"Proportions" in and around the Italian Abbacus Tradition

Høyrup, Jens

Published in:
Physis : Rivista Internazionale di Storia della Scienza

Publication date:
2010

Document Version
Early version, also known as pre-print

Citation for published version (APA):
Høyrup, J. (2010). "Proportions" in and around the Italian Abbacus Tradition. *Physis : Rivista Internazionale di Storia della Scienza*, *56*(1-2), 55-110.

“Proportions” in and around the Italian Abbacus Tradition*

Jens Høyrup
jensh@ruc.dk
http://akira.ruc.dk/~jensh/

Abstract

The language and notion of “proportions”, in the senses ascribed to the term during the epoch, are traced both in ordinary abbacus books and in those extensive works which were written in the vicinity of the abbacus culture by authors with erudite or Humanist ambitions, such as Fibonacci’s *Liber abaci*, Benedetto da Firenze’s *Trattato d’aritmetica* and Pacioli’s *Summa*. The very language turns out to have been initially absent from general abbacus culture as reflected in the ordinary books, but slowly and modestly crept in. The authors of the extensive works took up the topic, as indeed they had to if they wanted to connect to university and Humanist mathematics; but even in their case it generally remained isolated and did not penetrate their presentation of abbacus mathematics broadly to any noteworthy extent.

* A first version of this paper was presented as a contribution to the meeting “Proportions: Arts – Architecture – Musique – Mathématiques – Sciences”, at the Centre d’Études Supérieures de la Renaissance, Tours, 30 June to 4 July, 2008. Being much too long for the proceedings, these are to contain instead a revised version of the part on the *Liber abbaci*. That part is therefore reduced in the present paper.

1. Preliminaries

Before taking up the substance of my topic, I shall make three preliminary remarks: one on terminology, one on notation, and one on delimitation.

Terminology first. As other texts from the epoch, those I am going to consider speak of a ratio / λόγος (understood as a relation between two integers, not as a single number) as *proportio* / *proportione*. Some of them use *proportionalità* where we would speak of a proportion and the Greek mathematicians of ἀναλογία, that is, an affirmation that two ratios are "the same" or "similar"; others, however, use the term *proportio* / *proportione* even here, or speak of the numbers involved as *proportionales*. In the case of numbers being in continued proportion (ἑιῆς ἀνάλογον), moreover, our texts speak of *numeri continui proportionales*[1] / *numeros in continua proportione*, etc. An attempt to enforce a modern terminology would either divide the field in a way which does not correspond to the thought of the authors of the period, or it would force us to speak of "numbers in continued ratio" – which certainly makes sense, but is *not* modern terminology. It would also impose the modern conceptual confusion, more misleading than the medieval one, which uses "ratio" both in the historically proper sense, about the relation between *two* numbers, and about their quotient. I shall therefore translate *proportio* / *proportione* as "proportion", etc. – while still speaking in modern ways of ratio and proportion outside direct and indirect quotations when the relation between two numbers respectively the "similitude" between two such relations is meant; the single-number "ratio" I shall refer to as the "quotient".

Second, notation. When designating explicitly a proportion, our texts mostly say that "the first number is to the second, as the third to the fourth",[2] or use some equivalent expression. For typographical convenience, I shall use instead the notation $\frac{a}{b} : \frac{c}{d}$, which should be read as representing the frame

$$\begin{array}{|cc|}\hline a & c \\ b & d \\ \hline\end{array}$$

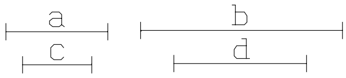
corresponding to what is found regularly in the margin in the *Liber abbaci*[3]

[1] Thus *Liber abbaci*, [ed. Boncompagni 1857: 171, 399].

[2] Thus *Liber abbaci*, [ed. Boncompagni 1857: 170]; as everywhere in the following, translations with no identified translator are mine.

[3] E.g., [ed. Boncompagni 1857: 170].

and (according to Rodet as cited in [Silberberg 1895: vi, 109]) consistently in a pre-1400 manuscript of Ibn Ezra's *Sefer ha-mispar* – whence probably more widespread.[4] The two notations – as well as the line diagram used both in the *Liber abbaci* [ed. Boncompagni 1857: 395 and *passim*] and by Campanus [ed. Busard 2005: 161 and *passim*]



are equally fit to serve the visualization and automation of the various operations that can be performed on the proportion:[5]

e contrario:	$\frac{b}{a} : \frac{d}{c}$	*conversa:*	$\frac{a}{a+b} : \frac{c}{c+d}$
permutata:	$\frac{a}{c} : \frac{b}{d}$	*eversa:*	$\frac{a}{a-b} : \frac{c}{c-d}$
conjuncta:	$\frac{a+b}{b} : \frac{c+d}{d}$	*aequa:*	$\frac{a}{b} : \frac{a+c}{b+d}$
disjuncta:	$\frac{a-b}{b} : \frac{c-d}{d}$		

and also of the equality of the products $a\cdot d = b\cdot c$ (to which I shall refer in the following as the "product rule"). The typographically convenient notation thus involves no serious anachronism – a:b::c:d, while fitting the phrase "the first to the second, as the third to the fourth", corresponds less well to the diagrams on which the medieval authors based their operational thinking. In order to distinguish, I shall write fractions (including "ratios" understood as quotients) as ${}^{a}/_{b}$. Ratios (not understood as quotients, and not constituents of a proportion) I shall denote a:b, and numbers in continued proportion will stand as a:b:c:....

Third, delimitation. Any applied arithmetic which goes beyond the simplest accounting runs into problems of proportionality – say, of the type "for *a* [coins], *b* [units], for *c* [coins], how much? In Near Eastern and Greek Antiquity, this would normally be solved in an intuitively transparent way: For *a* [coins], *b* [units], for 1 [coin] therefore ${}^{b}/_{a}$ [units], and for *c* therefore $c\cdot{}^{b}/_{a}$ [units]. Some Arabic reckoners[6] would prefer the argument "by *nisbah* ["ratio"]", for *a* [coins], *b* [units], for *c* therefore ${}^{c}/_{a}$ as much, that is, $({}^{c}/_{a})\cdot b$ [units]. From India, however, probably via the trade routes and possibly with ultimate roots in China, Arabic

---

[4] It is *not*, however, in the *Liber mahamaleth* (Paris, Bibliothèque Nationale, ms latin 7377A), even though this work makes use of the rectangular frame for other purposes.

[5] This way to present them is taken from the Campanus *Elements* [ed. Busard 2005: 171*f*].

[6] Thus Ibn Thabāt [ed., trans. Rebstock 1993: 43–45], and al-Karajī [ed., trans. Hochheim 1878: II, 17].

merchants and after them theoretically inclined Arabic mathematicians from al-Khwārizmī onward adopted the *rule of three*, stating that c must yield $(b{\cdot}c)/a$.[7] Indian practical reckoners appear to have used a formulation in the style "multiply the thing [whose counterpart] you want to know by that which is not similar [to it in kind] and divide by that which is similar". This is not the main formulation of the erudite Sanskrit writers (Āryabhaṭa, Brahmagupta, Mahāvīra, etc.), but the formulations of the latter two betray that they know it. Even in the Arabic world, it appears to have been the formulation of merchants. The theoretically trained Arabic mathematicians soon saw that the whole matter can be based on proportion theory as found in *Elements* VII – if only we forget about the numbers being concrete and indeed being of two different kinds (for instance, dinars and cloth), and not abstract. None the less, many of the Arabic mathematicians betray familiarity with the traditional formulation, in spite of its conflict with the Euclidean approach (which requires ratios to be between quantities of the same kind, e.g., abstract numbers[8]).

In the European (that is, Italian and Ibero-Provençal) abbacus environment, the rule also arrived in "non-Euclidean" interpretation (in Italy and perhaps in Provence in the traditional ("non-similar / similar") formulation, in Spain (as we shall see) apparently in a different shape); even in the Christian world, however, theoretically trained writers interacting with the abbacus environment, from Fibonacci to Chuquet, made use of the Euclidean formulation. This, however, I shall not discuss in any depth – not because it is not interesting but because it is a separate topic, and treated at best together with other aspects of the approach to the rule of three.

2. Fibonacci's *Liber abbaci*

I have argued on other occasions – for example in [Høyrup 2005] – that Fibonacci is not the founding father of abbacus culture but rather an early (towering) exponent of a culture which already flourished in his time, if not in Italy (which seems unlikely) then in Provence, Catalonia and the Maghreb and al-Andalus, perhaps even in Egypt, Syria and Byzantium, and which was connected to a culture of commercial arithmetic ranging at least as far as Iran and India; on the present occasion I shall refer to this as the "proto-abbacus

[7] This, and the remains of the paragraph, builds on [Høyrup 2007b: 1–8].

[8] Of course, the Euclidean approach is saved if only we use the equivalent proportion $\frac{a}{c} : \frac{b}{d}$. However, the sources never bother to perform this transformation.

culture".

However, the *Liber abbaci* is not just an early abbacus book. Fibonacci writes *in a mathematically educated perspective* about the kind of mathematics thriving in the environment in question; but his scope is much larger, encompassing not only what he encountered on business travels to Egypt, Syria, Constantinople, Sicily and Provence [ed. Boncompagni 1857: 1] but also topics which almost certainly fell outside the horizon of the proto-abbacus culture.[9] At least part of his treatment of proportions falls in that category (but see the beginning of Section 3 for a sharpening of this statement).

The first time numbers in proportion turn up in the *Liber abbaci* is in the explanation of the algorithm for multiplying multi-digit numbers [ed. Boncompagni 1857: 15]. Combining the product rule, for which he gives an unspecific reference to Euclid, with the observation that the "degrees" or decimal levels form an infinite continued proportion, Fibonacci concludes that multiplication of the first degree by the third gives as much as that of the second degree by itself, while the second by the third gives as much as the first by the fourth, etc.

The argument could be original; I do not remember having seen it in any earlier source, not even in hints.[10] Nice though it is, it also seems to have been a historical dead end, not to be repeated by any later writer.

A next passing reference [ed. Boncompagni 1857: 82] to (four) numbers in proportion and to the product rule turns up in the explanation of the decomposition of a fraction – once more with the unspecific reference to Euclid. This is followed closely by the presentation of the rule of three in simple and composite shape, which I shall not treat in depth (but see [Bartolozzi & Franci 1990: 5–7]). I shall merely mention

– that Fibonacci does *not* use what was to become the standard formulation of the abbacus school (the one which refers to the non-similar and the similar) – the formulations [ed. Boncompagni 1857: 83*f*] are likely to be his

9 Bartolozzi & Franci [1990: 5], though regarding the *Liber abbaci* as the archetype for abbacus books, align it more adequately with fifteenth-century encyclopediae like Benedetto da Firenze's *Praticha d'arismetricha* and the anonymous MS Florence, Palatino 573 – on both of which below.

10 It may have been inspired by analogous reasoning about the sequence of algebraic powers. The parallel between the powers of the algebraic *thing* and the powers of ten is pointed out by al-Karajī [Woepcke 1853: 48] and may have been common lore among Arabic writers 200 years later.

own;

– that Fibonacci makes use of the rectangular frame mentioned above, leaving the position for the unknown number empty and indicating the cross-multiplication by a diagonal;
– that the treatment of the non-composite rule is argued from the product rule "which has been proved in the arithmetical [books of the *Elements*] and in the geometry";
– that the composite rule (used in barter problems) is presented with a reference to *figura cata, scilicet sectoris* [Menelaos' theorem] "which Ptolemy teaches in the *Almagest*";

Whereas barter problems employ the rule of three "sequentially", partnership problems use it "in parallel"; in this case [ed. Boncompagni 1857: 114*f*, 135–143], however, Fibonacci speaks of neither "proportions" nor proportionality – nor indeed to the rule of three itself, but since in general he has no name for that rule this is not astonishing. However, in connection with a problem about alloying of three monies [ed. Boncompagni 1857: 149*f*], the first and the second in ratio 2:3, the second and the third in ratio 4:5, he speaks of "proportional alloying" and teaches how to harmonize these as easily composable ratios by means of multiplication. The idea of "proportional alloying" turns up repeatedly in the following pages. Proper interest in our topic only returns in Chapter 12, Part 2 [ed. Boncompagni 1857: 169–173].

This chapter starts by explaining equal, major and minor ratios, and gives the examples 3:3, 8:4, 9:3, 16:5, 4:8, 3:9 and 5:16 – providing them with names which are not in the Boethian tradition but come close to the "denomination" (though not using this word). For instance, 16:5 is a "triple proportion and a fifth". It goes on with the problem of finding the number to which 6 has the same "proportion" as 3 to 5, giving first the numerical solution $(5 \cdot 6)/3$ and saying then that this question is stated "in our vernacular" (*ex usu nostri vulgaris*[11]) in the phrase "if 3 were 5, what would then 6 be?". Similarly, it asks for the number to which 11 has the same ratio as 5 to 9, and gives it the vernacular formulation "if 5 were 9, what would 11 be?".

---

[11] A complete survey of the references to *modus vulgaris* and its cognates in the *Liber abbaci* shows that the genuine meaning is not the generic spoken vernacular but with one exception the simple ways of practical reckoners (the exception (p. 111) is the information that an alloy of silver and tin is called "false silver *vulgariter*"). Simple, stepwise calculation is meant in four places (pp. 115, 127, 204, 364). In the last place, the *modus vulgaris* is confronted explicitly with how one procedes *magistraliter*.

This formulation is remarkable (cf. full documentation in [Høyrup 2007a: 64–67]). Only one Italian abbacus treatise I know of identifies the rule of three by means of the same phrase, namely the *Columbia Algorism* [ed. Vogel 1977] – also untypical in other respects, almost certainly dated no later than 1290 [Høyrup 2007a: 31 n. 70] and thereby probably the earliest extant abbacus text (though known only from a fourteenth-century copy). Counterfactual questions – and even "counterfactual calculations" in the style "if 7 were the half of 12, what would the half of 10 be?" [ed. Boncompagni 1857: 10] – are certainly not absent from the Italian abbacus record, but they invariably turn up long after the rule of three is explained, or as secondary examples (the primary examples confronting either different currencies or goods and their monetary value). In all Ibero-Provençal treatises from before 1500 which I have inspected,[12] on the other hand, the rule of three is introduced first by counterfactual or abstract-number questions, "If 3 were 4, what would 5 be?" or "if $4\frac{1}{2}$ are worth $7\frac{2}{3}$, what are $13\frac{3}{4}$ worth?". All the Provençal specimens also know the formulation in terms of the non-similar and the similar, and so does Santcliment's Catalan *Summa* [ed. Malet 1998: 163]. Besides that, however, Santcliment informs us that this is spoken of "in our vernacular" by the phrase "if so much is worth so much, how much is so much worth" (*si tant val tant: que valra tant*). The same phrase (*sy tanto faze tanto, ¿qué sería tanto?*) is also used in the Castilian *Libro de arismética que es dicho alguarismo*.[13] Wherever Fibonacci encountered the vernacular tradition he refers to, it left no conspicuous traces in Italy, but many in the Ibero-Provençal orbit, most clearly in its Iberian section.

Next, Fibonacci presents the counterfactual calculation that was just quoted

---

[12] In chronological order

– the Castilian *Libro de arismética que es dicho alguarismo* [ed. Caunedo del Potro & Córdoba de la Llave 2000];
– the "Pamiers Algorism" [partial ed. Sesiano 1984];
– the mid–fifteenth-century Franco-Provençal *Traicté de la praticque d'algorisme* (I used the transcription in Stéphane Lamassé's unpublished dissertation, for access to which I am grateful).
– Barthélemy de Romans' Provençal *Compendy de la praticque des nombres* [ed. Spiesser 2003: 264];
– Francesc Santcliment's *Summa de l'art d'aritmètica* [ed. Malet 1998];
– Francés Pellos's *Compendion de l'abaco* [ed. Lafont & Tournerie 1967: 132–134].

I also looked at Chuquet's *Triparty en la science des nombres* [ed. Marre 1880], not strictly Provençal but in the Provencal tradition.

[13] Ed. Caunedo del Potro, in [Caunedo del Potro & Córdoba de la Llave 2000: 147].

("if 7 were the half of 12, what would the half of 10 be?"), and another counterfactual simple question. He goes on with procedures for finding four and six integers in proportion if the first two of them are given; shows how to divide 10 into four unequal parts in proportion – namely by scaling an arbitrary proportion $\frac{a}{b} : \frac{c}{d}$ by the factor $^{10}/_{(a+b+c+d)}$; explains how to construct a continued proportion with an arbitrary number of terms (explaining again the product rules); and finally demonstrates how to find two or three numbers so that $^1/_p n_1 = {}^1/_q n_2$ (and, in the case of three numbers, $^1/_r n_2 = {}^1/_s n_3$) – in a different formulation, not used by Fibonacci but common in later Italian abbacus algebra, $\frac{n_1}{n_2} : \frac{p}{q}$ (and $\frac{n_2}{n_3} : \frac{r}{s}$).

On the whole, what Fibonacci does in this chapter is thus to connect procedures and problem types belonging to the "vernacular" proto-abbacus tradition(s) he had encountered with the notion of "proportions". The theoretical field itself is not explored in any way.

Theoretical exploration of a kind comes in Chapter 15, Part 1 [ed. Boncompagni 1857: 387–397], which claims to treat of "the proportions of three and four quantities, to which the solution of many questions belonging to geometry are reduced" [ed. Boncompagni 1857: 387]. Actually it deals with problems about *numbers* in proportion. These numbers are spoken of as "the first / second / third / fourth number" (or, when the numbers are three numbers, "minor / middle / major"). In most cases, they are represented by letter-carrying line segments drawn in the margin – for brevity we may designate them *P*, *Q*, *R* and (when needed) *S*. At first proportions involving three numbers are presented, afterwards (much fewer) questions involving four numbers are dealt with. By means of conjunction, disjunction, permutation etc., the given proportion is transformed in such a way that the numbers can be found from the product rules by means of addition or subtraction or, more often, *Elements* II.5–6 (II.6 being sometimes preferred even in cases where II.5 would seem the obvious choice). Strikingly, Fibonacci never refers to Euclid here, which he is otherwise fond of doing [Folkerts 2006: IX].

We may divide into 50 sections, of which some 5 contain theorem-like observations (the delimitation is not quite sharp) and the remainder solve or show the solvability of problems.

At first ((1)–(3)) come questions about three numbers in continued proportion, *P:Q:R*. One of the numbers is given together with the sum of the other two. The naming of segments presupposes the alphabetic order *a*, *b*, *c*, ….

(4)–(38) still treat of three numbers, but now differences between the numbers are among the given magnitudes. The alphabetic order underlying naming changes to *a*, *b*, *g*, *d*, ….

In (4)–(5), the three numbers are still in continued proportion, but now one of the numbers and the difference between the two others are given. (7)–(38) are more astonishing. They fall in groups of three, divided by separate headings by Fibonacci. To each heading corresponds one of the non-arithmetical "means" of ancient Greek mathematics [Heath 1921: II, 85–88] – geometric, harmonic, their subcontraries, etc. – and it is shown how each mean can be found from the two extremes, or one of the extremes from the other extreme and the middle. Fibonacci deals with all the means defined by Nicomachos [ed. Hoche 1866: 124–144] (as followed by Boethius), but also with a mean defined by Pappos [Hultsch 1876: I, 70–73, 84–87] but left out by Nicomachos – see the scheme on the page 9. However, Fibonacci does not speak of means, even though he is likely to know about them from Boethius; his order is different from those of Nicomachos and Pappos; and he does not observe that his (27)–(29) represent the geometric mean, which he has already dealt with in (4)–(5). This, together with the change of underlying alphabetic order, suggests that he has not constructed this sequence on his own under inspiration from the ancients but borrowed for an Arabic or Greek treatise on the matter, which – in the interest of completeness – had also added the case (26) even though it defines no genuine mean (as pointed out by Fibonacci, the condition $\frac{R}{Q} : \frac{R-P}{Q-P}$ implies that $R \approx Q$).

(39)–(50) consider four numbers in proportion, $\frac{P}{Q} : \frac{R}{S}$. The underlying alphabetic order is still *a*, *b*, *g*, *d*, …. At first, the *e contrario* and *permutata* transformations are set out, and it is explained how any one of the numbers can be found from the three others via the product rule. Then follow problems where two of the numbers are given together with the sum of ((40)–(45)) respectively the difference between ((46)–(49)) the two others; finally, in (50), two numbers and the sum of the squares of the remaining two is given.

In (39)–(50) as in (4)–(5) but not in (7)–(38), some segments are labelled by a single letter, and the letter *c* is used during the manipulations. We may therefore assume that these sequences come from Fibonacci's own pen, or (less likely, I would say) from different source than the one for (7)–(38).

Chapter 15, Part 2 is claimed to deal with "questions concerning geometry". Actually, a number of its problems have nothing to do with geometry, apart from having solutions based on line diagrams; several of these – all dealing with composite gain – involve proportions.

The first of them [ed. Boncompagni 1857: 399] is very simple. Somebody goes

	Pappos	Nicomachos	*Liber abbaci*
$\frac{R-Q}{Q-P} : \frac{R}{R}$ (arithmet.)	P1	N1	
$\frac{R-Q}{Q-P} : \frac{R}{Q}$ or $\frac{R-Q}{Q-P} : \frac{Q}{P}$	P2	N2	27–29
$\frac{R-Q}{Q-P} : \frac{R}{P}$	P3	N3	7–9
$\frac{R-Q}{Q-P} : \frac{P}{R}$	P4	N4 (but inverted)	10–12 (inverted)
$\frac{R-Q}{Q-P} : \frac{P}{Q}$	P5	N5 (but inverted)	34–36 (inverted)
$\frac{R-Q}{Q-P} : \frac{Q}{R}$	P6	N6 (but inverted)	20–22 (inverted)
$\frac{R-P}{Q-P} : \frac{R}{P}$	absent	N7	16–18
$\frac{R-P}{R-Q} : \frac{R}{P}$	P9	N8	13–15
$\frac{R-P}{Q-P} : \frac{Q}{P}$	P10	N9	30–32
$\frac{R-P}{R-Q} : \frac{Q}{P}$	P7	N10	37–38
$\frac{R-P}{R-Q} : \frac{R}{Q}$	P8	absent	23–25
$\frac{R}{Q} : \frac{R-P}{Q-P}$	absent	absent	26

Means dealt with by Pappos and Nikomachos and in the *Liber abbaci* to one place of trade with 100 £ and earns, and afterwards earns proportionally in another place, and then has a total of 200 £. A continued proportion shows the possession after the first travel to be $\sqrt{(100\cdot200)} \approx$ £ 141, s. 8, d. $5\,{}^1\!/_8$.

The next case [ed. Boncompagni 1857: 399] is somewhat more tricky. The initial capital is still 100 £, but after the first travel a partner invests 100 £ in the enterprise, and after the second travel the total amounts to 299 £. This gives the proportion (represented by lines) $\frac{100}{Q} : \frac{Q+100}{299}$. The product rule and *Elements* II.6 (still unidentified) lead to the solution Q = 130 £. Interchange of left and right would reduce this to case (46) above, but Fibonacci does not establish the link.

Then follows [ed. Boncompagni 1857: 399*f*] an example with three travels (100 £ growing to 200 £) and no extra investments, which leads to a continued proportion with four terms and thus, with reference to Euclid (namely *Elements* VII.12), a solution expressible in cube roots. A digression follows discussing numbers allowing an exact solution (24 and 81) and the notions of duplicate and triplicate proportion. Fibonacci goes on to the case of four travels, involving five numbers in continued proportion and a quadruplicate proportion; and to the concepts of quintuple and sextuple proportion.

A final problem about composite gain [ed. Boncompagni 1857: 401] deals with two travels with initial capital P, final total R and intermediate possession Q = 80 £, with $\frac{P}{R} : \frac{5^2}{9^2}$. Fibonacci calculates $5\cdot9 = 45$ and claims without explanation that $\frac{45}{80} : \frac{25}{P}$, $\frac{45}{80} : \frac{81}{R}$. The trick is of course that $\frac{25}{45} : \frac{45}{81}$, while $\frac{P}{80} : \frac{80}{R}$; a scaling with the factor ${}^{45}\!/_{80}$ conserves the ratio between the extreme terms and adjusts the value of the middle term. Finally, Fibonacci explains it to be an equivalent problem to find two numbers p and q (namely, $p = \sqrt{P}$, $q = \sqrt{Q}$) so that ${}^1\!/_5 p = {}^1\!/_9 q$, $p\cdot q = 80$.[14] This is solved via a single false position, $p' = 5$, $q' = 9$, and subsequent scaling by the factor $\sqrt{{}^{80}\!/_{5\cdot9}}$.

The notion of "proportion" or proportionality turns up in two further places in this "geometric" section. In none of them, anything profound is meant. First, a rule is given [ed. Boncompagni 1857: 401] for producing "two integer roots whose squares together make the square of a number" – that is, for finding Pythagorean triples (triangles are *not* spoken of). Second, in the last problem of the section [ed. Boncompagni 1857: 405*f*], three numbers (say, a, b and c) are asked for, so that ${}^1\!/_2 a = {}^1\!/_3 b$, ${}^1\!/_4 b = {}^1\!/_5 c$, $abc = a+b+c$. This is solved by a single false position, $a' = 8$, $b' = 12$, $c' = 15$, with consecutive proportional scaling. Similarly to what he did in the last travel problem, Fibonacci goes on to discuss

[14] We recognize the structure ${}^1\!/_p n_1 = {}^1\!/_q n_2$, dealt with already in Chapter 12, Part 2 (see above, p. 7).

what to do when there are four, five and six numbers, using once again the notions of double, triple, quadruple and quintuple proportion.[15]

The third and final (and most famous) part of Chapter 15 [ed. Boncompagni 1857: 406–459] deals with "certain problems according to the method of algebra and almuchabala, that is, by proportion and restoration".[16] This identification of *algebra* with "proportion" and *almuchabala* with "restoration" is almost certainly Fibonacci's own invention.

Fibonacci knows the term "restoration" from Gherardo of Cremona's translation of al-Khwārizmī (with which he was familiar, see [Miura 1981: 60]) and also uses it himself quite often about the cancellation of a subtractive term by addition to both sides of an equation[17] (alternatively he employs a mere "add"); but Gherardo will not have helped him discover that it translates *al-jabr*.[18] On the other hand, the term used by Gherardo to translate *al-muqābalah* and the corresponding verb *qabila* – that is, *oppositio* / *opponere* – only occurs thrice in Fibonacci's algebra chapter [ed. Boncompagni 1857: 429, 436, 457], every time in the sense of confronting the two sides of an equation (in all probability the original function of the term, but not Gherardo's normal interpretation[19]).

This explains that there was space for Fibonacci's mistaken guess – he had two slots for only one technical operation. It does not explain why he used the other slot for "proportion", but at least this choice suggests him to have seen proportions as an important tool in the field. Why?

---

15 Most remarkable in this problem is presumably the use of *tetragonus* in the sense of a numerical square: everywhere else in the work this is spoken of as *quadratus*, while *tetragonus* invariably refers to a geometric square (often, [ed. Boncompagni 1857: 175*f*, 368, 408*f*, 421, 426*f*, 453]) or cube (once, [ed. Boncompagni 1857: 403]). We can presume that Fibonacci used a source written in Greek without bothering to adjust its style.

16 [...] *pars tertia de solutione quarumdam questionum secundum modum algebre et almuchabale, scilicet ad proportionem et restaurationem*.

17 The "equation" as a mathematical *object* is of course *our* concept and thus strictly speaking an anachronism. Fibonacci only has the action of equating – the isolated appearance of *equatio* [ed. Boncompagni 1857: 407] is to be understood as a corresponding verbal noun, *pace* Barnabas Hughes [2008: xxix, 361], who is seduced by Boncompagni's mistaken punctuation (*reddigi ad equationem. Vnius* (sic) *census per diuisionem* [...] should be simply *reddigi ad equationem unius census per diuisionem* [...]).

18 That Fibonacci does not discover on his own should downplay Fibonacci's Arabic skills, *pace* Barnabas Hughes [2008: xix].

19 There is one exception [ed. Hughes 1986: 255].

One hypothesis can be rejected straightaway. It has nothing to do with the proportional reduction of all coefficients when an equation is normalized. For this, Fibonacci uses *redigere*, as quoted in note 17, *reintegrare* [ed. Boncompagni 1857: 420], or performs the operation without naming it; neither "proportion" nor "proportional" ever occurs in this context.[20]

We may observe instead that Fibonacci inserts occasional pieces of reasoning based on proportion theory within algebraic or other calculations, and occasionally solves problems by means of proportion theory instead of algebra.

A simple example of the first type is found in the solution of the problem, to divide 60 *denarii* first among a number of men and then among 2 men more, by which the share of each man decreases by $2^1/_2$ *denarii*. Al-Khwārizmī [ed. Hughes 1986: 255; ed. Rashed 2007: 190–193] solves an analogous problem via (implicit) subtraction of fractions containing algebraic expressions in the denominator; Abū Kāmil [ed. Levey 1966: 106; ed. Chalhoub 2004: 76–78, 197; ed. Sesiano 1993: 370*f*] makes use of subtraction of areas within a geometric diagram; Fibonacci [ed. Boncompagni 1857: 413] replaces this "geometric arithmetic" by operations on a proportion.

A more advanced instance of the first type deals with the gains of a complex partnership: Somebody invests 12 £, and has a certain gain after 3 months. Then somebody else invests 11 £, and after another 12 months with gain at the same monthly rate, the total gain for the two is 9 £. This is expressed in line diagrams and treated *inter alia* by operations on proportions, which in the end allow the establishment of an algebraic equation.

A simple instance of the second type is an alternative solution to the problem to find two numbers with difference 6 and quotient $^1/_3$. The primary solution goes via algebra: the smaller number is posited as a *thing*, the larger is thus a *thing* plus 6, etc. Alternatively, the larger is a segment *ab*, the smaller the partial segment *ac*, whence $bc = 6$, $\frac{ab}{ac} : \frac{3}{1}$, and *disjunctim* $\frac{3}{ac} : \frac{2}{1}$, etc. For somebody as familiar with proportion techniques as Fibonacci, this may indeed have been

[20] Barnabas Hughes suggests [2004: 324 n. 43] that Fibonacci understood "*proportio* as a kind of operation" because "the two verbs *proportionari* and *equari* [...] are synonymous" in the Latin translation of Abū Kāmil's algebra. Hughes overlooks that the verb *equari* is used as an editorial explanation by Jacques Sesiano [1993: 325]. What is relevant is that the fourteenth-century translator uses the verb *proportionari* is the sense of "giving / having comparable size" a single time, in agreement with possible Italian usage (of *proporzionare*) of the late Middle Ages. It is not totally excluded – though quite improbable, given that there are no other traces of this meaning in Fibonacci's text – that Fibonacci did so too. Why should he coin a semantic neologism in the heading and never use it afterwards?

as easy as the primary solution, and for those not yet familiar with algebra it may have been easier.

Another alternative [ed. Boncompagni 1857: 423*f*], this time to an algebraic method which is mentioned but not presented, asks for a number which, when $^1/_3$ of it and 6 are removed and the remainder multiplied by itself, yields twice the original number – in symbols,

$$(x-{}^1/_3x-6)^2 = 2x \ .$$

In a line diagram, Fibonacci transforms this into a proportion which in symbols becomes

$$\frac{\frac{2}{3}x}{x-\frac{1}{3}x-6} = \frac{x-\frac{1}{3}x-6}{3} \ .$$

Disjunctim, this allows him to apply *Elements* II.6 (unidentified once again). This time, only a reader who had understood nothing of the algebra that precedes would be likely to prefer the alternative. If we observe that the underlying alphabetic order is *a*, *b*, *g*, *d* (which it rarely is in this section) and that the problem belongs to a family which was widespread in the "supra-utilitarian" stratum of proto-abbacus arithmetic inside as well as outside algebra – see [Høyrup 2007a: 131–133] – one may speculate whether Fibonacci found it in a source written in Greek and presented it for the sake of completeness (which would correspond to a general practice of his).

All in all, we may conclude that "proportions" had nothing to do with algebra as Fibonacci encountered it. He writes, however, as if he thought they *should* have. Nothing suggest him to have entertained the idea that existing algebra should be illegitimate because it was Arabic, nor that he had a consistent program to replace it with something more "magisterial", legitimately belonging within the realm of Greek [21] – but his global view of mathematics, coloured by his understanding of the *Elements*, and his possession of a level that enabled him to merge different approaches in a not fully eclectic manner, still made him go part of the way taken eventually with greater resolve by some Renaissance writers on algebra.

[21] That is, nothing like the ideal which shines through in Jordanus's *De numeris datis* and to which Regiomontanus, Viète and others paid lip service through references to Diophantos and *analysis* – see [Høyrup 1998].

3. Early abbacus books

Examination of early Italian abbacus books reveals that Fibonacci glued proportions not only onto algebra but also more generally to the proto-abbacus tradition, from which they were equally absent.

The *Columbia Algorism* – almost certainly the earliest extant abbacus text, cf. above, text before note 12 – does not speak of "proportions" a single time, not even in the sense of ratio; the rule of three, as explained, is referred to through the counterfactual "vernacular" structure – for instance [ed. Arrighi 1989: 32] "if 25 were 12, what would 12 be?".

Slightly but hardly much younger[22] is a *Livero de l'abbecho secondo la oppenione de maiestro Leonardo de la chasa degli figluogle Bonaçie da Pisa*, "Abbacus book according to the opinion of master Leonardo of Pisa from the house of the Fibonacci" [ed. Arrighi 1989]. This treatise is a mixed compilation – see [Høyrup 2005]. A little less than half (if we count lines, well over half if we count problems) has nothing at all to do with the *Liber abbaci*, the remainder is borrowed very closely but often demonstrably without understanding from that book. Apart from the contents of a final chapter containing mixed recreational problems, everything independent belongs on the basic level, the level corresponding to what should be taught in an abbacus school. What comes from Fibonacci is sophisticated, advanced – roughly speaking, adornment serving to show off (a purpose also ministered to by the reference to the famous predecessor in the title).

In the part of the text that is not borrowed from Fibonacci, the notion of "proportion" does not occur in any sense. The rule of three is presented in terms of the similar and the non-similar. The part copied from Fibonacci does borrow a number of references to the notion, translated either *propositione* (sometimes *prepositione*) or *proportione*. *Propositione* also occurs as translation of *petitio* or *propositio* (both referring to requests or propositions that somebody give part of his possessions to somebody else). The mix-up of *propositione* and *proportione* also turns up in other abbacus texts, facilitated probably by the possibility to abbreviate both in the same way, which might of course mislead a copyist who did not understand the text he copied. However, a global survey of the relevant passages suggests that the present compiler did not know he was sometimes speaking of proportions, glaringly misunderstood as they are occasionally (see the scheme on the next page). In any case, they were not part of his own

[22] For this revised dating, see [Høyrup 2007a: 31 n. 70].

Occurrences of *proportione/propositione* in the *Livero de l'abbecho* (right, with page numbers from [Arrighi 1989], and corresponding passages in the *Liber abbaci* (left, with page numbers from [Boncompagni 1857].

270	que multiplica per 6 de proportione superius inventa	49	el quale multiplica per 6 de la prepositione de sopre trovata
131	que proportio est composita ex duabus datis proportionibus. Et cum proportio aliqua est composita ex quotcumque proportionibus; tunc proportio proportionum ipsa appellatur: que compositio qualiter fiat, lucidius demonstrabo	30	la quale proportione ène conposta da doie prepositione e proportione dell propositione è chiamata perch'ella se mostra chiaramente
145	argenti uncias, que fuerint in omnibus prepositis monetis, addiscas	34	le onzie de l'argento che sonno en tutte le propositione e le monete en prende
229	positis petitionibus ipsorum [for the purchase of a horse]	69	noie devemo ponere le propositione
200	ex petitionibus ex proportionibus reliquorum hominum	78	de la petitone e de la proportone degl'altre huomene
201	ex petitionibus et ex propositionibus reliquorum	78	de la petitone e da propositione degl'altre
288	secunda aliquam datam proportionem [...] qui sunt in dicta proportione	81	secondo l'altra propositione [...] che sonno ella ditta propusitione
205	hec positio per priman regulam, hoc est per modum arborum, solui possit; tamen qualiter aliter soluatur demonstrare cupimus	87	quista propositione overo quistione se può fare per la regola del primo albero, el quale mostramo chusì
286	ut invenias proportionem, quam habent ad inuicem primum, et secundum uas	100	truova la propositione che àggiono emsieme el primo e'l sechondo vaso
133*f*	proportio uniuscuiusque numeri prime coniunctionis ad 6, qui est tertius ex numeris secunde, est composita ex duabus proportionibus quattuor reliquorum numerorum	137	la propositione de ciascuno numero de la prima congiontone a 6 el qual'è el terço numero

mathematical upbringing and culture as reflected in that part of the compilation which is not copied (or miscopied) from Fibonacci.

I shall leave aside for a moment Jacopo da Firenze's *Tractatus algorismi*, written in Montpellier in 1307, in which the notion of "proportion" does turn up a few times in particular contexts, and go on with other early abbacus treatises.

Two of these were also written in Provence: a *Liber habaci* from c. 1310, and Paolo Gherardi's *Libro di ragioni* from 1328.[23] None of them speaks of "proportions" at all, neither under this name nor as *propositioni* – with one specific kind of exception in Gherardi's book to which we shall return. The former gives the rule of three almost exactly as the *Livero de l'abbecho*, but differs from all other abbacus writings on one singular account: all its integer numbers are written with Roman numerals, and all its fractions are spelled out in full words. Even the brief explanation of the place value system [ed. Arrighi 1987b: 155] does not show a single Arabic numeral. This might (but need not) reflect a style preceding the *Columbia Algorism*.

A *Libro de molte ragioni d'abaco* written around 1330 in or around Lucca by three different hands [Van Egmond 1980: 163] (and thus, we must presume, fairly representative as a total of the linguistic habits of the local environment of the time) contains two passages of interest: for the digging of a well, the toil is said [ed. Arrighi 1973: 29] to be *aproportionata* to the depth; and it is said [ed. Arrighi 1973: 31] to be necessary for a certain problem solution to be valid that Florence and Lucca are either both *proportionata* as circles or both as squares. The latter request thus refers to geometric shape, considered generically as a "proportioning". In the former case it turns out in the following that the toil for each cubit is almost but not quite directly proportional to its depth, since the total work for depth n is as $1+2+...+n$.[24] However, this numerical specification comes afterwards, the word *aproportionata* seems to stand as an explanatory everyday term, meaning loosely "corresponding to". Apart from these two passages, due to the same hand, neither "proportion" not "proposition" (in any sense) can be found anywhere in the compilation. More or less specific notions of proportionality thus seem to have penetrated general language (perhaps coming

---

[23] Both are in [Arrighi 1987b]. Arrighi ascribes both to Gherardi, but gives no convincing reasons that the *Liber habaci* should come from his pen.

[24] The toil is thus supposed to be proportional to the depth of the bottom of the stratum, not to its average depth.

in particular from the visual arts[25]).

Only texts in modern print allow text recognition and thus (at least fairly reliable) complete search. Regarding the extensive *Trattato di tutta l'arte dell'abacho* (written in Avignon in c. 1334, as shown by Jean Cassinet [2001]), only existing in manuscript form, I am therefore not able to assert that "proportion", "proportionality" and "proposition" used for "proportion" are totally absent; however, I have consulted such passages in the earliest manuscript (Florence, Bibl. Naz. Centr., fond. prin. II,IX.57 – the author's draft autograph) where the concepts could be expected to turn up if they belonged to the standard vocabulary of the author, without finding any of them. All in all it seems a reasonable conclusion that the relation of early abbacus culture with proportions and proportionality was like that of Molière's Monsieur Jourdain with prose – he had spoken it for forty years without knowing anything about it. In other words: *We* may find reasoning based on proportions, but this observation of ours does not correspond to the conceptual world of the abbacus teachers.

It seems reasonable to infer that Fibonacci glued proportions not only onto algebra but onto the whole of proto-abbacus mathematics.

4. Jacopo da Firenze and early abbacus algebra

Three manuscripts exist which claim in identical colophons to contain Jacopo's *Tractatus algorismi*, written in Montpellier in September 1307: Milan, Trivulziana MS 90, dated by watermarks to c. 1410; Florence, Riccardiana MS 2236, undated; and Vatican, Vat. Lat. 4826, dated by watermarks to c. 1450.[26] Editions of all

---

[25] The word family derived from "proportion" has one representative in Dante's *Commedia divina*, namely a reference (XXX.56) to the *proporzione* of a giant, and one in Boccaccio's *Decameron* (sesta giornata, novella sesta), a reference to the duly *proporzionati* faces produced by a painter. Both have to do with geometric shape, the latter with being well-shaped.

Not clearly linked to shape and aesthetic proportionality, however, are the observation in Dante's *Convivio* IV, that the human intellect is *improporzionalmente* surpassed by the divine intellect, and the one in his *Vita nuova* XXV that "rhyme" in the vernacular is as much as "verse" in Latin, with the added proviso *secondo alcuna proporzione* – a *mutatis mutandis* with quantitative connotations. (I used the electronic versions of the texts on http://www.liberliber.it).

[26] For these datings, see [Van Egmond 1980: 225, 148, 166]. Van Egmond gives the date 1307 for the Florence manuscript, but this is merely the date stated in the colophon, common to all three manuscripts. Since this manuscript is close to the Trivulziana 90 (see imminently) but with more errors, a date later in the fifteenth century is plausible.

three are in [Høyrup 2007a][27]. The Vatican manuscript contains an algebra, a chapter with problems about wages in growing continued proportion and a final collection of mixed problems which is absent from the others. As I have argued in [Høyrup 2007a: 5–25], the Vatican manuscript is a faithful copy of a shared archetype at least antedating 1328 considerably (and thus likely to be Jacopo's original), whereas the other manuscripts, very close to each other, represent an abridgment adapted to the need of the abbacus school.[28]

Let us therefore look first at the Vatican manuscript. Its algebra contains rules for all "cases" (simplified equations) until the fourth degree which are either homogeneous or reducible to second-degree equations by division, and one of the three possible biquadratic equations. For the six equations of the first and the second degree, one or several examples are given – ten in total.

Theory of proportions is not used here, not even its simplest level.[29] However, the problem statements are of some interest. Five are sham commercial problems,[30] two number problems belong to classical types already found in al-Khwārizmī's *Algebra*. Two, however [ed. Høyrup 2007a: 307, 309], have a dress which appears not to be known from any earlier source:[31]

---

[27] The edition of the Riccardiana manuscript (as included in the critical edition with the Trivulziana manuscript) is a re-edition of Annalisa Simi's transcription [1995].

[28] In a brief inserted note in his [2008: 313], Van Egmond claims that the Vatican algebra belongs to a family descending from Benedetto da Firenze. He takes it for granted that the undated Florence manuscript is actually from 1307, refers to the Milan manuscript as "several later copies of it", and overlooks that verbatim repetition of the Vatican text in the *Trattato dell'alcibra amuchabile* from c. 1365 (with one improvement showing the model of the Vatical algebra to be earlier) excludes any date after 1365. His claim can be safely disregarded – as can much of his construction of "families", built exclusively on the appearance and order of equation *types*, with no regard for formulations, choice of examples, incipient symbolism, and almost none for terminology.

[29] Yet it *could* have served. In one problem, the partnership serves instead to establish the equation, in another one dealing with composite interest the rule of three is used.

[30] One of these [ed. Høyrup 2007a: 314*f*] deals with composite gain; it does not coincide with any of those found in the *Liber abbaci*, but it also leads to a problem of the second degree. In Jacopo's variant, the gain in the first travel (12 £) and the total possession after the second (54 £) are known. The problem of course involves a continued proportion, $\frac{C+12}{C} : \frac{54}{C+12}$ (C being the initial capital), but Jacopo deals with it by means of a rule of three – identified only as *la regola* – integrated in algebraic reasoning.

[31] It *may* be related, but then only distantly, to Fibonacci's repeated reference to numbers for which ${}^{1}/_{p}n_1 = {}^{1}/_{q}n_2$ (two examples above, paragraphs after note 13 and before note

> find me two numbers that are in proportion [*propositione*] as is 2 of 3 and when each (of them) is multiplied by itself, and one multiplication is detracted from the other, 20 remains

and

> Find me 2 numbers that are in proportion [*propositione*] as is 4 of 9. And when one is multiplied against the other, it makes as much as when they are joined together.

At first, these may look intricate, but at slightly closer inspection they are nothing but more complicated ways to ask for

> a number which, when multiplied by itself and by 5, gives 20

and

> a number which, when multiplied by itself and by 36, gives as much as when it is multiplied by 13.

As we shall see presently, later writers use the same principle to show off cheaply, but for long they mostly use the formulation "the first is such a part of the second as [say] 2 is of 3". Jacopo is thus not likely to have invented the mathematical principle, but the explicit use of the notion of proportion (expressed as *propositione*) *could* a priori have been his idea; see however note 34.

However that may be, the "proportion" concept turns up again slightly later, in a sequence of problems about the manager of a warehouse (a *fondaco*, written *fondicho* etc.) whose wages are supposed to increase in continued proportion. The statements run as follows:

> Somebody stays in a for warehouse 3 years, and in the first and third year together he gets in salary 20 *fiorini*. The second year he gets 8 *fiorini*. I want to know what he received precisely the first year and the third year, each one by itself. Do thus, and let this always be in your mind, that the second year multiplied by itself will make as much as the first in the third. [...].

> Somebody stays in a warehouse for 4 years, and in the first year he got 15 gold *fiorini*. The fourth he got 60 *fiorini*. I want to know how much he got the second year and the third at that same rate. Do thus, that you divide that which he got in the fourth year in that which he got in the first year. And you will say that what results from it is cube root. [...].

> Somebody stays in a warehouse for 4 years. And in the first year and the fourth together he got 90 gold *fiorini*. And in the second year and the third together he got 60 gold *fiorini*. I want to know what resulted for him, each one by itself. And let them

15).

be in proportion and let the first be such part of the second as the second of the third, and as the third of the fourth. And let it always stay in your mind this, that to multiply the first year in the fourth makes as much as the second year in the third. And it makes as much to divide the fourth year in the second as the third year in the first. [...]. And 40 *fiorini* he got the third year. And it is done, and you see well clearly that each of these numbers are in proportion. And such part is the first of the second as the second of the third, and as the third of the fourth: each is the half. [...].

Somebody stays in a warehouse for 4 years. And in the first year and the third together he got gold *fiorini* 20. And in the second and the fourth year he got gold *fiorini* 30. I want to know what was due to him the first year and the second and the third and the fourth. And that the first be such part of the second as the third is of the fourth. [...] .

As we see, the geometric proportion is taken for granted, as belonging tacitly to the dress. As soon as the procedure is explained, however, the necessary fundaments of proportion theory turn up, and in the third and fourth problem we even find the word (in the shape *propositione*[32]) together with the alternative formulation "to be such a part as".

The third problem goes beyond what can be found in the *Elements*, though it is based on knowledge which had been current in Arabic scientific mathematics since al-Karajī. If *a*, *b*, *c* and *d* designate the respective yearly wages, the first step of the solution is to state that

$$a \cdot d = b \cdot c = \frac{(b+c)^3}{3(b+c)+(a+d)}$$

This certainly goes beyond Jacopo's mathematical competence. *He* cannot have invented the problems. On his own he *may* have had the idea to introduce the term *propositione* – though no scholar he was not quite without scholarly pretensions, his five-line colophon is in Latin. On the whole, however, the appearances of the word in the algebra and in this quasi-algebraic chapter are more likely to have been borrowed together with the rest of that text: it turns up nowhere else in the treatise, and seems well integrated when appearing in the *fondaco*-problems. His pretensions may then have caused him not to eliminate it.[33]

Evidently, the references to proportions in the Vatican manuscript have no

[32] It should be noted that something which is proposed is spoken of as a *proponinento* by Jacopo [ed. Høyrup 2007a: 250, 425].

[33] Given the spelling *propositione* and the general conscientious precision of the manuscript we possess the term is not likely to have been inserted in the text during the transmission process.

counterparts in the Florence and Milan manuscripts, from which the very chapters where they should turn up are missing. None the less, the word appears a single time [ed. Høyrup 2007a: 420], namely in the counterfactual calculation "if 5 times 5 made 26, say me how much 7 times 7 would make in that same proportion [*in quela medesima proportione*]" – that is, in the sense of "rate"; the result is then stated with the words *diremo che 7 via 7 facia 50 et* $^{24}/_{25}$ *a quela medesima rasone*, "we shall say that 7 times 7 makes 50 and $^{24}/_{25}$ at this same rate". Further on, *rasone* is used in a counterfactual calculation which follows immediately, and in nine places where rates are spoken of. In the Vatican manuscript, the first counterfactual question [ed. Høyrup 2007a: 238] has *ragione* where the Milan and Florence manuscripts have *proportione*. This single appearance of the word is probably a secondary modification reflecting a general tendency in the later fourteenth century to absorb bits of the terminology of university mathematics.

A number of abbacus texts written between 1307 and 1345 (all mentioned above) contain a smaller or larger amount of algebra:

– Paolo Gherardi's *Libro di ragioni* from 1328;
– The *Libro de molte ragioni d'abaco* from c. 1330;
– The *Trattato di tutta l'arte dell'abacho* from c. 1334;

Gherardi has a systematic presentation of algebraic cases – all cases until the third degree treated by Jacopo, four more cases of the third degree which cannot be reduced to quadratic equations (the solutions for which are therefore false, produced by superficial imitation of second-degree solutions), and the case "cubes equal to square root of number". Ten of these are illustrated by problems of the type asking for numbers in given ratio (invariably, when more than two numbers are involved, as *n*:*m* and as *m*:*p*, etc., avoiding thus the need for composing ratios); but in seven of them the formulation of the matter is "such part ... as *m* is of *n*". Only three [ed. Arrighi 1987b: 102, 106, 107] ask for "3 numbers that are in position [*positione*] together, that is, the first of the second as 2 of 3, and the second of the third, as 3 of 4".[34]

[34] But still three, while one example which is shared with Jacopo has the alternative formulation. This decreases the likelihood that Jacopo should be the one who introduced the proportion formulation (and the respective *propositione*/*positione* suggests perhaps shared dependency on a source or environment where *proportione* was reinterpreted as one or the other, either because numbers may be *positioned* in ratio or *posited* (namely, as 2 *things* and 3 *things* if their ratio is 2:3), or because a specific ratio is *proposed* (but cf. note 32).

In the *Libro de molte ragioni d'abaco* and the *Trattato di tutta l'arte dell'abacho*, problems with the "such part" formulation are found, but never the "proportion" formulation.

In 1344, a certain Dardi of Pisa wrote the first treatise in the abbacus tradition dedicated exclusively to algebra.[35] This work contains several hundred problems, a large part of which deal with two or three numbers in given ratio. Mostly these use the formula "such part … as *m* is of *n*". In one case, however ([ed. Franci 2001: 89], similarly the manuscripts; counted as no. 10 by Dardi), a two-number problem asks for "two proportional numbers in continued proportion [*proportionali in continua proportione*] so that the first is such a part of the second as 4 is of 5". More meaningful is the question ([Franci 2001: 139], similarly the manuscripts; Dardi's no. 64) for "three numbers in continued proportion [*in continua proportione*], that is, that the first is of the second as the second of the third, and be such a proportion as 2 of 3" – but we notice that the final words of the question uses "proportion" as a synonym for "part". In three later problems of the same type (Dardi's nos. 67, 69 and 75), the Arizona manuscript replaces the information about the continued proportion by the phrase "that one is such a part of the other as ...", apparently meant as "each … of the following one". The Vatican and Siena manuscripts have the same construction in no. 75, but state "that the first is such a part of the second as ..." in no. 69, saying nothing about the ratio between the second and the third. So does the Vatican manuscript in no. 67, whereas the Siena manuscript adds "and that they are in continued proportion". It seems likely that the Arizona manuscript corresponds to the original on this point, and that Dardi has thus explained the notion of continued proportion in no. 64 (after having used it wrongly in no. 10), and afterwards just uses "one ...the other" as a way to indicate a repeated ratio; Siena and Vatican at first overlook this finesse, but in no. 69 Siena sees that information is then missing; in no. 75, both copyists have discovered.

Dardi, like Jacopo, has scholarly pretensions (and much higher mathematical competence and ambitions, but that is immaterial in this connection): his preface explains [ed. Franci 2001: 37] the four Aristotelian causes (*rispetti*) of his book, in the best scholastic manner. He *may* therefore have adopted a term from Latin university mathematics, without having much use for it (and, as we see in no. 10 and no. 64, without being quite sure of its use). His treatise is thus yet another

[35] I have consulted Vatican, Chigi M.VIII.170 from c. 1390; [Franci 2001], an edition of Siena, I.VII.17 (c. 1470); and Van Egmond's personal transcription of the Arizona manuscript written in 1429 (for access to which I am grateful).

piece of evidence that *proportione* and *proportionalità* did not yet belong to the standard terminology of the abbacus ambience.

Before c. 1340, a master Biagio known later as *il Vecchio*, "the Old", wrote an abbacus treatise which has been lost, but from which a collection of algebraic problems was copied by Benedetto da Firenze for his encyclopedia (see note 9). This collection confirms the picture, and adds some shades (with the proviso that we cannot be quite sure Benedetto did not change the precise wording of his model).

Firstly, we find again a large number of problems asking for numbers or quantities in given ratio – 19 in all.[36] Only the last of them [ed. Pieraccini 1983: 126] asks for "2 numbers, or 2 quantities, which are in proportion as 5 to 7, that is, that the first quantity is to the second as 5 to 7"; all the others ask either for quantities (10 of them) or numbers (8 of them), and all use the formula "such part ... as *m* is of *n*".[37] We may speculate that the first 18 occurrences are borrowed material, and the last one Biagio's own construction, in which he shows the applicability of the proportion concepts to this problem type (and points to the equivalence of number- and quantity-formulations).

This is not the first time Biagio refers to "proportions". In a problem about a loan with compound interest over three years [ed. Pieraccini 1983: 67–69] he introduces the notion of a continued proportion and uses the product rule to establish the equation. Later on, in Jacopo's fourth *fondaco problem* (still told about the manager of a *fondaco*, with the data 40 and 60) [ed. Pieraccini 1983: 89–91], he first shows that the product rule $ad = bc$ gives a tautology, and then that the rule $ac = b^2$ yields an equation.[38] In an indeterminate problem about three monies with unknown metal content [ed. Pieraccini 1983: 109*f*] he postulates that the quantities are in continued proportion with ratio 2:1, and thereby gets a single determinate equation. Finally, in a more intricate problem [ed. Pieraccini 1983: 119–121] about composite gain – given difference between the interest rates

[36] One of them [ed. Pieraccini 1983: 18*f*] asks for three number in ratios 2:3 and 2:5, but this does not lead to a presentation or investigation of the composition of ratios: Biagio simply posits the numbers to be 2 *things*, 3 *things*, and 7½ *things* without explanation. All others, as usually, have the ratios nicely fitting together.

[37] The homonymy should not mislead us into believing that the "quantities" are continuous magnitudes – lengths, areas, volumes, durations, weights – as they would have been in contemporary Aristotelian university discourse. None of the authors I treat before Pacioli uses the term in this way.

[38] Jacopo, in contrast, had only provided a rule for determining the solution.

and given ratio between the total interests of the first and the second year – this ratio is at first defined as being "as 2 to 3", but when it is used later we are told that "the proportion of the interest of the first year and that of the second is as 2 to 3". The last instance (perhaps also the second-last) sounds as if the idiom of proportions fell natural for Biagio; the formulations of the final problem about "2 numbers, or 2 quantities" may then indicate that he was aware that his public was less familiar with it. However that may be, 7 occurrences of the words *proportione* and *proportionalità* in a text of some 30000 words must be characterized as a modest intrusion in Biagio's language.

A final algebraic treatise, written after our limit 1345 but throwing light on the early epoch, is a *Trattato dell'alcibra amuchabile* from c. 1365 [ed. Simi 1994]. It consists of three parts

– rules for calculation with signs, square roots and binomials consisting of number and square root;
– a list of algebraic "cases", provided in part with examples;
– and a collection of problems.

Only the second of these concerns us at present [cf. Høyrup 2007a: 160, 163]. It contains all of Jacopo's cases including his examples almost verbatim – and where Jacopo has no example, the *Trattato* gives none. This segment of the second part almost certainly descends from Jacopo's text, and it is therefore no wonder that the examples with numbers in given ratio speak of "proportion", just like Jacopo. More interesting is that it also presents cases and examples which are in Gherardi's algebra but not in Jacopo's, moreover in a version which appears to antedate Gherardi's – seven examples in total, all constructed around numbers in given ratio(s). Four of these ask for numbers *in proporzione*, only three use the "such part" formulation. Of Gherardi's counterparts, 6 are of the latter type, only one asks for numbers *in positione*; this latter question corresponds to a "proportion"-formulation in the *Trattato*. It thus looks (but the statistics is not sufficient to allow any certainty) that Gherardi had a tendency to use "such part" even when his source had (we may presume) *propositione* or *positione*; this augments the likelihood that Jacopo did not introduce the "proportion" language on his own but took it over from his source.

5. Antonio de' Mazzinghi

Antonio de' Mazzinghi (probably c. 1355 to 1385 / 86, see [Ulivi 1996: 109–115]) is praised highly for his algebraic competence in three encyclopedias from the mid-fifteenth-century,[39] which are also our only sources for his mathematics. The largest extract is his *Fioretti* [ed. Arrighi 1967a].

This is an outstanding text, which fully confirms the praises heaped upon him. That is not what concerns us here, but it is good to know as a background to what follows.

The *Fioretti* do not contain a single problem of the kind asking for numbers or quantities in given ratio. We may guess that Antonio found it below his mathematical dignity to stoop to using such cheap tricks; alternatively, he may not have found them fitting for a collection of "blooms".

Two problems have a formal similarity with the cheap type [ed. Arrighi 1967a: 46–51], asking indeed for numbers in ratio – but this ratio is not given numerically but as that between two other numbers fulfilling algebraic conditions. Written in letter symbols, the respective structures are

$$ab = (a–b)^2\ ,\quad \tfrac{c}{d} : \tfrac{a}{b},\ 19 = c+d\ ,\quad c{\cdot}d = c^2+d^2$$

and

$$a^2+b^2 = 60\ ,\quad \tfrac{c}{d} : \tfrac{a}{b}\ ,\quad c{\cdot}d = 10\ ,\quad c^2+d^2 = ab$$

The first is the one where Antonio famously has to invent a trick that enables him to calculate with two unknowns.[40]

In problems about compound interest, Antonio points out with greater clarity than any predecessor [ed. Arrighi 1967a: 36] that interest *a chapo d'anno*, "[making

---

[39] Benedetto's *Praticha d'arismetricha*, see above, note 9; Vatican, Ottobon. lat. 3307; and Florence, Bibl. Naz., Palatino 573.

[40] If for example 10 has to be divided into two parts, it was often found convenient to take these as 5–*thing* and 5+*thing*. As we have seen, an unspecified number is regularly also spoken of as a "quantity". Antonio combines the two ideas, taking *a* to be "a *thing* minus a *quantity*", *b* to be "a *thing* plus a *quantity*". The intellectual jump involved in this seems to have gone almost unnoticed at the time and to have inspired little imitation, maybe because the use of habitual words made Antonio's readers (including Benedetto) overlook that something (potentially) important had occurred – exactly as had happened to Fibonacci's similar trick (using *res* and *causa* for the two unknowns) in the *Flos* [ed. Boncompagni 1862: 236].

Benedetto does use *quantity* as an algebraic unknown in his *Tractato d'abbaco* [ed. Arrighi 1974: 153, 168, 181] (Arrighi ascribes the treatise to Pier Maria Calandri), namely in solutions by means of *modo retto/repto/recto*, first-degree algebra designated *regula recta* by Fibonacci, who calls the unknown *res* [ed. Boncompagni 1857: 191 and *passim*].

up accounts] at the end of year" (that is, compound interest) "proceeds in continued proportionality", not only with correct calculations up to five years grounded explicitly in the product rules (thus no longer the rule of three) but also with the finding of equivalent rates of interest if accounts are made up every 8 or every 9 months. Belonging to the same field of theoretical interest is the problem [ed. Arrighi 1967a: 69] of finding a five-term continued proportion beginning with 16 and ending with 81.

A large number of problems ask for three or four numbers in continued proportion which fulfil other algebraic conditions of the first or the second degree. In symbolic abbreviation and with the numeration of Arrighi's edition (which is probably taken from the manuscript) they are:

#1 $19 = a+b+c\ ,\ \ a\cdot(b+c)+b\cdot(c+a)+c\cdot(a+b) = 228$

#2 $a\cdot(b+c)+b\cdot(c+a)+c\cdot(a+b) = 888\ ,\ \ a^2+b^2+c^2 = 481$

#3 $9\,{}^1\!/_2 = a+b+c\ ,\ \ a^2+b^2+c^2 = 33\,{}^1\!/_4$

#4 $19 = a+b+c\ ,\ \ 3a+4b+5c = 81$

#5 $a+c = 21\ ,\ \ b+d = 39$

#8 $a\cdot b\cdot c\cdot d = 2916\ ,\ \ a+b = 17\,{}^1\!/_2$

#23 $a+b+c = 14\ ,\ \ a\cdot b\cdot c = 64$

#25 $a^2+b^2+c^2 = 84\ ,\ \ {}^{20}\!/_a+{}^{20}\!/_b+{}^{20}\!/_c = 125$

#26 $10 = a+b+c\ ,\ \ 3a+4b = 5c$

#29[41] $c–a = 50\ ,\ \ d–b = 80$

Repeatedly, as can be expected, the solutions make use of the product rules. A couple of times, however, Antonio appeals to more advanced properties of proportions or continued proportions. In #25, he finds it "rather clear and obvious" (*è cosa assai chiara e manifesta*) that if a, b, and c are in continued proportion, then the same can be said about ${}^{20}\!/_a$, ${}^{20}\!/_b$ and ${}^{20}\!/_c$ [ed. Arrighi 1967a: 54]. In #29, the *disjuncta* mode is described and used [ed. Arrighi 1967a: 63].

The genre as such was not new, neither in general nor to abbacus mathematics. Abū Kāmil [ed., trans. Levey 1966: 186; Sesiano 1993: 405; Chalhoub 2004: 148] has a problem with the structure

$$10 = a+b+c\ ,\ \ a^2+b^2 = c^2\ ,$$

and there are two examples in the third part of the above-mentioned *Trattato*

[41] Actually, #29 starts with a problem $10 = a+b$, $a^2+b^2+\sqrt{a}+\sqrt{b} = 86$, and a position $a = 5–t$, $b = 5+t$. When this has been reduced to

$$\sqrt{5–t}+\sqrt{5+t} = 36–2t^2$$

Antonio comments "I do not like it, and therefore I do not complete it" – and goes on with the problem about three numbers in continued proportion.

dell'alcibra amuchabile [ed. Simi 1994: 39] – in symbolic abbreviation

$$10 = a+b+c\ ,\ \ a{\cdot}b = 4\ ,\ \ b{\cdot}c = 8$$

and

$$10 = a+b+c\ ,\ \ a^2+b^2+c^2 = 40$$

The former is overdetermined and impossible, and the solution which is proposed is wrong. The latter, we observe, has the structure of Antonio's #3. Antonio's #5, on its part, has the same structure as Jacopo's fourth *fondaco* problem, the one which was also solved by Biagio. Antonio solves it in the same algebraic way as Biagio, omitting however Biagio's pedagogical blind alley. I know of no evidence allowing to decide whether the number genre as found in the *Trattato dell'alcibra amuchabile* has the same ultimate origin as the *fondaco* genre, or Antonio fused the two.

What seems fairly certain is that the present type of number problems has no strong links to Fibonacci's division of 10 into four unequal parts in proportion [ed. Boncompagni 1857: 170]; Antonio certainly knew and appreciated Fibonacci,[42] but nothing suggests the same for the compiler of the *Trattato dell'alcibra amuchabile* or his source. Moreover, Fibonacci speaks of *any*, not a continued proportion (and uses the example $\frac{3}{7} : \frac{6}{14}$); afterwards he shows how to construct a sequence of any length of numbers in continued proportion, but now without constraint on their sum.

Towards the end of the *Fioretti* comes a section "Mirabile dictum" [ed. Arrighi 1967a: 81–87], showing how to divide a number (say, N[43]) into parts (say, a, b, c, d and e) in such a way that

$$^N/_a + {}^N/_b + {}^N/_c + {}^N/_d + {}^N/_e = N.$$

This section is analyzed in [Bartolozzi & Franci 1990: 10*f*]. Since it was certainly due to Antonio himself and had little further impact in the abbacus tradition which I know of (apart from being copied by an obviously impressed Benedetto and being used by Pacioli, see below), I shall not discuss it any further.

Antonio was familiar with Book 15, Part 1 of the *Liber abbaci* – Palatino 573 from the late 1450s quotes his *gran trattato* for presupposing "that the proportions from the first part of the 15th chapter [of the *Liber abbaci*] be clear to you" [Arrighi 2004 / 1967: 190]. But this familiarity left no trace in the *Fioretti*. Whether Antonio thought of the connection to the Boethian means cannot be decided with

---

42 Quotation in Ottobon. lat. 3307, ed. [Arrighi 2004 / 1968: 221].

43 Antonio builds his solution up around the core ($\sqrt{6}-\sqrt{2}$,$\sqrt{6}+\sqrt{2}$), but then states that any couple of binomials $\sqrt{b}\pm\sqrt{a}$ serves if $b{:}a$ is a multiple. Unfortunately, as pointed out by Bartolozzi & Franci [1990: 10 n. 16], Antonio's condition is insufficient.

certainty, but since those who read him have not notice it, it is unlikely.

6. Late fourteenth century otherwise

Like Fibonacci, Antonio knew theoretical mathematics well enough to adopt it creatively into his abbacus heritage. He was exceptional, and in consequence an exception; to what extent would his near-contemporaries make use of the notion of "proportion", in any of its possible senses? Two examples will have to suffice.

The first is Giovanni de' Danti d'Arezzo's *Tractato de algorisimo* [ed. Arrighi 1987a] from 1370. This is a decent but not sophisticated abbacus book, containing no systematic presentation of algebra but a short passage about the arithmetic of square roots and a few algebraic problems [ed. Arrighi 1987a: 52–57, 65–69]. Giovanni's remoteness from any scholarly mathematical environment is illustrated by his explanation [ed. Arrighi 1987a: 53] of the existence of surds: God does not want that anything but himself be perfect.

The word "proportion" (in any of the spellings we have encountered) is as absent from this treatise as from the non-Fibonacci parts of the *Livero*, from the *Liber habaci*, and from the non-algebraic parts of Jacopo's *Tractatus* and Gherardi's *Libro di ragioni*. *Propositione* is found often, but it means "a question which is proposed".

There are seven problems asking for numbers in given ratio; all three use the "such part" formulation. For once, the same formulation is also found in a non-algebraic business problem [ed. Arrighi 1987a: 34]: A loan, on which the interest in the first year is such a part of that in the second year as 3 is of 4. Since nothing similar is found in treatises from the first half of the century, this type is likely to be an offset from the analogous pure-number problems.

The second is a *Trattato d'algibra*, constituting the last fifth of a larger abbacus treatise (Florence, Bibl. Naz. Centr., fond. prin. II,V.152), according to internal evidence written in the 1390s – thus after Antonio's death, but apparently in the tradition after Biagio (or his source). Only the algebra has been published, namely in [Franci & Pancanti 1988]. My discussion is restricted to this published part, I have not seen the manuscript.

In this algebra, the word *proporzione* turns up in two different contexts – in the theoretical introduction, and in the problems.

The theoretical introduction [ed. Franci & Pancanti 1988: 3–6] is an investigation of the sequence of algebraic powers. This introduction is both interesting and puzzling – see [Høyrup 2008: 30–32]. What concerns us here, however, is merely that the sequence of powers is seen to be in continued proportion, which is used to show that *censo* times *censo* is the same as *thing* times *cube*, and *censo* times *cube* as much as *thing* times *censo di censo*.[44]

Proportions also turn up in four problem types: In problems about compound interest over three or four years (very similar to what Biagio does); in the *fondaco* problem also solved by Biagio;[45] in two problems about the division of ten in three parts in continued proportion – one analogous to Antonio's #3 and the second problem cited from the *Trattato dell'alcibra amuchabile*,

$$10 = a+b+c\ ,\ \ a^2+b^2+c^2 = 70\ ,$$

and one similar to Antonio's #4,

$$10 = a+b+c\ ,\ \ 3a+4b+5c = 35\ ;$$

and finally in a problem about three quantities of money (in the sense of coinable metal) in continued proportion, structurally identical with the problem shared with Antonio and the *Trattato dell'alcibra amuchabile*. There are also numerous problems about two, three or four numbers in given ratio, all in "such part" formulation. In an inverse variant of the well-digging problem from *Libro de molte ragioni d'abaco* nothing is said about the toil being in correspondence (*aproportio-nata* or otherwise) with the depth, we only get the solution by means of triangular numbers.

This treatise is mathematically very sophisticated. None the less, as we see, the use of proportion theory (or the very recourse to the terminology) expands only slightly beyond what was known at least since Biagio: proportions fully displace the rule of three in problems about compound interest, and they enter the explanation of the sequence of algebraic powers.

---

[44] According to Palatino 573 [Arrighi 2004 / 1967: 191], Antonio appears to have made the same observation in his *gran trattato*.

[45] The solution of this *fondaco* problem [Franci & Pancanti 1988: 80–82] runs almost exactly as Biagio's, but there is one telling difference. Biagio takes the wage of the first year to be two *things*, whereas the present author chooses 2 *censi*. He does not know, however, that this word translates Arabic *māl*, not only (namely in algebra) the square of the *thing* but also an amount of money (a capital, a dowry, etc.). In the end he therefore feels obliged to find the *thing* from the *censo* – only to square it again. This implies that the (direct or indirect) source cannot be Biagio's text; it must be traced to an ambience where the original meaning of the *censo* / *māl* was still alive.

7. The mid-fifteenth-century "abbacus encyclopediae"

Around 1460, three extensive works of encyclopedic character were produced in the Florentine abbacus environment, already listed together in note 39:

– Benedetto da Firenze's *Praticha d'arismetricha* (existing in many copies, see [Van Egmond 1980: 356]), described in [Arrighi 2004/1965];
– Palatino 573, described in [Arrighi 2004/1967];
– Ottobon. lat. 3307, described in [Arrighi 2004/1968].

All of them contain material which was foreign to the abbacus tradition, and extensive extracts from the writings of (mostly) *named* abbacus predecessors (quite unusual in the abbacus environment). They are indeed our only source for Biagio's and Antonio's mathematics, but also contain long translated extracts from the *Liber abbaci*. They *may* possibly have had a model in Antonio's lost *gran trattato* (see above).

Benedetto's work is divided into sixteen books. Three of these have to do specifically with proportions, and build for this on "foreign" material. Book II about "the nature and properties of numbers" (*la natura e propietà de' numeri*) is a presentation of speculative arithmetic in the Boethian tradition. It also offers an exposition and explanation of the way ratios were labelled in this tradition [ed. Arrighi 1967b: 324*f*]: multiple, submultiple, superparticular, superpartiens, sesquialtera, sesquitertia, etc. But it does not use the word *proportione* (nor *ragione*) but speaks of "a number which is referred to another number" (*numero che è riferito ad altro numero*). Book V is stated to deal with "the nature of numbers and proportional quantities" (*la nature de' numeri e quantità proportionali*) – see [Bartolozzi & Franci 1990: 12–14]. Its first part builds on the Campanus version of *Elements* V–IX and on Campanus's *De proportione et proportionalitate* on the composition of ratios; the second part concerns metrological conversions. The first part of Book XI presents *Elements* II (including the division in extreme and mean ratio); the second part is a translation of Book XV, Part I of the *Liber abbaci*.

How much does this general mathematical erudition influence what Benedetto did *within* abbacus mathematics? We may look at his own algebra, contained in Book XIII.

Firstly, it is said more clearly [ed. Salomone 1982: 20] than in the above-mentioned Florence manuscript, Bibl. Naz. Centr., fond. prin. II,V.152, that the algebraic powers (*termini dell'algebra*) are in geometric proportion; on the other hand, Benedetto does not *use* this observation for anything. The associative law for multiplication may have been too obvious for him, he sees no need to do so; alternatively, this should be said about Antonio, whom Benedetto seems to follow, cf. note 44.

Secondly, the word *proportione* turns up once [ed. Salomone 1982: 40] inside one of the many problems about numbers given in ratio, which however are all defined in terms of "such part".

Palatino 573 was written in the years preceding 1460 by a former student of one Domenico d'Agostino *vaiaio* ("the tanner" or "the furrier" – whether this profession was indeed his or that of an ancestor is not to be decided). Its author says that he uses Benedetto's homonymous treatise (which must hence be earlier) as a model, adding and removing matters as needed [ed. Arrighi 2004 / 1967: 168]. It falls in eleven parts, subdivided in chapters. I shall discuss the relevant aspects on the basis of the extracts in [Arrighi 2004 / 1967: 168–194] (introductions to parts and chapters) and the description and quotations in [Bartolozzi & Franci 1988: 14–16].

Chapter II.8 [Arrighi 2004 / 1967: 176; Bartolozzi & Franci 1990: 15], about "the way to express as part, and, first, the definition", starts by quoting Boethius's, Euclid's and Jordanus's definitions of a ratio (*proportione*) as a relation between two numbers or quantities, and goes on with the traditional names (*doppia*, *sexquialtera*, etc.). This is not unproblematic, according to the definition a ratio is *not* a (possibly broken) number, as is the "part" he wishes to express. The author glosses over the difficulty by regarding it merely as a questions of language (thus reminiscent of certain discussions within contemporary historiography of mathematics): "we in the schools do not use such terms [*vocaboli*] but say instead [...] that 8 is $^2/_3$ of 12 and 12 is $^3/_2$ of 8". The author also points to the necessity that the two magnitudes in a ratio be of the same kind, without noticing that this should create difficulties when, later, the concept is used to explain the rule of three. This is symptomatic of the whole project (as shared with Benedetto): abbacus mathematics is put into the framework of scholarly (in Fibonacci's word, "magisterial") mathematics, but the author reinterprets concepts as needed, and does not care much about the contradictions that may arise.

Part III [Arrighi 2004 / 1967: 176–178; Bartolozzi & Franci 1990: 15*f*] is similar in its aim. The introduction announces that its first chapter shall deal with "the 4 proportional quantities or numbers, in the vernacular called rule of three things". The chapter itself starts by defining the *proportionalità* as the equality of two ratios and explaining that such *proportionalità* may be continued or not continued, going on to present the product rule and using it to determine one term in a continued proportion from the two others. Chapter III.2 applies the rule to commercial examples, and ends by saying that

> it is true that many who want to shows this rule have said that one multiplies the quantity that one wants to know by the one which is not similar, and divides in the other quantity. And they actually say the truth. Because when you multiply a quantity by another one which is not similar, it is as multiplying the first by the fourth or the second by the third.

Nothing, as we see, is said about the impossibility to speak properly of ratios between dissimilar quantities; the author obviously thinks of nothing but the measuring numbers in the already established units of the problem formulation.

Chapter III.3 translates the second part of Chapter XII of the *Liber abbaci* (discussed above).

Chapter V.3 [Arrighi 2004 / 1967: 181] introduces problems about numbers in given ratio by giving once more the names of ratios. This time, however, it identifies ratios with numbers in abbacus manner ("5 to 16 are $^5/_{16}$ because from 5 divided by 16 comes $^5/_{16}$").

Part IX [Arrighi 2004 / 1967: 190*f*; Bartolozzi & Franci 1990: 16] translates Chapter XV, Part 1 of the *Liber abbaci*. The introduction refers to objections against the relevance of this topic for algebra ("many strain themselves to prove ..."). In defense of this relevance it cites Paolo dell'Abbaco's (otherwise unknown) *trattato delle quantità chontinue*, Antonio's *gran trattato*, and the oral injunctions of his own teacher, the *vaiaio*. No argument beyond those depending on authority is offered (nor could it probably be). The link to Boethius's presentation of the ten means once more goes unnoticed.

Part X [Arrighi 2004 / 1967: 191–194] is dedicated to algebra. The introduction states that in order to restore the almost lost *Maumetto arabicho* (that is, al-Khwārizmī), the presentation will be based on him.[46] Chapter 1 starts by quoting Fibonacci for the explanation that *algebra almuchabale* means "restoration and opposition, because the parts are opposed to each other, as you will see in the examples" – a misquotation, as we know, caused perhaps by al-Khwārizmī's text, perhaps (and rather) by earlier abbacus writers.[47] It then goes on with al-Khwārizmi's text in Gherardo's translation.

After this pious homage to tradition in Humanist style the author feels the need to be modern, and starts by explaining the algebraic powers as a continued

---

[46] Similarly, Benedetto [ed. Salomone 1982: 20] presents not Fibonacci's but al-Khwārizmī's geometric proofs "because more ancient".

[47] A third possibility is Guglielmo de Lunis's lost translation of (probably a revised version of) al-Khwārizmī's algebra, whose introduction (quoted for instance by Benedetto [ed. Salomone 1982: 1*f*]) has this interpretation of opposition; cf. also above, text before note 19.

proportion, following (as he says) Antonio's *trattato* (probably the *gran trattato* of which he has already spoken).

A large number of problems follow, in part taken from named predecessors.

Ottobon. lat. 3307 was written slightly later by another former student of the *vaiaio*. It is divided similarly to Palatino 573. Beyond the chapter headings as quoted in [Arrighi 2004/1968] I have a photocopy from a microfilm of the final algebraic part, which already allows me to say (I shall omit the documentation) that this third encyclopedia is of a markedly lower quality than the other two, but not to derive much of relevance for the topic of proportions.

Maybe, however, there is not much of relevance at all: the algebraic powers (for which this author has no general term like Benedetto's *termini dell'algebra*) are *not* explained to be in continued proportion; further, the 33 folios of problems contain 2 questions (fols 336^{r}, 342^{v}) about 4 respectively 3 numbers in continued proportion, and none about numbers in given ratio.

We should be aware that these three encyclopediae all come from a specific strand in the abbacus tradition, centred upon a specific group of Florentine abbacus schools and tracing its origins to Antonio, Paolo dell'Abbaco and Biagio. Its interest in precursors – outside those belonging to the strand itself al-Khwārizmī and Fibonacci – was not shared generally. All the more telling is it that their copying of Fibonacci's sections on proportions did not really influence what they did themselves in abbacus style, apart from the formulation of the rule of three as a rule of four quantities in proportion. The interests and orientation of abbacus mathematics proper, we may perhaps conclude, left no space for that.

8. Luca Pacioli

Even Luca Pacioli's *Summa de Arithmetica Geometria Proportioni et Proportionalita* [1494; 1523[48]], divided into nine *distinctiones*,[49] might be characterized as a kind of encyclopedia, *not* belonging to the strand discussed in the previous section. A symptom of this different affiliation is that Pacioli feels no need to give specific references for his borrowings – Pacioli instead has a general

[48] In general, the second edition is faithful to the first word for word and line for line. There are a few corrections (and different misprints), but the main difference is that a number of rare abbreviations are expanded in the second edition.

[49] I consider only the first, arithmetical part (1–224) and disregard the geometry (76 fols, 8 *distinctiones*).

acknowledgment in the initial unfoliated *Sommario* that most of his volume has been taken from Euclid, Boethius, Fibonacci, Jordanus, Blasius of Parma, Sacrobosco and Prosdocimo de' Beldomandi. Actually, most of it probably comes from less prestigious, anonymous abbacus sources – whatever the citation strategy in the encyclopedias, it was always a strategy.

Pacioli's title might make us predict that ratios and proportions play a greater role and are more integrated than in the preceding manuscript sources. The list of his confessed sources could make us expect the same. Actually, "proportions and proportionalities" are mainly dealt with in the Sixth Distinction (at great length, fols 67^{v}–98^{v}), but they do serve elsewhere.

The topic is taken up for the first time in connection with the rule of three.[50] Initially, this rule is presented in terms of the similar and dissimilar, and in a different but equally "a-theoretical" way (fol. 57^{r}).[51] After some examples, however, comes an explanation (fol. 57^{v}) *unde regula predicta procedat*, "where the said rule comes from", referring to *Elements* V. Actually, this explanation goes beyond what is needed for the purpose, as Pacioli points out, namely because he wants the reader to "better understand the fundaments" of the rule given. He starts by stating that the force of the rule of three "proceeds from the mutual proportionality of quantities, be they continuous or be they discrete, that is, be they numbers or be they measures, and be the proportionalities continued or not continued [*incontinua*]", with three respectively four terms. After pointing out that "in all the calculations of the trading and business world, 4 numbers always occur, of which 3 are always known and the fourth unknown" and repeating the need for three respectively four terms Pacioli makes a more interesting point: that in continued proportions all terms must be of the same nature because (except the extremes) all stand both as antecedent and as consequent, while the non-continued proportions require only pairwise identical nature. His explanation of these natures comes from the Aristotelian tradition – they are numbers, lines, surfaces and bodies. He then goes on with the product rule, with explanation of a schematic representation, and with examples of what to do if either of the four terms is unknown.

---

50 Continued proportions are mentioned but not really treated on fol. 37^{v}, during the treatment of progressions.

51 The second edition [1523] has this folio number misprinted as 64.

9. The Sixth Distinction

The Sixth Distinction is described by Bartolozzi & Franci [1990: 17–27], for which reason I shall elaborate on such things only as go beyond their work. It is subdivided into six *tractati*, the first of which (fols 67^{v}–72^{v}) deals with "proportions", argued initially to be necessary not only by means of a list of glorious *names* – Euclid, the Stoics, the Platonists, the Peripatetics, Boethius, Jordanus, etc. – but also with reference to the prestigious uses of the concept: in Archimedes' *De mensura circuli*,[52] in law, in medicine (namely in composite drugs and the determination of diets), in mechanical artifices, in the painter's mixing of colours and in the canonical proportions of the human body,[53] in rhetoric, in architecture, in carpentry, in music etc. Only afterwards (fol. 69^{r}) come the "definitions of the various proportions", said to be preceded in Euclid by the definition of parts "as we did in the definition of fractions" (namely fol. 48^{r}). A ratio can, with Plato and Boethius, be determined for any two magnitudes of the same kind – but not, as "in a certain abuse of common speech", between the sharpness of a voice and that of a knife.

After this very general "definition" comes the subdivision into geometrical, arithmetical, and harmonic proportion, the first of which is supposed at this point to be applicable to continuous quantities only, the second to discrete as well as continuous quantities,[54] the third to sound and song. As can also be seen from the examples, Pacioli primarily links the three types to the disciplines from which they have their name, even though he does explain later on that the arithmetical proportion has to do with "excesses or differences" and says on fol. 75^{v} that this linking is what "certain blunt minds" (*alcuni roçi*) think. The fact that a harmonic proportion has to involve three terms leads to a digression (provided one can distinguish digressions from the rest in Pacioli's almost Borgesian prose) about

---

[52] It may be taken note of that Pacioli follows the abbacus tradition and not the Archimedean treatise in the value of these ratios; abbacus geometry had always taken the quotient between perimeter and diameter to be exactly $^{22}/_{7}$ (not 22:7, since it was not interested in ratios), and that between the areas and the circumscript square to be $^{11}/_{14}$. Pacioli's reading of Archimedes's treatise must (at best) have been superficial.

[53] In this connection there is a laudatory reference to *maestro Pietro de li Franceschi nostro conterraneo del Borgo san Sepolchro* and his *De prospectiva pingendi*.

[54] Pacioli even gives an argument for this claim: discrete quantities can only enter in rational ratios, continuous indifferently in rational and irrational ratios.

Everywhere else Pacioli evidently takes "geometric proportions", that is, ratios, between numbers. *Interdum dormit Homerus* – and lesser spirits too.

the sloppy habit to say *proportione* where the precise word would be *proportionalità*, and about the subdivision of such proportions into continued and discontinued.[55]

The presumed observation that only geometrical proportions can be between rational as well as irrational quantities (*viz* because arithmetic *as a discipline* considers only rational magnitudes) leads to a discussion of commensurability and incommensurability, with a reference to *Elements* X. Since Pacioli's example is the diagonal in a square with side 10, this is superfluous sophistication, and in fact he goes on with an (unreferenced) borrowing from the scholastic theory of ratios, namely when speaking of the ratio diameter:side as *meççadoppia*, "half of double", explaining (with reference to *Elements* VII–VIII) that this ratio duplicated is the double ratio.

Follows a long presentation (fols 71^{r}–72^{v}) of the Boethian subdivisions of the category of "rational proportions" – now obviously only geometrical, but that goes unmentioned: equal, major, minor, multiplex, simple and multiplex superparticular and superpartient, submultiplex, etc. In the end comes the wonderful admission that "these terms which serve to denominate these many kinds of proportions serve (for you, practitioner [*a te pratico*]) no other purpose than speaking solemnly [*proferire*] about the species you have found". Fol. 82^{r} presents the Boethian categories in a scheme.

The second treatise (fols 72^{v}–76^{r}) takes up *proportionalità* for good, defining these (with *Elements* V) as similitude of ratios.[56] Once more we get the geometrical, arithmetical and harmonic proportions, this time with numerical examples for the former two,[57] the harmonic proportion being left explicitly aside. Those still considered may be continued or discontinued; the need for similarity of kind is repeated. For continued proportions of both remaining types, the necessity of equal kind for all members is pointed out.

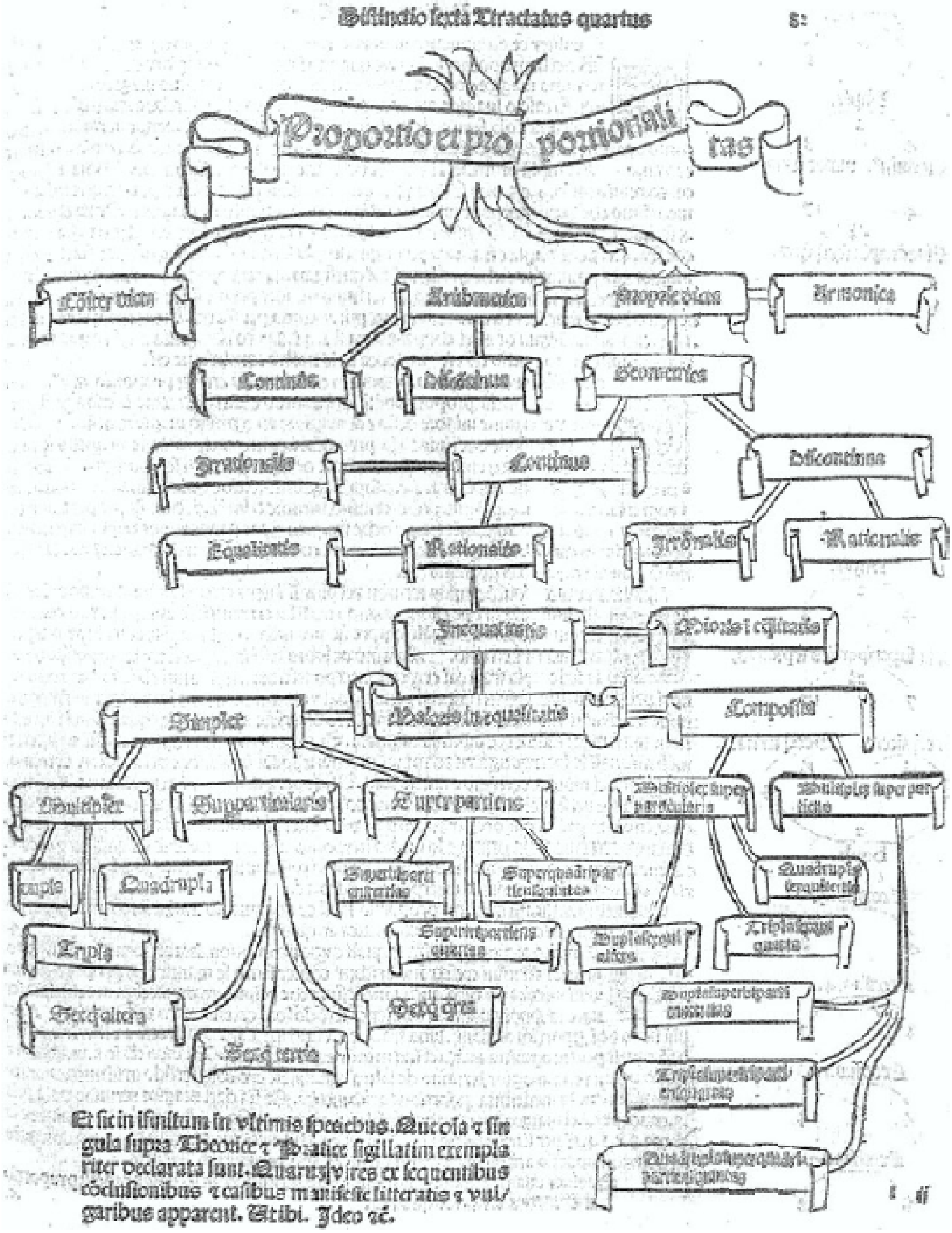
After a discussion of *disproportionality* (fol. 74^{r}) – the first "proportion" being either larger or smaller than the second – Pacioli deals with the six ways to come to grips with proportions (*de sex specibus sive modis arguendi proportionalitatum*),

---

55 Pacioli's term has changed since fol. 57^{v}, now he uses *discontinua* instead on *incontinua*.

56 Bartolozzi & Franci [1990: 19] reproach Pacioli that this similitude is meaningless without its definition via equimultiples, forgetting that this is not only absent from Pacioli's *Summa* but also from the Campanus *Elements*.

57 For geometric proportionality, the example is that 6 is to 3 as 4 to 2 – no problem with numbers, cf. note 54 and the preceding text.



Pacioli's schematic presentation of the Boethian categories [1494: 82r].

repeating Campanus's seven modes (see text before note 5) and coming down to six by conflating *e contrario* with *e conversa*. Pacioli says afterwards (fol. 75^{v}) that he now deals with geometric proportions only, at which point he also asks how much of it holds for arithmetical proportions. He shows the *permutatim* mode to be valid, and then generalizes that it should hold for all – obviously without calculating, since only the *ex aequa* mode is true.

The third treatise (fols 76^{r}–80^{r}) begins by explaining the *denominations* of ratios, (not to be mixed up with the Boethian *names*), the number resulting from the division of one term by the other, in agreement with the terminology of Jordanus and Campanus in their treatises about proportions [ed. Busard 1971: 205, 213; Busard 2005: 230]. After the dismissive remark about the utility of the Boethian terminology in the first treatise (followed up here with references to *phylosophi*), Pacioli thus chooses not do as Palatino 573, which identifies ratios with numbers (or replaces them with numbers); what he does is equivalent, but in the dress of established theory. The cost (which the loquacious Pacioli may not have seen as a cost) is that what Palatino 573 does in a couple of lines now needs two dense pages (fols 76^{r}–77^{r}) to be explained.

Procured with the denomination concept and in agreement with the Campanus *Elements* VII [ed, Busard 2005: 230], Pacioli can return (fol. 77^{r}–77^{v}) to the question of whether one ratio (among numbers, which he does not say) is equal to, greater than or smaller than another one. He uses the occasion to show how this can be done also for the Boethian names, translating them into denominating numbers. He can also take up the composition of ratios (fols 77^{v}–78^{r}), "without comparison much more difficult" than the operations on integers, fractions and roots, and (once more) necessary for instance for the physician in his preparation of composite drugs. First he deals with continued proportions (fol. 78^{r}), where we see that Pacioli spontaneously tends to forget the distinction between the ratio and its denomination: in order to find the ratio between the first and the third term "it is sufficient to multiply [that between the first and the second term] by itself, or its denomination by itself, and it will make the denomination of the proportion between the first and the third". The same tendency underlies an explanatory observation on Campanus, "by duplicated [ratio] Campanus understands (as true is) multiplied by itself" and in the corresponding reference to the "multiplication of the double [ratio] by itself" and in the general claim that "as multiplying a proportion by itself makes a third proportion, thus to multiply the denomination of the said proportion by itself will make the denomination of that third proportion". The composition of unequal ratios comes briefly on fol. 79$^{\mathrm{v}}$. Now Pacioli is more faithful to his theoretical base, speaks of “joining” the ratios 2:1, 6:2 and 24:6; the way is of course to multiply the denominations 2, 3 and 4.

Next follows (fols 79$^{\mathrm{v}}$–80$^{\mathrm{r}}$) the problem, how to divide a given ratio in several ratios from which it is composed.[58] It is correctly said, and demonstrated by examples, that this can be done in many ways, by insertion of intermediate terms ad libitum.[59] Finally (fol. 80$^{\mathrm{r}}$) Pacioli teaches how to determine one term of a ratio if the denomination and the other term is known (or the other chosen freely, if none is fixed).

The fourth treatise (fols 80$^{\mathrm{r}}$–81$^{\mathrm{v}}$) is an attempted *Algorismus proportionum*, based once again on Witelo. It teaches how to add (that is, compose) and subtract ratios and how to “multiply” and “divide” ratios. “Multiplication”, however, is simply composition of several not necessarily equal ratios, and “division” is the splitting of a ratio into several not necessarily equal ratios (the examples for both use unequal ratios). Oresme’s work is clearly no inspiration.

The fifth treatise (fols 832$^{\mathrm{v}}$–84$^{\mathrm{r}}$) examines what happens to ratios and arithmetical proportions if they are changed in various ways (examples in the margin combine denominations and Boethian names).[60] Namely that

– a ratio grows if the major term is augmented (all ratios are supposed to have the major term first) or the minor diminished;
– the same happens if to both terms something is added, to the major something larger than itself, to the minor something smaller than itself;
– if between the extreme terms of a ratio one or several others are inserted, then any ratio between any two intermediate terms or one extreme and an intermediate term is smaller than the original ratio;
– the increase of both major terms or both minor terms or the decrease or increase of all four terms in an arithmetical proportion by the same amount conserves the proportionality;

[58] Actually, the title asks for several equal ratios, but that does not correspond to what actually follows. In [1523: 79$^{\mathrm{v}}$] the word “equal” has justly been removed – the publisher Paganinus de Paganino may have had the assistance of somebody who understood the matter, or based himself on a copy with corrections inserted. The latter seems plausible; in the first line of fol. 81$^{\mathrm{v}}$, an erroneous $^2/_3$ is not corrected into $1\,^2/_3$.

[59] Pacioli refers for this to Witelo’s *Perspectiva*, which he has already said on fol. 79$^{\mathrm{r}}$ to have consulted years ago.

[60] [Bartolozzi & Franci 1990: 23] offers a translation into modern mathematical symbols.

– a ratio does not change if both terms increase *geometrice* – further explained as increase by the same part;
– a ratio diminishes if to both terms the same absolute (*arithmetice*) amount is added, and it increases if the same absolute amount is subtracted from both;
– if both terms of a ratio are increased geometrically, then their "arithmetical proportion" (that is, their difference) increases;
– if both are diminished geometrically, then their "arithmetical proportion" decreases;
– if both terms of a ratio are equal, increasing or decreasing both arithmetically equally is the same as increasing them geometrically equally, and their ratio is conserved.

Three corollaries follow which are related to the Peripatetic theory of motion.

According to its title, the sixth treatise (fols 84^r–98^v) deals with the "seven marvels [*mirabiles*] from the proportions between two quantities". Actually, it *begins* with seven "marvels" involving two quantities and then considers others which concern three or more. The first marvel is that

> any two quantities you want in any proportion joined together, and then the sum divided by each of the said quantities; the results then joined together, and then the sum of the said results equally divided by the each of the said results; and again these latter two results joined together, will always be the sum of the first two results, and it never fails.

In symbols,[61]

(1) $$\frac{\frac{a+b}{a}+\frac{a+b}{b}}{\frac{a+b}{a}}+\frac{\frac{a+b}{a}+\frac{a+b}{b}}{\frac{a+b}{b}}=\frac{a+b}{a}+\frac{a+b}{b}$$

I shall not go through all seven marvels (all are rendered in symbols in [Bartolozzi & Franci 1990: 23–24][62]), but two are noteworthy – in symbols,

[61] The fraction lines stand for the operation which Pacioli speaks of as "division"; *denom* in (5) stands for "denomination of" the ensuing "proportion".

[62] There is a (mathematical as well as translational) error in the fourth, which should be

(4) $$\frac{\frac{a+b}{a}+\frac{a+b}{b}}{\frac{a+b}{a}}=\frac{a+b}{b}\quad\text{and}\quad\frac{\frac{a+b}{a}+\frac{a+b}{b}}{\frac{a+b}{b}}=\frac{a+b}{a}$$

respectively,[63]

(3) $$\frac{a+b}{a} \times \frac{a+b}{b} = \frac{a+b}{a} + \frac{a+b}{b}$$

and

(5) $$\frac{a+b}{a} + \frac{a+b}{b} = 2 + denom(a:b) + denom(b:a)$$

The marvels seem to have to do to with the connection between problems about the splitting of 10 into two parts *a* and *b*, where $a/_b + b/_a$ respectively $10/_a + 10/_b$ is given. Such problems are known since the beginning of the algebra tradition,[64] and they had also been taken up by Jordanus in *De numeris datis*.[65] Even though I do not remember having seen Pacioli's rules in earlier sources, I therefore suspect him to have borrowed at least some of them.

After the seven, as told, others marvels follow (fol. $85^{r–v}$) regarding three, four or five numbers in continued proportions, first of which is that if three numbers are in continued proportion, then division of their sum by the single numbers produces another continued proportion. This was (for an arbitrary dividend) what Antonio considered as "rather clear and obvious" (text after note 41) and in fact a theorem which is useful for certain of the problems about the splitting of a number into a sum of numbers in continuous proportion. We may take it for granted that Pacioli took it from the tradition – perhaps indeed directly or indirectly from Antonio, since he goes on (fols 85^v–86^r) to apply the rules to binomials in the way Antonio had done in his "Mirabile dictum" (above, note 43 and surrounding text). As pointed out by Bartolozzi & Franci [1990: 24], Pacioli generalizes Antonio's method further than the Antonio himself had done (asking only for a rational ratio *b*:*a*) without controlling – and errs (or so it seems – the text is not fully clear as to how many conditions Pacioli wants to fulfil).

Next (fols 86^v–87^v) come a number of rules about three, four or more numbers in continued or (occasionally) non-continued proportion. Most, as Pacioli states,

---

The authors have overlooked that the equality between the first and second results are said to be *econverso*. As Pacioli points out, the first marvel follows from this, as do the second versions of (3) and (5), not given by Bartolozzi & Franci, in which the right-hand sides of (4) are replaced by the left-hand sides.

63 In both cases Pacioli also points out that the rules hold for the "second results" as well.

64 See [Rosen 1831: 44–46; Rashed 2007: 167–165] (al-Khwārizmī), [Levey 1966: 94–102, cf. 132–140; Sesiano 1993: 365–369, cf. 382–388; Chalhoub 2004: 58–65, 103–109] (Abū Kāmil), and [Woepcke 1853: 91*f*] (al-Karajī). All three give general rules for the behaviour of the quotients, e.g., $a/_b \cdot b/_a = 1$ and $(a/_b + b/_a) \cdot ab = a^2 + b^2$.

65 I.20 and I.21α, [ed. Hughes 1981: 64].

follow from *Elements* VI.15–16 and VII.20 (our VI.16–17 and VII.19 – the product rule for three or four segments or numbers in proportion / continued proportion): how, if two (or, an overdetermined case, three) neighbouring quantities in a continued proportion are known, to find the remaining one(s).

Slightly more intricate are the cases where the first and the last of four or five quantities in continued proportion are known. In the case of four quantities, this coincides mathematically with Jacopo's second *fondaco* problem, but whereas Jacopo merely prescribes the extraction of the cube root of the ratio between the fourth and the first quantity without explaining why, Pacioli uses algebra, without which he finds it difficult to solve the problem. In the case of five quantities, the middle quantity is found first from the product rule.[66]

Between these two cases, Pacioli gives the abstract analogue of Jacopo's third *fondaco* problem. Without explanation Pacioli gives the same rule as Jacopo (see text after note 32); he certainly does not know how it has come about (if so, the algebraic solution of the preceding problem shows that he would have explained). However, the last step of his procedure (how to find two numbers from their sum and their product) suggest that Jacopo is not his source: it contains a hint of an underlying geometric procedure (a reference to operation with two *different* halves of a quantity) which is absent from Jacopo's text, and which Pacioli is not likely to have introduced himself.[67]

Pacioli now (fol. 88[r]) supplies a number of "keys", likened (nothing less!) to the two spiritual keys of gold and silver by which "in our Catholic Militant Church the first shepherd Saint Peter" opens and closes the doors of Paradise and Hell for us.

The keys – fifteen in total – are theorems (not labelled so by Pacioli), in part

---

[66] In generic terms, Pacioli says that the same method can be used for "6, 7, 8, etc." terms, but he abstains (maybe wisely) from implementing this insight – fol 182[r] he speaks of the sixth root as the "cube root of the cube root" and of the seventh root as the "root of the root of the cube root". Possibly, these composite expressions indicate that Pacioli believed they could be found by stepwise calculation. This is not quite certain, however: as we have seen above, Fibonacci speaks in the *Liber abbaci* [ed. Boncompagni 1857: 400] of the quintupled proportion as "cube of the square, or square of the cube" and of the sextupled ratio as the "cube of the cube", but his numerical examples (32:1 as the quintuple of 2:1, 729:1 as the sextuple of 3:1) show he was not misled.

[67] When solving in the second part of the *Summa* the corresponding geometric problem, Pacioli [1494: II, fol. 18[r]] merely refers to the contents of *Elements* II.5, as does his ultimate source (Fibonacci's *Pratica geometrie* [ed. Boncompagni 1862: 63]). Similarly also (with explicit citation of Euclid's proposition) in the arithmetical part, fol. 93[v].

near- or full repetitions of what he has already explained before or easy corollaries of familiar stuff, in part new to the book and not easily guessed without symbolic manipulation. All are illustrated by numerical examples. I list them in symbolic translation, indicating the beginning of new pages:

(1)[88r] If $a{:}b{:}c{:}d$, then $\frac{b+c}{a+b+c+d} : \frac{b}{a+c}$.

(2) If $a{:}b{:}c{:}d$, then $\frac{a+b}{c+d} : \frac{a}{c}$.

(3) If $a{:}b{:}c{:}d$, then $\frac{a+c}{b+d} : \frac{a}{b}$.

(4) If $a{:}b{:}c{:}d$ and $S = a+b+c+d$, then ${}^{S}/_{a}{:}{}^{S}/_{b}{:}{}^{S}/_{c}{:}{}^{S}/_{d}$; with three members, this was the first three-number "marvel" on fol. 85[r].

(5) If $\frac{a}{b} : \frac{c}{d}$, then $ad = bc$; the product rule, amply used before.

(6)[88v] If $\frac{a}{b} : \frac{c}{d}$ and if $c^2+d^2 = a{\cdot}b$, then $\sqrt{(a^2+b^2)(cd)}$ has the same value. Actually, given only the proportion, $(a^2+b^2){\cdot}cd = ab{\cdot}(c^2+d^2)$.

(7) If $\frac{a}{b} : \frac{c}{d}$, then $([a{\cdot}b]{\cdot}c){\cdot}d = (a{\cdot}d){\cdot}(b{\cdot}c)$; evidently, this does not depend on the proportionality.

(8) If $a{:}b{:}c{:}d$, then $(a+b+c+d)^2 = a{\cdot}(b+c+d)+b{\cdot}(a+c+d)+d{\cdot}(a+b+c)+c{\cdot}(a+b+d)+a^2+b^2+c^2+d^2$; this time, Pacioli himself points out that the rule does not depend on the proportionality.

(9) If $a{:}b{:}c$, then $(a{\cdot}b){\cdot}c = b^3$.

(10) If $a{:}b{:}c$, and if for some quantity Q ${}^{Q}/_{a}+{}^{Q}/_{b}+{}^{Q}/_{c} = a+b+c$, then $b = \sqrt{Q}$.

(11)[89r] If $a{:}b{:}c$, then ${}^{(a\cdot b)\cdot c}/_{a} = b{\cdot}c$, ${}^{(a\cdot b)\cdot c}/_{b} = a{\cdot}c$, ${}^{(a\cdot b)\cdot c}/_{c} = a{\cdot}b$, and ${}^{(a\cdot b)\cdot c}/_{a\cdot b} = c$, ${}^{(a\cdot b)\cdot c}/_{a\cdot c} = b$, ${}^{(a\cdot b)\cdot c}/_{b\cdot c} = a$; Pacioli points out that this does not depend on the proportionality.

(12) If $a{:}b{:}c$ and further $\frac{a}{b} : \frac{p}{q}$, then $p{\cdot}(b+c) = q{\cdot}(a+b)$.

(13) If $a{:}b{:}c$, then $2{\cdot}(a{\cdot}c+b{\cdot}[a+c]) = a(b+c)+b(a+c)+c(a+b)$. With references to *Elements* II.2 and the formulations "in other words" in *Elements* VI and IX Pacioli points out that this does not depend on the proportionality.

(14)[89v] If $a{:}b{:}c$, then ${}^{a(b+c)+b(a+c)+c(a+b)}/_{2\cdot(a+b+c)} = b$.

(15) If $a{:}b{:}c$, then $\frac{a^2}{b^2} : \frac{a}{c}$.

This is the last key. Under the heading "to find mean proportionals between two quantities", two sophisticated counterfactual calculations follow (fol. 89[v]) which I guess are Pacioli's own invention: If 2 is the arithmetical respectively geometric mean between 5 and 11, what is then the corresponding mean between 7 and 13? In both cases, the true means between 5 and 11 and 7 and 13 are found (8 and 10 respectively $\sqrt{55}$ and $\sqrt{91}$), and the rule of three is applied. In the arithmetical case, a proof is performed, consisting in corresponding proportional change of the limits, after which the true means between these limits are shown to coincide with what was found before; in the geometrical case, a similar proof is sketched but not performed.

The "second case" under the same heading is a traditional question "Three is (too) little and 4 is (too) much". The "just or due" amount is said to be $\sqrt{12}$, the geometric mean; this – not the arithmetical mean – is then stated to be what is used in all commercial matters (*in omnibus mercantiis*). Primarily, this probably extrapolates from the observation that the rule of three is based on geometric *proportionality*. But Pacioli may also think of the use of the geometric mean in certain mathematical *problems* in commercial disguise.

In any case, such a problem, about three pearls, follows as the "third case". The first pearl weighs 1 carat and is worth 200 *ducati*, the second weighs 2 carats and is worth 1000 *ducati*, the third weighs 3 carats. What is its just price?

Pacioli posits a fourth pearl with weight 4 carats. To the weights 1:2:4 in continued proportion must correspond prices in continued proportion, i.e., 200:1000:5000. Therefore the price of the 4-carat pearl must be 5000 *ducati*. 3 carats being the (arithmetical) mean between 2 and 4, the price of the 3-carat pearl must be $\sqrt{(1000 \cdot 5000)}$.

A fourth case is also about justice. The Holy Father, Innocent VIII, orders that 10000 *ducati* be distributed justly between the citizens of Perugia for service rendered. This gives rise to a long discourse (more than 500 words) about Aristotle's two kinds of justice from the [Nicomachean] *Ethics* V.2–5 [Barnes 1984: II, 1784–1789]: "commutative"[68], applicable to commercial exchange, and distributive. Both, according to Pacioli, "can, broadly speaking, be understood in two ways, geometrically and arithmetically, though, strictly and properly speaking, the maximal distributive sort can only be geometrical".[69] After the digression into ethical theory it is then explained that the money is justly distributed if made in geometric proportion to the "quality" (*bontà*) of each.

The sixth distinction ends (fols 90^{v}–98^{r}) with 35 problems[70] and an epilogue (fol. 98$^{r–v}$). The final two have nothing to do with proportions – #34 is "Bachet's weight problem", and #35 belongs to the same family; parallels in the wording suggest that they are borrowed from the *Liber abbaci* [ed. Boncompagni 1857:

68 Nowadays normally translated "rectificatory", but Pacioli follows his fellow friar Thomas Aquinas (*Summa theologiae* 1^{a} q. 21 a. 1 s 1 co, see [*Corpus thomisticum*]), whom he cites.

69 This point comes from Aristotle, whose Chapter 3 also contains as discourse on proportion theory. Mathematical proportions (represented by lines and letters) are used further in Chapters 4 and 5.

70 Pacioli also counts until 35, but has two #18, skips #19 and #28 and has two #29.

297*f*]. In all the others, "proportions" play a role.

First come 23 problems about three numbers in continued proportion. In seven of them, a number (19, 19, 14, 10, "a number",[71] 10, 10) is split into these constituents; towards the end of the sequence, four are dressed as dealing with economic life.[72] In #1–6, specified "keys" are used as a first step in the procedure, which in these and the other cases often makes use of algebra or (in #5, #6 and #18) of *Elements* II.[73]

Next follows a sequence of ten problems about four magnitudes in continued proportion, none of them in concrete dress. Once again, the first ones make use of specified "keys" (#24–27 – but also #31–32). Most interesting are probably #31–33: #31 and #33 are pure-number versions of Jacopo's third and fourth *fondaco* problems, #32 of a similar problem where the sums of the wages for the first two and for the last two years are given. In #31, key (1) is used to reduce the problem; then the second number is taken as the *thing* and found by second-degree algebra to be $12\frac{1}{2}+\sqrt{7\frac{37}{84}}$ – at which point Pacioli cautiously leaves it to the reader to continue.[74] Since his present method does not lead easily[75]

---

[71] This problem (#15) is indeterminate. Afterwards, the number is chosen to be 10, whereby it is made determinate.

[72] #18bis deals with a gambler's gains, where the product rule is explained once again, suggesting perhaps the text to be borrowed (but Pacioli is too fond of repeating to make the inference certain); #21, which "was proposed to me in Florence in 1480, the 22nd of June", deals with a purchase of saffron, cinnamon and mastic, and #22–23 with alloys.

[73] Algebra is thus *used* by Pacioli well before he presents it systematically. Often, this algebra is quite complex. In #4, for instance, Pacioli has to operate with two unknowns in the same way as Antonio, that is, with "a *thing* less a quantity" and "a *thing* plus a quantity". The problem in which this is used is not the same as the one where Antonio introduces it in the *Fioretti*, nor with the one from the *Flos* where Fibonacci employs it, cf. note 40.

[74] The solution is correct, but corresponds to a decreasing sequence, which is certainly not what Pacioli intended; in order to have an increasing sequence, he should have chosen the other root of the equation, $12\frac{1}{2}-\sqrt{7\frac{37}{84}}$. Since Pacioli did not discover, he cannot have finished the calculations.

When applying later the same method to an analogous wage problem with rational solutions, Pacioli makes the complete calculation and chooses the correct solution – see presently.

[75] Of course it *can* lead to it, but only if one is able to express the double root (the second and the third number, respectively) as

to the formula used by Jacopo and by Pacioli in the first presentation of the abstract problem just before the "keys" (above, text before note 67), Pacioli appears not to have noticed the connection.

The use of the "keys" in problem reductions leaves little doubt that these *new* theorems about the behaviour of proportions were created as tools for the solution of problems – but apparently only problems formulated in terms of proportions or proportionality, for whose initial reduction they served. Pacioli's way to add observations about (8), (11) and (13) strongly suggests that the basic set was not his own. It is likely to have been created during the fifteenth century and *seems to reflect a more intimate integration between algebra and proportions than other sources would make us expect*.

10. Further "proportions" in Pacioli's *Summa*

Proportionality turns up in (at least) three other contexts in the *Summa* – in the general presentation of algebra, and in two sets of problems.

Fol. 143r lists a sequence of 30 algebraic powers (*dignità*) in two different terminologies and observes that the reader may go on *proceeding proportionally* "as long as you want". On fol. 145v, the same insight (which as we know was not new) is hinted at in the statement that all solvable cases are *proportionati* to the six basic cases.[76] It becomes more explicit (and somewhat more innovative) on fols 149v–150r, after a short list of select possible and impossible cases. In order to find out to which basic case a given equation reduces, one shall locate the *dignità* in the ordered sequence and reduce[77] geometrically equally to the lowest possible degree by counting downwards. However, if the *intervalli* between the three powers in the equation (the only equations Pacioli considers) are not equal, it has "so far not been possible to form general rules because of their

---

$$\frac{P}{2}\pm\sqrt{\frac{P^2}{4}-\frac{P^3}{3P+Q}}$$

(P being the sum of the second and the third number, Q that of the first and the fourth). The product of these is indeed

$$\frac{P^3}{3P+Q}$$

as required; but this will have been far too complicated for Pacioli.

76 This, certainly, is not true *stricto sensu* if we consider as solvable, e.g., the case "*cubes* equal to number"; but Pacioli's target is the proliferation of false solutions to non-homogeneous higher-degree equations.

77 The verb is *schizzare*, which mostly refers to the reduction of a fraction through division of numerator and denominator by the same divisor.

disproportionality".

On fols 186^r–187^v, a number of problems deal with gain (occasionally loss) "at the same rate" in two or more travels. Mostly, the proportionality leads to the application of the rule of three, but once, in an alternative ("and more beautiful" / *pulchrius*) solution (fol. 186^r), proportionality is mentioned explicitly, and the product rule applied. This evidently gives the same calculations as the rule of three; the aesthetic advantage is solely in the use of "magisterial" terminology – "speaking solemnly", in Pacioli's earlier words.

Finally, fol. 194^r brings six problems about the wages of a servant, in two of which the wage is supposed to increase "at the same rate" each year (four years in total). In the first of them the wage of the first year is 10, that of the last year is 60; apart from a change of the wage of the first year, this coincides with Jacopo's second *fondaco* problem. This has to be done "according to what I showed you in the proportions, and I shall say no more, except that there are four proportional numbers, and the first is 10, and the last is 60. I ask for the means" – which are then stated to be $\sqrt[3]{6000}$ and $\sqrt[3]{36000}$.

The second coincides with Jacopo's third *fondaco* problem, even in its choice of parameters. For the solution, Pacioli refers to the "first key" and to what he has already taught. This time, the numbers are convenient, and Pacioli makes the complete calculation, finding the second number to be $30-\sqrt{100}$ and the third to be $\sqrt{100}+30$ (an order which suggest he has not used the double solution but subtracted from 60).

There may be other scattered references to the concept of proportionality in the work. All in all, however, "proportions and proportionalities" are mainly treated in the sixth distinction, which is indeed extensive and profound enough to justify the appearance of the terms in the title of the work; to this distinction are also moved traditional abbacus problem types about numbers in proportion, abstract as well as in commercial dress. Outside the sixth distinction, "proportions" play as modest a role as in most abbacus treatises.

11. Summing up

In this way, Pacioli's *opus magnum* suggests the general summary we may draw up. Abbacus mathematics, based in practical arithmetic, was always centred around problems of simple (direct or inverse) proportionality; to this came a strand of algebraic thought with hight prestige due both to its efficiency and to its character of “theoretical level of practical arithmetic”.[78] Initially, neither the language nor the theory of proportions had anything to do in either; gradually, but hardly to a larger extent than its penetration in daily discourse, would the proportion language pop up. Problems might also be formulated in terms of quantities or numbers in proportion. *Theory* beyond the product rules remained outside.

To this, only writers with “magisterial” pretensions – Fibonacci, Antonio, Benedetto, the author of Palatino 573, Pacioli – and the shadowy inventor of Pacioli’s “keys” constitute exceptions. Apart from the inventor of the keys, who to some extent made *new* theory, what they offer in terms of theory and technicalities beyond the product rules are isolated chapters, in some cases just copied from Fibonacci (and never understanding more about these than Fibonacci himself). They are there rather by pious duty than by mathematical necessity.

As regards the Boethian terminology for ratios, the situation is even more blatant. Since late Carolingian times, this categorization had been the almost sacred core of the mathematics of Latin schools and universities. Benedetto and Palatino 573 introduce them, but the latter dismisses them immediately, and the former makes no use of them. Only Pacioli employs this “solemn speech” rather consistently when explaining the composition of ratios in Distinction 6, 4th Treatise. The general tendency is to speak “at best” (as judged from the perspective of the schools) of the denominations of ratios, “at worst” to come close to identifying the ratio with the quotient (as when Pacioli identifies composition and multiplication of ratios).

All in all, abbacus mathematics is much more modern on this account (and on several others) than scholarly mathematics of the late Renaissance. As scholars digested the abbacus heritage, they took over norms, not only technical algebraic knowledge.

---

[78] Two quotations may suffice to illustrate this prestige. Palatino 573 [ed. Arrighi 2004/1967: 191] opens the part on algebra (as we remember, this encyclopedia falls in eleven “parts”) with the words “every part would be in vain if this [part] was left out; because [...] this is the one that gives solution to all cases”. Pacioli [1494: 144ʳ] observes in the corresponding place that we have now “arrived with the help of God to the much desired place: that is, to the mother of all the cases popularly called the *regola della cosa* or *Arte magiore*, that is, *pratica speculativa*, otherwise called *algebra & almucabala* in the Arabic or Chaldean tongue”. The words *pratica speculativa* mean exactly “theoretical [level of] practical arithmetic”.

Bibliography

Arrighi, Gino (ed.), 1967a. Antonio de' Mazzinghi, *Trattato di Fioretti* nella trascelta a cura di M° Benedetto secondo la lezione del Codice L.IV.21 (sec. XV) della Biblioteca degl'Intronati di Siena. Siena: Domus Galilaeana.

Arrighi, Gino, 1967b. "L'aritmetica speculativa nel 'Trattato' di M° Benedetto". *Physis* **9**, 311–336.

Arrighi, Gino (ed.), 1973. *Libro d'abaco.* Dal Codice 1754 (sec. XIV) della Biblioteca Statale di Lucca. Lucca: Cassa di Risparmio di Lucca.

Arrighi, Gino (ed.), 1974. Pier Maria Calandri, *Tractato d'abbacho.* Dal codice Acq. e doni 154 (sec. XV) della Biblioteca Medicea Laurenziana di Firenze. Pisa: Domus Galiaeana.

Arrighi, Gino (ed.), 1987a. Giovanni de' Danti Aretino, *Tractato de l'algorisimo.* Dal Cod. Plut. 30. 26 (sec. XIV) della Biblioteca Medicea Laurenziana di Firenze. *Atti e Memorie della Accademia Petrarca di Lettere, Arti e Scienze*, nuova serie **47**, 3–91.

Arrighi, Gino (ed.), 1987b. Paolo Gherardi, *Opera mathematica: Libro di ragioni – Liber habaci*. Codici Magliabechiani Classe XI, nn. 87 e 88 (sec. XIV) della Biblioteca Nazionale di Firenze. Lucca: Pacini-Fazzi.

Arrighi, Gino (ed.), 1989. "Maestro Umbro (sec. XIII), *Livero de l'abbecho.* (Cod. 2404 della Biblioteca Riccardiana di Firenze)". *Bollettino della Deputazione di Storia Patria per l'Umbria* **86**, 5–140.

Arrighi, Gino, 2004/1965. "Il codice L.IV.21 della Biblioteca degl'Intronati di Siena e la «bottega dell'abaco a Santa Trinita di Firenze»". pp. 129–159 *in* Gino Arrighi, *La matematica nell'Età di Mezzo. Scritti scelti*, a cura di F. Barberini, R. Franci & L. Toti Rigatelli Pisa: Edizioni ETS, 2004. First published in *Physis* **7**, 369–400.

Arrighi, Gino, 2004/1967. "Nuovi contributi per la storia della matematica in Firenze nell'età di mezzo: Il codice Palatino 573 della Biblioteca Nazionale di Firenze", pp. 159–194 *in* Gino Arrighi, *La matematica nell'Età di Mezzo. Scritti scelti*, a cura di F. Barberini, R. Franci & L. Toti Rigatelli Pisa: Edizioni ETS, 2004. First published in *Istituto Lombardo. Accademia di scienze e lettere. Rendiconti, Classe di scienze (A)* **101**, 395–437.

Arrighi, Gino, 2004/1968. "La matematica a Firenze nel Rinascimento: Il codice ottoboniano 3307 della Biblioteca Apostolica Vaticana", pp. 209–222 *in* Gino Arrighi, *La matematica nell'Età di Mezzo. Scritti scelti*, a cura di F. Barberini, R. Franci & L. Toti Rigatelli Pisa: Edizioni ETS, 2004. First published in *Physis* **10**, 70–82.

Barnes, Jonathan (ed.), 1984. The Complete *Works* of Aristotle. The Revised Oxford Translation. 2 vols. (Bollingen Series, 71:2). Princeton: Princeton University Press.

Bartolozzi, Margherita, & Raffaella Franci, 1990. "La teoria delle proporzioni nella matematica dell'abaco da Leonardo Pisano a Luca Pacioli" *Bollettino di Storia delle Scienze Matematiche* **10**, 3–28.

Boncompagni, Baldassare (ed.), 1857. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. I. Il *Liber abbaci* di Leonardo Pisano. Roma: Tipografia delle Scienze Matematiche e Fisiche.

Boncompagni, Baldassare (ed.), 1862. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. II. *Practica geometriae* et *Opusculi*. Roma: Tipografia delle Scienze

Matematiche e Fisiche.
Busard, Hubert L. L. (ed.), 1971. “Die Traktate *De proportionibus* von Jordanus Nemorarius und Campanus”. *Centaurus* **15**, 193–227.
Busard, H. L. L., 2005. *Campanus of Novara and Euclid’s Elements*. 2 vols. (Boethius, 51,1–2). Stuttgart: Franz Steiner.
Cassinet, Jean, 2001. “Une arithmétique toscane en 1334 en Avignon dans la citè des papes et de leurs banquiers florentins”, pp. 105–128 *in Commerce et mathématiques du moyen âge à la renaissance, autour de la Méditerranée*. Actes du Colloque International du Centre International d’Histoire des Sciences Occitanes (Beaumont de Lomagne, 13–16 mai 1999). Toulouse: Éditions du C.I.H.S.O..
Caunedo del Potro, Betsabé, & Ricardo Córdoba de la Llave (eds), 2000. *El arte del alguarismo*. Un libro castellano de aritmética comercial y de ensayo de moneda del siglo XIV. (Ms. 46 de la Real Colegiato de San Isidoro de León). Salamanca: Junta de Castilla y León, Consejeria de Educación y Cultura.
Chalhoub, Sami (ed., trans.), 2004. *Die Algebra, Kitab al-Gabr wal-muqabala* des Aby Kamil Soga ibn Aslam. (Quellen und Studien über die Geschichte der Arabischen Mathematik, 7). Aleppo: University of Aleppo, Institute for the History of Arabic Science.
Corpus thomisticum. http://www.corpusthomisticum.org/ (29.1.2006).
Folkerts, Menso, 2006. *The Development of Mathematics in Medieval Europe: The Arabs, Euclid, Regiomontanus*. (Variorum Collected Studies Series, CS811). Aldershot: Ashgate.
Franci, Raffaella, & Marisa Pancanti (eds), 1988. Anonimo (sec. XIV), *Il trattato d’algibra* dal manoscritto Fond. Prin. II. V. 152 della Biblioteca Nazionale di Firenze. (Quaderni del Centro Studi della Matematica Medioevale, 18). Siena: Servizio Editoriale dell’Università di Siena.
Franci, Raffaella (ed.), 2001. Maestro Dardi, *Aliabraa argibra*, dal manoscritto I.VII.17 della Biblioteca Comunale di Siena. (Quaderni del Centro Studi della Matematica Medioevale, 26). Siena: Università degli Studi di Siena.
Heath, Thomas L., 1921. *A History of Greek Mathematics*. 2 vols. Oxford: The Clarendon Press.
Hoche, Richard (ed.), 1866. Nicomachi Geraseni Pythagorei *Introductionis arithmeticae* libri II. Leipzig: Teubner.
Hochheim, Adolf (ed., trans.), 1878. *Kâfî fîl Hisâb (Genügendes über Arithmetik)* des Abu Bekr Muhammed ben Alhusein Alkarkhi. I-III. Halle: Louis Nebert.
Høyrup, Jens, 1998. “A New Art in Ancient Clothes. Itineraries Chosen between Scholasticism and Baroque in Order to Make *Algebra* Appear Legitimate, and Their Impact on the Substance of the Discipline”. *Physis*, n.s. **35**, 11–50.
Høyrup, Jens, 2005. “Leonardo Fibonacci and *Abbaco* Culture: a Proposal to Invert the Roles”. *Revue d’Histoire des Mathématiques* **11**, 23–56.
Høyrup, Jens, 2007a. *Jacopo da Firenze’s Tractatus Algorismi and Early Italian Abbacus Culture*. (Science Networks. Historical Studies, 34). Basel etc.: Birkhäuser.
Høyrup, Jens, 2007b. “Further questions to the historiography of Arabic (but not only Arabic) mathematics from the perspective of Romance abbacus mathematics”. Contribution to the “9ième Colloque Maghrébin sur l’Histoire des Mathématiques

Arabes", Tipaza, 12–13–14 mai 2007. (Slightly revised by still preliminary version). *Manuscript*. For the time being (but that will probably change) at the address http://www.akira.ruc.dk/~jensh/Work%20in%20progress/FurtherQuestions.pdf.

Høyrup, Jens, 2008. "Über den italienischen Hintergrund der Rechenmeister-Mathematik". Beitrag zur Tagung "Die Rechenmeister in der Renaissance und der frühen Neuzeit: Stand der Forschung und Perspektiven". München, Deutsches Museum, 29. Februar 2008 *Max-Planck-Institut für Wissenschaftsgeschichte, Preprint* 349 (Berlin). http://www.mpiwg-berlin.mpg.de/Preprints/P349.PDF.

Hughes, Barnabas B. (ed.), 1981. Jordanus de Nemore, *De numeris datis*. A Critical Edition and Translation. (Publications of the Center for Medieval and Renaissance Studies, UCLA, 14). University of California Press.

Hughes, Barnabas, O.F.M., 1986. "Gerard of Cremona's Translation of al-Khwārizmī's *Al-Jabr*: A Critical Edition". *Mediaeval Studies* **48**, 211–263.

Hughes, Barnabas B., 2004. "Fibonacci, Teacher of Algebra. An Analysis of Chapter 15.3 of *Liber Abbaci*". *Mediaeval Studies* **66**, 313–361.

Hughes, Barnabas (ed., trans.), 2008. Fibonacci' *De practica geometrie*. New York: Springer.

Hultsch, Friedrich (ed., trans.), 1876. Pappi Alexandrini *Collectionis* quae supersunt. E libris manu scriptis edidit et commentariis instruxit Fridericus Hultsch. I, II, III.1-2. Berlin: Weidmann, 1876, 1877, 1878.

Lafont, R., & G. Tournerie (eds), 1967. Francès Pellos, *Compendion de l'abaco*. Montpellier: Édition de la Revue des Langues Romanes.

Levey, Martin (ed., trans.), 1966. The *Algebra* of Abū Kāmil, *Kitāb fī al-jābr* (*sic*) *wa'l-muqābala*, in a Commentary by Mordechai Finzi. Hebrew Text, Translation, and Commentary with Special Reference to the Arabic Text. Madison etc: University of Wisconsin Press.

Malet, Antoni (ed.), 1998. Francesc Santcliment, *Summa de l'art d'Aritmètica*. Vic: Eumo Editorial.

Marre, Aristide (ed.), 1880. "Le Triparty en la science des nombres par Maistre Nicolas Chuquet Parisien". *Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche* **13**, 593–659, 693–814.

Miura, Nobuo, 1981. "The Algebra in the *Liber abaci* of Leonardo Pisano". *Historia Scientiarum* **21**, 57–65.

Pacioli, Luca, 1494. *Summa de Arithmetica Geometria Proportioni et Proportionalita*. Venezia: Paganino de Paganini.

Pacioli, Luca, 1523. *Summa de Arithmetica geometria. Proportioni: et proportionalita*. Novamente impressa. Toscolano: Paganinus de Paganino.

Pieraccini, Lucia (ed.), 1983. M° Biagio, *Chasi exenplari all regola dell'algibra nella trascelta a cura di M° Benedetto* dal Codice L. VII. 2Q della Biblioteca Comunale di Siena. (Quaderni del Centro Studi della Matematica Medioevale, 5). Siena: Servizio Editoriale dell'Università di Siena.

Rashed, Roshdi (ed., trans.), 2007. Al-Khwārizmī, *Le Commencement de l'algèbre*. (Collections Sciences dans l'histoire). Paris: Blanchard.

Rebstock, Ulrich, 1993. *Die Reichtümer der Rechner (Ġunyat al-Hussāb) von Ahmad b. Ṯabāt* *(gest. 631/1234). Die Araber – Vorläufer der Rechenkunst*. (Beiträge zur Sprach- und Kulturgeschichte des Orients, 32). Walldorf-Hessen: Verlag für Orientkunde Dr. H. Vorndran.

Rosen, Frederic (ed., trans.), 1831. The *Algebra* of Muhammad ben Musa, Edited and Translated. London: The Oriental Translation Fund.

Salomone, Lucia (ed.), 1982. M° Benedetto da Firenze, *La reghola de algebra amuchabale* dal Codice L.IV.21 della Biblioteca Comunale de Siena. (Quaderni del Centro Studi della Matematica Medioevale, 2). Siena: Servizio Editoriale dell'Università di Siena.

Sesiano, Jacques, 1984. "Une arithmétique médiévale en langue provençale". *Centaurus* **27**, 26–75.

Sesiano, Jacques (ed.), 1993. "La version latine médiévale de l'Algèbre d'Abū Kāmil", pp. 315–452 *in* M. Folkerts & J. P. Hogendijk (eds), *Vestigia Mathematica. Studies in Medieval and* Early Modern Mathematics in Honour of H. L. L. Busard. Amsterdam & Atlanta: Rodopi.

Silberberg, M. (ed., trans.), 1895. *Sefer ha-mispar. Das Buch der Zahl*, ein hebräisch-arithmetisches Werk des R. Abraham ibn Esra. Zum ersten Male herausgegeben, ins Deutsche übersetzt und erläutert. Frankfurt a.M.: J. Kaufmann.

Simi, Annalisa (ed.), 1994. Anonimo (sec. XIV), *Trattato dell'alcibra amuchabile* dal Codice Ricc. 2263 della Biblioteca Riccardiana di Firenze. (Quaderni del Centro Studi della Matematica Medioevale, 22). Siena: Servizio Editoriale dell' Università di Siena.

Simi, Annalisa, 1995. "Trascrizione ed analisi del manoscritto Ricc. 2236 della Biblioteca Riccardiana di Firenze". *Università degli Studi di Siena, Dipartimento di Matematica. Rapporto Matematico* N° 287.

Spiesser, Maryvonne (ed.), 2003. *Une arithmétique commerciale du XV^{e} siècle*. Le *Compendy de la praticque des nombres* de Barthélemy de Romans. (De Diversis artibus, 70) Turnhout: Brepols.

Ulivi, Elisabetta, 1996. "Per una biografia di Antonio Mazzinghi, maestro d'abaco del XIV secolo". *Bollettino di Storia delle Scienze Matematiche* **16**, 101-150.

Van Egmond, Warren, 1980. *Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abbacus Manuscripts and Printed Books to 1600*. (Istituto e Museo di Storia della Scienza, Firenze. Monografia N. 4). Firenze: Istituto e Museo di Storia della Scienza.

Van Egmond, Warren, 2008. "The Study of Higher-Order Equations in Italy before Pacioli", pp. 303–320 *in* Joseph W. Dauben et al (eds), *Mathematics Celestial and Terrestrial*: Festschrift für Menso Folkerts zun 65. Geburtstag. (Acta Leopoldina, 54). Halle: Wissenschaftliche Vergalsgesellschaft

Vogel, Kurt, 1977. *Ein italienisches Rechenbuch aus dem 14. Jahrhundert (Columbia X 511 A13)*. (Veröffentlichungen des Deutschen Museums für die Geschichte der Wissenschaften und der Technik. Reihe C, Quellentexte und Übersetzungen, Nr. 33). München.

Woepcke, Franz, 1853. *Extrait du Fakhrî, traité d'algèbre* par Aboû Bekr Mohammed ben Alhaçan Alkarkhî; *précédé d'un mémoire sur l'algèbre indéterminé chez les Arabes*. Paris: L'Imprimerie Impériale. Available at http://gallica.bnf.fr/.