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Abstract

We formulate and prove a precise statement of asymptotic shrinking of "Carrot-fields" around the Mandelbrot set $\mathbf{M}$. This phenomenom had been suggested in the founder text "On polynomial-like mappings" [DH2]. This is helpful for understanding how the copies of $\mathbf{M}$ sit in the bifurcations loci of families of rational maps.

Introduction

For a general analytic family of rational maps $(f_t)_{t\in X}$, the *bifurcation locus*, is the set of parameters $t \in X$ such that the Julia set does not move continuously (in the Hausdorff topology) over any neighborhood of t. For instance, the bifurcation locus of the quadratic family $(Q_c(z) = z^2 + c)_{c\in\mathbb{C}}$ is the boundary of the *Mandelbrot set* $\mathbf{M}$ (*i.e.* $\mathbf{M} \setminus int(\mathbf{M})$), where $\mathbf{M}$ is defined as the set of parameters $c \in \mathbb{C}$ such that the Julia set $J(Q_c)$ is connected (see Figure 1).

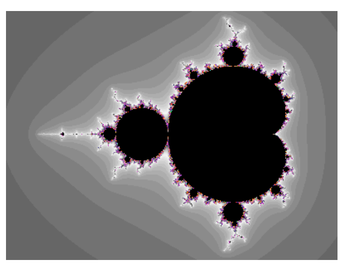

Figure 1: The Mandelbrot set.

McMullen discovered in [McM] the following universality property of $\mathbf{M}$: the boundaries of small Mandelbrot sets are dense in the bifurcation locus of *any* non trivial holomorphic family of rational maps. A *small* Mandelbrot set (or a *copy of* $\mathbf{M}$) is the image of $\mathbf{M}$ by a homeomorphism h that preserves the dynamics: some restriction of an iterate of $h(c)$ and Q_c are topologically conjugated on their Julia sets. This result of McMullen motivates the study of how those copies are embedded in the bifurcation locus of a general family.

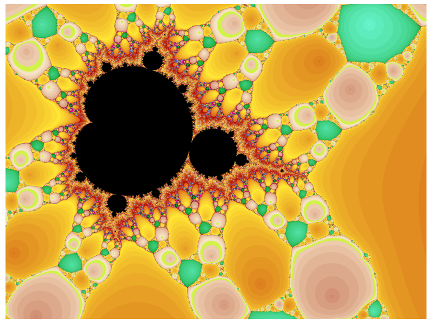

Figure 2: The picture 16 of "carrots for dessert" in [DH2], carrots are in orange.

In their pioneering paper " On polynomial-like mappings" [DH2], Douady and Hubbard exhibited for the first time such an image of $\mathbf{M}$ interlocked in the bifurcation locus of a particular family of rational maps (see Figure 2). In the paper, they developed the theory of what they called "polynomial-like mappings". This theory allows to recognise copies of $\mathbf{M}$ in the parameter space of analytic families of rational maps. Moreover, they suggest that the copies of $\mathbf{M}$ come with some additional structure. Namely, on their particular example, they *observe on the picture* (Figure 16 of section "Carrots for dessert" of [DH2]) that there are connected components of the complement of the Bifurcation locus which are attached to the tips of the copy and that their diameter tends to 0 (see Figure 2). Note that those components are hyperbolic components and that it is not known, even for the quadratic family, that the diameter of hyperbolic components tends to 0. Douady and Hubbard called them "carrots". They also give a notion of carrot fields in the setting of polynomial-like mappings.

In the present paper, we give a reformulation of Douady-Hubbard definition of "carrots" and of "carrot fields" around $\mathbf{M}$. We prove in Theorem 1 a general statement concerning the asymptotic shrinking of "Carrot-fields" around $\mathbf{M}$.

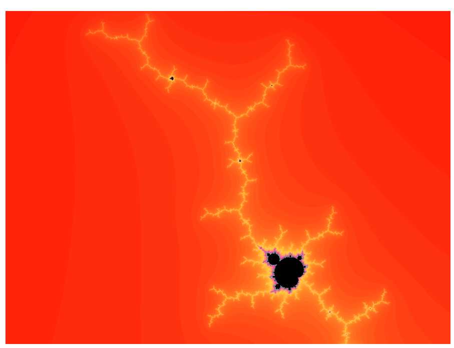

Figure 3: A Mandelbrot copy in $\mathbf{M}$.

We give in Theorem 3 the simplest possible application: namely to the copies of $\mathbf{M}$ inside $\mathbf{M}$ (see Figure 3). We prove that , if $\mathbf{M}_0$ is a copy of $\mathbf{M}$ in $\mathbf{M}$, only finitely many connected components of $\mathbf{M}\setminus\mathbf{M}_0$ have diameter greater than some $\epsilon > 0$. For proving this we show that any component of $\mathbf{M}\setminus\mathbf{M}_0$ can be included inside a carrot. By the way, we prove that Theorem 1 is a necessary condition for the Mandelbrot set to be locally connected (the famous MLC conjecture).

Theorem 1 provides a tool that can be applied to lots of holomorphic families. For instance, in a forthcomming paper [PR3], to the case of the parabolic Mandelbrot set.

The proof uses two fundamental tools : the generalized Yoccoz-Levin inequality and the combination of two kind of puzzles in the parameter plane (Yoccoz puzzle together with the puzzle defined by the carrots).

1 Statement and results

• Carrots for $Q_0 : z \mapsto z^2$.

Let z_1, z_2 be two points of $\mathbb{H}_+ = \{z \in \mathbb{C} \mid \Re e(z) > 0\}$ such that the euclidean closed triangle Δ supported by $0, z_1, z_2$ is disjoint from $\Delta + 2i\pi$. Then, the *carrot* $\Delta_0 = \exp(\Delta)$ is homeomorphic to Δ. For any $n \in \mathbb{N}$, any $p \in \mathbb{N}^*$ with $2^n \wedge p = 1$, we can consider the *carrot* $\Delta_{p/2^n}$ which is the connected component of $Q_0^{-n}(\Delta_0)$ containing the point $e^{2i\pi p/2^n}$ of the unit circle $\mathbf{S}^1$. The carrots are all homeomorphic images of Δ.

• Carrots for $\mathbf{M}$.

Douady and Hubbard gave in [DH1] a dynamical parameterization of the complement of $\mathbf{M}$ denoted by $\Phi : \mathbb{C} \setminus \mathbf{M} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ which is the conformal representation tangent to identity at ∞. It has the property that its inverse $\Psi = \Phi^{-1}$ admits a limit $\gamma(p/q)$ on every ray of rational angle p/q (*i.e.* $\gamma(p/q) = \lim_{r \to 1^+} \Psi(re^{2i\pi p/q})$) and more over in any Stolz angle based at $e^{2i\pi p/q}$. Therefore one can take the image $\Psi(\Delta_{p/2^n})$ which is a compact set in $\mathbb{C}$.

The *dyadic carrot field around* $\mathbf{M}$ generated by Δ is the union of all the "carrots" $\Delta^{\mathbf{M}}_{p/2^n} := \Psi(\Delta_{p/2^n})$ where $p/2^n \in \mathbb{Q}$. A first remark is that $\Delta^{\mathbf{M}}_{p/2^n}$ is homeomorphic to Δ. With this terminology the Theorem of shrinking of dyadic carrots of $\mathbf{M}$ is

Theorem 1. *For any dyadic carrot field* Δ *of* $\mathbf{M}$

$$\lim_{n\to\infty} \operatorname{diam}(\Delta^{\mathbf{M}}_{p/2^n}) = 0.$$

• Corollary for a copy $\mathbf{M}_0$ of $\mathbf{M}$.

Let $\mathbf{M}_0$ be a copy of $\mathbf{M}$ and denote by $\chi_{\mathbf{M}_0}$ the homeomorphim from $\mathbf{M}_0$ to $\mathbf{M}$. The *tips* $\gamma_0(p/2^n)$ of $\mathbf{M}_0$ are the images of the tips $\gamma(p/2^n)$ of $\mathbf{M}$ by $\chi^{-1}_{\mathbf{M}_0}$. The *tips* of $\mathbf{M}$ correspond to the limit $\gamma(p/2^n)$ of the map Ψ at the point $e^{2i\pi p/2^n}$ in $\mathbf{S}^1$. Denote by $\Delta^{\mathbf{M}_0}_{p/2^n}$ the connected component of $\mathbf{M} \setminus \mathbf{M}_0$ containing the tip $\gamma_0(p/2^n)$ in its closure and disjoint from $\mathbf{M}_0$.

Theorem 2 (Douady-Hubbard, Yoccoz)**.** *For any copy* $\mathbf{M}_0$ *of* $\mathbf{M}$ *in* $\mathbf{M}$ *:*

$$\mathbf{M} = \mathbf{M}_0 \cup \bigcup_{n\geq 0} \bigcup_{p/2^n \in \mathbb{Q}} \Delta_{p/2^n}^{\mathbf{M}_0}.$$

The Shrinking decorations Theorem for strict copies $\mathbf{M}_0$ of $\mathbf{M}$ in $\mathbf{M}$ can then be stated as

Theorem 3. *For any strict copy* $\mathbf{M}_0$ *of* $\mathbf{M}$ *in* $\mathbf{M}$

$$\lim_{n\to\infty} \mathrm{diam}(\Delta_{p/2^n}^{\mathbf{M}_0}) = 0.$$

The two theorems Theorem 1 and Theorem 3 have very similar proofs, the proof of the first being slighly more complicated. We shall detail the proof of the first and sketch the difference to the proof of the second.

Dzmitry Dudko presents a different and independent proof of the Shrinking decorations Theorem for strict copies M' of $\mathbf{M}$ in $\mathbf{M}$ in [Du]. His statement includes more generally strict copies of the Multibrot set inside the Multibrot set of the same degree. The proof we give here would also easily extend to the Multibrot case. In a forthcomming paper [PR4], we shall treat the more general case of Mandelbrot-like families.

2 Framework

2.1 Independence on Δ.

The degenerate case where the triangle Δ is defined by $z_1 = z_2 > 0$ is in fact the only one we need to consider. Let us call the carrot a stick when the triangle is degenerate. We shall prove the following :

Theorem 4. *For the dyadic stick field of* $\mathbf{M}$ *generated* $\Delta = [0,1]$

$$\lim_{n\to\infty} \mathrm{diam}(\Delta_{p/2^n}^{\mathbf{M}}) = 0.$$

Lemma 5. *Theorem 4 implies Theorem 1.*

Proof. We will prove in fact that the result is independent of the non trivial triangle chosen. Let Δ^1 and Δ^2 be any two possibly degenerate triangles in $\mathbb{H}_+ \cup \{0\}$. Then there exists $\delta > 0$ such that $\Delta^1 \setminus \{0\}$ is contained in a hyperbolic δ-neighbourhood of $\Delta^2 \setminus \{0\}$ in $\mathbb{H}_+$ and vice versa. As $\exp : \mathbb{H}_+ \longrightarrow \mathbb{C}\setminus\overline{\mathbb{D}}$, $Q_0 : \mathbb{C}\setminus\overline{\mathbb{D}} \longrightarrow \mathbb{C}\setminus\overline{\mathbb{D}}$ and $\Phi : \mathbb{C}\setminus\mathbf{M} \longrightarrow \mathbb{C}\setminus\overline{\mathbb{D}}$ are local hyperbolic isometries the same statement holds with the same δ for each $\Delta_{p/2^n}^{j\mathbf{M}}$, $j = 1, 2$. Thus, by elementary estimates on hyberbolic metrics, there exists $k = k(\delta) > 1$ such that for any reduced $\theta = p/2^n$

$$\frac{1}{k} \leq \frac{\mathrm{diam}(\Delta_\theta^{1\mathbf{M}})}{\mathrm{diam}(\Delta_\theta^{2\mathbf{M}})}, \leq k$$

where diam$(\cdot)$ denotes euclidean diameter.

This follows from the following inequality (see [Po]) where $K \subset \widehat{\mathbb{C}}$ is a compact set and B_H denotes the ball for the hyperbolic metric in $\widehat{\mathbb{C}} \setminus K$:

$$\mathrm{diam}(B_H(z,d)) \leq C(d)\mathrm{dist}(z,K).$$

Indeed, if D is the diameter of $\Delta_\theta^{1\mathbf{M}}$, then $\mathrm{dist}(z,K) \leq D$ for $z \in \Delta_\theta^{1\mathbf{M}}$ and there exists $z' \in \Delta_\theta^{2\mathbf{M}}$ such that $z \in B_H(z',d)$, do that $|z - z'| \leq DC(d)$. Hence $\mathrm{diam}\Delta_\theta^{2\mathbf{M}} \leq (C(d)+1)D$. Computing the reverse inequality yields the comparison.

To prove Theorem 1 it suffices to consider a particular stick field, say the field for $z_1 = z_2 = 1$, which is Theorem 4. □

The choice of this particular degenerate stick field allows to avoid discussions on the position of the stick (or carrot) with respect to the parameter rays (see section 2.4.1).

2.2 Argument on the toy exemple

One of the main argument in the proof will be that for any strict copy M' of $\mathbf{M}$ in $\mathbf{M}$ the stick field defined by $\Delta^{\mathbf{M}}_{p/2^n} = \Psi(\Delta_{p/2^n})$ does not touch M'. Thus we can put any stick $\Delta^{\mathbf{M}}_{p/2^n}$ in a annulus that is disjoint from M' of modulus bounded from below.

We can visualize this argument first on the trivial toy example. We replace Δ_0 that was define by the interval $[0,1]$ simply by the compact set $T_0 = \exp([1/2,1])$. This completely trivialises the problem by considerations on the comparison of hyperbolic and euclidean distance similar to above. In this simpler case T_0 has finite hyperbolic diameter *diam* and moreover this bound is an upper bound on the hyperbolic diameter of any of the preimages $T_{p/2^n}$. Hence the euclidean diameter of any $T^{\mathbf{M}}_{p/2^n}$ is bounded uniformly from above by a universal constant $k = k(diam)$ times the euclidean distance between $T^{\mathbf{M}}_{p/2^n}$ and $\mathbf{M}$. Since the later tends to zero uniformly as $n \to \infty$ we have in the toy case

$$\lim_{n\to\infty} \mathrm{diam}(T_{p/2^n}) = 0.$$

With this in mind let us proceed to the proof of Theorem 1. Then as mentioned above Theorem 3 will follow by using the same proof.

2.3 Proof of Theorem 1.

We prove the following result :

Proposition 6. *For the dyadic stick field of* $\mathbf{M}$ *generated by* $\Delta = [0,1]$*, let* $\{\Delta_i = \Delta^{\mathbf{M}}_{p_i/2^{r_i}}\}_{i\in\mathbb{N}}$ *be any sequence of sticks, with* $r_{i+1} > r_i$ *and with roots* c_i *tending to* c_∞ *then*

$$\mathrm{diam}(\Delta_i) \underset{i\to\infty}{\longrightarrow} 0.$$

Remark 7. *Theorem 1 is an easy corollary of this proposition by compactness of the Mandelbrot set. (The details are left to the reader.)*

Sketch of the Proof of Proposition 6: Note that necessarily $c_\infty \in \partial\mathbf{M}$.

1. First of all, the Yoccoz-Levin Parameter inequality (Theorem 15) implies that if the limit point c_∞ belongs to the main cardioid then the diameter of the carrots has to tend to 0. Therefore we can assume in the following that the limit point c_∞ belongs to some limb $L^{H_0}_{p/q}$ of the main cardioid H_0 of $\mathbf{M}$.

2. Secondly we apply Yoccoz Parameter Puzzle Theorem (Theorem 19). There exists a non decreasing and non eventually constant sub-sequence n_i such that for $i \geq i_0$ the carrots $\Delta_i \subset \overline{\mathcal{P}}_{n_i}(c_\infty)$ (the Parameter Puzzle Piece). Hence either the diameter tends to 0 or the limiting parameter c_∞ is renormalizable, that is $c_\infty \in M'$ for some first renormalization copy M' of $\mathbf{M}$ in $L^{H_0}_{p/q}$, of period k where $q \leq k$.

3. The key point in the proof of Proposition 6 is that all of the dyadic carrots $\Delta^{\mathbf{M}}_{p/2^n}$ are disjoint from M', because their root points $\Psi(p/2^n)$ are disjoint from M'.

   Then we can wrap each $T^{\mathbf{M}}_{p/2^n}$ in some annulus. We shall use the fact that if $T^{\mathbf{M}}_{p/2^n} \subset V' \subset V \subset U$, with U a hyperbolic domain and $\mathrm{mod}(V \setminus \overline{V'}) \geq \delta > 0$, then the hyperbolic diameter of $T^{\mathbf{M}}_{p/2^n}$ in U satisfies $\mathrm{diam}_U(T^{\mathbf{M}}_{p/2^n}) \leq d(\delta)$. Therefore, if $T^{\mathbf{M}}_{p/2^n}$ tends to M', it follows that the euclidean diameter has to go to zero.

4. To obtain these annuli we shall set the dynamical counter part. There we obtain annuli that are univalent preimage of some annulus that admits a holomorphic motion in a large domain. We shall then use the observation by Shishikura, that holomorphic motions can be used to transfer bounds for (locally) persistent annuli in dynamical space to bounds for corresponding annuli in parameter space (see [R]).

5. We shall follow slightly different paths according to wether M' is a primitive copy or the satelite copy $M_{p/q}$ with root on the cardioid. We start with the primitive case and afterwards indicate the changes which make the proof in the satelite case.

2.4 The set-up

Before to start, we recall some classical facts from [DH1] (see also [Mi]) that will be used at different places of the paper.

2.4.1 Rays

The polynomial Q_c is conjugated to z^2 near ∞. The conjugacy ϕ_c that is tangent to identity at infinity is called the Böttcher-coordinate of Q_c at ∞. It has the property that

$$\Phi(c) = \phi_c(c).$$

We shall use also the Green's functions for $\mathbf{M}$ and $K_c = \{z \mid Q_c^n(z) \text{is bounded}\}$, i.e. the subharmonic functions $g_{\mathbf{M}}(c) = \log^+ |\phi_c(c)|$ and

$$g_c(z) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ |Q_c^n(z)|.$$

Moreover we shall use the notation $E^{\mathbf{M}}(h)$ and $E^c(h)$ for the *equipotentials* for $g_{\mathbf{M}}$ and g_c of level $h \geq 0$. Similarly we shall use the notation $F^{\mathbf{M}}(h)$ and $F^c(h)$ for the*closed filled equipotentials* of level or height h:

$$F^{\mathbf{M}}(h) = \{c \mid g_{\mathbf{M}}(c) \leq h\}, \qquad F^c(h) = \{z \mid g_c(z) \leq h\}.$$

The *external ray* of argument θ for $\mathbf{M}$ or K_c is the field line of $g_{\mathbf{M}}$ or g_c, which is asymptotic to the halfline $\exp(t + i2\pi\theta)$ at ∞. By *parameter ray* of angle θ we mean the external ray for $\mathbf{M}$, which is simply the image $R_\theta^{\mathbf{M}} = \Psi(]1, \infty[e^{2i\pi\theta})$. The *dynamical ray* can also be defined as $R_\theta^c = \phi_c^{-1}(]1, \infty[e^{2i\pi\theta})$ when the Julia set is connected, else this curve might be broken on the preimages of the critical point.

2.4.2 Copies of M

Definition 8. *We say that $f : U \to U'$ is quadratic-like if f is a proper holomorphic map between discs U, U' such that $\overline{U} \subset U'$. The filled Julia set associated is $K_f = \cap f^{-n}(U)$. A map g is said to be l-renormalizable if g^l admits a restriction which is quadratic-like with connected filled Julia set.*

Definition 9. *Two quadratic-like maps f and g are said hybrid equivalent if there is a quasi-conformal conjugacy ϕ between f and g defined on a neighborhood of their respective filled Julia sets such that $\overline{\partial}\phi = 0$ on $K(f)$.*

Theorem 10. *Every quadratic-like map f is hybrid equivalent to a polynomial of the same degree. When $K(f)$ is connected, the polynomial g is unique up to affine conjugation.*

Definition 11. *A copy $\mathbf{M}_0$ of the Mandelbrot set $\mathbf{M}$, is the image of $\mathbf{M}$ by a homeomorphism h that preserves the dynamics in the following sense. For the map $h(c) = Q_{c'}$, there exists an iterate k minimal and a restriction of $Q_{c'}^k$ to an open disk containing 0 which is quadratic-like and hybrid equivalent to Q_c. The minimal k above is called the period of the copy.*

2.4.3 Wakes of H

We denote by H_0 the main cardioid in the Mandelbrot set. It corresponds to the set of parameters c such that Q_c has an attracting fixed point. Its closure is parameterized by the multiplier λ_a at the non repelling fixed point. At each $\lambda_a(e^{2i\pi p/q})$ with $p \wedge q = 1$, there are two external parameter rays $R^{\mathbf{M}}_{\eta^{p/q}_{\pm}}$ landing. We call *p/q-wake of* H_0 the region $W^{H_0}_{p/q}$ bounded by these two rays and disjoint from H_0. The part of $\mathbf{M}$ in $W^{H_0}_{p/q}$ is usually called *p/q-Limb*: $L^{H_0}_{p/q} = \mathbf{M} \cap W^{H_0}_{p/q}$.

Now consider any hyperbolic component H of $\mathbf{M}$. It can be seen as the principal hyperbolic component of a copy M' of $\mathbf{M}$. Namely, there exists a dynamical homeomorphism denoted χ_H or $\chi_{M'} : M' \to \mathbf{M}$ called the Douady-Hubbard straightening map (for a precise definition see [DH2, Chap. II, l-4]) that sends H to H_0. Note that the period of the copy is the period of H namely the period of the attracting cycle in H. One can defined similarly for H a parameterization by the multiplier of the cycle (it can just be $\lambda_a \circ \chi_H$). For each p/q reduced rational we denote by *p/q-wake of* H the region $W^H_{p/q}$ bounded by the two parameter rays co-landing at $\lambda_H(p/q)$ and avoiding H.

The root of H (or root of M') is the parameter $\lambda_H(0)$ of multiplier 1. Let $\theta_\pm$ be the arguments of the pair of external rays co-landing at the root of H. We denote by *tuning interval* for M' (or equivalently for H), the interval $I(M') = I(H) = [\theta_-, \theta_+]$.

2.4.4 Wake of M'

Let M' be a copy of $\mathbf{M}$ in $\mathbf{M}$ with tuning interval I. Let k be the period of M', or equivalently of H, the central hyperbolic component of M' (*i.e.* the period of the attracting cycle in H). Let $\widehat{\theta}_+ < \widehat{\theta}_-$ be the points in I such that each of the subintervals $I_0 = [\theta_-, \widehat{\theta}_+]$ and $I_1 = [\widehat{\theta}_-, \theta_+]$ map diffeomorphically onto I under σ^k, where $\sigma(\theta) = 2\theta \bmod 1$. Let $I^{M'}$ denote the corresponding σ^k-invariant Cantor set and let $\kappa = \kappa_{M'} : I^{M'} \longrightarrow \Sigma_2$ denote the conjugacy of $\sigma^k : I^{M'} \longrightarrow I^{M'}$ to the shift on $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ with $\kappa(\theta_-) = \overline{0}$ and $\kappa(\theta_+) = \overline{1}$.

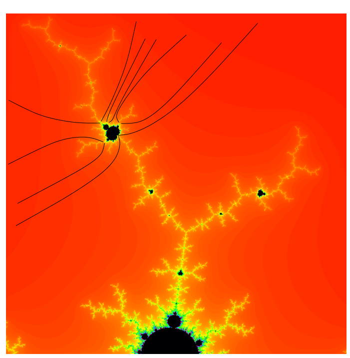

Figure 4: Wakes of M' copy of $\mathbf{M}$

Then the pair of rays with arguments $\widehat{\theta}_\pm$ coland at the principal tip $c'_{1/2} = \chi_{M'}^{-1}(\Psi(1/2))$ of M'. The sector $W'_{1/2}$ bounded by these rays and disjoint from M' is called the *principal wake* of M'.

More generally for p odd with binary expansion $p = \epsilon_1 \dots \epsilon_n$, $\epsilon_n \neq 0$ the dyadic number $p/2^n$ has two binary expansions $0.\epsilon_1 \dots \epsilon_n \overline{0}$ and $0.\epsilon_1 \dots \epsilon_{n-1} 0 \overline{1}$. According to the Douady tuning algorithm $\theta^-_{p/2^n} = \kappa^{-1}(\epsilon_1 \dots \epsilon_n \overline{0})$ and $\theta^+_{p/2^n} = \kappa^{-1}(\epsilon_1 \dots \epsilon_{n-1} 0 \overline{1})$ are the two endpoints of a complementary interval of $I^{M'}$. The corresponding external rays of $\mathbf{M}$ co-land at the relatively dyadic tip $c'_{p/2^n} = \chi_{M'}^{-1}(\Psi(p/2^n))$ of M'. The $p/2^n$*-wake* $W^{M'}_{p/2^n}$ is the sector bounded by these rays and disjoint from M'. Moreover, for any parameter $c \in M'$ the corresponding dynamical rays co-land on a point, which is preperiodic to the relative β fixed point (the non-attracting fixed point of the renomalized map).

Denote by W'_0 the sector bounded by the rays of arguments $\theta_\pm$ and not containing M'. Note that for each $p/2^n$ the root $c'_{p/2^n}$ of the corresponding wake is the only point of intersection between M' and (the closure of the) wake. Note also that any two wakes are disjoint. Note that with notation of Theorem 2 the *"limbs"* $\Delta^{M'}_{p/2^n}$ coincide with the part of $\mathbf{M}$ in the wake: $\overline{W^{M'}_{p/2^n}} \cap \mathbf{M}$.

3 Yoccoz-Levin inequality

For the version of Proposition 6 leading to a proof of Theorem 3 the simpler Yoccoz (rather than Levin-Yoccoz) parameter space inequality suffices, but for Proposition 6 we need the extension due to Levin.

3.1 Dynamical Inequality

Theorem 12 (The Yoccoz-Levin Dynamical Inequality)**.** *Let H be any hyperbolic component of $\mathbf{M}$ of period k. Let p/q be any non zero reduced rational and let $W^H_{p/q}$ denote the relative p/q wake of H, bounded by parameter rays with arguments $0 < \eta_- < \eta_+ < 1$. For any $c \in W^H_{p/q}$, the kq periodic rays $R^c_{\eta_\pm}$ land at a common point α', which is a repelling k-periodic point. Thus α' has combinatorial rotation number p/q. Let λ denote the multiplier of α'. It admits a logarithm Λ such that:*

$$|\Lambda - \frac{p}{q} 2\pi i| \leq \frac{2k \log 2 \cos\theta}{q} \frac{\pi}{\omega(c)},$$

where $\theta \in \,]-\pi/2, \pi/2[$ is the argument of $\Lambda - 2\pi i \frac{p}{q}$ and $\omega(c)$ is the angle of vision of the interval $i2\pi[\eta_-, \eta_+]$ from $\operatorname{Log}\Phi(c) \in \{z = x + iy | 0 < y < 2\pi\}$.

Proof. Levin proved the fixed point case $k = 1$ in [L, TH. 5.1], the general case is similar. For completeness we give a proof in the Appendix, page 25. $\square$

We shall use this Theorem when c belongs to $\mathbf{M} \cup \Delta^{\mathbf{M}}$, so we need to estimate the quantity $\omega(c)$ for $c \in \Delta^{\mathbf{M}}$. For this we will estimates the largest potential of $\Delta^{\mathbf{M}}$, which corresponds to the dyadic parameter rays of lowest denominator entering the wake.

Proposition 13. *Let H be a period k hyperbolic component of $\mathbf{M}$ with tuning interval $I(H) = [\theta_-, \theta_+]$ and let $p'/2^{m'} \in I(H)$, $1 \leq m' < k$ be the dyadic with the smallest denominator. For any irreducible rational p/q, let $0 < \eta_- < \eta_+ < 1$ be the arguments of the co-landing parameter rays bounding $W^H_{p/q}$ and let $p/2^m \in [\eta_-, \eta_+]$ be the dyadic with the smallest denominator. We have*

$$2^{-kq} \leq \eta_+ - \eta_- \qquad \textit{and} \qquad m = m' + k(q-2).$$

Proof. As $\eta_- < \eta_+$ are periodic of exact period kq, we have $\eta_+ - \eta_- \geq 1/(2^{kq} - 1) > 2^{-kq}$. For the second inequality let M' denote the copy of $\mathbf{M}$ with H as central hyperbolic component. Let $\theta_- < \theta_+ \in I^{M'}$ denote the arguments of the parameter rays colanding at the root of M'. Let $I = [\theta_-, \theta_+] \supset I_0, I_1$, $I^{M'} \subset I_0 \cup I_1$ and $\kappa : I^{M'} \longrightarrow \Sigma_2$ be as above and write π for the binary projection of Σ_2 onto $\mathbb{T}$ and set $\widehat{\kappa} = \pi \circ \kappa$. Then $\tau_\pm = \widehat{\kappa}(\eta_\pm)$ are the arguments of the parameter rays co-landing at the root of the wake $W^{H_0}_{p/q}$. It is well known that the intervals $\sigma^j([\tau_-, \tau_+])$, $0 \leq j < q$ are interiorly disjoint and injective images. Moreover $0 \in \sigma^{(q-1)}([\tau_-, \tau_+])$ and thus $1/2 \in \sigma^{(q-2)}([\tau_-, \tau_+])$. Consequently $\sigma^{k(q-2)}$ maps $[\eta_-, \eta_+]$ injectively into I. Morever $I \supset \sigma^{k(q-2)}([\eta_-, \eta_+]) \supset (I \setminus (I_0 \cup I_1))$. Let $p'/2^{m'} \in I$ be the dyadic with smallest denominator then $p'/2^{m'} \in I \setminus (I_0 \cup I_1)$ and $1 \leq m' \leq k$. Thus $m = m' + k(q-2)$. $\square$

Corollary 14. *For any $c \in W^H_{p/q} \cap (\mathbf{M} \cup \Delta^{\mathbf{M}})$ the angle of vision $\omega(c)$ of $i2\pi[\eta_-, \eta_+]$ from $\mathrm{Log}(\Phi(c))$ is bounded from below by*

$$\arctan(2\pi 2^{m'-2k}).$$

Proof. The angle is bounded from below by the angle obtained, when c belongs to one of the two bounding rays of $W^H_{p/q}$ and on the largest potential of $\Delta^{\mathbf{M}}$. This largest potential is obtained when $g_{\mathbf{M}}(c) = 1/2^m$ with $p/2^m$ the angle of the parameter ray entering the Wake with smallest denominator, hence $\log|\Phi(c)| \leq 1/2^m$.

$$\begin{aligned}\arctan(2\pi(\eta_+ - \eta_-)/\log|\Phi(c)|) &\geq \arctan(2\pi 2^{-kq}/2^{-(m'+k(q-2))}) \\ &= \arctan(2\pi 2^{m'-2k}).\end{aligned}$$

$\square$

3.2 Parameter Inequality

Theorem 15 (The Yoccoz-Levin Parameter Inequality)**.** *For any hyperbolic component H of $\mathbf{M}$ there exists a constant $C = C_H > 0$ such that for any relative p/q wake $W^H_{p/q}$*

$$\mathrm{diam}(W^H_{p/q} \cap (\mathbf{M} \cup \Delta^{\mathbf{M}})) \leq \frac{C}{q}.$$

Proof. For $h = 0$ i.e. for the limbs $\mathbf{M} \cap W^H_{p/q}$, this is essentially proved by Hubbard in [H]. Note that for primitive hyperbolic components he obtains an inequality with $C/\sqrt{q}$ instead of C/q. Whereas the bounds actually gives C/q^2 assymptotically when p/q tend to 0 or 1. For the extension we use the Levin-Yoccoz dynamical inequality above instead of the Yoccoz inequality. By Corollary 14 the angle $\omega(c)$ for $c \in W^H_{p/q} \cap (\mathbf{M} \cup \Delta^{\mathbf{M}})$ is bounded from below by the angle $\omega_H = \arctan(2\pi 2^{m'-2k})$. The argument is then identical to the argument in Hubbard's paper [H], except using the Levin-Yoccoz dynamical inequality with the fixed value ω_H. Thus asymptotically for q large we can take

$$C_H = \frac{\pi}{\omega_H} C_H^{\mathrm{Yoccoz}}$$

where C_H^{Yoccoz} is the corresponding assymptotic value for Yoccoz parameter inequality. □

Corollary 16. *We can assume that the sequence Δ_k of carrots belong to $W^{H_0}_{p/q}$ for some fixed p/q.*

Proof. Let us first apply the Yoccoz-Levin parameter inequality. This gives a constant $C = C_{H_0} > 0$ such that for all r/s

$$\mathrm{diam}(W^{H_0}_{r/s} \cap (\mathbf{M} \cup \Delta^{\mathbf{M}})) \leq \frac{C}{s}. \tag{1}$$

The sequence $\{\Delta_k\}$ of carrots is included in a sequence of Wakes $W^{H_0}_{p_k/q_k}$ and the roots belongs to the corresponding limbs $L^{H_0}_{p_k/q_k}$. If q_k tends to ∞ then the diameter of Δ_k tends to 0 by (1). On the other hand if the sequence $\{p_k/q_k\}$ contains a bounded subsequence. Then it contains a constant subsequence $p_{k_m}/q_{k_m} = p/q$. Hence the limit point $c_\infty \in L^{H_0}_{p/q}$. For any Q the limb $L^{H_0}_{p/q}$ is strongly separated from any of the limbs $L^{H_0}_{p'/q'}$, $p/q \neq p'/q'$ with $q' \leq Q$. Hence either $p_k/q_k = p/q$ for sufficiently large k or for the remaining elements in the sequence q_k tend to infinity, and thus $\mathrm{diam}(\Delta_k)$ tend to zero for these k. In both cases we can suppose $p_k/q_k = p/q$ for some $p/q \in]0,1[$. □

4 Yoccoz Puzzle and holomorphic motions

4.1 Definition of Yoccoz Puzzles

For $c \in W^{H_0}_{p/q}$ the q cycle of external rays with arguments $\theta_0 < \cdots < \theta_{q-1}$ land together at the fixed point α_c, where $\Theta = \{\theta_0, \cdots, \theta_{q-1}\}$ is the unique p/q-cycle under angle doubling. They thus assign the combinatorial rotation number p/q to α_c.

Definition 17. *The p/q-"model" graphs are*

$$\mathcal{G}_0 = \{z \mid |z| = e\} \cup \bigcup_{0 \leq j \leq q-1} [1, e]e^{2i\pi\theta_j}, \quad \mathcal{G}_n = Q_0^{-n}(\mathcal{G}_0).$$

For $c \in L^{H_0}_{p/q}$*, the* p/q*-Yoccoz graphs are* $\mathcal{Y}^c_0 = \phi_c^{-1}(\mathcal{G}_0)$*,* $\mathcal{Y}^c_n = \phi_c^{-1}(\mathcal{G}_n) = Q_c^{-n}(\mathcal{Y}^c_0)$*.*

The puzzle pieces of level n *are the closures of the bounded connected components of* $\mathbb{C} \setminus \mathcal{Y}^c_n$*, where* $\mathcal{Y}^c_n$ *is the level* n *Yoccoz graph.*

Definition 18. *The* p/q*-Yoccoz parameter graphs are* $\mathcal{Y}_0 = \Psi(\mathcal{G}_0)$ *and* $\mathcal{Y}_n = \Psi(\mathcal{G}_n)$*. The parameter puzzle pieces of level* n *are the closures of the bounded connected components of* $\mathbb{C} \setminus \mathcal{Y}_n$*, where* $\mathcal{Y}_n$ *is the level* n *Yoccoz parameter graph.*

The level n parameter puzzle piece(s) containing c is the closed subset of $\mathbb{C}$ bounded by the closure of $\Psi \circ \phi_c(\partial P^c_n)$, where P^c_n denotes the (or possibly any of the q) level n puzzle piece(s) in the dynamical plane containing c. (If c, c' belong to the same level n parameter puzzle piece $\mathcal{P}_n$, then $\phi_{c'}^{-1} \circ \phi_c$ induces a homemorphism between the Yoccoz graphs for c and c' at least up to and including level n.)

4.2 Yoccoz Theorem and its application

Theorem 19 (Yoccoz)**.** *For any* p/q *and any* $c \in L^{H_0}_{p/q}$ *there are two possibilities; either* Q_c *is not renormalizable and* $\cap \mathcal{P}_n = \{c\}$ *for any nest* $(\mathcal{P}_n)$ *of parameter puzzle pieces containing* c*, or* c *is at least once renormalizable, say with first renormalization period* k *and there is a first level* n *such that for the dynamical puzzle pieces* $Int(P^c_n) =: U$ *and* $Int(P^c_{n+k}) =: U'$*,* $Q^k_c : U \longrightarrow U'$ *is quadratic like with connected filled-in Julia set (fattening* U *and* U' *if* $k = q$*.)*

For a proof see [H].

We provide a rough proof of Theorem 2 using the puzzles, the reader shall easily supply the details:

Theorem (Douady-Hubbard, Yoccoz)**.** *For any copy* $\mathbf{M}_0$ *of* $\mathbf{M}$ *in* $\mathbf{M}$ *:*

$$\mathbf{M} = \mathbf{M}_0 \cup \bigcup_{n \geq 0} \bigcup_{p/2^n \in \mathbb{Q}} \Delta^{\mathbf{M}_0}_{p/2^n}.$$

Proof. The copy $\mathbf{M}_0$ of $\mathbf{M}$ belongs to the limb $L^{H_0}_{p'/q'}$ of the central hyperbolic component H_0 of $\mathbf{M}$, for some $p', q' \in \mathbb{N}$ with $(p', q') = 1$. Let $c \in \mathbf{M}_0$ be the center of $\mathbf{M}_0$, i.e. $Q^k_c(c) = c$, where k is the period of $\mathbf{M}_0$. Let P_n, $n \in \mathbb{N}$ be the level n (p'/q')-Yoccoz puzzle piece containing the critical value c of Q_c and let $\mathcal{P}_n$ denote (p'/q')-Parameter Yoccoz puzzle piece containing the parameter c. Then for each n the map $\Psi \circ \phi_c(z)$ restricted to $\partial P_n \smallsetminus J_c$ extends to a homeomorphism of ∂P_n onto $\partial \mathcal{P}_n$ preserving argument and potential. Also for each n the closed puzzle piece P_n contains the ends from potential 2^{-n} and down of the external rays with arguments in $I^{\mathbf{M}_0}$. Hence the same holds for the corresponding parameter rays and $\mathcal{P}_n$. It follows that any other level n parameter puzzle piece as well as $\mathbf{M} \smallsetminus L^{H_0}_{p'/q'}$ is contained in one of the relatively dyadic wakes $W'_{p/2^m}$ of $\mathbf{M}_0$. The theorem then follows from Yoccoz parameter puzzle theorem for renormalizable parameters, which states that

$$\mathbf{M}_0 = \bigcap_{n\geq 0} \overline{\mathcal{P}}_n.$$

$\square$

Corollary 20. *Either the diameter of the sequence* Δ_k *tends to* 0 *or* c_∞ *is in a strict copy* M' *of* $\mathbf{M}$ *in* $\mathbf{M}$*, i.e. that* Q_{c_∞} *is renormalizable.*

Proof. For any p/q the corresponding rotation orbit $0 < \theta_0 < \ldots < \theta_{q-1}$ is disjoint from the set of dyadic arguments. Thus for any p/q the graph defining the associated p/q puzzle for $L^{H_0}_{p/q}$ is disjoint from $\Delta^{\mathbf{M}}$. Therefore, there exists an increasing sequence n_k such that for $k \geq k_0$ the carrots are in the nest of puzzle pieces: $\Delta_k \subset \overline{\mathcal{P}}_{n_k}(c_\infty)$ (the Parameter Puzzle Piece containing c_∞).

Hence by Yoccoz Theorem 19 either the diameter tends to 0 or the limiting parameter c_∞ is renormalizable, that is $c_\infty \in M'$ for some period k first renormalization copy M' of $\mathbf{M}$ in $L^{H_0}_{p/q}$, where $q \leq k$. $\square$

For the rest of the proof, it is enough now to consider parameters c_∞ in a copy $\mathbf{M}'$ of $\mathbf{M}$.

Remark 21. *The key point in the proof of Proposition 6 is that all of the dyadic carrots* $\Delta_{p/2^n}$ *are disjoint from* M'*, because their root points* $\Psi(p/2^n)$ *are disjoint from* M'*.*

5 Dynamical Estimates

We make here the assumption that the limit point $c_\infty \in M' \subset L^{H_0}_{p/q}$ and that M' is a primitive copy of $\mathbf{M}$ of period k. and will treat the satellite case later. Let c_b be the center of the hyperbolic component H'_0 of period k of $\mathbf{M}'$. We first define, for this parameter c_b, a slicing of some neighborhood of the Julia set that allows to wrap the carrots in annuli. It will yields a lower bound on the modulus of these annuli. Then we will deform this picture by holomorphic motion on some neighborhood of $\mathbf{M}'$, keeping the lower bound on the moduli.

5.1 Slicing of some part of the Julia set

Since $M' \subset L^{H_0}_{p/q}$, one can define a Yoccoz puzzle as in Definition 17. Then Theorem 19 says that for $c \in M'$ there is a first level n such that the puzzle piece P^c_n containing the critical value c defines a quadratic like map as $Q^k_c : Int(P^c_{n+k}) \to Int(P^c_n)$.

We give now some notations related to the puzzle structure, written for $c = c_b$:

- Let $\eta^- < \eta^+$ denote the (rational) arguments of the co-landing pair of external rays for Q_c, which are on the boundary of $P = P_n^c$ and which separates c from 0. Then by Douady and Hubbard (see [DH1]) the parameter rays $R^{\mathbf{M}}_{\eta\pm}$ co-land at some Misiurewicz point in $\mathbf{M}$.

- Denote by γ_0^c the set $\overline{R^c_{\eta_-} \cup R^c_{\eta_+}} \cap F^c(1)$ and let U_0^c denote the component not containing 0 and which is bounded by γ_0^c and a subarc of the equipotential $E^c(1)$.

- Let U_1^c denote the connected component of $Q_c^{-k}(U_0^c)$ contained in P^c_{n+k}. It is a disk and $f_c := Q_c^k : U_1^c \to U_0^c$ is quadratic-like, with critical point ω_c satisfying $f_c(\omega_c) = c$.

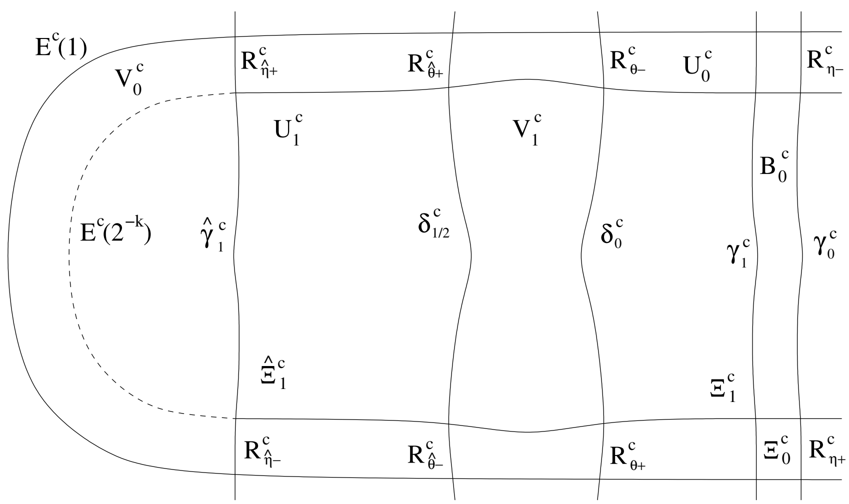

Figure 5: The disks U_0^c, U_1^c, and $\widehat{\Xi}_1^c$. The set U_0^c is the disk insided $E^c(1)$ and to the left of γ_0^c. The set U_1^c is the disk inside $E^c(2^{-k})$ and bounded to the right and left by γ_1^c and $\widehat{\gamma}_1^c$ respectively. The arc δ_0^c separates the two subdisks V_0^c and Ξ_0^c of U_0^c, V_0^c to the left and Ξ_0^c to the right of δ_0^c. The subsets $\widehat{\Xi}_1^c$, V_1^c and Ξ_1^c of U_1^c are to the left of $\delta_{1/2}^c$, between $\delta_{1/2}^c$ and δ_0 and to the right of δ_0 repsectively.

We give now some notations related to the copy $\mathbf{M}'$, expressed for $c = c_b$:

- Let $\theta_\pm$ denote the arguments of the external rays of $\mathbf{M}$ co-landing at the root c_0' of M'.

- Let δ_0^c denote the subarc of $R_{\theta_-} \cup \{\beta_c'\} \cup R_{\theta_+}$ consisting of points with potential up to and including 1, *i.e.* $\delta_0^c = \overline{R_{\theta_-} \cup R_{\theta_+}} \cap F^c(1)$.

- Similarly let $\delta_{1/2}^c$ denote the subarc of $\overline{R_{\widehat{\theta}_-} \cup R_{\widehat{\theta}_+}} \cap F^c(1)$.

- Let V_0^c be the connected component of $U_0^c \setminus \delta_0^c$ containing ω_c. Let Ξ_0^c denote the other connected component. Define recursively $V_n^c = f_c^{-n}(V_0^c)$ and $\Xi_n^c = f_c^{-n}(\Xi_0^c) \cap \Xi_0^c$ (see also Fig. 5). The restriction $f_c : V_{n+1}^c \longrightarrow V_n^c$ is a $2:1$ branched covering, whereas $f_c : \Xi_{n+1}^c \longrightarrow \Xi_n^c$ is an isomorphism.

- Let γ_1^c denote the extension to potential level 1 of $Q_c^{-k}(\gamma_0^c) \cap \partial \Xi_0^c$ and let $B_0^c \subset \Xi_0^c$ denote the quadrilateral bounded by γ_0^c, γ_1^c and subarcs of $E^c(1)$. Define recursively the univalently iterated preinages $B_{n+1}^c = f_c^{-1}(B_n^c) \cap \Xi_n^c$.

- For each $n \geq 1$ let $\widehat{B}_n^c \subset \widehat{\Xi}_n^c$ denote the "twin" of B_n^c, i.e. the connected component of $f_c^{-1}(B_{n-1}^c) \cap \widehat{\Xi}_n^c$.

- Let $\widehat{\gamma}_1 = Q_c^{-k}(\gamma_0^c) \cap \partial \widehat{\Xi}_1^c$ extended to equipotential level $3/4$ and let Ω^c denote the open disk bounded by $\widehat{\gamma}_1^c$ and the subarc of $E^c(3/4) \cap U_0^c$ connecting the endpoints of $\widehat{\gamma}_1^c$.

- Let $D^c \subset V_0^c$, denote the disc bounded by $\delta_{1/2}^c$ union the subarc of $E^c(1)$ connecting the endpoints of $\delta_{1/2}^c$.

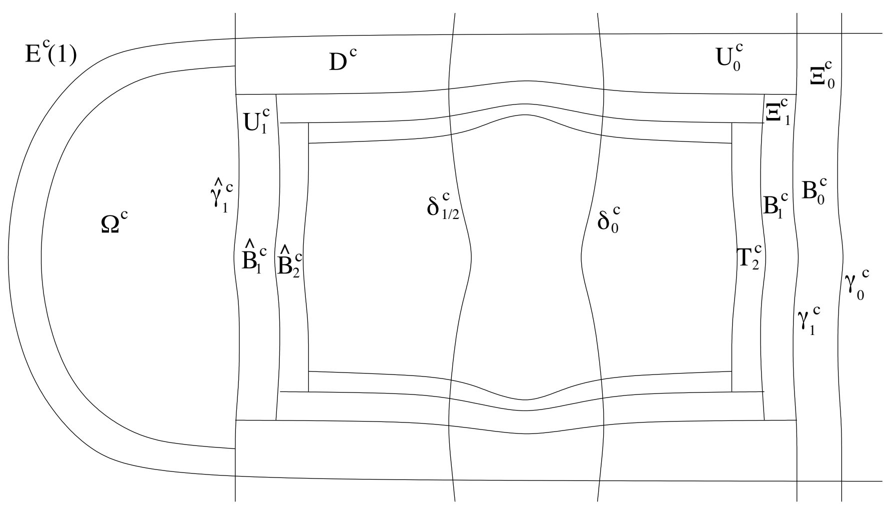

Figure 6: The decomposition of the disk D^c, which is bounded by the equipotential $E^c(1)$ and the arc $\delta_{1/2}^c$. The disk Ξ_0^c is the subset of U_0^c to the right of δ_0^c. The disk $\widehat{\Xi}_1^c$ (not labelled) is the subset of U_1^c to the left of $\delta_{1/2}^c$. The disc Ω^c is to the left of $\widehat{\gamma}_1^c$ and inside the equipotential $E^c(3/4)$ (not labelled).

Lemma 22. *By construction each of the sets, for $c = c_b$, Ω^c and $\widehat{B}_n^c$, $n \geq 1$ are relatively compact in D^c. Moreover, there exists $m = m(c_b) > 0$ such that*

$$\operatorname{mod}(D^c \setminus \Omega^c) \geq m, \quad \text{and} \quad \forall\, n \geq 1 \quad \operatorname{mod}(D^c \setminus \overline{\widehat{B}_n^c}) \geq m$$

so that

$$\forall\, n \geq 1 \quad \mathrm{mod}(\Xi_0^c \smallsetminus \overline{B_n^c}) \geq m.$$

Proof. The restriction $f_c : \widehat{\Xi}_1^c \longrightarrow \Xi_0^c$ is biholomorphic so that for all $n \geq 1$

$$\begin{aligned}\mathrm{mod}(D^c \smallsetminus \overline{\widehat{B}_{n+1}^c}) &\geq \mathrm{mod}(\widehat{\Xi}_1^c \smallsetminus \overline{\widehat{B}_{n+1}^c}) = \mathrm{mod}(\Xi_0^c \smallsetminus \overline{B_n^c}) \\ &\geq \mathrm{mod}(\Xi_{n-1}^c \smallsetminus \overline{B_n^c}) = \mathrm{mod}(\Xi_0^c \smallsetminus \overline{B_1^c})\end{aligned}$$

Thus we may define

$$m(c) = \min\{\mathrm{mod}(D^c \smallsetminus \Omega^c), \mathrm{mod}(D^c \smallsetminus \overline{\widehat{B}_1^c}), \mathrm{mod}(\Xi_0^c \smallsetminus \overline{B_1^c})\}.$$

□

5.2 Holomorphic motion of graphs

A holomorphic motion of a set $X \subset \mathbb{C}$ parameterized by $\mathbb{D}$ is a map $H(\lambda, z)$ from $\mathbb{D} \times X$ to $\mathbb{C}$ that is holomorphic in $\lambda \in \mathbb{D}$ injective in $z \in \mathbb{C}$ and the identity for $\lambda_0 = 0$.

Douady and Hubbard proved (in [DH1]) that the closure $\overline{R}$ of an external rays $R_t^{c_b}$ admits a holomorphic motion when t is enventually periodic and if the orbit of $\overline{R}$ does not meet neither a critical point nor a parabolic point.

Definition 23. *Let $\theta_\pm$ denote the arguments of the external rays of $\mathbf{M}$ co-landing at the root c_0' of M'.*

- *Let Λ_0, resp. Λ_1, denote the parameter disk whose closure contains M' and which is bounded by the segments of the rays $R_{\theta_\pm}^{\mathbf{M}}$ with potential up to and including 2, resp 1, union a connecting subarc of the level 2 equipotential $E_{\mathbf{M}}(2)$, resp. $E_{\mathbf{M}}(1)$.*
- *Denote by Λ_0^P, resp. Λ_1^P, the parameter disk which contains Λ_0, resp. Λ_1, and which is bounded by a subarc of $\overline{R_{\eta-}^{\mathbf{M}} \cup R_{\eta+}^{\mathbf{M}}}$ union a subarc of $E^{\mathbf{M}}(2)$, resp. $E^{\mathbf{M}}(1)$.*

We need the following result on $\mathbf{M}$ presumably due to Douady, Hubbard and Lavaurs. This theorem is at least folklore. But because we do not have a precise reference, we have for completeness provided a proof in the Appendix, on page 27.

Theorem 24. *Let $0 < t_- < t_+ < 1$ be rationals for which the parameter rays $R_{t_\pm}^{\mathbf{M}}$ coland at some point $c_0 \in \mathbf{M}$ and let $W_{t_\pm}$ denote the parameter sector bounded by $R_{t_-}^{\mathbf{M}} \cup \{c_0\} \cup R_{t_+}^{\mathbf{M}}$ and not containing the main cardioid H_0. Then we have the following properties:*

- *the forward orbits of $t_\pm$ do not enter the interval $]t_-, t_+[$;*
- *for any $c \in W_{t_\pm}$ the clousre of the pair of dynamical rays $R_{t_\pm}^c$ move homorphically with c;*

- *they co-land at some repelling (pre)periodic point $z(c)$ with $Q_c^{k'+l}(z(c)) = Q_c^l(z(c))$, where $l \geq 0$ is the common preperiod of $t_\pm$ and $k' > 0$ divides the common period $k > 0$ of $\sigma^l(t_\pm)$;*
- *the set $R^c_{t_-} \cup \{z(c)\} \cup R^c_{t_+}$ bounds a sector W^c containing c, but not 0.*

Lemma 25.

- *For $c \in \Lambda_0^P$, the set γ_0^c is an arc that moves holomorphically (within Λ_0^P). The component U^c, not containing 0 and which is bounded by γ_0^c and a subarc of the equipotential $E^c(1)$ is a disk whose boundary moves holomorphically over Λ_0^P.*
- *Let U_1^c denote the connected components of $Q_c^{-k}(U_0^c)$ containing $(R^c_{\theta_-} \cup R^c_{\theta_+}) \cap F^c(2^{-k})$.*

  The restriction f_c
  $$f_c := Q_c^k : U_1^c \longrightarrow U_0^c \tag{2}$$
  is a degree two ramified covering.

  For $c \in \Lambda_0^P \setminus \Lambda_1^P$, U_1^c is the union of two disks and f_c is non ramified.

  For $c \in \Lambda_1^P$, U_1^c is a disk whose boundary ∂U_1^c moves holomorphically. Moreover f_c is quadratic like with critical point ω_c. The filled Julia set K'_c is connected, if and only if $c \in M'$.

Proof. By definition, γ_0^c is the set $\overline{R^c_{\eta_-} \cup R^c_{\eta_+}} \cap F^c(1)$ and U_0^c denote the component not containing 0. It follows from Theorem 24 that the dynamical rays $R^c_{\eta_\pm}$ co-land for every $c \in \Lambda_0^P$ and admits a holomorphic motion. Moreover by stability, they separate 0, from c. The rest follows. $\square$

Notice that Λ_1^P is precisely the set of parameters for which $c \in U_0^c$, in fact $c \in \partial U_0^c$ if and only if $c \in \partial \Lambda_1^P$.

Lemma 26.

$$E_c(1) \bigcup \bigcup_{j=0}^{k} Q_c^{-j}(R^c_{\theta_-} \cup \{\beta'_c\} \cup R^c_{\theta_+})$$

moves holomorphically with $c \in \Lambda_0$.

Proof. Here we use Theorem 24. For any $c \in \Lambda_0$ the dynamical rays $R^c_{\theta_\pm}$ co-land at a repelling k-periodic point β'_c and the rays $R^c_{\widehat{\theta}_\pm}$ co-land at the Q_c^k-preimage coβ'_c of β'_c all of which moves holomorphically with $c \in \Lambda_0$. Moreover the set

$$\bigcup_{j=0}^{k-1} Q_c^{-j}(R^c_{\theta_-} \cup \{\beta'_c\} \cup R^c_{\theta_+})$$

does not enter the sector $W^c_{\theta_-,\theta_+}$ bounded by the closure of the colanding pair of rays $R^c_{\theta_-}$ and $R^c_{\theta_+}$ and containing c. Hence

$$E_c(1) \bigcup \bigcup_{j=0}^{k} Q_c^{-j}(R^c_{\theta_-} \cup \{\beta'_c\} \cup R^c_{\theta_+})$$

moves holomorphically with $c \in \Lambda_0$. $\square$

Corollary 27. *In particular, $\delta_0^c := \overline{R_{\theta_-} \cup R_{\theta_+}} \cap F^c(1)$ and $\delta_{1/2}^c := \overline{R_{\widehat{\theta}_-} \cup R_{\widehat{\theta}_+}} \cap F^c(1)$ move holomorphically in Λ_0.*

For $c \in \Lambda_0$ the set ∂U_1^c does not admit a holomorphic motion. Thus,

Lemma 28. *For $c \in \Lambda_0 \smallsetminus \Lambda_1$, consider $\tilde{V}^c$ the disk included in U_0^c bounded by δ_0^c and $\delta_{1/2}^c$. Then the subset of ∂U_1^c outside this disk moves holomorphically with $c \in \Lambda_0$.*

Proof. Indeed, the only place where the boundary of ∂U_0^c does not admits a holomorphic motion is when the parameter c crosses the equipotential $E(1)$ in the region bounded by δ_0^c. At this parameter, the equipotential of ∂U_0^c makes a figure eight in the region bounded by δ_0^c and $\delta_{1/2}^c$. $\square$

Corollary 29. *Let V_0^c be the connected component of $U_0^c \setminus \delta_0^c$ containing $\delta_{\frac{1}{2}}^c$. Then ∂V_0^c admits a holomorphic motion in Λ_0. The preimage $V_1^c = f_c^{-1}(V_0^c)$ is a disk for $c \in \Lambda_1$ whose boundary ∂V_1^c admits a holomorphic motion in Λ_1.*
For $c \in \Lambda_0 \setminus \Lambda_1$, the preimage V_1^c is the union of two disks.
Moroever, for $c \in \Lambda_0$, $f_c : V_1^c \to V_0^c$ is a degree 2 (maybe) ramified covering.

Proof. Since ω_c is only defined for $c \in \Lambda_1$, we have to use $\delta_{\frac{1}{2}}^c \cap F^c(1)$ here to defined V_0^c. By the Lemma, the arcs $\delta_0^c, \delta_{\frac{1}{2}}^c$ move holomorphically in Λ_0. The part of the equipotential $E^c(1)$ bounding V_0^c moves holomorphically in Λ_0, and its preimages moves holomrophically if $c \notin E^c(1)$, *i.e.* for $c \in \Lambda_1$. $\square$

Note that for $c \in \Lambda_0 \setminus \Lambda_1$, the preimage V_1^c is the union of two disks. They ly in the disk $\tilde{V}^c$ bounded by δ_0^c and $\delta_{1/2}^c$.

Lemma 30. *Now for $c \in \Lambda_0$ we can extend the definitions of V_n^c, Ξ_n^c, B_n^c, $\widehat{B}_n^c$, $\widehat{\Xi}_n^c$, since they are just pull back by f_c of sets like U_0^c and V_0^c that are well defined in Λ_0 and whose boundary admit a holomorphic motion.*

We recall here the definitions for $c \in \Lambda_0$ that extends the one given for c_b :

- Define recursively $V_n^c = f_c^{-n}(V_0^c)$, $\Xi_0^c := U_0^c \smallsetminus V_0^c$ and $\Xi_n^c = f_c^{-n}(\Xi_0^c) \cap \Xi_0^c$ (see also Fig. 5).

  Then the restriction $f_c : V_{n+1}^c \longrightarrow V_n^c$ is a $2:1$ branched covering whereas $f_c : \Xi_{n+1}^c \longrightarrow \Xi_n^c$ is an isomorphism.

- Let γ_1^c denote the extension to potential level 1 of $Q_c^{-k}(\gamma_0^c) \cap \partial\Xi_0^c$ and let $B_0^c \subset \Xi_0^c$ denote the quadrilateral bounded by γ_0^c, γ_1^c and subarcs of $E^c(1)$. Define recursively the univalently iterated preinages $B_{n+1}^c = f_c^{-1}(B_n^c) \cap \Xi_n^c$.

- For each $n \geq 1$ let $\widehat{\Xi}_n^c$ denote the "other" connected component of $f_c^{-1}(\Xi_{n-1}^c)$, having a boundary arc in $\delta_{1/2}^c$.

- For each $n \geq 1$ let $\widehat{B}_n^c \subset \widehat{\Xi}_n^c$ denote the "twin" of B_n^c, i.e. the connected component of $f_c^{-1}(B_{n-1}^c) \cap \widehat{\Xi}_n^c$.
- Let $\widehat{\gamma}_1 = Q_c^{-k}(\gamma_0^c) \cap \partial\widehat{\Xi}_1^c$ extended to equipotential level 3/4 and let Ω^c denote the open disk bounded by $\widehat{\gamma}_1^c$ and the subarc of $E^c(3/4) \cap U_0^c$ connecting the endpoints of $\widehat{\gamma}_1^c$.
- Let $D^c \subset V_0^c$, denote the disc bounded by $\delta_{1/2}^c$ union the subarc of $E^c(1)$ connecting the endpoints of $\delta_{1/2}^c$.

Then it follows that

Lemma 31. *The graph*

$$G^c = \partial D^c \cup \partial\Omega^c \cup \bigcup_{n=1}^{\infty}(\partial\widehat{\Xi}_n^c \cup \partial\widehat{B}_n^c) \cup \bigcup_{n=0}^{\infty}(\partial\Xi_n^c \cup \partial B_n^c) \cup \partial U_0^c$$

moves holomorphically with $c \in \Lambda_0$

Proof. Note that $Q_c^n(G^c)$ does not meet the critical point 0 for any $n \geq 0$ or $c \in \Lambda_0$: When the parameter $c \in \Lambda_0$ the critical value c does not belong to $\overline{\Xi_0^c}$. Hence the Böttcher-coordinate is defined and depends holomorphically on c, on the dense subset $(\partial\Xi_0^c \cup \partial B_0^c)\smallsetminus K_c$ of $(\partial\Xi_0^c \cup \partial B_0^c)$, so that the later moves holomorphically with c. Secondly f_c depends holomorphically on c and its critical value c again still does not belong to $\overline{\Xi_0^c}$. Hence the iterated univalent preimages of $(\partial\Xi_0^c \cup \partial B_0^c)$ inside Ξ_0^c and $\overline{\widehat{\Xi}_1^c}$ depend also holomorphically on c. This takes care of

$$\bigcup_{n=1}^{\infty}(\partial\widehat{\Xi}_n^c \cup \partial\widehat{B}_n^c) \cup \bigcup_{n=0}^{\infty}(\partial\Xi_n^c \cup \partial B_n^c).$$

Finally ∂D^c moves holomorphically with c, because $\delta_{1/2}^c$ does and $\partial\Omega^c$ moves holomorphically with c, because $\widehat{\gamma}_1^c$ does. □

Corollary 32. *There exists $m = m(c) > 0$ such that* $\forall\, n \geq 1$

$$\mathrm{mod}(D^c\smallsetminus\Omega^c) \geq m, \quad \mathrm{mod}(D^c\smallsetminus\overline{\widehat{B}_n^c}) \geq m \quad \textit{and} \quad \mathrm{mod}(\Xi_0^c\smallsetminus\overline{B_n^c}) \geq m.$$

Proof. It follows from the fact that these sets admit a holomorphic motion and from Lemma 22. □

5.3 Dynamical carrots

We define the relatively dyadic wakes W_0^c as the open set not containing c and bounded by the closure of the rays $R_{\theta_\pm}^c$ and $W_{1/2}^c = W_{\widehat{\theta}_+,\widehat{\theta}_-}^c$ as the open set bounded by the closure of the rays $R_{\widehat{\theta}_\pm}^c$ and disjoint from W_0^c. For any $c \in \Lambda_0$ the filled-in Julia set $K_c' \subset K_c$, defined above, coincides with the points in the filled-in Julia set of K_c, whose orbits never enters the relatively dyadic wakes W_0^c and $W_{1/2}^c$ (see (2)).

The key point in the proof of Proposition 6 is that if a carrot intersects Λ_0, then it is entirely contained in Λ_0 and its dynamical counter part in the dynamical planes of Q_c is either contained in $W_{1/2}^c$ or has a univalent forward image, which is. In order to prove the theorem we shall wrap the dynamical counter part of each dyadic carrot inside the relatively dyadic wake $W_{1/2}^c$ in an annulus in $W_{1/2}^c$ moving holomorphically with $c \in \Lambda_0$ and of modulus bounded uniformly from below. We do this in the next subsection.

Let $\Delta_{p/2^m}^c$ denotes the dynamical filed of dyadic carrots. They are defined as $\phi_c^{-1}(\Delta_{p/2^m})$, where we use for ϕ_c^{-1} the maximal radial extension. (We can then view the carrots Δ of $\mathbf{M}$ as the set of parameters for which c belong to the corresponding carrot Δ^c of the Filled Julia set K_c.)

Lemma 33. *The graph*

$$G^c = \partial D^c \cup \partial \Omega^c \cup \bigcup_{n=1}^{\infty} (\partial \widehat{\Xi}_n^c \cup \partial \widehat{B}_n^c) \cup \bigcup_{n=0}^{\infty} (\partial \Xi_n^c \cup \partial B_n^c) \cup \partial U_0^c$$

does not intersect any of the dynamical dyadic carrots $\Delta_{p/2^m}^c$.

Proof. All such carrots are at dyadic angles and only Δ_0^c extends further than potential $1/2$. □

Corollary 34. *Such a carrot is thus wrapped by an annulus of modulus at least* $m(c) > 0$*, contained in* D^c *and thus disjoint from* K_c'.

Proof. No dynamical plane dyadic carrot $\Delta_{p/2^n}^c$ intersects G^c. Hence any such dyadic carrot in the relative dyadic wake ${W^c}_{1/2}$ of the filled Julia set K_c' for $f_c : U_1^c \longrightarrow U_0^c$ is contained in one of the sets Ω^c or $\widehat{B}_n^c$ for some $n \geq 1$. It then follows from Lemma 22. □

6 Transfer to the parameter plane

We shall use an argument to transfer bounds on moduli of dynamical annuli to bounds on moduli of corresponding parameter annuli. This argument was pioneered by Shishikura (see also [R]).

We have fixed a base point namely $c_b \in \Lambda_1 \subset \Lambda_0$ the center of the central hyperbolic component of M'. Note that on $F(2) \supset \Lambda_0$ the Böttcher-coordinate at ∞ defines a holomorphic motion of the set $\overline{\mathbb{C}} \setminus F^{c_b}(1)$ extending the Böttcher-motion of G^{c_b}. By Slodkowski's extension theorem there exists a global holomorphic motion $H : \Lambda_0 \times \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ over Λ_0 with base point c_b and extending the Böttcher-motion of the graph G^{c_b} union $\overline{\mathbb{C}} \setminus F^{c_b}(1)$, in particular we obtain holomorphic motions of $\overline{U_0^{c_b}} \supset \overline{D^{c_b}}$. As usual for $c \in \Lambda_0$ write $H_c(\cdot) = H(c, \cdot)$, then each map $H_c : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ is a quasi-conformal homeomorphism with a dilatation bounded uniformly above by $\log \mathrm{d}_\Lambda(c, c_b)$, where $\mathrm{d}_\Lambda(\cdot, \cdot)$ denotes hyperbolic distance in Λ_0.

Let $\delta_0^{M'}, \delta_{1/2}^{M'}$ denote the parameter rays of the same angle as $\delta_0^c, \delta_{1/2}^c$ for $c \in M'$ Define similarly to D^c the parameter disk $D^{M'} \subset \Lambda_1 \setminus M'$ as the disc bounded by $\delta_{1/2}^{M'}$ union the subarc of $E^{\mathbf{M}}(1)$ connecting the endpoints of $\delta_{1/2}^{M'}$. Then $D^{M'}$ is relatively compact in Λ_0. Let $\chi : \Lambda_0 \longrightarrow \overline{\mathbb{C}}$ be the map $\chi(c) = H_c^{-1}(c)$. Then χ is locally quasi-regular with dilatation $K(c)$ bounded by $\log \mathrm{d}_\Lambda(c, c_b)$. By construction the restriction $\chi : \partial D^{M'}_{\longrightarrow} \partial D^{c_b}$ is a homeomorphism. Hence the restriction $\chi : \overline{D^{M'}} \longrightarrow \overline{D^{c_b}}$ is the restriction of a quasi-conformal homeomorphism with dilatation bounded by $K = K_{1/2} = \max\{\log \mathrm{d}_\Lambda(c, c_b) | c \in \overline{D^{M'}}\}$.

Define $\Omega^{M'} = \chi^{-1}(\Omega^{c_b}) \subset D^{M'}$ and $\widehat{B}_n^{M'} = \chi^{-1}(\widehat{B}_n^{M'}) \subset D^{M'}$, $n \geq 1$. Then any dyadic carrot $\Delta_{p/2^n} \subset W_{1/2}^{M'}$ is contained in one of the disks $\Omega^{M'}$ or $\widehat{B}_n^{M'}$ and is thus wrapped in an annulus with a modulus bounded from below by $K_{1/2} \cdot m(c_b)$ according to Lemma 22.

Rename $D^c =: D_{1/2}^c$, $D^{M'} =: D_{1/2}^{M'}$ and $\chi =: \chi_{1/2}$. We have proved that any dyadic carrot $\Delta_{p/2^n}$ in the relative $1/2$ wake $W_{1/2}^{M'}$ of M' is wrapped in an annulus of modulus uniformly bounded from below and contained in the disk $D_{1/2}^{M'}$, which is disjoint from M'. Moreover the annuli are q-c images of corresponding annuli in the dynamical plane of Q_{c_b}. We shall prove by induction the similar statements for any other relative dyadic wake $W_{r/2^s}^{M'}$ of M'. The only difference is that the bounds on the dilatation of the q-c homeomorphisms $\chi_{r/2^s}$ and hence on the moduli of annuli in the $W_{r/2^s}^{M'}$ wake depends on $r/2^s$. As a remedy for this we shall apply the Levin-Yoccoz parameter inequality once more. Here follow the details.

Recall that V_0^c is the connected component of $U_0^c \setminus \delta_0^c$ containing the critical point ω_c and $V_n^c = f_c^{-n}(V_0^c)$. We shall need also the extension $\widetilde{V}_1^c = V_0^c \setminus \overline{D^c}$ of V_1^c and its iterated preimages $\widetilde{V}_n^c = f_c^{-(n-1)}(\widetilde{V}_1^c)$. Define parameter disks Λ_s, $s > 1$ by

$$\Lambda_s = \{c | c \in \widetilde{V}_{s-1}^c\}.$$

Evidently $\Lambda_s \supset \Lambda_{s+1}$. Note that the condition $c \in \widetilde{V}_s^c$ is equivalent to $f_c^s(\omega_c) \in \widetilde{V}_1^c$. Rename $G^c =: {G^c}_1$ and define recursively , ${G^c}_{s+1} = f_c^{-1}({G^c}_s) \cup {G^c}_1$ for $s \geq 1$. Then as noted above G_1^c moves holomorphically in $\Lambda_0 \supset \Lambda_1$ and we shall prove as part of the induction on $s \geq 2$, that for $c \in \Lambda_s$ the critical value $c \notin {G^c}_{s-1}$, so that ${G^c}_s$ moves holomorphically over Λ_s.

For $s = 2$ notice that, by the above $c \in \overline{D_{1/2}^c}$ if and only if $c \in \overline{D_{1/2}^{M'}}$. Thus $c \notin {G^c}_1$ for any $c \in \Lambda_2$, so that G_2^c moves holomorphically with $c \in \Lambda_2$. For $c \in \Lambda_2$ let $D_{rj/2^2}^c$ for $r = 1, 3$ denote the connected components of $f_c^{-1}(D_{1/2}^c)$ containing the $r/2^2$ dyadic decorations and define $D_{r/2^2}^{M'} \subset \Lambda_2$ as the parameter disks bounded by the corresponding parameter ray segments and equipotential level.

Rename the previous holomorphic motion H to H_1 and let H_2 denote the restriction of H_1 to $\Lambda_2 \times (\overline{\mathbb{C}} \setminus \widetilde{V}_1^{c_b})$. Extend H_2 to a motion including $\widetilde{V}_1^{c_b} \setminus {V^{c_b}}_1$ using the Böttcher-motion and extend H_2 further to $\Lambda_2 \times f_{c_b}^{-1}(\overline{D_{1/2}^{c_b}})$ by $f_c^{-1} \circ$

$H_1(c, f_{c_b}(z))$, where the inverse branches are taken so as to map $D^{c_b}_{r/2^2}$ quasi-conformically onto $D^c_{r/2^2}$. Finally use Slodkowski's extension theorem to extend this holomorphic motion to a holomorphic motion of $\overline{\mathbb{C}}$ over the disk Λ_2 (i.e. extend the motion by a motion of $V^{c_b}{}_2$). By the same argument as above the map $\chi_2 : \Lambda_2 \longrightarrow \widetilde{V}^{c_b}_1$ given by $\chi_2(c) = (H_2)^{-1}_c(c)$ is a locally quasi regular map. Again by construction $\chi_2 : \partial D^{M'}_{r/2^2} \longrightarrow \partial D^{c_b}_{r/2^2}$ are homeomorphisms so that the restrictions $\chi_2 : D^{M'}_{r/2^2} \longrightarrow \partial D^{c_b}_{r/2^2}$ are quasi-conformal. However on the sets $D^{M'}_{r/2^2}$ the holomorphic motion H_2 is a conjugacy between the holomorphic maps f_{c_b} and f_c. Hence the dilatation of $(H_2)_c$ at z equals that of $(H_1)_c$ at $f_{c_b}(z)$. Hence again the dilatation of χ_2 on $D^{M'}_{r/2^2}$ is again bounded by bound given by $K_{r/2^s} = \max\{\log \mathrm{d}_\Lambda(c, c_b) | c \in \overline{D^{M'}_{r/2^2}}\}$. Arguing as in the initial case corresponding to $s = 1$ completes the case $s = 2$. The inductive step is completely similar and is left to the reader.

Let H' denote the central hyperbolic component of M'. Then for k sufficiently large c_k belongs to the p'_k/q'_k limb $L^{H'}_{p'_k/q'_k}$ of H'. Applying the Yoccoz-Levin parameter inequality Theorem 15 to H' we find that the diameter of $L^{H'}_{p'/q'}$ union its attached dyadic carrots is bounded uniformly by C/q' for some constant $C = C_{H'}$. Arguing as in the beginning of the first reduction we can assume that $p'_k/q'_k = p'/q'$ for all large k. The set $W^{H'}_{p'/q'} \cap \Lambda_1$ is relatively compact in Λ_0 so that

$$\sup\{\log \mathrm{d}_\Lambda(c, c_b) | c \in W^{H'}_{p'/q'} \cap \Lambda_1\} = K = K^{H'}_{p'/q'} < \infty.$$

Hence the dyadic carrots Δ_k either has a diamter which a priori tends to zero or such carrots are separated from M' by an annulus in $\Lambda_0 \setminus M'$ of modulus at least $m(c_b)/K$. And in the latter case their diameters are forced to converge to zero a posteriori. Because the roots $c_k \in \Delta_k$ converge to $c_\infty \in M'$,

This completes the proof that if $\Lambda_0 \ni c_k \to c_\infty \in M'$, then the diameter of Δ_k converge to zero. For the case $c_k \in \Lambda_P \setminus \Lambda_0$ we necessarily have $c_\infty = c_r$, where c_r denotes the root of M'. To prove that the diameter of Δ_k converge to zero also in this case let

$$\Lambda^P_n = \{c \in \Lambda_P | c \in U^c_n\}.$$

Where $U^c_n = f_c^{-n}(U^c_0)$, (it may or may not be connected) for $n > 1$.

For any $c \in \Lambda_0$ the sets $\partial \Xi^c_n$, $n \geq 0$ move holomorphically with c. Define $A^c_n = \Xi^c_n \setminus \Xi^c_{n+1}$, then the A^c_n are quadrilaterals with a-sides the boundary arcs $\partial A^c_n \cap R^c_{\theta_-}$ and $\partial A^c_n \cap R^c_{\theta_+}$. Moreover ∂A^c_n even move holomorphically with $c \in (\Lambda_0 \cup \Lambda^P_n)$. Let

$$A^{M'}_n = \{c \in \Lambda^P_n | c \in A^c_n\}$$

denote the corresponding parameter quadrilaterals. Then the root c_r of M' belong to Λ^P_n, for all n. Choose by Slodkowski's extension theorem a holomorphic motion

$$H^0 : \Lambda^P_0 \times A^{c_r}_0 \longrightarrow \mathbb{C}$$

over Λ_0^P with base point c_r of the quadrilateral $A_0^{c_r}$ extending the Böttcher motion of its boundary.

For $c \in \Lambda_n^P$ the restriction $f_c^n : A_n^c \longrightarrow A_0^c$ is biholomorphic. Hence we may lift the motion H^0 to a holomorphic motion

$$H^n : \Lambda_n^P \times A_n^{c_r} \longrightarrow \mathbb{C}.$$

As with the annuli above define quasi-conformal homeomorphisms

$$\rho_n : A_n^{M'} \longrightarrow A_n^{c_r}$$

by $\rho_n(c) = (H_c^n)^{-1}(c)$. Then as above these have q.-c. distortion bounded by the distortion of the q.-c. homeomorphisms $H_c^0(\cdot)$, $c \in A_n^{M'}$. That is bounded by

$$K = \sup\{\log \mathrm{d}_{\Lambda_0^P}(c, c_r) | c \in A_n^{M'}\}$$

which is uniformly bounded, because $A_n^{M'} \subset \Lambda_1^P \subset\subset \Lambda_0^P$. Thus all the quadrilaterals $A_n^{M'}$ have modulus bounded uniformly from below by $\mathrm{mod}(A_0^{c_r})/K$. Moreover the a-sides of these quadrilaterals are all contained in the two rays $R_{\theta_-}^{M'}$ and $R_{\theta_+}^{M'}$ co-landing at c_r. By the Grötzsch-inequality for annuli the euclidean diameter of $A_n^{M'}$ tend to zero and the closures converge to c_r. By construction no dyadic carrot intersects the boundary of any of the $A_n^{M'}$. Thus also in this case the diameter of Δ_k converge to zero as $k \to \infty$. This completes the proof in the case c_∞ belongs to a primitive first renormalization copy.

7 The satelite case

In the complementary satelite case $M' = M_{p/q}$ with central hyperbolic component $H_{p/q}$ attached at internal argument $\exp(i2\pi p/q)$ from the central hyperbolic component H_0 of $\mathbf{M}$. Let as above $\theta_- < \theta_+$ be the arguments of the parameter rays co-landing at the root and bounding the wake $W_{p/q}^{H_0}$. Recall that $c_\infty \in M_{p/q}$ is the limiting parameter of the roots of dyadic carrots and that these dyadic carrots are eventually contained in $W_{p/q}^{H_0}$.

We apply the Yoccoz-Levin parameter inequality Corollary 15 similarly as we have done twice above. This reduces the problem to the case where c_∞ belongs to a relative p'/q'-limb $L_{p'/q'}^{H_{p/q}}$ of $H_{p/q}$ for some $p'/q' \neq 0/1$ and the dyadic carrots Δ_k are subsets of the corresponding wake $W_{p'/q'}^{H_{p/q}}$ for large k. Denote by $\tau_- < \tau_+$ the arguments of the parameter rays bounding $W_{p'/q'}^{H_{p/q}}$ and define $\Lambda = \Lambda_0 = W_{p'/q'}^{H_{p/q}} \cap F^{\mathbf{M}}(2)$. Let $I^{M'} = I^{M_{p/q}}$ denote the Cantor set of arguments of parameter rays accumulating M' as given by the Douady tuning algorithm. Then $\tau_\pm \in I^{M'}$ and each has a unique preimage $\widehat{\tau}_\pm = \sigma^{-q}(\tau_\pm) \cap I^{M'}$ different from itself. For $c \in \Lambda$ let U_0^c denote the disk containg the fixed point α^c of Q_c and bounded by the segments $\overline{(R_{\sigma^i(\widehat{\tau}_-)}^c \cup R_{\sigma^i(\widehat{\tau}_+)}^c) \cap F^c(1)}$ for $0 < i < q$ union the connecting subarcs of $E^c(1)$. Denote by ι^c the open subarc of $\partial U_0^c \cap E^c(1)$

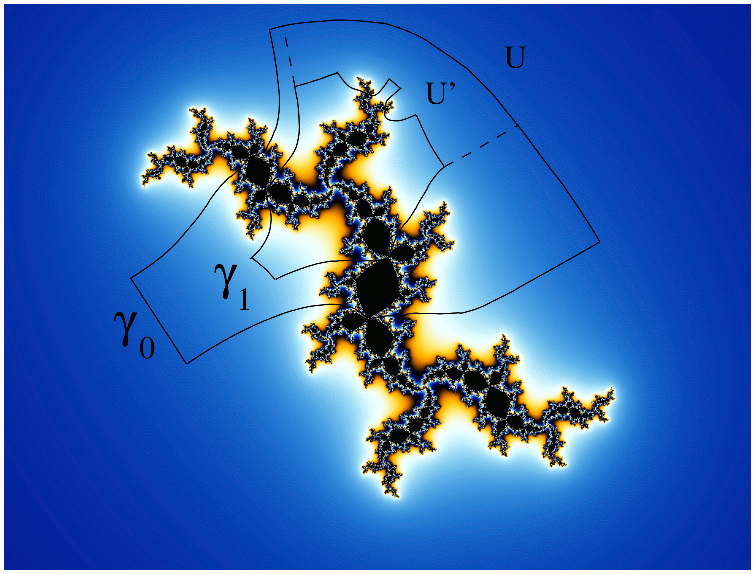

Figure 7: The disks U_0^c, U_1^c, B_0^c and Ξ_0^c.

intersecting the rays $R^c_{\theta_\pm}$ and let $\gamma_0^c = \partial U_0^c \setminus \iota^c$. As in the primitive case write $\delta_0^c = \overline{(R^c_{\theta_-} \cup R^c_{\theta_+}) \cap F^c(1)}$ for $c \in \Lambda$.

Then the whole setup is similar to the primitive case. We can thus define Ω^c, $\Xi_n^c, {B^c}_n, \widehat{\Xi}^c_{n+1}, \widehat{B}^c_{n+1}$ for $n \geq 0$ and G^c, all of which moves holomorphically with $c \in \Lambda$. There are however two differences: The first is that the center of $H_{p/q}$ does not belong to Λ. The arguments we used in the primitive case are in-sensitive to a change of base point c_b to another point in the interior of $\mathbf{M}$. We shall thus take as base point $c_b \in \Lambda$ the center of the central hyperbolic component $H^{H_{p/q}}_{p'/q'} \subset W^{H_{p/q}}_{p'/q'}$. The second difference is that the Yoccoz-Levin parameter inequality is applied to the sublimbs of the hyperbolic component $H^{H_{p/q}}_{p'/q'}$. We leave the details to the reader.

This completes the satelite case and thus completes the proof of Proposition 6.

8 Proving Theorem 3

The proof of Theorem 3 is completely analogous to the proof above of Theorem 1. Let M' be any copy of $\mathbf{M}$ inside $\mathbf{M}$ or $\mathbf{M_1}$. In the arguments above replace the carrot field Δ of $\mathbf{M}$ by the dyadic decorations Δ' of M'. As any decoration ${\Delta'}_{r/2^n}$ is contained in the relatively dyadic wake $W^{M'}_{r/2^n}$ bounded by the $r/2^n$ pair of co-landing relatively dyadic rays of M'. And as external rays do not cross the above arguments applies with the rays replaced by the corresponding rays relative to M'. That is use Yoccoz parameter inequality and the iterated

Yoccoz parameter puzzle theorem relative to M' to prove that: $\mathrm{diam}(\Delta'_k) \to 0$ for any sequence of decorations $(\Delta'_k)_k$ with roots c_k converging to a relatively non renormalizable parameter $c_\infty \in M'$. Secondly consider the case $c_\infty \in M''$, where $M'' \subset M'$ is a relative to M' first renormalizable copy of $\mathbf{M}$ belonging to some p/q limb of the central hyperbolic component H' of M'. Use again the p'/q' Yoccoz puzzle relative to M' to define the parameter disk Λ containing M'' similarly as we defined Λ for M' above. And define also Λ^P and γ_n analogously, i.e. with the aid of the p'/q' puzzle piece P and rays relative to M' given by Theorem 19 for M'. Similarly let δ_n be defined in terms the pairs co-landing rays which are dyadic relative to M''. From here the proof proceeds analogously.

Appendix

In this appendix we supply for completeness proofs of the two theorems we refer to, but for which we have not been able to find either adequate or complete proofs in the litterature.

Theorem 12 (The Levin-Yoccoz Dynamical Inequality)**.** *Let H be any hyperbolic component of $\mathbf{M}$ of period k. Let p/q be any non zero reduced rational and let $W^H_{p/q}$ denote the relative p/q wake of H, bounded by parameter rays with arguments $0 < \eta_- < \eta_+ < 1$. For any $c \in W^H_{p/q}$ let λ denote the multiplier of the repelling k-periodic common landing point α' of the kq periodic rays $R^c_{\eta_\pm}$. Then α' has combinatorial rotation number p/q and λ has a logarithm Λ such that:*

$$|\Lambda - p/q 2\pi i| \leq \frac{2k \log 2 \cos\theta}{q} \frac{\pi}{\omega(c)},$$

where $\theta \in\,]-\pi/2, \pi/2[$ is the argument of $\Lambda - p/q2\pi i$ and $\omega(c)$ is the angle of vision of the interval $i2\pi[\eta_-, \eta_+]$ from $\mathrm{Log}\,\phi_c(c) \in \{z = x + iy | 0 < y < 2\pi\}$.

Proof. Levin proved the fixed point case $k = 1$ in [L, TH. 5.1], the general case is similar. For completeness we sketch here a proof. Let us first recall the proof of Yoccoz inequality (or the Pommerenke-Levin-Yoccoz inequaltiy), full details can be found in [P]. Let T denote the quotient torus $T = D^*/Q^k_c$, where $D^* = \{z | 0 < |z - \alpha'| < r\}$ and $r > 0$ is chosen so small that Q^k_c is univalent on $D = D^* \cup \{\alpha'\}$ and $D \subset\subset Q^k_c(D)$. Let $\Pi : D^* \longrightarrow T$ denote the natural projection. The two rays $R^c_{\eta_\pm}$ belong to the same orbit and define combinatorial rotation number p/q for α'. Let $\gamma = \Pi(D^* \cap R^c_{\eta_-}) = \Pi(D^* \cap R^c_{\eta_+})$. Then γ is a Jordan curve and thus the pair (T, γ) has a conformal modulus which satisfies a Grötzsch inequality.

Let $w_\pm = \exp(i2\pi\eta_\pm)$. Then $Q^k_0(w_-) = w_+$ and $Q^{kq}_c(w_\pm) = w_\pm$. Similarly to T above let $\widehat{T}$ denote the quotient torus $\widehat{T} = \widehat{D}^*/Q^k_c$, where $\widehat{D}^*$ is a small punctured disk centered at say w_- and let $\widehat{\Pi} : \widehat{D}^* \longrightarrow \widehat{T}$ denote the natural projection. Then $\widehat{\Pi}(\widehat{D}^* \cap \mathbb{S}^1)$ are two disjoint Jordan curves in $\widehat{T}$, with complement two disjoint, symmetric and straight annuli A_i and A_o. Moreover

$\widehat{\gamma} = \widehat{\Pi}(\widehat{D}^* \cap R_{\eta_-})$ is the Jordan equator of A_o and

$$\mathrm{mod}(A_i) + \mathrm{mod}(A_o) = 2\mathrm{mod}(A_o) = \mathrm{mod}(\widehat{T}, \widehat{\gamma}).$$

If $c \in \mathbf{M}$ so that K_c is connected, the Böttcher coordinate at ∞ induces an isomorphism between A_o and $\Pi(S)$ where S is the connected component of $D^* \cap B^c(\infty)$ containing the end of $R^c_{\eta_-}$. Hence the Grötzsch inequality for (T, γ) implies that

$$\mathrm{mod}(A_o) \leq \mathrm{mod}(T, \gamma). \tag{3}$$

Writting out the values of these two numbers explicitly yields the Yoccoz dynamical inequality: The torus T is isomorphic to $\mathbb{C}^*/\lambda z$ via the linearizer for Q_c^k at α', or equivalently to $\mathbb{C}/(\mathbb{Z}\Lambda + \mathbb{Z}i2\pi)$ via the log-linearizer. Let $\Pi_u : \mathbb{C} \longrightarrow T$ denote the universal covering corresponding to the latter isomorphism. Then the Jordan curve $\gamma = \Pi(D^* \cap R^c_{\eta_-})$ lifts under Π_u to an arc Γ, which is invariant under the translation $z \mapsto z + L$, where $L = q\Lambda - pi2\pi$ for some appropriate logarithm Λ of λ. A simple computation shows that

$$\mathrm{mod}(T, \gamma) = \frac{2\pi \cos\theta}{q|L|}$$

where θ is the angle between the vector L and the positive real axis. A similar computation shows that

$$2\mathrm{mod}(A_o) = \mathrm{mod}(\widehat{T}, \widehat{\gamma}) = \frac{2\pi}{kq \log 2}.$$

Hence (3) is equivalent to

$$|\Lambda - \frac{p}{q} i2\pi| \leq \frac{2k \log 2 \cos\theta}{q}, \tag{4}$$

which is Yoccoz inequality.

If $c \notin \mathbf{M}$ let $0 \leq \theta < 1$ denote the argument of c, i.e. $c \in R^c_\theta$. Then the Böttcher coordinate ϕ_c at infinity does not extend to a biholomorphic map between $B^c(\infty)$ and $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$, but almost: It extends to a biholomorphic map of $\overline{\mathbb{C}}\setminus F^c(h)$ onto $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}(\mathrm{e}^h)$ where $h = g_c(c)/2$. Let ψ_c denote the inverse of this extension, then ψ_c extends continuously to $C(0, \mathrm{e}^h)$, but this extension is not injective because $0 = \psi_c(\exp(h + 2\pi i\theta/2)) = \psi_c(\exp(h + i2\pi(\theta+1)/2))$. Let $C = g_c(c) + i2\pi\theta$, $N_0 = [\mathrm{e}^{i2\pi\theta}, \phi_c(c)]$ and $N_n = Q_0^{-n}(N_0)$. Define $N^0_\theta = \cup_{n\geq 0} N_n$ and $N^1_\theta = \cup_{n\geq 0} N_1$. Then $Q_0(N^1_\theta) = N^0_\theta$ and ψ_c is easily seen to extend by iterated lifting to a univalent map from $\overline{\mathbb{C}}_\theta := \overline{\mathbb{C}}\setminus(\overline{\mathbb{D}} \cup N^1_\theta)$ into $B^c(\infty)$. The map Q_0 lifts under $\exp(z)$ to the map $z \mapsto 2z$ on $\mathbb{C}$. That is exp is a simultanuous linearizer for all the repelling periodic points of Q_0. The corresponding lifted sets $\widetilde{N}^j_\theta = \log(N^j_\theta)$, $j = 0, 1$ are invariant under translation by $i2\pi$ and $2\widetilde{N}^1_\theta = 2\widetilde{N}^0_\theta$. Thus if $w = \exp(i2\pi\tau) \in \mathbb{S}^1$ is periodic and if $0 \leq \tau < 1$ does not belong to the orbit of θ, then $\mathbb{C}_\theta$ contains a definite sector around the horizontal $\widetilde{R}_\tau = \{t + i2\pi\tau | t > 0\}$, which projects to R^0_τ under exp: Let $\tau_l < \tau < \tau_r$ be the arguments closest to θ of points in the orbit of w. Then the sectors $\widetilde{S}_l$ between $\widetilde{R}_{\tau_l} = \{t + i2\pi\tau_l | t > 0\}$ and the oblique line through $i2\pi\tau_l$ in the direction $v_l = C - i2\pi\tau_l$, and $\widetilde{S}_r$ between $\widetilde{R}_{\tau_r} = \{t + i2\pi\tau_r | t > 0\}$ and the oblique line through $i2\pi\tau_r$ in the direction $v_r = C - i2\pi\tau_r$ are contained in $\mathbb{C}_\theta$: If not some line segment L with $\exp(L) \in N_n$ for some $n \geq 1$ intersects say $\widetilde{S}_l$. But then $2^n L$ intersects the sector $2^n \widetilde{S}_l$ with top point $2^n \tau_l$, and is also congruent modulo $i2\pi$ to $L_0 = [i2\pi\theta, C]$ with $\exp(L_0) = N_0$. Since the $2^n \tau_l$ is an argument for a point in the orbit of w this contradicts that τ_l is the closest such argument for points in the orbit of w.

Consequently the sector $\widetilde{S}$ around $\widetilde{R}_\tau$ bounded by the two lines through $i2\pi\tau$ and of directions v_l and v_r is contained in $\mathbb{C}_\theta$.

In the case at hand $c \in W^H_{p/q}$ implies that $\eta_< \theta < \eta_+$ and for $\eta = \eta_-$ we have $\eta_- = \eta_l$, $\eta_+ = \eta_r$. Let ω_l and ω_r denote the angle of inclination of the vectors $C - i2\pi\tau_l$ and $C - i2\pi\tau_r$ respectively. Then the opening angle ω of $\widetilde{S}$ equals $\omega_r - \omega_l$ and the sector $\widetilde{S}$ projects to a straight subannulus A_o^θ of A_o with

$$\operatorname{mod}(A_o^\theta) = \frac{\omega}{\pi} \operatorname{mod}(A_o).$$

Arguing as for the proof of the Yoccoz inequality we obtain

$$\operatorname{mod}(A_o^\theta) = \frac{\omega}{\pi} \operatorname{mod}(A_o) \leq \operatorname{mod}(T, \gamma).$$

Properly rewritten as with (4) above, this is the Levin-Yoccoz inequality except for the interpretation of ω. This interpretation is however an elementary exercise in planar geometry and is left to the reader. By continuity the inequality even holds on $\partial W^H_{p/q}$, where either ω_l or ω_r but not both is zero. $\square$

Theorem 23. *Let $0 < \eta_- < \eta_+ < 1$ be rationals for which the parameter rays $R^{\mathbf{M}}_{\eta_\pm}$ coland at some point $c_0 \in \mathbf{M}$ and let $W^{\mathbf{M}}_{\eta_-,\eta_+}$ denote the parameter sector bounded by $R^{\mathbf{M}}_{\eta_-} \cup \{c_0\} \cup R^{\mathbf{M}}_{\eta_+}$ and not containing 0. Then the forward orbits of $\eta_\pm$ do not enter the interval $]\eta_-, \eta_+[$. And for any $c \in W^{\mathbf{M}}_{\eta_-,\eta_+}$ the pair of dynamical rays $R^c_{\eta_\pm}$ move homorphically with c, co-land at some repelling (pre)periodic point $z(c)$ with $Q_c^{k'+l}(z(c)) = Q_c^l(z(c))$, where $l \geq 0$ is the common preperiod of $\eta_\pm$ and $k' > 0$ divides the common period $k > 0$ of $\sigma^l(\eta_\pm)$ and the set $R^c_{\eta_-} \cup \{z(c)\} \cup R^c_{\eta_+}$ bounds a sector W^c containing c, but not 0.*

Proof. This theorem is at least folklore. We supply a proof here for completeness. We shall treat separately the strictly preperiodic case $l > 0$ and the periodic case $l = 0$. For the strictly preperiodic case we have $k = qk'$ with $q > 1$ and c_0 admits precisely q external arguments $0 < \theta_0 < \ldots < \theta_{q-1} < 1$ both in dynamical plane and in parameter plane by the Douady-Hubbard ray landing theorem. The arguments $\eta_- < \eta_+$ are amongst these. The set

$$R^c = \bigcup_{i=0}^{q-1} \overline{R^c_{\theta_i}}$$

moves holomorphically with c in $\mathbb{C}\smallsetminus\widehat{R}^{\mathbf{M}}$, where

$$\widehat{R}^{\mathbf{M}} = \bigcup_{i=0}^{q-1}\bigcup_{j=1}^{k+l}\overline{R^{\mathbf{M}}_{\sigma^j(\theta_i)}}.$$

Because the Böttcher coordinate ϕ_c depends holomorphically on c and thus R^c moves holomorphically with c as long as c does not belong to the strict forward orbit of R^c. Write $W^{\mathbf{M}}_{c_0}$ for the sector bounded by $\overline{R^{\mathbf{M}}_{\theta_0}\cup R^{\mathbf{M}}_{\theta_{q-1}}}$. Then $W^{\mathbf{M}}_{\eta_-,\eta_+}\subseteq W^{\mathbf{M}}_{c_0}$ and it suffices to prove that $W^{\mathbf{M}}_{c_0}\cap\widehat{R}^{\mathbf{M}}=\emptyset$. For the later it is enough to prove that

$$\left(\bigcup_{i=0}^{q-1}\bigcup_{j=1}^{k+l}\sigma^j(\theta_i)\right)\cap[\theta_0,\theta_{q-1}]=\emptyset. \tag{5}$$

To this end let us consider the Hubbard tree T^{c_0} for Q_{c_0}. In this strictly preperiodic case T^{c_0} is the minimal connected subset of $K_c=J_c$ containing the orbit

$$\mathcal{O}^{c_0}(0)=\bigcup_{j=0}^{k+l}Q^j_{c_0}(0).$$

As the orbit $\mathcal{O}^{c_0}(0)$ is forward invariant, so is T^{c_0}. Moreover any extremal point of T^{c_0} belongs to $\mathcal{O}^{c_0}(0)$ by minimality. As $Q^j_{c_0}$ is a local homeomorphism for all j the critical value $c_0=Q_{c_0}(0)$ is necessarily an extremal point. This implies (5). Notice that the conclusion of the theorem holds in this case even for c in a neighbourhood of $\overline{W^{\mathbf{M}}_{c_0}}$.

The periodic case is similar and yet slighly different. The common landing point c_0 of the two parameter rays $R^{\mathbf{M}}_{\eta_\pm}$ is the root of a hyperbolic component $H\neq H_0$. Let us rename c_0 to c_1 and use c_0 to denote the center of H. As above the dynamical rays $R^c_{\eta_\pm}$ moves holomorphically on $\mathbb{C}\smallsetminus\widehat{R}^{\mathbf{M}}$, where

$$\widehat{R}^{\mathbf{M}} = \bigcup_{j=0}^{k-1}\overline{R^{\mathbf{M}}_{\sigma^j(\eta_-)}\cup R^{\mathbf{M}}_{\sigma^j(\eta_+)}}$$

And to prove the theorem it suffices to prove that $R^{c_0}_{\eta_-}$ and $R^{c_0}_{\eta_+}$ coland at a repelling periodic point $z(c_0)$ in the dynamical plane of Q_{c_0} and that

$$\left(\bigcup_{j=0}^{k-1}\sigma^j(\eta_-)\cup\sigma^j(\eta_+)\right)\cap\,]\eta_-,\eta_+[\,=\emptyset.$$

Again the proof is that c_0 is extremal in the Hubbard tree T^{c_0} for Q_{c_0} and that $R^{c_0}_{\eta_\pm}$ coland at a k' periodic point $z(c_0)$, $k'|k$ on the boundary of the Fatou component F_0 of c_0. Notice that in this case the Hubbard tree is defined as the minimal D-H regulated set. Where D-H regulated means that for any Fatou component F the image $\phi(F\cap T^{c_0})$ under the extended Böttcher coordinate consists of radial lines. The proof of extremality of c_0 in T_{c_0} is the same as in the preperiodic case. Also by minimality $z(c_0) = \partial F_0 \cap T^{c_0}$ is the unique periodic point on the boundary of F_0 whose period divides k. By the Douady-Hubbard ray landing theorem $z(c_0)$ is the common landing point of $R^{c_0}_{\eta\pm}$. $\square$

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