques dans le Maghreb des XIIIe–XIVe siècles (étude partielle). Publications mathématiques d'Orsay, 81–102. Université de Paris-Sud, Orsay.

Djebbar, A., 1988. Quelques aspects de l'algèbre dans la tradition mathématique arabe de l'Occident musulman. In: Histoire des Mathématiques Arabes. Premier colloque international sur l'histoire des mathématiques arabes, Alger, 1.2.3 décembre 1986. Actes. La Maison des Livres, Algiers, pp. 101–123.

Eneström, Georg, 1906. Über zwei angebliche mathematische Schulen im christlichen Mittelalter. Bibliotheca Mathematica 3 (Folge 7), 252–262.

Franci, Raffaella (Ed.), 2001. Maestro Dardi, Aliabraa argibra, dal manoscritto I. VII.17 della Biblioteca Comunale di Siena. Quaderni del Centro Studi della Matematica Medioevale, vol. 26. Università degli Studi di Siena, Siena.

Franci, Raffaella, 2002. Trends in fourteenth-century Italian algebra. Oriens–Occidens (4), 81–105.

Goldthwaite, Richard A., 1972. Schools and teachers of commercial arithmetic in Renaissance Florence. Journal of European Economic History 1, 418–433.

Gregori, Silvano, Grugnetti, Lucia (Eds.), 1998. Anonimo (sec. XV), Libro di conti e mercatanzie. Università degli Studi di Parma, Dipartimento di Matematica, Parma.

Hochheim, Adolph (Ed., trans.), 1878. Kâfî fîl Hisâb (Genügendes über Arithmetik) des Abu Bekr Muhammed ben Alhusein Alkarkhi, 3 vols. Louis Nebert, Halle.

Høyrup, Jens, 1998a. "Oxford" and "Cremona": On the relations between two versions of al-Khwārizmī's Algebra. In: Actes du 3me Colloque Maghrébin sur l'Histoire des Mathématiques Arabes, Tipaza (Alger, Algérie), 1–3 Décembre 1990, vol. II. Association Algérienne d'Histoire des Mathématiques, Algiers, pp. 159–178.

Høyrup, Jens, 1998b. A new art in ancient clothes. Itineraries chosen between Scholasticism and Baroque in order to make *Algebra* appear legitimate, and their impact on the substance of the discipline. Physis N.S. 35, 11–50.

Høyrup Jens (Ed., trans.), 2000. Jacobus de Florentia, Tractatus algorismi (1307), the chapter on algebra (Vat. Lat. 4826, fols 36^v–45^v). Centaurus 42, 21–69.

Høyrup, Jens, 2001. The founding of Italian vernacular algebra. In: Commerce et mathématiques du moyen âge à la renaissance, autour de la Méditerranée. Actes du Colloque International du Centre International d'Histoire des Sciences Occitanes (Beaumont de Lomagne, 13–16 May 1999). Éditions du C.I.S.H.O., Toulouse, pp. 129–156.

Hughes, Barnabas O.F.M., 1986. Gerard of Cremona's translation of al-Khwārizmī's Al-Jabr: A critical edition. Mediaeval Stud. 48, 211–263.

Hughes, Barnabas, 1987. An early 15th-century algebra codex: A description. Historia Math. 14, 167–172.

Karpinski, Louis C., 1910. An Italian algebra of the fifteenth century. Bibliotheca Math. 3 (Folge 11), 209–219.

Karpinski, Louis C., 1929. The Italian arithmetic and algebra of Master Jacob of Florence, 1307. Archeion 11, 170–177.

Kaunzner, Wolfgang, 1985. Über eine frühe lateinische Bearbeitung der Algebra al-Khwārizmīs in MS Lyell 52 der Bodleian Library Oxford. Arch. Hist. Exact Sci. 32, 1–16.

Kaunzner, Wolfgang, 1986. Die lateinische Algebra in MS Lyell 52 der Bodleian Library, Oxford, früher MS Admont 612, pp. 47–89. In: Hamann, G. (Ed.), Aufsätze zur Geschichte der Naturwissenschaften und Geographie. (Österreichische Akademie der Wissenschaften, Phil.-Hist. Klasse, Sitzungsberichte, Bd. 475). Österreichische Akademie der Wissenschaften, Vienna.

Libri, Guillaume, 1838–1841. Histoire des mathématiques en Italie. 4 vols. Paris.

Luckey, Paul, 1941. T̲ābit b. Qurra über den geometrischen Richtigkeitsnachweis der Auflösung der quadratischen Gleichungen. Sächsische Akademie der Wissenschaften zu Leipzig. Mathematisch-Physische Klasse. Berichte 93, 93–114.

Malet, Antoni (Ed.), 1998. Francesc Santcliment, summa de l'art d'Aritmètica. Eumo Editorial, Vic.

Musharrafa, 'Alī Muṣṭafā, Ahmad, Mursī Muḥammad (Eds.), 1939. Al-Khwārizmī, Kitāb al-muḫtaṣar fī ḥisāb al-jabr wa'l-muqābala. Cairo.

Nesselmann, G.H.F. (Ed., trans.), 1843. Mohammed Beha-eddin ben Alhossain aus Amul, Essenz der Rechenkunst, arabisch und deutsch. Reimer, Berlin.

Procissi, Angiolo (Ed.), 1954. I Ragionamenti d'Algebra di R. Canacci. Bollettino Unione Matematica Italiana, serie III, 9, 300–326, 420–451.

Raṅgācārya, M. (Ed., trans.), 1912. The Gaṇita-sāra-sangraha of Mahāvīrācārya with English translation and notes. Government Press, Madras.

Rashed, Roshdi, Djebbar, Ahmed (Eds.), 1981. L'Oeuvre algébrique d'al-Khayyām. Établie, traduite et analysée. Sources and Studies in the History of Arabic Mathematics, 3. University of Aleppo, Institute for the History of Arabic Science, Aleppo.

Rosen, Frederic (Ed., trans.), 1831. The Algebra of Muhammad ben Musa. The Oriental Translation Fund, London.

Salomone, Lucia (Ed.), 1982. M^o Benedetto da Firenze, La reghola de algebra amuchabale dal Codice L. IV.21 della Biblioteca Comunale de Siena. (Quaderni del Centro Studi della Matematica Medioevale, 2). Servizio Editoriale dell'Università di Siena, Siena.

Saliba, George, 1972. The Meaning of al-jabr wa'l-muqābalah. Centaurus 17, 189–204.

Sánchez Pérez, José A. (Ed., trans.), 1916. Compendio de algebra de Abenbéder. Texto arabo, traduccion y estudio. In: Junta para Ampliación de Estudios e Investigaciones Científicas, 47. Centro de Estudios Históricos, Madrid.

Sarton, George, 1931. Introduction to the History of Science. II. From Rabbi ben Ezra to Roger Bacon. Carnegie Institution of Washington, Publication, vol. 376. William & Wilkins, Baltimore.

Sayılı, Aydın, 1962. Abdülhamid ibn Türk'ün katışık denklemlerde mantıkî zaruretler adlı yazısı ve zamanın cebri (Logical Necessities in Mixed Equations by 'Abd al Ḥamîd ibn Turk and the Algebra of his Time). Publications of the Turkish Historical Society, Series VII, N^o 41. Türk Tarih Kurumu Basımevi, Ankara.

Sesiano, Jacques, 1988. Le Liber Mahamaleth, un traité mathématique latin composé au XII^e siècle en Espagne. Histoire des Mathématiques Arabes. Premier colloque international sur l'histoire des mathématiques arabes, Alger, 1.2.3 décembre 1986. Actes. La Maison des Livres, Algiers. pp. 69–98.

Sesiano, Jacques, 1993. La version latine médiévale de l'Algèbre d'Abū Kāmil. In: Folkerts, M., Hogendijk, J.P. (Eds.), Vestigia Mathematica. Studies in Medieval and Early Modern Mathematics in Honour of H.L.L. Busard. Rodopi, Amsterdam/Atlanta, pp. 315–452.

Simi, Annalisa (Ed.), 1994. Anonimo (sec. XIV), Trattato dell'alcibra amuchabile dal Codice Ricc. 2263 della Biblioteca Riccardiana di Firenze. Quaderni del Centro Studi della Matematica Medioevale, vol. 22. Servizio Editoriale dell'Università di Siena, Siena.

Simi, Annalisa, 1995. Trascrizione ed analisi del manoscritto Ricc. 2236 della Biblioteca Riccardiana di Firenze. Dipartimento di Matematica, Università degli Studi di Siena. Rapporto Matematico N^o 287.

Souissi, Mohamed (Ed., trans.), 1969. Ibn al-Bannā’, Talkhīṣ a‘māl al-ḥisāb. Texte établi, annoté et traduit. L’Université de Tunis, Tunis.
Souissi, Mohamed (Ed., trans.), 1988. Qalaṣādī, Kashf al-asrār ‘an ‘ilm ḥurūf al-ġubār. Maison Arabe du Livre, Carthage.
Ulivi, Elisabetta, 2002. Benedetto da Firenze (1429–1479), un maestro d’abbaco del XV secolo. Con documenti inediti e con un’Appendice su abacisti e scuole d’abaco a Firenze nei secoli XIII–XVI. Bollettino di Storia delle Scienze Matematiche 22 (1), 3–243.
Van Egmond, Warren, 1976. The commercial revolution and the beginnings of Western mathematics in Renaissance Florence. Dissertation, Indiana University.
Van Egmond, Warren, 1978. The earliest vernacular treatment of algebra: The Libro di ragioni of Paolo Gerardi (1328). Physis 20, 155–189.
Van Egmond, Warren, 1980. Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abbacus Manuscripts and Printed Books to 1600. Monografia N. 4. Istituto e Museo di Storia della Scienza, Florence.
Van Egmond, Warren, 1983. The algebra of Master Dardi of Pisa. Historia Math. 10, 399–421.
Woepcke, Franz, 1853. Extrait du Fakhrî, traité d’algèbre par Aboû Bekr Mohammed ben Alhaçan Alkarkhî; précédé d’un mémoire sur l’algèbre indéterminé chez les Arabes. L’Imprimerie Impériale, Paris.