Generosity

No doubt, but at times excessive and delusive

Høyrup, Jens

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ROSKILDE UNIVERSITETSCENTER
Faggruppen for filosofi og videnskabsteori

ROSKILDE UNIVERSITY
Section for philosophy and science studies

Generosity: no doubt, but at times excessive and delusive

JENS HØYRUP

FILOSOFI OG VIDENSKABSTEORI PÅ ROSKILDE UNIVERSITETSCENTER

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Referee: Arne Thing Mortensen

Jens Høyrup
jensh@ruc.dk
http://www.akira.ruc.dk/~jensh

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Abstract

One of the ways in which the artificial languages of mathematics are "generous", that is, in which they assist the advance of thought, is through their establishment of advanced operatory structures that permit an even further advance of intuition. However, this generosity may be delusive, suggest ideas which in the longer run turn out to be untenable. The paper analyses two cases of "honest generosity", namely a "proof" of the sign rule "less times less makes plus" from the 1340s and a result in partition theory obtained by Euler by means of rash manipulations of infinite series and products, one perilous case – Cantor's introduction of transfinite numbers from 1895 – and (in modern terms) a failed attempt to extend the semi-group of algebraic powers into a complete group, also from c. 1340.

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Language, operatory thought and intuition

Language is generous: it allows us to communicate and thereby to undertake tasks we could not even image when left on our non-communicative own. Even when left on our own, the tool of auto-communication gives us insights we would never be able to attain without the support of language. In Piagetian terms, language is essential for the passage from sensori-motor to operatory cognition; moreover, the possession of operatory thought at one level allows our intuition to work at further levels it could never reach when based on sensori-motor cognition and non-operatory language alone (as the situation was, for all of us, between roughly two and six years).[1]

[1] For my interpretation of Piaget's thought (and for the particular emphasis on the notions of familiarity and concreteness) I refer to [Høyrup 2000a: 235–255].

A short example may serve to familiarize with the essential notions. Time and again, Piaget refers to this problem:

(1) Edith is fairer than Susan.
(2) Edith is darker than Lily.
Who is darkest?

The problem was one of those which appeared in the IQ-tests designed by Cyril Burt (tests which the young Piaget re-standardized for a Swizz environment) and one of those which made Piaget discover that the "errors" of young children were not mere errors but symptoms of a different way to organize thought. As told by Piaget, the child of ten will typically argue that both Edith and Susan are fair, while both Edith and Lily are dark. Edith must thus be in between, Lily must be darkest, and Susan is fairest. For such complex matters, the thought of the child is still intuitive, and it argues from the way language distributes objects in categories ("fair" / "dark"). Only at (typically) twelve will the child be as familiar with the mutual relations between the categories expressed by language as it is since the age of 2–3 with those between (say) boxes of different size that can be fitted into each other. Now it may, for example, invert the first statement into "Susan is darker than Edith" – its thinking has become operatory even for questions of this kind.

Language creates a wholly new ontology, consisting of entities that can only be experienced, together with their properties and the network of relations within which they are spun, through their linguistic realization. Though they receive part of their emotional loading from non-linguistic sources – in part non-linguistic symbols, in part directly lived experience – none of those ideals and imagined communities for which (and in whose *name*) some of us kill and others die exist without the mediation of language.

Mathematics, an artificial language *par excellence*,[2] is also generous; directly, and through its service for sciences concerned with material reality, it has increased the efficiency of this same killing (and of most of human technological practice in general) immensely;[3] it has done so in particular since the innovations in symbolism of the Early Modern epoch allowed mathematical thought to emulate the grammar of natural language freely [Høyrup 2000b].

A different aspect of the efficiency of the mathematical language is that the familiarity with very abstract objects with which it provides us allows the intuition of the mathematically trained to go at least one step beyond that which is safely established and meaningful, and often to do so in a way which is vindicated by later developments.

Oft-discussed instances of this are the introduction of complex numbers and the infinitesimal calculus in Leibnizian formulation, the former "guaranteed" by the applicability of the rules of algebra as known to Cardano (and, in more mature versions, by later generations), the second by the arithmetic of fractions and sums. They have the disadvantage as objects for our discussion that the full development in both cases took centuries, and that pushing-together of this long development into an instant picture might make us believe that the resulting

---

Piaget used observations of this kind to distinguish a stage of "concrete operations" from one of "formal thought". My own observation of professional mathematicians ("formal animals" *par excellence* if such creatures exist) confronted with the Edith-Susan-Lily problem is that they immediately begin, either to represent the girls by three fingers or by writing things down, using a symbol which imitates the nestability of boxes (namely "<"). From the point of view of practical cognitive psychology, "formal thought" is a philosophical hoax, what is in play is assimilation of new objects – often manipulable on paper – to the realm of that with which we are so familiar that it has become "concrete".

[2] At least if we accept the broad delineation of the term given in [Staal 2006: 89–91], as I shall do here.

[3] If anybody should feel that a reference is needed, [Booß-Bavnbek & Høyrup (eds) 2003] may do.

sham snapshot really represent exactly that which was vindicated by later formally based insights. Instead we may look at two real snapshots, one from the fourteenth and one from the eighteenth century. Afterwards we shall address two cases of excessive or delusive generosity, illustrating that the quotations from Goethe and Tom Lehrer apply also to artificial languages.

Honest generosity I: Dardi of Pisa

Our first case is Dardi of Pisa’s proof of the least obvious of the sign rules, “less times less makes plus”, from his *Aliabraa argibra*, apparently written in 1344, and probably borrowing from earlier sources even part of that which we do not recognize from elsewhere.[4]

> Ora te voio demostrar per numero como men via m̃ fa più a zo che ogni fiada ch’el te vegniva in costruzion a moltiplicar m̃e via m̃e che tu vezi zertamente ch’el fa più, com’io te darò l’exenplo manifesto. 8 via 8 fa 64, e questo 8 si è 2 m̃ de 10, e a moltiplicar lo per l’altro 8, lo qual è ancora 2 m̃ de 10, die similmente far 64. Echo la prova. Moltiplica 10 via 10, fa 100, e 10 via 2 m̃ fa 20 m̃, e l’altro 10 via 2 m̃ fa 40 m̃, lo qual 40 m̃ traze da 100, e roman 60. Ora manca per conplir la moltiplication a moltiplicar 2 m̃ via 2 m̃, monta 4 più, lo qual 4 più zonzi sovra 60, monta 64. E se 2 m̃ via 2 m̃ fosse 4 m̃, questo 4 m̃ se doverave trazer der 60, resterave 56, e così parerave che 10 m̃ 2 via 10 m̃ 2 fesse 56, la qual cosa non è vera. E così etiandio se 2 m̃ via 2 m̃ fesse niente, la moltiplication de 10 m̃ 2 via 10 m̃ 2 vegnirave a far 60, la qual cosa è ancora false. Adunqua m̃ via m̃ de necesità vien a far più.

In fairly literal translation but without abbreviations:[5]

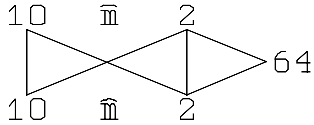
> Now I want to demonstrate by number how less times less makes plus, so that every times you have in a construction to multiply less times less you see with certainty that it makes plus, of which I shall give you an obvious example. 8 times 8 makes 64, and this 8 is 2 less than 10, and to multiply by the other 8, which is still 2 less than 10, it should similarly make 64. This is the proof. Multiply 10 by 10, it makes 100, and 10 times 2 less makes 20 less, and the other 10 times 2 less makes 40 less, which 40 less detract from 100, and there remains 60. Now it is left for the completion of the multiplication to multiply 2 less times 2 less, it amounts to 4 plus, which 4 plus join above 60, it amounts to 64. And if 2 less times two less had been 4 less, this 4 less should have been detracted from 60, and 56 would remain, and thus it would appear that 10 less 2 times 10 less two had been 56, which is not true. And

[4] See [Høyrup 2006: 25–31] both for this, for a discussion of the quality of the manuscripts, and for references to earlier work on Dardi and his treatise. The present quotation is from the manuscript Vatican, Chigi M.VIII.170, fol. 5^{v}, from c. 1395 (the earliest of the known manuscripts), with diacritics and punctuation added. The corresponding passage in the Siena manuscript (c. 1470) is in [Franci 2001: 44].

[5] My translation, as everywhere in the following where nothing else is said.

> so also if 2 less times 2 less had been nothing, then the multiplication of 10 less 2 times 10 less 2 would come to be 60, which is still false. Hence less times less by necessity comes to be plus.

This almost looks like a natural-language discussion, but the passage is followed by a diagram representing the genuine medium of the argument:



We do not know whether the particular proof is Dardi's invention,[6] but the diagram for cross-multiplication (used regularly by Dardi) recurs in the anonymous *Trattato dell'alcibra amuchabile* [ed. Simi 1994] from c. 1365, a treatise that does not appear to borrow from Dardi. We may thus legitimately assume it to represent a shared heritage. A first guess at the source might be the Maghreb school, where algebraic symbolism of a similar kind had been on its way since the twelfth century (the symbolic expressions appearing in particular "windows" and not as part of the running text, exactly like here).[7] In the Maghreb writings we also find formal operations on fractions quite similar to operations turning up in the *Trattato dell'alcibra amuchabile* and other fourteenth-century Italian works (operations where one or more denominators are polynomials instead of numbers). However, the Maghreb writers express the multiplication of polynomials in a different scheme (with both factors on the same line, and the partial products written below in a way reminding more of the multiplication of multi-place numbers); although Dardi's ultimate dependency on Maghreb sources for the use of symbols is a reasonable assumption, the particular scheme he uses here appears to have been developed somewhere on the road from the Maghreb to Pisa.[8]

---

[6] In any case, Ibn al-Bannāʾ had used a somewhat similar argument some decades before in the *Rafʿ al-ḥijāb ʿan wujūh aʿmāl al-ḥisāb* – see [Djebbar 2005: 91].

[7] A convenient summary, including important new material, is [Abdeljaouad 2002].

[8] Since a somewhat similar diagram appears once in the margin of the *Liber mahamaleth* (Paris, Bibliothèque Nationale, ms latin 7377A, fol. 109r), an Iberian location is plausible. It illustrates the sign rules (see also [Sesiano 1988: 96 n. 14]):

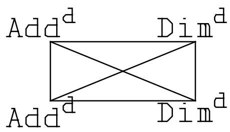
> From the multiplication of added in added comes nothing but added. From the multiplication of added in diminished comes nothing but diminished, and from the multiplication of diminished in added diminished. From the multiplication of diminished in diminished comes nothing but added.

The diagram is equally abstract, and involves no numbers. No result therefore occurs, nor the two lines pointing to it; on the other hand, two horizontal lines that do not

Whether it be his own or a borrowing, Dardi's argument for the sign rule was found convincing and was not forgotten. Luca Pacioli repeats it in the *Summa de arithmetica* [1494: 113^r], now with the diagram in the margin, and with an explicit reference to the cross-multiplication; apart from Luca's verbosity (which allows us to see that he finds the very concept *absurda* and an abuse but none the less necessary – Pacioli, indeed, thinks in terms of *negative*, not merely subtractive numbers as does Dardi[9]), the only innovation is that the alternative to the alternative is now (–2)·(–2) = –2, not (–2)·(–2) = 0. We may guess that the attempt of both authors to cover "all" possibilities and not only the ones that first had come intuitively to their mind (*viz* plus and less 4) reflects a wish to make "real" mathematics (we know nothing about Dardi's training, but Pacioli knew his Euclid well, he even published an edition of the *Elements*). As a matter of fact, even these attempted exhaustions of all possibilities are based on intuitions, inspired by things that sometimes happen in the kind of computations they knew (as revealed by the words in which they express them): either when a multiplication by something plus nothing is performed, or in a multiplication by 1.[10]

Dardi probably did not know, but his argument is almost in the style of classical *analysis:* in analysis, we assume that a solution exists and that it behaves as the objects with which we are already familiar; from this we derive what it *must* be. The only shortcoming (the typical shortcoming of intuitive thought) with respect to this methodological yardstick is that Dardi does not show that (–2)·(–2) *has to* be +4, instead he shows that +4 fits while alternatives that come to his mind do not. He does not perform the ensuing *synthesis* (the step, we may say, from intuition to critique), which in this case would indeed imply a complete

correspond to a multiplication (but rather to the m̂ in Dardi's diagram) have been added:



No comparable diagram is found anywhere in the *Liber abbaci* – at least not in the manuscript published in [Boncompagni 1857].

[9] This is also visible when he explains that a number of this kind is "less than zero and in consequence a debt".

[10] The history of the argument does not stop here, Pacioli's diagram is repeated in the margin of Feliciano de Latezio's *Libro di arithmetica et geometria speculativa et practicale*, first published in 1526 [Feliciano de Latezio 1563: K3^r], now accompanying the example (14–4)·(14–4) and without rejection of alternatives – but still only showing that the sign rule *meno via meno fa più* fits, not deriving it as a necessity (cf. imminently).

justification of the possibility of negative numbers. Only the nineteenth century provided that proof, but long before that it had been seen that reliance on the sign rules (and the ensuing operations with negative numbers) never led to contradictions.[11]

Honest generosity II: Euler

As a second example of generosity that turned out to be reliable we may look at two paragraphs from Euler's *Introductio in analysin infinitorum* [1748: I, 271–272]:

> 325. Quod si autem hoc productum
>
> $$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6), \&c.,$$
>
> evolvatur, sequens prodibit series
>
> $$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+\&c,$$
>
> in qua quisque coefficiens indicat, quot variis modis exponens ipsius x per additionem numerorum inaequalium oriri possit. Sic numerus 9 octo variis modis per additionem ex numeris inaequalibus formari potest:
>
> | | |
> |---|---|
> | 9 = 9 | 9 = 6+2+1 |
> | 9 = 8+1 | 9 = 5+4 |
> | 9 = 7+2 | 9 = 5+3+1 |
> | 9 = 6+3 | 9 = 4+3+2 |
>
> 326. Ut comparationem inter has formas instituamus, sit
>
> $$P = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\&c.,$$
>
> &
>
> $$Q = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\&c.,$$
>
> erit
>
> $$PQ = (1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\&c.,$$
>
> qui factores cum omnes in P contineantur, dividatur P per PQ; erit
>
> $$\frac{1}{Q} = (1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\&c.,$$
>
> ideoque
>
> $$Q = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)etc.}$$
>
> quae fractio si evolvatur, prodibit series, in qua quisque coëfficiens indicabit, quot variis modis exponens ipsius *x* per additionem ex numeris imparibus produci possit. Cum igitur haec expressio aequalis sit illi, quam in §. praecedente contemplati sumus,

[11] Strictly speaking, the nineteenth century only proved that integers are just as well-founded as natural numbers. To my knowledge, the acceptability of the latter still relies on the experience that paradoxes do not turn up.

sequitur hinc istud theorema:

Quot modis datus numerus per additionem formari potest ex omnibus numeris integris inter se inaequalibus, totidem modis idem numerus formari poterit per additionem ex numeris tantum imparibus, sive aequalibus sive inaequalibus.

In translation:

§ 325. If, however,[12] one develops the product

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\cdots$$

as a series, then this will be:

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+\cdots\ ,$$

and here every coefficient indicates in how many ways the exponent of the corresponding power of x may arise from addition of unequal integer numbers. Thus the number 9 can be composed in 8 different ways by addition of unequal numbers, namely:

9 = 9	9 = 6+2+1
9 = 8+1	9 = 5+4
9 = 7+2	9 = 5+3+1
9 = 6+3	9 = 4+3+2

§ 326. In order to compare now these two forms, let:

$$P = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\cdots$$

$$Q = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\cdots\ ,$$

thus:

$$PQ = (1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\cdots$$

Since all factors of this product are also contained in P, then it follows, if P is divided by PQ:

$$\frac{1}{Q} = (1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\cdots\ ,$$

and hence

$$Q = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\cdots}$$

from which fraction, if it is developed, one gets a series in which every coefficient indicates in how may different ways the exponent of the power of x in question may be produced from addition of odd numbers. As now this expression is the same as the one which we considered in the preceding paragraph, then this theorem follows:

[12] The previous paragraph has developed the unending fraction

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots}$$

as an infinite series $1+a_1x+a_2x^2+a_3x^3+\cdots$ (my formulation), and shown that a_n coincides with the number of different ways n can be written as a sum (3 thus as 3, as 2+1, and as 1+1+1).

Every number can be composed from odd numbers alone, be they equal, be they unequal, in exactly as many ways, as it can be formed by addition of all integer but mutually different numbers.

At first we observe (with some amazement, if we have not studied such matters before) that Euler's theorem is correct; secondly, that none of those who would have loved to know such wonderful properties of numbers in Antiquity or the Middle Ages (Islamic as well as Latin) had ever noticed them; thirdly, that only Euler's highly developed mathematics allowed their derivation – no everyday intuition in the style of

If I had contracted a debt of 2 florins 2 times less, my total net possession would have been larger by 4 florins

would ever lead forward. The symbolic language – now far removed from the inspiration once received from the Maghreb mathematicians – had proven really generous. As a matter of facts, it gave Euler and his readers much more astonishing results by quite analogous means, for instance concerning the connection between the sequence of prime numbers and the ζ-function.[13]

On the other hand, it is obvious that Euler's arguments are not rigorous. Euler does not himself present the intuitive arguments that are needed, in a kind interpretation he leaves it to the intelligent reader to find them (but see below, note 14). If the very first step is pinned out, the argument has to run more or less as follows:

(1) The result of the multiplication must be of the form $\sum_{i=0}^{\infty} a_i \cdot x^i$.

(2) The power x^0 results only if the term 1 is chosen in all factors, whence $a_0 = 1$.

(3) The power x^1 results only if x is chosen from the first factor, and 1 from all the others, whence $a_1 = 1$.

(4) The power x^2 results only if x^2 is chosen from the second factor, and 1

---

[13] $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\prod_{m\ prime} (1-\frac{1}{m^s})}$ [Euler 1748: I, 225*f*]. The trick is simple: Each fraction $\frac{1}{1-\frac{1}{m^s}}$ is developed as a power series $1 + \frac{1}{m^s} + (\frac{1}{m^s})^2 + (\frac{1}{m^s})^3 + \ldots$. Multiplying these series together, keeping only those partial products where the number of factors different from 1 is finite, and using that each number can be expressed in a unique way as a product of powers of primes, we get the result. Provided, evidently, that the operations are meaningful, which on its part is *not* immediately evident.

from all the others, whence $a_2 = 1$.

(5) The power x^3 results *either* if x is chosen from the first and x^2 from the second factor, and 1 from all the others, *or* if x^3 is chosen from the third factor, and 1 from all the others; hence $a_3 = 2$.

.
.
.

(*n*) Similarly, the power x^n results if we take the x^i-term from such terms that $\sum i = n$ and 1 from all the other factors, that is, in as many ways as n can be written as a sum $\sum i$ where all i are different.

A very nice piece of analysis, also in the classical sense: we assume that the result of the infinite product exists, and that it can be found as a sum of partial products just in the way we are accustomed to – and that the infinity of the number of partial products where the sum of exponents is infinite is overruled by the infinitely small value which these terms have if only x is small. Finally, it is needed that two infinite power series can only be identical if all coefficients are pairwise identical – just as is the case for finite series, that is, for polynomials.[14]

The operations needed for showing the theorem about the partitions as well as the one about primes and ζ-function turned out to be justifiable in the nineteenth century; but in 1748 it asked for Euler's fine intuition not to end up in absurdities (and as we see in note 14, even Euler's intuition was not always sufficient).

Perilous generosity, or, set theory saved *in extremis* from addiction

Absurdities did result (together with wonderful results) from another intercourse with the infinite. Apparently, Georg Cantor's "Beiträge zur Begründung der transfiniten Mengenlehre" [1895] moves on well-trodden paths. A "set" is explained (p. 481) to be any collection of well-determined, distinct

[14] But woe, at an earlier point [Euler 1748: I, 222] a similar operation is performed on the product $\prod_{m\ prime}(1+m)$ = (1+2)(1+3)(1+5)(1+7)(1+11)(1+13)... . Even here, all partial products which contain more than a finite number of factors different from 1 (and which are thus infinitely great!) are disregarded, and the product is "shown" to equal the sum 1+2+3+5+6+7+10+11+13+14+15+17+... of all natural numbers that are not divisible by the square of a prime number. In this case it is difficult to see what meaning to attribute to the result – which is hence beyond the categories of *true* and *false*. Only set theory allows an interpretation – *viz* $\aleph = \aleph_0$!

objects, and its "potency" ("Mächtigkeit") or "cardinal number" results when our active intellect removes the particular characteristics and the order of the elements by an act of abstraction; the cardinal number is thus (p. 482) a set consisting of mere units deprived of all characteristics (except that of being distinct). All of this sounds perfectly Aristotelian, and should lead to no more problems than ancient Greek arithmetic.

Evidently Cantor knew the techniques by which his nineteenth-century predecessors had derived the integers, the rational numbers and the real numbers from the natural numbers, that is, essentially from the Greek *Anzahlen,* and his arguments surrounding these are perfectly sound, in the sense that even shortcuts of the type "it is easily seen that" can be filled out by anybody possessing familiarity with these techniques; if intuition plays a role, then it is an intuition that can easily be made operational. Cantor is also quite aware that care has to be taken; he points out (p. 501) that a vicious circle in Veronese's definition of "equality" of transfinite numbers leads the latter to invent objects which "have no existence, if not that on the paper".

None the less, this first version of Cantor's theory of sets did not avoid the trouble of vicious circles, not apparent in the definitions but arising as unsuspected consequences. What happened?

Firstly, Cantor leaves behind the familiar ground of ancient arithmetic when he allows the objects which make up a set to be objects "for our perception or our thought". This seems to be an innocuous generalization of the monads making up an ancient Greek ἀριθμός (or no generalization at all, since no Greek philosopher seems ever to have excluded the counting of mental objects explicitly). As Cantor's reader finds out soon, however, even sets and transfinite cardinal numbers are meant to be objects that can be elements in sets; in consequence, Cantor is able to find the actual infinite, emphatically rejected by Aristotle. The reader also finds it to be supposed always possible (p. 493), if elements t_1, t_2, ..., t_{n-1} have been taken from a transfinite set T, "to take out yet another element t_n" – in complete agreement with our experience with natural numbers (where *a rule* for doing so is easily invented, for instance to take $t_n = t_1+t_2+...+t_{n-1}+1$). In both cases, intuition based on previous experience turned out to be problematic. As only too well known, sets of sets lead to paradoxes; Russell and Whitehead's *theory of types* tried to avoid them by prohibiting the collection of objects of different "type" into the same set, but if this is taken into account several of Cantor's proofs have to be reshaped with great care.[15] The possibility "to take out yet another element t_n" without any need for specifying a rule for choosing it also turned out at Zermelo's closer investigation to be a postulate, the "axiom of choice", and no matter of course.

Delusive generosity, or, the failed extension of a semi-group

Though that was not his aim, Dardi's proof of the sign rule can be seen from a later point of view to allow the extension of the natural numbers organized with addition and multiplication into the ring of (similarly organized) integers. In this case, the generosity of his language gave him even more than he asked for (and Pacioli as much as he asked for since it was "necessary" albeit "absurd").

Almost contemporary with Dardi is an attempt to extend the semi-group of non-negative algebraic powers (organized with multiplication) into a group. In the Vatican manuscript Vat. Lat. 10488 we find the following:[16]

fol. 29v

Algebra

Queste sono alchune ragioni assemprate di 1 libro fato per mano di Giovanni di Davizo de l'Abacho da Firenza schrito su quelo ad 15 setembre ani 1339, è questo 1424.

Sapi che moltiplicare numero per chubo fa chubo
e numero per per zenso fa zenso
e numero per chosa fa chosa.

E più via più fa più
e meno via meno fa più
e più via meno fa meno

fol. 29v

Algebra

These are some computations collected from a book made by the hand of Giovanni di Davizzo dell'Abacho from Florence written the 15th of September 1339, this is 1424.

Know that to multiply number by cube makes a cube
and number by censo makes censo
and number by thing makes thing

And plus times plus makes plus
and less times less makes plus
and plus times less makes less

[15] Aristotle's barring of a broad category of sophisms through the introduction of ontologically based syllogistic logic comes to mind as a parallel. It achieved its purpose, or largely so; but in consequence most of Euclid's demonstrations are not based on "logic" – see, e.g., [Harari 2003].

[16] The non-mathematical abbreviations are resolved (but not ℞, here rendered ℞, standing for *radice*), and diacritics as well as punctuation are added. I use the most recent of the two discordant foliations. Words inserted above the line are marked ^ ^, deletions with a stroke, forgotten words and passages are in ⟨ ⟩.

e meno via più fa meno.	and less times plus makes less.
E sapi che una chose via una chosa fa 1 zenso	And know that a thing times a thing makes a censo
e zenso via zenso fa zenso di zenso	and censo times censo makes censo of censo
e cosa via zenso fa chubo	and thing times censo makes cube
e chubo via chubo fa chubo di chubo	and cube times cube makes cube of cube
e zenso via chubo fa zenso di chubo.	and censo times cube makes censo of cube.
E sapi partire numero per chosa viene numero	And know that dividing number by thing gives number
e partire numero per zenso viene R̸	and dividing number by censo gives root
e partire chosa per zenso viene numero	and dividing thing by censo gives number
e partire numero per chubo viene R̸ chubica	and dividing number by cube gives cube root
e partire chosa per chubo viene R̸	and dividing thing by cube gives root
e partire zenso per chubo viene numero	and dividing censo by cube gives number
e partire numero per zenso di zenso viene R̸ di R̸	and dividing number by censo of censo gives root of root
e partire chosa per zenso di zenso viene R̸ chubica	and dividing thing by censo of censo gives cube root
e partire zenso per zenso di zenso viene R̸	and dividing censo by censo of censo gives root
e partire chubo per zenso di zenso viene numero	and dividing cube by censo of censo gives number
fol. 30r	**fol. 30r**
e partire numero per chubo di chubo viene R̸ chubicha di R̸ chubica	and dividing number by cube of cube gives cube root of cube root
e partire chosa per chubo di chubo viene R̸ di R̸ chubica	and dividing thing by cube of cube gives root of cube root
e partire zenso per chubo di chubo viene R̸ di R̸	and dividing censo by cube of cube gives root of root
e partire chubo per chubo di chubo viene R̸ chubicha	and dividing cube by cube of cube gives cube root
e partire zenso di zenso per chubo di chubo viene R̸	and dividing censo of censo by cube of cube gives root
e partire zenso di chubo per chubo viene ~~numero~~ zenso*	and dividing censo of cube by cube gives ~~number~~ censo*
e partire numero per zenso di zenso di zenso di zenso viene R̸ di R̸ di R̸ di R̸	and dividing number by censo of censo of censo of censo gives root of root of

e partire numero per chubo di chubo di chubo di chubo viene ℞ chubicha di ℞ chubicha di ℞ chubicha di ℞ chubicha.

Se voi moltiplichare ℞ per ℞, moltiplicha ℞ di 9 via ℞ di 9, dì 9 via 9 fa 81, e farà ℞ di 81, et e fata.

Dividere ℞ di 40 per ℞ di 8, parti 40 per 8, viene 5, e ℞ di 5 sia.

Parti ℞ di 25 per ℞ di 9, parti 25 per 9, viene ℞ di 2 e $^{7}/_{9}$, fata.

Se voi moltiplichare 7 ~~via~~ ^meno^ ℞ di 6 per se, fa 7 via 7, fa 49, giungi 6 chon ⟨49, fa⟩ 55, e 7 via 6 fa 42, poi moltiplicha 7 via 42, fa 294, e moltiplicha poi 4 via 294, fa 1176, dicho che 55 meno ℞ di 1176 farà moltiplichato 7 meno ℞ di 6 ~~fata~~ per se, fata.

fol. 30v

Se voi trarre ℞ di 8 per [sic] ℞ di 18 fa 8 via 18, fa 144, sua ℞ è 12, e di, 12 e 12 fa 24, e dì, 8 e 18 fa 26, trai 24 di 26, ~~resta~~ e ℞ di 2 rimarà, fata.

Se voi giungere ℞ di 8 chon ℞ di 18, fa 8 via 18, fa 144, sua ℞ è 12, e dì, 12 e 12 fa 24, e dì, 8 e 18 fa 26, e giungni 24 e 26, fa 50, e ℞ di 50 sarà il numero.

Se voi moltiplichare 5 e ℞ di 4 via 5 meno ℞ di 4, fa chosì e dì, 5 via 5 fa 25, e dì, 5 via ℞ di 4, fa chosì, recha 5 a ℞, fa 25, e fa ℞ di 25 via ℞ di 4, fa ℞ di 100, e fa 5 via meno ℞ di 4, fa meno ℞ di 100, resta pure 25, ora tra 4 di 25, resta 21, e 21 fano.

root of root

and dividing number by cube of cube of cube of cube gives cube root of cube root of cube root of cube root.

If you want to multiply root by root, multiply root of 9 times root of 9, say, 9 times 9 makes 81, and it will make the root of 81, and it is done.

To divide root of 40 by root of 8, divide 40 by 8, it gives 5, and root of 5 let it be.

To divide root of 25 by root of 9, divide 25 by 9, it gives root of $2^{7}/9$, done.

If you want to multiply 7 less root of 6 by itself, do 7 times 7, it makes 49, join 6 with ⟨49, it makes⟩ 55, and 7 times 6 makes 42, then multiply 7 times 42, it makes 294, and multiply then 4 times 294, it makes 1176, I say that 55 less root of 1176 will it make when 7 less root of 6 is multiplied by itself, done.

fol. 30v

If you want to detract root of 8 from root of 18, do 8 times 18, it makes 144, its root is 12, and say, 8 and 18 makes 26, detract 24 from 26, and root of 2 will remain, done.

It you want to join root of 8 with root of 18, do 8 times 18, it makes 144, its root is 12, and say, 12 and 12 makes 24, and say, 8 and 18 makes 26, and join 24 and 26, it makes 50, and root of 50 will the number be.

If you want to multiply 5 and root of 4 times 5 less root of 4, do thus and say, 5 times 5 makes 25, and say, 5 times root of 4, do thus, bring 5 to root, it makes 25, and do root of 25 times root of 4, it makes root of 100, and make 5 times less root of 4, it makes less root of 100, 25 still remains, now detract 4 from 25, 21 remains, and 21 they make.

Se voi moltiplichare 7 e ℞ di 9 via 7 e ℞ di 9, fa 7 via 7, fa 49, poni quel 9, ài 58, e 9 via 49 fa 441, moltiplicha per 4, fa 1764, ài che farà 58 e ℞ di 1764 che è 42, fata.

If you want to multiply 7 and root of 9 times 7 and root of 9, do 7 times 7, it makes 49, put (above) this 9, you have 58, and 9 times 49 makes 441, multiply by 4, it makes 1764, you have that it will make 58 and root of 1764, which is 42, done.

Se voi partire 35 per ℞ di 4 e per ℞ di 9 fa chosì, da 4 a 9 si à 5, moltiplicha 5 via 5 fa 25, e dì, rechare 35 a ℞ fa 1225, ora dì, 4 via 1225 fa 4900, parti per 25, viene 196, e fa 9 via 1225, fa 11025, parti in 25, viene 441. Abiamo che partire 35 per ℞ di 4 e per ℞ di 9 viene ℞ di 441 meno ℞ di 196, et è fata.

If you want to divide 35 by root of 4 and by root of 9, do thus, from 4 to 9 there is 5, multiply 5 times 5, it makes 25, and say, bring 35 to root, it makes 1225, now say, 4 times 1225 makes 4900, divide by 25, it makes 196, and do 9 times 1225, it makes 11025, divide by 25, it gives 441. We have that dividing 35 by root of 4 and by root of 9 gives root of 441 less root of 196, and it is done.

* Correction apparently made in the same hand. Presumably Giovanni di Davizzo's original text was (or was intended to be):

e partire zenso di chubo per chubo di chubo viene numero.

Somewhere in the process, this had become

e partire zenso di chubo per chubo viene numero.

Noticing the error, somebody – presumably the writer of the manuscript – discovered that this was wrong, and stated the correct result.

This is followed by 19 rules for solving reduced equations of the first, second, third and fourth degree.

The extract starts, in modern terms, by presenting through examples[17] the semi-group of positive powers of the unknown, designated a *thing / cosa* (which we may represent by t). Its square is a *censo* – which we (taking advantage of the arithmetization of powers introduced by Chuquet and Bombelli) are able to designate by t^2, whereas t^3 is a *cube*. The designations for the higher powers are composed multiplicatively, without nesting (even though nesting is suggested

[17] Not very orderly, as we observe; and in the middle of the presentation we find the sign rules.

by the use of the genitive[18]). The *censo of censo* is thus $t^2 \cdot t^2 = t^4$, the *cube of cube* is $t^3 \cdot t^3 = t^6$, the *censo of cube* is $t^2 \cdot t^3$, etc.

Once we know that, all products $t^m \cdot t^n$ are seen to be expressed correctly, and so is the only division $t^m \div t^n$ where $m > n \geq 0$ (*censo of cube* by *cube*). The semi-group of non-negative powers is thus treated coherently. In contrast, divisions $t^m \div t^n$ where $m < n$ look strange. However, they are quite systematic. If we forget about the meaning of the words "root" and "cube root" and take ℞ to be the inverse of a *censo* (i.e., to be t^{-2}), ℞ *cubica* to be that of a *cube* (i.e., to be t^{-3}) and their composition to be multiplicative, everything turns out to be meaningful, except in cases where $n = m+1$ and where the outcome of the division is thus t^{-1}. Here, Giovanni and his copyist assert that the result is *number*, that is, of power t^0. Apparently, a hunch of arithmetization interferes, in the sense that ℞ is associated with the number 2, and conceptualized as a "second root", and ℞ *cubica* with "third root". "First root", not known as a standard operation, is then identified with the number.

Giovanni thus has not thought through the matter; since every abbacus master knew, then even he must have been aware that if $a \div b = c$, then $c \cdot b = a$, and therefore, if "dividing number by thing gives number", then "number by thing" makes *number*, not *thing* as he has stated earlier. Moreover, he can be assumed to have known well what *root* really means, since the subsequent section of the arithmetic of roots is fully correct and makes use in several places of the rule $\sqrt{n} \cdot \sqrt{n} = n$. He can never have used his daring extension of the semi-group for any practical purpose, since then its inherent contradictions would have exploded it. In the most literal (but also in the Piagetian) sense, it stays at the intuitive level, and never reaches that of operatory thought.[19]

This holds not only for Giovanni. As we see, his text was copied in 1424 by

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[18] Evidently, Giovanni's Italian terminology follows traditions shaped in other languages and grammatical structures. Both Diophantos and the Arabic traditions (for whose probably connection in the designation of higher powers I shall not argue in this place) also use multiplication, not nesting; actually, this choice facilitates the production of names for all powers – when nesting without arithmetization took over (for instance in Pacioli's *Summa*), t^5, t^7, t^{11} etc. asked for the introduction of independent names (*primo relato, secondo relato*, etc.).

[19] Tools to go beyond this level were at hand and used by other abbacus writers. In the algebra section of an anonymous *Tractato sopra l'arte della arismetricha volgaramente chiamata abacho* [ed. Franci & Pancanti 1988: 3–6] from c. 1390–1400, the rules for multiplying the positive powers (and for multiplying one of these by the square root of a number) are illustrated perfectly by numerical examples; if Giovanni had tried this, his construction would have collapsed immediately.

somebody who understood it well enough to repair correctly (though hardly as intended by Giovanni) an error that had crept in during copying. Moreover, in Piero della Francesca's *Trattato d'abaco* (earlier than c. 1480) [ed. Arrighi 1970: 84*f*, emphasis added] we find the following:

> Sappi che cosa via cosa fa censo, et cosa via censo fa cubo, et censo via censo fa censo di censo, et cosa via cubo fa censo di censo, et cubo via cubo fa cubo di censo [*sic*], et censo via cubo fa censo di cubo.
>
> Sappi che a partire numero per cosa ne vene numero, et partire cosa per censo ne vene numero, e partire numero per censo ne ve' radici, et partire censo per cubo ne vene numero, et partire numero per censo di censo ne vene radici de radici, et partire cosa per censo di censo ne vene radici cuba, et partire censo per censo di censo ne vene radici, et partire cubo per censo di censo ne vene numero, e partire numero per cubo di cubo ne vene radici cubicha ⟨di radice cubica⟩, e partire cosa per cubo di cubo ne vene radici ⟨di radice⟩ cubica, et partire censo per cubo di cubo ne vene radici de radici, et a partire chubo per cubo di cubo ne vene radici cubica, et partire censo di censo per cubo di cubo ne vene radici, *et partire censo de cubo per cubo di chubo ne v*[*e*]*ne numero*, et a partire numero per cubo ven radici cuba, e a partire cosa per cubo ven radici, et partire numero per censo di censo di censo ven radici de radici de radici, et partire numero per cubo de cubo di cubo ven radici cuba de radici cuba de radici cuba.

As we see, the rules are almost exactly as those in the quotation from Giovanni di Davizzo (confirming, by the way, the hypothesis of note (*)). This leaves little doubt that Giovanni di Davizzo is Piero's ultimate source.[20] But as we also see, the order of the rules is not the same. Either Piero himself or, rather, some intermediary[21] must thus have understood the project well enough to tinker with it, though not well enough to see that it was a nonsensical blind alley. One strand of abbacus mathematics thus walked up and down this alley for more than a century without discovering that it led nowhere.[22] Piero, claiming to

[20] Other early Italian algebra writings speak against the possibility that Giovanni himself has copied the passage wholesale from a predecessor although they do not exclude it. What can apparently be excluded is a borrowing from Arabic algebra – at least since al-Karajī, the inverses of algebraic powers had been dealt with coherently.

[21] Indeed, the passage in question is found almost verbatim in Giovanni Guiducci's *Libro d'arismetricha* from c. 1465 – see [Giusti 1993: 205].

[22] It was during the same century that erroneous solutions to non-reducible higher-degree equations were repeated uncritically by a large number of algebra authors, including Piero. Luca Pacioli, probably less gifted as a geometer than Piero but obviously a keener explorer of the algebra writings he made use of, discovered the error, just as he avoided Giovanni di Davizzo's paradoxes. Others too, like the anonymous author referred to in note 19, kept clear of the fallacies.

write a treatise about “certain abbacus things that are necessary for merchants” [ed. Arrighi 1970: 39], could do so because neither he nor any merchant had the least operatory need for it.

Final meditations

What do these examples teach us? First of all, that the artificial languages of mathematics confirm another maxim pronounced by Goethe’s Mephisto (concerning theology, also presented by him as a kind of artificial language) (*Faust*, I, 1995-99):

> [...] wo Begriffe fehlen,
> Da stellt ein Wort zur rechten Zeit sich ein.
> Mit Worten läßt sich trefflich streiten,
> Mit Worten ein System bereiten,
> An Worte läßt sich trefflich glauben,
> [...][23]

that is, they allow us to go beyond that which we grasp in fully operatory concepts (*Begriffe*) and project our thought into the unknown. This is not all there is to their generosity, but it may be their most creative as well as their most treacherous aspect. In cases where no real jump is made because no operations are implied that are not already tested (as in the case of Dardi’s proof), no danger is involved. When the jump is really one, only a process of making operational the suggested concepts, of trying them out in (theoretical) practice (thus doing with the artificial “vocabulary” and its grammar what we all did with those of basic natural language between the ages of five and ten years) can distinguish such entities which “exist only on paper” from such as are or can be made

None the less, the mistaken extension of the semi-group survived Pacioli’s *Summa*. Giovanni’s first 15 rules are still found, in exactly the same order, in Bento Fernandes’ *Tratado da arte de arismetica* from 1555; some of the wrong rules for solving higher-degree equations are also repeated by Fernandes. For both, see Silva 2006: 14, 30–33].

[23] For just where fails the comprehension,
A word steps promptly in as deputy.
With words ’tis excellent disputing;
Systems to words ’tis easy suiting;
On words ’tis excellent believing;
[...].
(trans. Bayard Taylor, http://www.gutenberg.org/files/14591/14591-h/14591-h.htm#III).

coherently meaningful;[24] where no such "active mathematician's practice" takes place, inventions like that of Giovanni di Davizzo (or that of Euler referred to in note 14) may survive indefinitely long, guaranteed as they seem to be by their expressibility in the venerated language of mathematics. Mephisto was certainly right that the same may happen in theology – not to speak of philosophy, yet another venerated artificial language (or cluster of languages) which has served to legitimate much addictive nonsense; but these are different topics.

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[24] Eventually, Tullio Levi-Civita did precisely this with Veronese's infinities, which led to their ultimate vindication in non-standard analysis.

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