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# Perturbation of Sectorial Projections of Elliptic Pseudo-differential Operators

Bernhelm Booß–Bavnbek, Guoyuan Chen, Matthias Lesch, and Chaofeng Zhu

ABSTRACT. Let $\mathrm{L}^m_{\mathrm{sc}}(M,E)$ denote the space of semi–classical pseudo-differential operators of order $m$, acting between sections of a Hermitian vector bundle $E$ over a closed Riemannian manifold $M$. Let $A \in \mathrm{L}^m_{\mathrm{sc}}(M,E)$ be elliptic with principal symbol $a_m$ and $m > 0$. We assume that there exist two rays $L_{\alpha_j}, j = 1,2$ with $\operatorname{spec}(a_m(x,\xi)) \cap L_{\alpha_j} = \emptyset$ for all $x \in M$ and all cotangent vectors $\xi \neq 0$. We choose an arc around zero connecting the two rays and making a path $\Gamma_+$ such that $\operatorname{spec}(A) \cap \Gamma_+ = \emptyset$, as well. Then the sectorial projection $P_{\Gamma_+}(A)$ is a well-defined bounded operator on the Sobolev spaces $H^s(M;E), s \in \mathbb{R}$. We show that $P_{\Gamma_+}(A)$ varies continuously as bounded operator in $H^s(M;E)$, if $A$ is continuously varying in a specific sense, depending on a strong topology of the leading symbol and a weaker topology of the lower order parts.

## 1. Introduction and formulation of the result

In this introductory section, we explain the goal, the background, and the place of our present work.

**1.1. The perturbation problem for sectorial projections.** We recall the general knowledge about our perturbation problem and present our new results.

1.1.1. *The bounded case.* Let $\mathcal{B}(H)$ denote the space of bounded operators in a complex separable Hilbert space $H$ and let $A \in \mathcal{B}(H)$. Assume that there exists a curve $\Gamma_+ \subset \mathbb{C} \setminus \operatorname{spec} A$ that divides $\mathbb{C}$ into two sectors $\Lambda_\pm$ as in Figure 1a. Then we can

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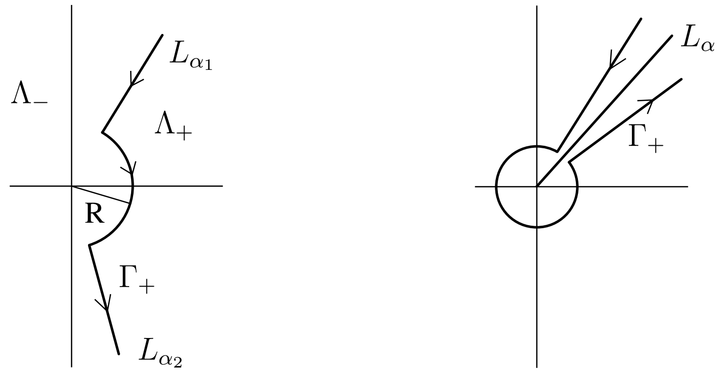


FIGURE 1. Left: *Two* rays of minimal growth and an arc, making the spectral cut curve $\Gamma_+$ in *our situation* with spectrum of $a_m$ on *both sides* $\Lambda_\pm$ of $\Gamma_+$. Right: *One* ray of minimal growth and a small arc, making the spectral cut curve $\Gamma_+$ in *Seeley's situation* with *no* spectrum of $a_m$ inside $\Gamma_+$ for $|\xi| = 1$, see [**See67**].

encircle all spectral points in the positive sector $\Lambda_+$ by a closed curve $\Gamma_0$, as in Figure 2, and so get a well-defined projection by setting

$$P_{\Gamma_+}(A) := \frac{-1}{2\pi i}\int_{\Gamma_0}(A-\lambda)^{-1}d\lambda\,. \tag{1.1}$$

If $\dim H < \infty$, then $\operatorname{spec} A$ is discrete and $P_{\Gamma_+}(A)$ is a projection onto the root spaces (generalized eigenspaces) corresponding to the eigenvalues in $\Lambda_+$.

From the integral it is clear that

$$\|P_{\Gamma_+}(A+B) - P_{\Gamma_+}(A)\| < C_A\|B\| \text{ for any small bounded perturbation } B, \tag{1.2}$$

i.e., the map $P_{\Gamma_+} : A \mapsto P_{\Gamma_+}(A)$ is continuous in the operator norm of $\mathcal{B}(H)$.

1.1.2. *Spectral projections of self-adjoint elliptic operators on closed manifolds.* Self-adjoint elliptic operators on closed manifolds have a discrete spectrum of finite multiplicity contained in $\mathbb{R}$ and a complete set of eigenvectors. Then the imaginary axis (or a parallel $\{c + ri \mid r \in \mathbb{R}\}$ with $c \notin \operatorname{spec} A$) becomes a suitable separating curve $\Gamma_+$ and we obtain $P_{\Gamma_+}(A) = 1_{[c,\infty)}(A) = 1_{[F(c),\infty)}(F(A))$ as a pseudo-differential projection by applying the integral representation of (1.1) to the bounded Riesz transform $F(A) := (I + A^2)^{-1/2}A$ of $A$. Note that $F(A)$ has its spectrum contained in the interval $(-1, 1)$, but has the same eigenspaces and sectorial projection as $A$. We refer to our [**BBLZ09**, Propositions 7.14-7.15] (see also [**BBFu98**, Thm. 4.8] for a wider purely functional analytic setting) for a proof of the continuity of the Riesz transformation $A \mapsto F(A)$ on the space of self-adjoint elliptic differential operators. That yields the

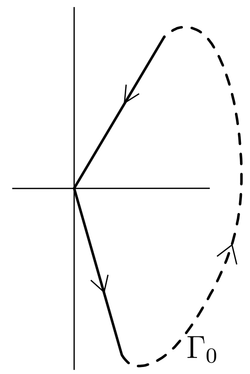


FIGURE 2. Specifying a bounded set of eigenvalues by a separating curve $\Gamma_+$ made of two rays and capturing it by a closed contour $\Gamma_0$

continuity of the map $A \mapsto P_{\Gamma_+}(A)$, when we take the operator norm $L^2 \to L^2$ for $P_{\Gamma_+}(A)$ and the operator norm $H^m \to L^2$ for $A$, where $m$ denotes the order of $A$.

1.1.3. *Seeley's approach to spectral integrals for elliptic pseudo-differential operators of positive order.* As explained in our [**BBLZ09**, Section 3.2], a semigroup $\{Q_+(x,A)\}_{x>0}$ of sectorial operators can be defined by inserting a weight $e^{-\lambda x}$ into the integral (1.1). Then *sectorial projections* can be defined asymptotically in an abstract Hilbert space framework. More precisely, for a closed, not necessarily self-adjoint operator $A$ in separable Hilbert space with compact resolvent and minimal growth of the resolvent in a cone we may take the closure of the densely defined $\lim_{x\to 0+} Q_+(x,A)$. However, such projections are unbounded operators, in general, and do not necessarily vary continuously under perturbation of the underlying operator, see, in particular, [**BBLZ09**, Example 3.13]. Consequently, we shall apply much deeper analysis and exploit the symbolic calculus for the investigation of sectorial projections of not necessarily self-adjoint elliptic pseudo-differential operators of positive order with two rays of minimal growth of the resolvent.

Like our predecessors BURAK [**Bur70**], WOJCIECHOWSKI [**Woj85**] and NAZAIKINSKII ET AL. [**NSSS98**] (for a recent presentation see also PONGE [**Pon06**] and our [**BBLZ09**]), we adapt R. T. SEELEY's methods of symbolic calculus from his study of the complex powers of an elliptic pseudo-differential operator $A$ of positive order $m$, [**See67**]. Roughly speaking, Seeley replaced the resolvent $(A-\lambda)^{-1}$ by a parametrix $B(\lambda)$ defined by the symbol $b(\lambda) := (a_m - \lambda)^{-1}$, possibly after a slight deformation of $a_m$ to avoid singularities. Then, exploiting the symbolic calculus yields sharp estimates and permits the integration of $\lambda^{-1} B(\lambda)$ over $\Gamma_+$.

Unfortunately (and contrary to defective arguments in [**Woj85**], [**NSSS98**] and [**Pon06**]), this "slight deformation of $a_m$ to avoid singularities" is not possible in general for two rays of minimal growth due to topological obstructions (see Section 3.1 below). To obtain our *perturbation results*, we need, anyway, a more precise recapitulation of Seeley's method and some sharper estimates at some points than the preceding references. As a side result, that permits us to repair the defective arguments of the three mentioned papers. Hence, their results and theorems building on them (like the recent GAARDE and GRUBB [**GaGr08**, Theorem 4.6]) remain valid. As we shall show, some of the necessary symbolic calculus and the required estimates of the approximations have been provided already by Seeley in germ.

**1.2. The geometric meaning of sectorial projections and uniform structures.** We point to various geometric contexts where sectorial projections and uniform structures play an important role. A common set-up is the following: Let $X$ be a compact smooth Riemannian manifold with boundary $M$, and $E$ and $F$ be two Hermitian vector bundles over $X$. Let $D : H^1(X; E) \to L^2(X; F)$ be a first order elliptic differential operator and let $A : H^1(M; E|_M) \to L^2(M; E|_M)$ denote the tangential operator of $D$ on $M$ relative to the fixed metric structures.

1.2.1. *Index correction formulas.* The sectorial projections are significant in the celebrated Atiyah-Patodi-Singer Index Theorem. In the classical works [**APS75a, APS75b, APS76**], M. ATIYAH, V. PATODI AND I. SINGER assumed that $D$ is of Dirac type; all metric structures near $M$ are product; hence, the coefficients of $D$ in normal direction close to $M$ are constant and the tangential operator $A$ is self-adjoint. Imposing a spectral projection condition $P^+(A)u|_{\partial X} = 0$ on the boundary, they proved that the resulting (densely defined) operator $D_{P^+(A)}$ over $X$ is Fredholm. Furthermore, they gave an index formula, comprising topological, spectral and differential terms. The arguments of [**APS76**, p. 95] (worked out in detail in [**LMP10**, Thm. 7.6]) lead to the index correction formula

$$\operatorname{ind}(D_0)_{P^+(A_0)} - \operatorname{ind}(D_1)_{P^+(A_1)} = \operatorname{sf}\{A_t\}_{t\in[0,1]}, \tag{1.3}$$

where $\{D_t,\, t \in [0, 1]\}$ is a smooth homotopy, and $\{A_t\}$ denotes its corresponding family of tangential operators. It is also called the *Spectral Flow Theorem*. The continuous dependence of $P^+(A_t)$ on $A_t$ (in the sense that $P^+(A_t)$ has the same jumps as $1_{(-\varepsilon,\varepsilon)}(A_t)$, if $\pm\varepsilon \notin \operatorname{spec} A_t$) is important in this theorem. When $A_t$ is self-adjoint, it can be proved by standard techniques of functional analysis (cf. [**BBWo93**, Chapter 17]).

It is natural to consider a more general case. In [**SSS99**], A. SAVIN, B.-W. SCHULZE and B. STERNIN gave a similar formula for the case that the tangential family $A_t$ is non-self-adjoint. However, they did not give the details of a proof of the continuous dependence of $P^+(A_t)$ on $A_t$ when $A_t$ has no spectral points on the imaginary axis for all $t \in [0, 1]$.

1.2.2. *Continuous variation of the Calderón projection.* In [**BBFu98**, Theorem 3.8], K. Furutani and the first present author obtained the continuous variation of the Cauchy data spaces in a purely functional analytic setting for closed symmetric operators in Hilbert space admitting a self-adjoint Fredholm extension and inner unique continuation property - under the condition of bounded perturbation. In [**BBLZ09**], three of the present authors elaborated an alternative approach. They gave a new definition of Calderón projections also in the case of not necessarily symmetric tangential operators. To obtain the continuous variation of the Cauchy data spaces for a curve of elliptic operators on a compact manifold with smooth boundary, the continuous variation of the Calderón projection had to be established in that generality. In loc.cit., Theorem 7.2, it was shown that the continuous variation of the Calderón projection follows from the continuous variation of the positive sectorial projections. Recall: the principal symbol of the tangential operator $A$ over $M = \partial X$ of an elliptic operator $D$ of first order over $X$ has no purely imaginary eigenvalues. Hence, the imaginary axis forms two natural rays of minimal growth.

More recently, two of us sketched a research program in [**BBLe09**] for confining, respectively closing, the last remaining gaps between the geometric Dirac operator type situation and the general linear elliptic case. We listed some problems and conjectures in the last part of that paper. Now, in the present paper, we shall solve the first of these problems (Problem 4.1 in loc.cit.).

**1.3. Main result.** Before presenting our precise definitions, estimates and applications, we shall point to various characteristic challenges to overcome with our approach.

1.3.1. *Formulation of the main result.* Let $M$ be an $n$-dimensional closed Riemannian manifold and $\pi : E \to M$ a Hermitian vector bundle. Let $A : C^\infty(M; E) \to C^\infty(M; E)$ be a semi–classical elliptic pseudo-differential operator of order $m > 0$. See Subsections 2.1, 2.2 and 2.3 for a short review of the notations regarding various algebras of pseudo-differential operators and their symbols.

To begin with, we recall that the spectrum of $A$ regarded as an operator in $L^2(M; E)$ with the Sobolev space $H^m(M; E)$ as its domain, is clearly either the whole complex plane or a discrete subset of $\mathbb{C}$. The reason is simply that the resolvent, if it exists, is compact (see Shubin [**Shu01**, Theorem 8.4], similarly already in Agmon [**Agm62**, Section 2] for well-posed elliptic boundary value problems). Clearly, $\operatorname{ind} A \neq 0$ implies $\operatorname{spec} A = \mathbb{C}$.

Let $L_{\alpha_1} = \{\lambda \in \mathbb{C} \mid \arg \lambda = \alpha_1\}$ and $L_{\alpha_2} = \{\lambda \in \mathbb{C} \mid \arg \lambda = \alpha_2 \equiv \alpha_1 - \theta \mod 2\pi\}$ $(0 < \theta < 2\pi)$ be two rays. We assume that the principal symbol $a_m(x, \xi)$ of $A$ has no eigenvalues on the rays $L_{\alpha_j}, j = 1, 2$ for each point $x \in M$ and covector $\xi \in T^*_x M, \xi \neq 0$.

Let $\Lambda := \{re^{i\alpha} \mid r < 2\rho \text{ or } |\alpha - \alpha_j| < \varepsilon, j = 1, 2\}$ for $\rho, \varepsilon > 0$ and $\varepsilon$ sufficiently small. We can choose $\rho$ in such a way that there exists an $R \in [0, \rho]$ such that $A - \lambda$ is invertible for $\lambda \in \Lambda$ with $|\lambda| \geq R$, and there is only a finite number of eigenvalues in the region $\Lambda_R := \{\lambda \in \Lambda \mid |\lambda| < R\}$. For an elaboration of the meaning of such spectral cuttings, also called *rays of minimal growth* (of the resolvent $(A-\lambda)^{-1}$), see Subsection 2.4 below. If $A$ is differential, then $A-\lambda$ is *elliptic with respect to the parameter* $\lambda \in \Lambda$ for sufficiently small $\rho, \varepsilon > 0$ (for that concept c.f. [**See67**] or [**Shu01**]).

Now we choose the curve

$$\Gamma_+ = \left\{re^{i\alpha_1} \mid \infty > r \geq R\right\} \cup \left\{Re^{i(\alpha_1 - t)} \mid 0 \leq t \leq \theta\right\} \cup \left\{re^{i\alpha_2} \mid R \leq r < \infty\right\} \tag{1.4}$$

in the resolvent set of $A$, see Figure 1a.

We define an operator in the following form:

$$P_{\Gamma_+}(A) = -\frac{1}{2\pi i} A \int_{\Gamma_+} \lambda^{-1}(A-\lambda)^{-1} d\lambda. \tag{1.5}$$

Let $\mathrm{Ell}^m_{\Gamma_+}(M,E)$ denote the space of all elliptic *semi-classical* (see below Section 2.3) pseudo-differential operators $A$ of order $m > 0$ on $M$ acting on sections of the bundle $E$ such that the leading symbol $a_m$ of $A$ has no eigenvalues on the two rays $L_{\alpha_j}, j = 1,2$ and $A$ no eigenvalues on the small arc between the two rays. We equip the space $\mathrm{Ell}^m_{\Gamma_+}(M,E)$ with the locally convex topology $\mathcal{T}$ induced by continuous variation of the principal symbol and all its derivatives and continuous variation of the lower order symbol (the difference between the total and the principal symbol), made precise in Section 6.2 below. The following theorem is our main result:

**Theorem 1.1.** (a) *For each $A \in \mathrm{Ell}^m_{\Gamma_+}(M,E)$ the operator $P_{\Gamma_+}(A)$ is well defined by* (1.5) *as a bounded operator on $H^s(M;E), s \in \mathbb{R}$.*
(b) *The set $\mathrm{Ell}^m_{\Gamma_+}(M,E)$ is open in $\mathcal{B}(H^s(M;E))$ and the map*

$$P_{\Gamma_+} : \mathrm{Ell}^m_{\Gamma_+}(M,E) \to \mathcal{B}(H^s(M;E)), \quad A \mapsto P_{\Gamma_+}(A) \tag{1.6}$$

*is continuous. Here $\mathcal{B}(H^s(M;E))$ denotes the set of bounded linear operators on $H^s(M;E), s \in \mathbb{R}$.*

1.3.2. *Challenges met.* Recall that in the non-self-adjoint case, even an elliptic differential operator $A$ may not have a spectral decomposition. For a discussion of this issue and its history see [**Pon06**, Section 3 and Appendix]. See also Seeley [**See86**] and Agranovich and Markus [**AgMa89**] for simple examples of elliptic differential operators without a complete set of root vectors. In these examples, however, the principal symbols do not admit a spectral cutting. Our $A$ has a spectral cutting. Unfortunately, as explained above, this is not enough to obtain a positive result by purely functional analytic methods alone (see also [**Pon06**, Appendix] for additional details). While the break-down of the functional analytic method forces us to exploit special features of the concrete cases, the delicacy of A. Axelsson, S. Keith and A. McIntosh [**AKM06**] indicates that there is no easy way through to be expected . They studied the Hodge-Dirac operator $D_g$ defined on a closed Riemannian manifold with metric $g$. In general, $D_g$ is non-self-adjoint, and its spectrum is contained in an open double sector which includes the real line. They showed – by harmonic analysis methods – that the spectral projections of the Hodge-Dirac operator $D_g$ depend analytically on $L_\infty$ changes in the metric $g$.

Basically, we have to overcome three difficulties: Firstly, we have $L^2(M;E) \neq \overline{\sum_{\lambda\in\operatorname{spec}(A)} E_\lambda}$ in general.

Secondly, Seeley deals only with one ray of minimal growth. In our situation, we have *two rays of minimal growth*, meeting and overcoming certain topological obstructions. To apply the symbolic calculus, one would wish that $a(x,\xi)$ has no eigenvalues on the whole of $\Gamma_+$, or that it can be deformed to a symbol with that property. That can be done in Seeley's case, but not in our case. More precisely, both in Seeley's case and in our case, the path $\Gamma_+$ separates $\mathbb{C}$ into two regions $\Lambda_\pm$. In Seeley's case, however, we have $\operatorname{spec} a_m(x,\xi) \cap \Lambda_+ = \emptyset$ for $|\xi|$ sufficiently large. Note that the space of matrices $\{a \mid \operatorname{spec}(a) \cap \Lambda_+ = \emptyset\}$ is contractible. Consequently, $a_m$ can be continuously extended and deformed for $|\xi| < 1$ such that $\operatorname{spec} a_m(x,\xi) \cap \Lambda_+ = \emptyset$ is maintained. On the other hand, in our case we have the spectrum of $a_m$ on both sides $\Lambda_+$ and $\Lambda_-$ of the curve $\Gamma_+$. So, topological obstructions (seemingly not noticed by the quoted references) do not allow such extension, respectively deformation in our case, in general. The reason is that the corresponding space of square matrices is not contractible, see also Figure 1b and compare to Figure 1a. Differently put, we must take into regard that there is no continuous family $\{P_t\}_{t\in[0,1]}$ of projections with $P_0 = I$ and $P_1 \neq I$. In Section 3.1, we elaborate on the topological obstruction.

Thirdly, and most demanding, a priori, the variational properties of the symbol do not suffice for establishing the continuous variation of $P_{\Gamma_+}(A)$ in the topology of the operator norm of a suitable Sobolev space. A smoothing operator may have a large operator norm defined on any Sobolev space. Therefore, our approach requires slightly sharper estimates than Seeley's original work.

**1.4. The idea of the proof of our main theorem.** Now we shall give the idea of the proof of Theorem 1.1.

While the resolvent $(A-\lambda)^{-1}$ is well defined on $\Gamma_+$, the symbol $a_m - \lambda$ can vanish, by definition not on the rays, but by homogeneity of the symbol on the circular arc with radius $R$ for small $|\xi|$. That is, there is no arc of radius $R$, no matter how small $R$ is chosen, where we can be sure that no $a_m(x,\xi)$ has an eigenvalue - in spite of the regularity of $a_m(x,\xi)$ for $\xi \neq 0$. Neither is it possible, in general, to deform $a_m(x,\xi)$ for $|\xi| < 1$ in a continuous way so that no eigenvalues remain on $\Gamma_+$, by topological obstruction (to be explained in Section 3.1 below). Unfortunately, that topological obstruction was disregarded in [**Woj85**], [**NSSS98**] and [**Pon06**].

Therefore, we must follow a slightly different path. In Section 3.2, we follow [**Shu01**, Sections 11.3-11.4] and define a smoothed resolvent symbol $\psi(\xi)\,(a_m(x,\xi) - \lambda)^{-1}$

bounded away from $R$ in a suitable way, for instance, with a cut-off function

$$\psi(\xi) = \begin{cases} 1, & \text{for } |\xi| \text{ large}, \\ 0, & \text{for } |\xi| < 1. \end{cases}$$

Then our approach builds on the following sequence of constructions, replacements, and estimates:

I. In Section 2.3, we investigate the space $\mathrm{L}^m_{\mathrm{sc}}(M,E)$ of semi-classical pseudo-differential operators and establish an exact sequence

$$0 \longrightarrow \mathrm{L}^{m-1}(M,E) \hookrightarrow \mathrm{L}^m_{\mathrm{sc}}(M,E) \xrightarrow{\sigma_m} C^\infty(S^*M; \mathrm{End}(\pi^*E)) \longrightarrow 0.$$

In Section 6.2, we describe a locally convex topology $\mathcal{T}$ on $\mathrm{Ell}^m_{\Gamma_+}(M,E) \subset \mathrm{L}^m_{\mathrm{sc}}(M,E)$ derived from this exact sequence.

II. In Section 2.5, we factorize

$$P_{\Gamma_+}(A) = \frac{A}{-2\pi i}\,\Phi(A) \quad \text{with } \Phi(A) := \int_{\Gamma_+} \lambda^{-1}(A-\lambda)^{-1}\,d\lambda \tag{1.7}$$

for better transparency. The operator $\Phi(A)$ is well defined as a bounded operator $H^s(M;E) \to H^s(M;E)$ on all Sobolev spaces, $s \in \mathbb{R}$. Indeed, we have $\|(A-\lambda)^{-1}\|_{s,s} \le C|\lambda|^{-1}$ by assumption (ray of minimal growth). Here, $\|\cdot\|_{s,s}$ denotes the norm on the space $\mathcal{B}(H^s,H^s)$ of bounded operators from the Sobolev space $H^s(M;E)$ into itself.

III. Consequently, $\|\Phi(A)\|_{s,s} \le C\int_{\Gamma_+}|\lambda|^{-2}\,|d\lambda|$. Estimating a bit more carefully, one sees

$$\|\Phi(A)\|_{s,s+\delta} \le C\int_{\Gamma_+}|\lambda|^{-2+\delta/m}\,|d\lambda|.$$

The integral on the right side converges for $\delta < m$. Hence, that does not *quite* suffice to show that

$$P_{\Gamma_+}(A) : H^s(M;E) \xrightarrow{\Phi(A)} H^{s+\delta}(M;E) \xrightarrow{A} H^{s+\delta-m}(M;E) \tag{1.8}$$

is bounded $H^s \to H^s$. To do that and to investigate its dependence on $A$, we will prove in Sections 3-5 an array of estimates in the *symbolic calculus* that may be of independent interest.

IV. We replace the true resolvent $(A-\lambda)^{-1}$ by a parametrix

$$\mathrm{Op}\Big(\psi(\cdot)\big(a_m(\cdot,\cdot)-\lambda\big)^{-1}\Big)$$

given by the symbolic calculus and obtain (in Lemma 5.2) the estimate

$$\|\,\mathrm{Op}\Big(\psi(\cdot)\big(a_m(\cdot,\cdot)-\lambda\big)^{-1}\Big) - (A-\lambda)^{-1}\|_{s,s+m} \le C|\lambda|^{-\min(\frac{1}{m},1)} \tag{1.9}$$

for $\lambda \in \Gamma_+$. Here, similarly as above, $\|\cdot\|_{s,s+m}$ denotes the norm on the space $\mathcal{B}(H^s, H^{s+m})$ of bounded operators from the Sobolev space $H^s(M;E)$ to $H^{s+m}(M;E)$.

V. In Proposition 3.1a, we approximate the factor $\Phi(A)$, defined in (1.7), by

$$\begin{aligned}\Phi_0(A) &:= \int_{\Gamma_+} \lambda^{-1} \operatorname{Op}\Big(\psi(\cdot)\big(a_m(\cdot,\cdot)-\lambda\big)^{-1}\Big) d\lambda \\ &= \operatorname{Op}\Big(\int_{\Gamma_+} \lambda^{-1}\, \psi(\cdot)\big(a_m(\cdot,\cdot)-\lambda\big)^{-1}\, d\lambda\Big) : H^s \to H^{s+m}. \quad (1.10)\end{aligned}$$

Note that $\int_{\Gamma_+} \lambda^{-1}\, \psi(\cdot)\big(a_m(\cdot,\cdot)-\lambda\big)^{-1}\, d\lambda$ is a $(-m)$th order homogeneous symbol for large $|\xi|$. That is the easy part. It involves only the leading symbol $a_m$, its derivatives and the cut-off function.

VI. The main part of our paper is devoted to investigate the error term

$$\Phi_0(A) - \Phi(A) = \int_{\Gamma_+} \lambda^{-1}\Big( \operatorname{Op}\Big(\psi(\cdot)\big(a_m(\cdot,\cdot)-\lambda\big)^{-1}\Big) - (A-\lambda)^{-1}\Big)\, d\lambda\,. \quad (1.11)$$

Our applications will follow from our composition formulas for symbolic calculus Lemmata 4.2, 5.1, 5.2. In Section 6.3, they yield the estimate

$$\begin{aligned}&\| \operatorname{Op}\Big(\psi(\cdot)\big(\widetilde{a}_m(\cdot,\cdot)-\lambda\big)^{-1}\Big) - (\widetilde{A}-\lambda)^{-1} \\ &\qquad - \operatorname{Op}\Big(\psi(\cdot)\big(a_m(\cdot,\cdot)-\lambda\big)^{-1}\Big) + (A-\lambda)^{-1}\|_{s,s+m} \le \varepsilon|\lambda|^{-\min(\frac{1}{m},1)}\,, \quad (1.12)\end{aligned}$$

where $\widetilde{A}$ is a perturbation of $A$ with principal symbol $\widetilde{a}_m$. More precisely, for fixed $A \in \mathrm{Ell}^m_{\Gamma_+}(M,E)$ and $\varepsilon > 0$ there is an open neighborhood $U_\varepsilon$ of $A$ with regard to the topology $\mathcal{T}$ such that (1.12) holds for $\widetilde{A} \in U_\varepsilon$.

## 2. Definitions and notations

To fix the notation, we recall the basic concepts of symbolic calculus and introduce semi-classical symbols and semi-classical pseudo-differential operators on closed manifolds. For elliptic semi-classical pseudo-differential operators of positive order and for a fixed contour $\Gamma_+$ we define the sectorial projections and discuss the natural factorization.

**2.1. Classes of symbols.** Let $U \subset \mathbb{R}^n$ be an open subset. We denote by $\mathrm{S}^m(U \times \mathbb{R}^n)$, $m \in \mathbb{R}$, the space of (complex valued) symbols (the generalization for matrix valued symbols is straight forward) of Hörmander type $(1,0)$ (Hörmander [**Hör71**], Grigis–Sjöstrand [**GrSj94**]). More precisely, $\mathrm{S}^m(U \times \mathbb{R}^n)$ consists of those $a \in \mathrm{C}^\infty(U \times \mathbb{R}^n)$

such that for multi–indices $\alpha, \gamma \in \mathbb{Z}_+^n$ and compact subsets $K \subset U$ we have an estimate

$$\left|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi)\right| \le C_{\alpha,\gamma,K}(1+|\xi|)^{m-|\gamma|}, \quad x \in K. \tag{2.1}$$

The best constants in (2.1) provide a set of semi-norms which endow $\mathrm{S}^\infty(U\times\mathbb{R}^n) := \bigcup_{m\in\mathbb{R}} \mathrm{S}^m(U\times\mathbb{R}^n)$ with the structure of a Fréchet algebra.

The space $\mathrm{CS}^m(U\times\mathbb{R}^n)$ of *classical symbols* consists of all $a \in \mathrm{S}^m(U\times\mathbb{R}^n)$ that admit sequences $a_{m-j} \in \mathrm{C}^\infty(U\times\mathbb{R}^n), j \in \mathbb{Z}_+$ with

$$a_{m-j}(x, r\xi) = r^{m-j} a_{m-j}(x,\xi), \quad r \ge 1, |\xi| \ge 1, \tag{2.2}$$

such that

$$a - \sum_{j=0}^{N-1} a_{m-j} \in \mathrm{S}^{m-N}(U\times\mathbb{R}^n) \quad \text{for all } N \in \mathbb{Z}_+. \tag{2.3}$$

The latter property is usually abbreviated $a \sim \sum_{j=0}^{\infty} a_{m-j}$.

Homogeneity and smoothness at 0 contradict each other except for monomials. Our convention is that symbols should always be smooth functions, thus the $a_{m-j}$ are smooth everywhere but homogeneous only in the restricted sense of Eq. (2.2).

Furthermore, we denote by $\mathrm{S}^{-\infty}(U\times\mathbb{R}^n) := \bigcap_{a\in\mathbb{R}} \mathrm{S}^a(U\times\mathbb{R}^n)$ the space of *smoothing symbols.*

**2.2. (Classical) pseudo-differential operators.** We fix our notation for the various algebras of pseudo-differential operators and recall the elements of the corresponding symbolic calculus.

Let $M$ be a smooth manifold of dimension $n$. For convenience and to have an $L^2$–structure at our disposal, we assume that $M$ is equipped with a Riemannian metric. We denote by $\mathrm{L}^\bullet(M)$ the algebra of *pseudo-differential operators* with symbols of Hörmander type $(1,0)$ ([**Hör71**], [**Shu01**]), see Subsection 2.1. The subalgebra of *classical pseudo-differential operators* is denoted by $\mathrm{CL}^\bullet(M)$. These operator algebras are naturally defined on the manifold $M$ by localizing in coordinate patches in the following way:

Let $U \subset \mathbb{R}^n$ be an open subset. Recall that for a symbol $a \in \mathrm{S}^m(U\times\mathbb{R}^n)$, the *canonical* pseudo-differential operator *associated* to $a$ is defined by

$$\begin{aligned}\big(\mathrm{Op}(a)\,u\big)(x) &:= \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\, a(x,\xi)\,\hat{u}(\xi)\, đ\xi \\ &= \int_{\mathbb{R}^n}\int_U e^{i\langle x-y,\xi\rangle}\, a(x,\xi)\, u(y) dy đ\xi,\end{aligned} \qquad đ\xi := (2\pi)^{-n} d\xi. \tag{2.4}$$

For a manifold $M$, elements of $\mathrm{L}^\bullet(M)$ (resp. $\mathrm{CL}^\bullet(M)$) can locally be written as $\mathrm{Op}(\sigma)$ with $\sigma \in \mathrm{S}^\bullet(U\times\mathbb{R}^n)$ (resp. $\mathrm{CS}^\bullet(U\times\mathbb{R}^n)$).

Recall that there is an exact sequence

$$0 \longrightarrow \mathrm{CL}^{m-1}(M) \hookrightarrow \mathrm{CL}^m(M) \xrightarrow{\sigma_m} C^\infty(S^*M) \longrightarrow 0\,, \tag{2.5}$$

where $\sigma_m(A)$ denotes the *principal* (homogeneous leading) symbol of $A \in \mathrm{CL}^m(M)$. Here, $S^*M$ denotes the cosphere bundle, i.e., the unit sphere bundle $\subset T^*M$. As usual, the principal symbol is locally defined as a map $\sigma_m : \mathrm{S}^m(U \times \mathbb{R}^n) \to C^\infty(U \times S^{n-1})$ by putting

$$\sigma_m(x,\xi) := \lim_{r\to\infty} r^{-m} a(x, r\xi). \tag{2.6}$$

Note that $\sigma_m(A)$ is a homogeneous function on the symplectic cone $T^*M \setminus M$. We will tacitly identify the homogeneous functions on $T^*M \setminus M$ by restriction with $\mathrm{C}^\infty(S^*M)$.

Recall that the principal symbol map is multiplicative in the sense that

$$\sigma_{a+b}(A \circ B) = \sigma_a(A)\sigma_b(B) \tag{2.7}$$

for $A \in \mathrm{CL}^a(M), B \in \mathrm{CL}^b(M)$.

**2.3. Semi–classical symbols and operators.** As mentioned in the Introduction, continuous variation of the operator $A$ by bounded $L^2 \to L^2$ perturbation is sufficient to obtain continuous variation of the Cauchy data space, of the Calderón projection and of the sectorial projection in various cases (see [**BBFu98, BBLZ09**]). However, we have a hunch that continuous variation of the operator $A$ in the operator norm, say from $H^m(M)$ to $L^2(M)$ will not always yield continuous variation of the sectorial projection $P_{\Gamma_+}(A)$ in the operator norm from $L^2(M)$ to $L^2(M)$. These are our intuitive arguments:

We know that general functional analysis does not suffice to obtain the boundedness of the sectorial projection. The more refined structure of differential or pseudo-differential operators is required. Apparently, for variation in the highest order (lower order variation can be treated by spectral theory as shown in [**BBFu98**, Theorem 3.8] and [**BBLZ09**, Proposition 7.13]), the principal symbol must be singled out. All that indicates that variation in the operator norm hardly will suffice for continuous variation of the sectorial projection. More is required. But what? Not necessarily so much as in the delicate estimates of [**AKM06**] for the Hodge-Dirac operator.

As we shall see below, we must be rather *restrictive* regarding the topology of the *leading terms* of the perturbation. Fortunately, much less will suffice for the lower order terms of the perturbation. This corresponds to our mentioned previous results that the sectorial projection depends continuously on lower order variations in the respective operator norm. Consequently, we shall *weaken* the continuity demands for *lower order* perturbations also here.

A suitable way to do that is by working with algebras of semi–classical pseudo-differential operators, where the principal symbol is singled out, but all the lower order terms are treated in bulk:

Let $U \subset \mathbb{R}^n$ be an open subset. Let $\mathrm{S}^m_{\mathrm{sc}}(U \times \mathbb{R}^n)$ denote the space of *semi–classical symbols* consisting of all $a \in \mathrm{S}^m(U \times \mathbb{R}^n)$ such that there is $a_m \in C^\infty(U \times \mathbb{R}^n)$ with

$$a_m(x, r\xi) = r^m a_m(x,\xi) \quad \text{for} \quad r \geq 1, |\xi| \geq 1$$

and such that

$$a - a_m \in \mathrm{S}^{m-1}(U \times \mathbb{R}^n).$$

**Lemma 2.1.** *There is a (non-canonically) isomorphism*

$$\mathrm{S}^m_{\mathrm{sc}}(U \times \mathbb{R}^n) \cong C^\infty(U \times S^{n-1}) \oplus \mathrm{S}^{m-1}(U \times \mathbb{R}^n).$$

Proof. Let $a \in \mathrm{S}^m_{\mathrm{sc}}(U \times \mathbb{R}^n)$. By (2.6), we have $\sigma_m(a) \in C^\infty(U \times S^{n-1})$. Fix a cut-off function $\psi \in C^\infty(\mathbb{R}^n)$ with

$$\psi(\xi) = \begin{cases} 0, & |\xi| \leq 1/4, \\ 1, & |\xi| \geq 1/2. \end{cases}$$

Then the map

$$(x,\xi) \longmapsto a(x,\xi) - \psi(\xi)\sigma_m(a)(x, \frac{\xi}{|\xi|})|\xi|^m =: \pi_{m-1}(a)(x,\xi)$$

is in $\mathrm{S}^{m-1}(U \times \mathbb{R}^n)$. The map

$$\begin{aligned} \mathrm{S}^m_{\mathrm{sc}}(U \times \mathbb{R}^n) &\to C^\infty(U \times S^{n-1}) \oplus \mathrm{S}^{m-1}(U \times \mathbb{R}^n), \\ a &\longmapsto \sigma_m(a) \oplus \pi_{m-1}(a) \end{aligned} \tag{2.8}$$

is invertible with inverse

$$(f,g) \longmapsto \big((x,\xi) \longmapsto \psi(\xi) f(x, \frac{\xi}{|\xi|})|\xi|^m + g(x,\xi)\big). \qquad \square$$

We equip $\mathrm{S}^m_{\mathrm{sc}}(U \times \mathbb{R}^n)$ with the Fréchet topology induced by the isomorphism constructed in the proof of the lemma. The topology is obviously independent of $\psi$.

Semi–classical operators on manifolds are now defined in the usual way. To check that this notion is coordinate invariant is standard and we omit the details.

Let $\mathrm{L}^m_{\mathrm{sc}}(M)$ denote the algebra of semi–classical pseudo-differential operators on $M$. The principal symbol map

$$\sigma_m : \mathrm{L}^m_{\mathrm{sc}}(M) \to C^\infty(S^*M)$$

is invariantly defined and in view of Lemma 2.1 and (2.5), we have an exact sequence

$$0 \longrightarrow \mathrm{L}^{m-1}(M) \hookrightarrow \mathrm{L}^m_{\mathrm{sc}}(M) \xrightarrow{\sigma_m} C^\infty(S^*M) \longrightarrow 0. \tag{2.9}$$

Recall the basic inclusions $\mathrm{CL}^m(M) \hookrightarrow \mathrm{L}^m_{\mathrm{sc}}(M) \hookrightarrow \mathrm{L}^m(M)$ and note that on the left of (2.9) we have the inclusion of the "usual" pseudo-differential operators of order $m-1$ into $\mathrm{L}^m_{\mathrm{sc}}(M)$.

**Remark 2.2.** It is straight–forward to check that the preceding definitions and observations carry over to matrix valued symbols and, correspondingly, to algebras of pseudo-differential operators acting on sections of vector bundles over smooth manifolds. We omit the details. Adjusted due to the presence of a vector bundle $E$, the exact sequence (2.9) will define the locally convex topology $\mathcal{T}$ on $\mathrm{L}^m_{\mathrm{sc}}(M,E)$ in Section 6.2 below.

**2.4. The definition of $P_{\Gamma_+}(A)$.** Formally, our definition of sectorial projections of elliptic operators may resemble the definitions given in Burak [**Bur70**], Wojciechowski [**Woj85**] and Nazaikinskii et al [**NSSS98**] or more recently in Ponge [**Pon06**]. However, [**Bur70**] is in a different context (studying well-posed boundary value problems). Moreover, as mentioned in the Introduction and detailed below in Subsection 3.1, [**Woj85**] is defective by ignoring a topological obstruction (unfortunately that error was reproduced in [**NSSS98**] and [**Pon06**]). Therefore, we shall give our definition in some detail. These details will be decisive for proving the perturbation results, as well.

2.4.1. *Our data.* Let $M$ be an $n$-dimensional closed smooth manifold and $\pi : E \to M$ a smooth complex vector bundle of fibre dimension $N$. Fix a Hermitian metric on $E$ and a Riemannian metric on $M$. So we can form the Sobolev spaces $H^s(M;E), s \in \mathbb{R}$ as usual.

Let $A \in \mathrm{L}^m_{\mathrm{sc}}(M,E)$ be elliptic and of order $m > 0$. For simplicity, denote the principal symbol $\sigma^m_A(x,\xi)$ by $a_m = a_m(x,\xi)$.

**Definition 2.3.** Let $0 \le \alpha < 2\pi$ and $L_\alpha := \{\lambda \in \mathbb{C} \mid \arg \lambda = \alpha\}$. We say that the ray $L_\alpha$ is a *ray of minimal growth* for $A$, if the principal symbol $a_m$ of $A$ has no spectral points on $L_\alpha$.

Every ray of minimal growth has a cone-shaped neighborhood $\Lambda$ such that any ray contained in $\Lambda$ is also a ray of minimal growth for $A$. Then there exists $R > 0$ such that $A - \lambda$ is invertible for $\lambda \in \Lambda$, $|\lambda| > R$, and we have

$$\|(A-\lambda)^{-1}\|_{s,s+p} \le C|\lambda|^{-1+\frac{p}{m}}, \qquad 0 \le p \le m,\ s \in \mathbb{R}, \tag{2.10}$$

for any such $\lambda$. Here, we use the following convention which will be in effect for the rest of the paper:

**Convention 2.4.** We denote the norm on the space $\mathcal{B}(H^s, H^t)$ of bounded operators from the Sobolev space $H^s(M;E)$ to $H^t(M;E)$ by $\|\cdot\|_{s,t}$.

For the proof of (2.10) see [**See67**, Corollary 1]. For differential operators see also [**Shu01**, Theorem 9.3]. Equation (2.10) explains the common usage of "ray of minimal growth of the resolvent" for such spectral cutting rays.

Assume that $L_{\alpha_1} = \{\lambda \in \mathbb{C} \mid \arg \lambda = \alpha_1\}$ and $L_{\alpha_2} = \{\lambda \in \mathbb{C} \mid \arg \lambda = \alpha_2\}$ are two rays of minimal growth with $\alpha_1 < \alpha_2 < \alpha_1 + 2\pi$.

Let $\Gamma_+$ denote the curve in the resolvent set of $A$ specified in (1.4). Recall that $R$ is a positive constant such that $A - \lambda$ is invertible for all $\lambda \in L_{\alpha_1} \cup L_{\alpha_2}$ with $|\lambda| \geq R$. Since $\operatorname{spec} A$ is discrete, we can choose such an $R$ that $A - \lambda$ is invertible also for all $\lambda$ on the arc of radius $R$ leading clockwise from $L_{\alpha_1}$ to $L_{\alpha_2}$. We fix this contour $\Gamma_+$ and let $\Lambda_+$ and $\Lambda_-$ denote the regions *inside*, respectively, *outside* $\Gamma_+$ (see Figure 1a).

2.4.2. *Definition of the sectorial projection and our goal.* Equation (2.10) explains why we cannot expect convergence of the integral $\int_{\Gamma_+} (A-\lambda)^{-1}\, d\lambda$, which is familiar in the bounded case presented above in (1.1). The common way to get something finite is to guarantee convergence of the integral by inserting a factor $\lambda^{-1}$ and to compensate by multiplying the integral by $A$. Alternative formulas for the sectorial projection are derived by other authors, see [**Bur70**] and more recently [**GaGr08**]. However, they do not seem to offer an advantage for treating our perturbation problem and we shall not follow that line.

**Definition 2.5.** For the preceding data, we define

$$P_{\Gamma_+}(A) := \frac{-1}{2\pi i} A\, \Phi(A), \qquad \Phi(A) := \int_{\Gamma_+} \lambda^{-1} (A-\lambda)^{-1}\, d\lambda. \tag{2.11}$$

**Remark 2.6.** (a) In view of the estimate (2.10) (see also (1.8)), the composition of $A$ with the integral $\Phi(A)$ *a priori* gives rise to an unbounded operator on $L^2(M;E)$ with domain $\cup_{s>0} H^s(M;E)$. The nice fact, however, is that $P_{\Gamma_+}(A)$ truly is a bounded operator. We postpone the proof to Section 6.3.
(b) Our goal is to specify a locally convex topology $\mathcal{T}$ on $\mathrm{L}^m_{\mathrm{sc}}(M,E)$ such that

$$P_{\Gamma_+} : \mathrm{Ell}^m_{\Gamma_+}(M,E) \longrightarrow \mathcal{B}(H^s(M;E)) \quad \text{ is continuous for all } s \in \mathbb{R}. \tag{2.12}$$

Here we keep the rays $L_{\alpha_j}, j = 1,2$ and the contour $\Gamma_+$ fixed and set

$$\begin{aligned} \mathrm{Ell}^m_{\Gamma_+}(M,E) := \{A \in \mathrm{L}^m_{\mathrm{sc}}(M,E) \mid A \text{ elliptic, } \operatorname{spec} A \cap \Gamma_+ = \emptyset \\ \text{and } L_{\alpha_j}, j = 1,2 \text{ rays of minimal growth}\}. \end{aligned} \tag{2.13}$$

(c) As a side result, we shall show under what conditions $P_{\Gamma_+}(A)$ becomes a pseudo-differential operator. We consider that of minor importance. The proof of (b) will anyway show that $P_{\Gamma_+}(A)$ is of the form $P_{\Gamma_+,0}(A) + K$ with $P_{\Gamma_+,0}(A) \in \mathrm{L}^0_{\mathrm{sc}}(M,E)$ and $K$ a compact operator.

**2.5. First reduction.** The factorization of $P_{\Gamma_+}(A) = \frac{-1}{2\pi i} A\Phi(A)$ in Equation (2.11) of Definition 2.5 permits a first reduction of our problem.

**Lemma 2.7.** *Let $\mathcal{T}$ be a locally convex topology on $\mathrm{L}^m_{\mathrm{sc}}(M,E)$. Suppose that the map*

$$\Phi: \ \mathrm{Ell}^m_{\Gamma_+}(M,E) \ni A \ \mapsto \ \int_{\Gamma_+} \lambda^{-1}\,(A-\lambda)^{-1}\,d\lambda \in \mathcal{B}(H^s, H^{s+m})$$

*is continuous and that $\|\cdot\|_{s,s+m}$ is a continuous (semi-)norm with respect to $\mathcal{T}$. Then our claim (2.12) holds.*

Proof. Given $A \in \mathrm{Ell}^m_{\Gamma_+}(M,E)$. Then there is a neighborhood $U$ of $A$ such that $\|\cdot\|_{s+m,s}$ is bounded on $U$. Hence we reach the conclusion from

$$\begin{aligned}\|P_{\Gamma_+}(A) - P_{\Gamma_+}(B)\|_{s,s} \\ &\le \|A-B\|_{s+m,s}\,\|\Phi(A)\|_{s,s+m} + \|B\|_{s+m,s}\,\|\Phi(A)-\Phi(B)\|_{s,s+m}\,. \quad \square\end{aligned}$$

This Lemma reduces the problem to the task of considering $\int \lambda^{-1}\,(A-\lambda)^{-1}\,d\lambda$, which is more convenient.

## 3. Local considerations

We shall not specify the topology $\mathcal{T}$ of Remark 2.6b now. Rather we shall successively identify the (semi-)norms we need on $\mathrm{L}^m_{\mathrm{sc}}$ to ensure that $P_{\Gamma_+}$ is continuous.

Before continuing, we shall explain a topological obstruction which excludes repeating Seeley's construction literally and which was overlooked by various authors (see above).

**3.1. The topological obstruction.** Given the two rays of minimal growth $L_{\alpha_j}, j=1,2$ with $\operatorname{spec} a_m(x,\xi) \cap L_{\alpha_j} = \emptyset$ for $x \in M, \xi \in T^*_xM, \xi \neq 0, j=1,2$, we are guaranteed a symbol "ingredient" $(a_m(x,\xi)-\lambda)^{-1}$ of order $-m$ for each $\lambda \in L_{\alpha_1} \cup L_{\alpha_2}$ and for $\xi \neq 0$. Moreover, we can find a small arc of radius $R$ connecting the two rays such that the resulting curve $\Gamma_+$ belongs to the resolvent set of $A$, as explained above.

3.1.1. *The problem.* It might be tempting to look for a smooth deformation and extension $\widetilde{a}$ of $a(x,\xi)$ to $\xi = 0$ in such a way that for *all* $(x,\xi) \in T^*M$ one has

$$\operatorname{spec} \widetilde{a}(x,\xi) \cap \Gamma_+ = \emptyset.$$

Actually, we may choose $R > 0$ such that $\operatorname{spec} a(x,\xi) \cap \Gamma_+ = \emptyset$ for, say, $|\xi| = 1$. Then the problem arises whether such map

$$a(x,\cdot): S^{n-1} \to \mathcal{M}(N,\Gamma_+), \tag{3.1}$$

$$\mathcal{M}(N,V) := \{a \in \mathcal{M}(N) \mid \operatorname{spec} a \cap V = \emptyset\}, V \subset \mathbb{C} \tag{3.2}$$

can be extended over the whole $n$-dimensional ball to a map $\widetilde{a}: B^n \to \mathcal{M}(N,\Gamma_+)$ in a continuous way. In the preceding, $x \in M$ is fixed, $\dim M = n$, the fibre dimension of the Hermitian bundle is $\dim E_x = N$, $\mathcal{M}(N)$ denotes the space of $N \times N$ matrices with complex entries, and the matrix spaces inherit the topology of $\mathbb{C}^{N^2}$. We assume that we are given a trivialization of the cotangent bundle $T_x^* M = \mathbb{R}^n$ and of the fibre $E_x = \mathbb{C}^N$.

3.1.2. *A one-dimensional counterexample.* The most simple one-dimensional example $A := -i\frac{d}{d\theta}$ on $M = S^1$, $N = 1$ refutes that naive hope. $A$ is the tangential operator for the Cauchy-Riemann operator on the 2-ball $\{|z| \leq 1\}$. We have $a(\theta, \xi) = \xi$ with $\operatorname{spec} a(\theta, \xi) = \{\xi\}$, $\operatorname{spec} A = \mathbb{Z}$, and the imaginary line $i\mathbb{R} = L_{\pi/2} \cup L_{3\pi/2}$ as spectral cut for $a(\theta, \xi), \xi \neq 0$. Clearly, we cannot get anything useful, if we multiply $a$ just by a cut-off function leading to

$$\widetilde{a}(\theta, \xi) = \begin{cases} \xi \text{ for } |\xi| \geq 1, \\ 0 \text{ for } |\xi| \leq \varepsilon. \end{cases}$$

By the Intermediate Value Theorem, for each $R \in (0, 1)$ there will always be a $\hat{\xi} \in (\varepsilon, 1)$ such that $\widetilde{a}(\theta, \hat{\xi}) = R$. However, if we exempt only one ray, say $L_{\pi/2}$ instead of the whole imaginary line, we *can* deform the given $a(\cdot, \cdot) : S^1 \times (\mathbb{R} \setminus (-1, 1)) \to \mathcal{M}(1, L_{\pi/2})$ into

$$\begin{aligned} \widetilde{a}(\cdot, \cdot) \quad : \quad S^1 \times \mathbb{R} \quad &\longrightarrow \quad \mathcal{M}(1, L_{\pi/2}), \\ (\theta, \xi) \quad &\mapsto \quad \begin{cases} \xi, & \text{for } |\xi| \geq 1, \\ e^{-i(1-\xi)\frac{\pi}{2}}, & \text{for } 0 \leq |\xi| < 1. \end{cases} \end{aligned} \tag{3.3}$$

Here the point is that we only require that $\widetilde{a}(\theta, \xi)$ has no purely non-negative eigenvalues. What we did was a spectral deformation of the original matrices (here complex numbers) into the point $\{-i\}$. Clearly, that deformation breaks down, if we have two rays of minimal growth forming a separating curve in $\mathbb{C}$: There is no continuous path connecting $\{1\}$ and $\{-1\}$ that is not crossing the imaginary line. The topological obstruction for $n = 1$ is simply that the space $\mathcal{M}(1, i\mathbb{R})$ has two connected components, $(-\infty, 0), (0, \infty)$ and that $a(\theta, 1), a(\theta, -1)$ belong to different components.

3.1.3. *The essence of the topological obstruction.* Let us muse upon the cases $n, N > 1$. Shortly, the essence of the topological difficulties overlooked by our predecessors is the following: Without loss of generality, let $\Gamma_+$ be the imaginary line $i\mathbb{R}$. Fix a *non-trivial* smooth complex vector bundle $G$ on the sphere $S^{n-1}$ (or on the sphere cotangent bundle $S^*M$ over the $n$-dimensional manifold $M$ – for simplicity, however, we shall ignore the spatial variables). Next, we embed $G$ into a trivial bundle $S^{n-1} \times \mathbb{C}^k$ for $k$ sufficiently large. Let $\{P_\xi\}_{\xi \in S^{n-1}}$ denote the smooth family of self-adjoint projections of $\mathbb{C}^k$ onto the fibers $G_\xi, \xi \in S^{n-1}$.

Set $a(\xi) := 2P_\xi - I : \mathbb{C}^k \to \mathbb{C}^k$ and extend it, say by homogeneity 1 to $\mathbb{R}^n$ and smooth it out in 0. Then this is an elliptic symbol with the two imaginary half-axes being rays of minimal growth. More precisely, we have $\operatorname{spec} a(\xi) = \{-1, 1\}, \xi \in S^{n-1}$, and $E_{1,\xi} = G_\xi$ and $E_{-1,\xi} = G_\xi^\perp$, where $E_{\lambda,\xi}$ denotes the linear span of the eigenvectors of $a(\xi)$ for $\lambda \in \operatorname{spec} a(\xi)$.

Then it is impossible to find a $k \times k$ matrix valued function $\widetilde{a}$ on the whole $\mathbb{R}^n$ which coincides with $a$ outside a large ball such that $\operatorname{spec} \widetilde{a}(\xi) \cap \Gamma_+ = \emptyset$ for all $\xi \in \mathbb{R}^n$: Let us assume we could. Let $\widetilde{E}_{\Lambda_+,\xi} = \operatorname{im} \widetilde{P}_+(\xi)$ denote the linear span of all root vectors of $\widetilde{a}(\xi)$ for eigenvalues in the positive half plane $\Lambda_+ \subset \mathbb{C}$. The family of vector subspaces of $\mathbb{C}^k$ is continuous and forms a vector bundle over the unit ball $B^n$. It is trivial because the base space is contractible, but its restriction on the $n-1$ sphere is $G$ which is by assumption non-trivial. That is a contradiction. So, we have a necessary condition for the construction to work.

Since Seeley only dealt with one ray of minimal growth, this problem did not occur there.

Therefore, we cannot expect to be able to make the wanted extension, respectively deformation in general. Instead of the direct (and futile) search for a suitable modification of the principal symbol to get a well-defined resolvent for $A$ along the spectral cut $\Gamma_+$ we shall apply the symbolic calculus solely to obtain a parametrix for $A - \lambda$.

3.1.4. *The topology of the underlying space of hyperbolic matrices.* As a service to the reader we determine the precise homotopy type of the matrix space $\mathcal{M}(N, \Gamma_+)$. By deformation, we may assume that the imaginary line is the given spectral cut for all matrices $a(x,\xi)$ for $\xi \neq 0$. In $\mathbb{C} \setminus \Gamma_+$, we denote the two complementary sectors by $\Lambda_\pm$. Then the space $\mathcal{M}(N, i\mathbb{R})$ of $N \times N$ matrices with no purely imaginary (generalized) eigenvalues decomposes into $N+1$ connected components

$$\mathcal{M}_k(N, i\mathbb{R}) := \{a \in \mathcal{M}(N, i\mathbb{R}) \mid \dim \operatorname{im} P^+(a) = k\}, \quad k = 0, 1, \dots N, \tag{3.4}$$

where

$$\begin{array}{rcccc} P^+ & : & \mathcal{M}(N, i\mathbb{R}) =: \mathcal{E} & \longrightarrow & \mathcal{P}(N) \\ & & a & \mapsto & -\frac{1}{2\pi i} \int_{\Gamma_+} (a - \lambda I)^{-1} \, d\lambda. \end{array} \tag{3.5}$$

Here $\mathcal{P}(N) = \cup_{k=0}^N \mathcal{P}_k(N)$ denotes the space of projections (idempotent $N \times N$ matrices, fibred according to the dimension of their ranges) and $P^+(a)$ denotes the projection onto the generalized eigenspaces of $a$ for generalized eigenvalues in the positive sector $\Lambda_+$.

For $k = 0$ and $k = N$, the spaces $\mathcal{M}_k(N, i\mathbb{R})$ are homeomorphic to the full space $\mathcal{M}(N)$ of all square matrices and hence contractible. That explains why Seeley's deformation is always possible for one ray of minimal growth, dividing $\mathbb{C}$ into one sector without spectrum and one sector with all the eigenvalues, see once again Fig. 1b.

To investigate the homotopy type of $\mathcal{M}_k(N, i\mathbb{R})$ for $k = 1, \dots N-1$, we restrict the map (3.5) to a single component $\mathcal{M}_k(N, i\mathbb{R})$. We obtain a fibration of the total space $\mathcal{M}_k(N, i\mathbb{R})$ as a fibre bundle over the base $\mathcal{P}$ with contractible fibre

$$(P^+)^{-1}\{P_0\} = \{a \in \mathcal{M}(\operatorname{im} P_0) \mid \operatorname{spec} a \subset \Lambda_+\} \times \{a \in \mathcal{M}(\ker P_0) \mid \operatorname{spec} a \subset \Lambda_-\}$$

for any $P_0 \in \mathcal{P}_k(N)$. Hence, the topological spaces, the base $\mathcal{P}_k(N)$ and the total space $\mathcal{M}_k(N, i\mathbb{R})$ have the same homotopy type. By orthogonalization, it suffices to consider a projection space made of orthogonal projections which easily can be identified with the subspaces of $\mathbb{C}^N$ of dimension $k$. So we arrive at the complex Grassmannian $\mathrm{Gr}_{\mathbb{C}}(N, k)$, which is known for non-trivial homotopy, if $0 < k < N$.

**3.2. Cut-off symbols.** While we cannot deform and extend $a_m(x, \xi)$ in a suitable way, we can easily deform and extend $\big(a_m(x,\xi) - \lambda\big)^{-1}$ in the usual way as a *smoothed resolvent symbol* [**Shu01**, Sections 11.3-11.4] (similarly, e.g., Bilyj, Schrohe, and Seiler in the recent [**BSS10**, Definition 2.5]). Recall that we denote the principal symbol of $A$ by $a_m(x, \xi)$ and that we have assumed that

$$\operatorname{spec} a_m(x,\xi) \cap L_{\alpha_j} = \emptyset \qquad \text{for } (x,\xi) \in T^*M, \xi \neq 0, j = 1, 2. \tag{3.6}$$

Thus, there is a constant $\rho > 0$ such that $a_m(x,\xi) - \lambda$ is invertible for $(x, \xi) \in T^*M$, $|\xi| \geq \rho$ and $\lambda \in \Gamma_+$. Hence, for any cut-off function $\psi \in C^\infty(\mathbb{R}^n)$ with

$$\psi(\xi) = \begin{cases} 0, & \text{for } |\xi| \leq \rho, \\ 1, & \text{for } |\xi| \gg 0, \end{cases} \tag{3.7}$$

(that is, the function $1 - \psi$ is compactly supported) and for each $\lambda \in \Gamma_+$ the symbol

$$(x, \xi) \mapsto \psi(\xi)(a_m(x,\xi) - \lambda)^{-1} \tag{3.8}$$

is a classical symbol of order $-m$.

**3.3. Symbol estimates and semi-norms.** From now on we shall switch forward and backward between argueing locally (in the open domain $U \subset \mathbb{R}^n$) and globally (on $M$). With the preceding symbol $a_m$ and cut-off function $\psi$, we shall write

$$r^\psi(x,\xi,\lambda) := \psi(\xi)(a_m(x,\xi) - \lambda)^{-1}\,. \tag{3.9}$$

For fixed $\lambda$ we have $r^\psi(\cdot,\cdot,\lambda) \in \mathrm{CS}^{-m}(U \times \mathbb{R}^n, E)$. Considered as a $\lambda$-dependent symbol, it does not necessarily belong to the usual parameter dependent calculus. Actually, the cut-off $\psi$ prevents this.

However, we have the following symbol estimates, which are uniform in $\lambda \in \Gamma_+$:

$$\begin{aligned} &\big|\partial_x^\alpha \partial_\xi^\beta\, r^\psi(x,\xi,\lambda)\big| \\ &\qquad \leq \begin{cases} C_{0,0}(1 + |\xi| + |\lambda|^{1/m})^{-m}\,, & \alpha = \beta = 0, \\ C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}\,(1 + |\xi| + |\lambda|^{1/m})^{-2m}\,, & (\alpha,\beta) \neq (0,0), \end{cases} \\ &\qquad \leq C_{\alpha,\beta}(1+|\xi|)^{-|\beta|}\,(1 + |\xi| + |\lambda|^{1/m})^{-m}\,. \end{aligned} \tag{3.10}$$

The proof is an exercise in induction and Leibniz rule.

What is important is that the best constants $C_{\alpha,\beta}$ in (3.10), as functions of $a_m$, are continuous semi-norms on the space of sections $C^k(S^*M; \mathrm{End}(\pi^*E)), k = |\alpha| + |\beta|$. In particular, they are continuous semi-norms on $C^\infty(S^*M; \mathrm{End}(\pi^*E))$.

As a consequence, we have the following: for fixed $a_m \in C^\infty(S^*M; \mathrm{End}(\pi^*E))$ and fixed $k$ there is an open neighborhood $U$ of $a_m$ such that $C_{\alpha,\beta}, |\alpha|+|\beta| \leq k$, are bounded on $U$ and such that each $b_m \in U$ is "invertible" on $\Gamma_+$, that is, it satisfies the same $\mathrm{Ell}_{\Gamma_+}$–conditions as $a_m$.

*Note.* We have to fix $k$ and cannot bound infinitely many semi-norms simultaneously: the intersection of infinitely many open $U_{\alpha,\beta}$ might be non-open.

**3.4. A first approximation.** The symbolic calculus yields the following first approximation result.

**Proposition 3.1.** (a) *For $a_m$ and $r^\psi$ as above, the operator*

$$\Phi_0(a_m) := \int_{\Gamma_+} \lambda^{-1}\ \mathrm{Op}(r^\psi(\cdot,\cdot,\lambda)\, d\lambda \tag{3.11}$$

*belongs to the class $\mathrm{CL}^{-m}(U,E)$.*
(b) *If $\mathcal{T}$ is a locally convex topology on $\mathrm{L}^m_{\mathrm{sc}}(M,E)$ such that*

$$\sigma_m\ :\ \mathrm{L}^m_{\mathrm{sc}}(M,E) \longrightarrow C^\infty(S^*M; \mathrm{End}(\pi^*E)) \tag{3.12}$$

*is continuous, then $\Phi_0$ is continuous on $\mathrm{Ell}^m_{\Gamma_+}(M,E)$ with regard to $\mathcal{T}$.*

Proof. For (a) we see that

$$\psi(\xi) \int_{\Gamma_+} \lambda^{-1}\,(a_m(x,\xi)-\lambda)^{-1}\, d\lambda = \int_{\Gamma_+} \lambda^{-1}\, r^\psi(x,\xi,\lambda)\, d\lambda$$

is homogeneous of degree $-m$ outside a compact set, and smooth otherwise. Recall that principal symbols are determined by their values in $\{(x,\xi) \in T^*M \mid |\xi| \geq C\}$ where $C$ is any positive constant. That proves (a).
For (b) we denote the symbol analogue of the operator space $\mathrm{Ell}^m_{\Gamma_+}$ by $C^\infty_{\Gamma_+}(S^*M, \mathrm{End}(\pi^*E))$. Certainly,

$$\begin{array}{ccc} C^\infty_{\Gamma_+}(S^*M, \mathrm{End}(\pi^*E)) & \longrightarrow & \mathrm{CS}^{-m}(T^*M, \mathrm{End}(\pi^*E)) \\ a_m & \mapsto & \int_{\Gamma_+} \lambda^{-1}\,\psi(\xi)\,(a_m(x,\xi)-\lambda)^{-1}\, d\lambda \end{array}$$

and

$$\mathrm{Op} :\ \mathrm{CS}^{-m}(T^*M, \mathrm{End}(\pi^*E)) \longrightarrow \mathcal{B}(H^s, H^{s+m})$$

are continuous. That proves (b). $\square$

## 4. The main technical lemma

In this section, we shall prove a technical lemma which is crucial in the proof of our main theorem. Our arguments are local for a fixed open coordinate patch $U \subset \mathbb{R}^n$.

**Definition 4.1.** For a compact subset $K \subset U$ we denote by $\mathrm{S}^m_K(U \times \mathbb{R}^n) \subset \mathrm{S}^m(U \times \mathbb{R}^n)$ those $a \in \mathrm{S}^m(U \times \mathbb{R}^n)$ such that $a(x,\xi) \neq 0$ implies $x \in K$. $\mathrm{CS}^m_K(U \times \mathbb{R}^n)$ is defined accordingly. A typical example is $a(x,\xi) = \theta(x)b(x,\xi)$ with $b \in \mathrm{S}^m(U \times \mathbb{R}^n)$ and a cut-off function $\theta \in C^\infty_c(U)$.

Clearly, the preceding definitions carry over to matrix valued symbols and to globally defined symbols with values in bundle endomorphisms.

We shall prove the following product formula. Note that all symbols are matrix valued.

**Lemma 4.2** (Main Technical Lemma)**.** *Let $m > 0$, $0 \le r \le m$. Let $f, g \in C^\infty(U \times \mathbb{R}^n \times \Gamma_+)$ such that for $\lambda \in \Gamma_+$*

$$f(\cdot,\cdot,\lambda) \in \mathrm{S}^r_K(U \times \mathbb{R}^n), \qquad g(\cdot,\cdot,\lambda) \in \mathrm{S}^{-m}_K(U \times \mathbb{R}^n).$$

*Assume that*

$$|\partial^\alpha_x \partial^\beta_\xi f(x,\xi,\lambda)| \le \begin{cases} C_{0,0}(f)(1+|\xi|+|\lambda|^{1/m})^r, & \alpha=\beta=0, \\ C_{\alpha,\beta}(f)(1+|\xi|)^{m-|\beta|}(1+|\xi|+|\lambda|^{1/m})^{r-m}, & |\alpha|+|\beta|>0, \end{cases} \tag{4.1}$$

*and*

$$|\partial^\alpha_x \partial^\beta_\xi g(x,\xi,\lambda)| \le \widetilde{C}_{\alpha,\beta}(g)(1+|\xi|)^{-|\beta|}(1+|\xi|+|\lambda|^{1/m})^{-m}, \tag{4.2}$$

*where $C_{\cdot,\cdot}(\cdot), \widetilde{C}_{\cdot,\cdot}(\cdot)$ are constants depending on certain datas in the dots' positions. Set $C_N(f) = \sum_{|\alpha|,|\beta|\le N} C_{\alpha,\beta}(f)$ and $\widetilde{C}_N(g) = \sum_{|\alpha|,|\beta|\le N} \widetilde{C}_{\alpha,\beta}(g)$. Then for $s \in \mathbb{R}$, there is an $N(s) \in \mathbb{N}$ and $C > 0$ such that*

$$\|\mathrm{Op}(g(\cdot,\cdot,\lambda))\mathrm{Op}(f(\cdot,\cdot,\lambda)) - \mathrm{Op}(gf(\cdot,\cdot,\lambda))\|_{s,s+m-r} \le C C_{N(s)}(f)\widetilde{C}_{N(s)}(g)|\lambda|^{-\min(\frac{1}{m},1)}.$$

**Remark 4.3.** We should notice that $C_N(\cdot)$ and $\widetilde{C}_N(\cdot)$ are semi-norms if we choose the smallest constants $C_{\alpha,\beta}(\cdot)$ and $\widetilde{C}_{\alpha,\beta}(\cdot)$ in (4.1), (4.2). Moreover, $C_N(f)$ and $\widetilde{C}_N(g)$ are dominated by the finitely many constants $C_{\alpha,\beta}(f)$ and $\widetilde{C}_{\alpha,\beta}(g)$, $|\alpha|$, $|\beta| \le N$, respectively.

Before proving the lemma, we give some additional examples and lemmata.

**Example 4.4.** $g(x,\xi,\lambda) := \psi(\xi)(a_m(x,\xi)-\lambda)^{-1}$ satisfies (4.2). See (3.10).

**Example 4.5.** $f(x, \xi, \lambda) := a(x, \xi) - \lambda$ satisfies (4.1) with $r = m$. If $b \in \mathrm{CS}^m_K(U \times \mathbb{R}^n)$ is a symbol of order $m$, then $f(x, \xi, \lambda) := \psi(\xi)(a_m(x, \xi) - \lambda)^{-1} b(x, \xi) = r^\psi(x, \xi) b(x, \xi)$ also satisfies (4.1) with $r = 0$. Note that in this case

$$\sum_{|\alpha|,|\beta| \le N} C_{\alpha,\beta}(f) \le \left( \sum_{|\alpha|,|\beta| \le N} C_{\alpha,\beta}(r^\psi) \right) \left( \sum_{|\alpha|,|\beta| \le N} C_{\alpha,\beta}(b) \right).$$

Here $C_{\alpha,\beta}(b)$ denotes the best constant in the symbol estimate for $\partial_x^\alpha \partial_\xi^\beta b(x, \xi)$ and $C_{\alpha,\beta}(r^\psi)$ is of similar meaning.

**Remark 4.6.** Note that in the examples above, $C_{\alpha,\beta}(f)$ and $C_{\alpha,\beta}(g)$ are bounded by a $C^k$-norm on $a_m$ (and $b_m$ in the preceding example) for sufficiently large $k$.

As always with product formulae for pseudo-differential operators, the proof of the Main Technical Lemma is somewhat intricate. To avoid overloaded notation, we shall translate the wanted estimates into statements about integral operators.

**4.1. $L^2$-estimates for integral operators and other estimates.** We recall the well-known and very convenient Schur's Test for integral operators (see, e.g., Halmos and Sunder [**HaSu78**, Theorem 5.2]):

**Lemma 4.7** (Schur's Test)**.** *Let $K$ be an integral operator with measurable kernel $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$. Assume that*

$$C_1 := \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |k(x, y)| dy < +\infty \text{ and } C_2 := \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} |k(x, y)| dx < +\infty.$$

*Then $K$ is bounded $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\|K\|_{L^2 \to L^2} \le \sqrt{C_1 C_2}$.*

*In particular, if for some $p > n$*

$$|k(x, y)| \le C_3 (1 + |x - y|)^{-p},$$

*then the criterion is fulfilled with*

$$C_1 = C_2 = C_3 \int_{\mathbb{R}^n} (1 + |\xi|)^{-p} d\xi.$$

Now fix $U \subset \mathbb{R}^n$ open, $K \subset U$ compact and $a \in \mathrm{S}^m_K(U \times \mathbb{R}^n)$. Then Schur's Test yields a very operational estimate for $\|\operatorname{Op}(a)\|_{s,s-m}$.

To explain that, we introduce various useful notations. For the Fourier transform, we shall follow Hörmander's convention

$$\big(\mathcal{F} f\big)(\xi) := \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} f(x) dx, \quad \big(\mathcal{F}^{-1} u\big)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} u(\xi) d\xi$$

$$đ\xi := (2\pi)^{-n} d\xi.$$

Then we have

$$\big(\mathcal{F}\operatorname{Op}(a)u\big)(\eta) := \int_{\mathbb{R}^n} e^{-i\langle x,\eta\rangle}\big(\operatorname{Op}(a)u\big)(x)dx = \int_{\mathbb{R}^n}\Big[\int_U e^{i\langle x,\xi-\eta\rangle}a(x,\xi)đx\Big]\widehat{u}(\xi)d\xi.$$

We set $q_a(\xi-\eta,\xi) := [\cdots]$ in the preceding formula and define

**Definition 4.8.** For $a \in \mathrm{S}^m_K(U\times\mathbb{R}^n)$, we set

$$q_a(\zeta,\xi) := \big(\mathcal{F}^{-1}_{x\to\xi}a(x,\xi)\big)(\zeta) = \int_{\mathbb{R}^n} e^{i\langle\zeta,\xi\rangle}a(x,\xi)đx.$$

Consequently, the kernel of $\mathcal{F}\operatorname{Op}(a)\mathcal{F}^{-1}$ is given by $k_a(\tau,\xi) := q_a(\xi-\tau,\xi)$. To estimate the operator norm $\|\cdot\|_{s,s-m}$ of $\operatorname{Op}(a)$ it suffices therefore to estimate the norm of the integral operator $\mathcal{F}\operatorname{Op}(a)\mathcal{F}^{-1}$ as a map from the weighted $L^2$-space $L^2(\mathbb{R}^n,(1+\|\xi\|^2)^s)$ into $L^2(\mathbb{R}^n,(1+\|\xi\|^2)^{s-m})$. By Schur's test an estimate of the form

$$\big|(1+|\tau|)^{m-s}k_a(\tau,\xi)(1+|\xi|)^{-s}\big| \le C(a)C(p)(1+|\tau-\xi|)^{-p} \qquad \text{for some } p>n \quad (4.3)$$

implies

$$\|\operatorname{Op}(a)\|_{s,s-m} \le C(a)\widetilde{C}(p) \qquad \text{with } \widetilde{C}(p) := C(p)\int_{\mathbb{R}^n}(1+|x|)^{-p}dx.$$

Proof of the Main Technical Lemma. Let $f(\cdot,\cdot,\lambda) \in \mathrm{S}^r_K(U\times\mathbb{R}^n)$, $g(\cdot,\cdot,\lambda) \in \mathrm{S}^m_K(U\times\mathbb{R}^n)$ satisfying (4.1), (4.2) be given. In the sequel we will suppress the argument $\lambda$ from the notation for simplicity. We should be aware that all expressions will depend on $\lambda$ unless otherwise stated. The kernel of $\mathcal{F}\operatorname{Op}(f)\operatorname{Op}(g)\mathcal{F}^{-1}$ is given by

$$k_{f\cdot g}(\tau,\xi) = \int_{\mathbb{R}^n} k_f(\tau,\eta)k_g(\eta,\xi)d\eta = \int_{\mathbb{R}^n} q_f(\eta-\tau,\eta)q_g(\xi-\eta,\xi)d\eta.$$

On the other hand

$$\begin{aligned} q_{f\cdot g}(\zeta,\xi) &= \int_{\mathbb{R}^n} e^{i\langle\zeta,x\rangle}f(x,\xi)g(x,\xi)đx \\ &= \mathcal{F}^{-1}\big(f(\cdot,\xi)g(\cdot,\xi)\big)(\zeta) \\ &= \int_{\mathbb{R}^n} q_f(\zeta-\eta,\xi)q_g(\eta,\xi)d\eta, \end{aligned}$$

respectively,

$$\begin{aligned} k_{f\cdot g}(\tau,\xi) &= q_{f\cdot g}(\xi-\tau,\xi) \\ &= \int_{\mathbb{R}^n} q_f(\xi-\tau-\eta,\xi)q_g(\eta,\xi)d\eta; \quad \xi-\eta \rightsquigarrow \eta \\ &= \int_{\mathbb{R}^n} q_f(\eta-\tau,\xi)q_g(\xi-\eta,\xi)d\eta. \end{aligned}$$

Thus the kernel of $\mathcal{F}\Big\{\mathrm{Op}(f)\,\mathrm{Op}(g)-\mathrm{Op}(f\cdot g)\Big\}\mathcal{F}^{-1}$ is given by

$$k(\tau,\xi,\lambda):=\int_{\mathbb{R}^n}\Big\{q_f(\eta-\tau,\eta)-q_f(\eta-\tau,\xi)\Big\}q_g(\xi-\eta,\xi)d\eta. \tag{4.4}$$

We are now going to estimate this kernel. The estimate of $q_g$ is standard: for any multiindex $\alpha$, $\zeta\in\mathbb{R}^n$ we have (for $D_x^\alpha:=-i\partial^{\alpha_1+\ldots\alpha_n}/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}$, as usual):

$$|\zeta^\alpha q_g(\zeta,\xi,\lambda)|=\left|\int_K e^{i\langle\zeta,x\rangle}D_x^\alpha g(x,\xi,\lambda)đx\right|\le \mathrm{vol}(K)C_\alpha(g)(1+|\xi|+|\lambda|^{1/m})^{-m}.$$

Since $\alpha$ is arbitrary, we see that for any $N\in\mathbb{N}$

$$|q_g(\zeta,\xi,\lambda)|\le\widetilde{C}_N(g)(1+|\zeta|)^{-N}(1+|\xi|+|\lambda|^{1/m})^{-m}. \tag{4.5}$$

Next we discuss the difference $q_f(\zeta,\eta,\lambda)-q_f(\zeta,\xi,\lambda)$. Again for a multiindex $\alpha$ we have

$$\begin{aligned}|\zeta^\alpha(q_f(\zeta,\eta,\lambda)-q_f(\zeta,\xi,\lambda))|&=\left|\int_K e^{i\langle\zeta,x\rangle}\{D_x^\alpha(f(x,\eta,\lambda)-f(x,\xi,\lambda))\}đx\right|\\ \le&\int_K\sup_{t\in[0,1],\,|\beta|=1}|D_x^\alpha\partial_\xi^\beta f(x,\xi+t(\eta-\xi),\lambda)|đx\,|\xi-\eta|\\ \le&\ \mathrm{vol}(K)C_N(f)\sup_{t\in[0,1]}(1+|\xi+t(\eta-\xi)|)^{m-1}(1+|\xi+t(\eta-\xi)|+|\lambda|^{\frac1m})^{r-m}\,|\xi-\eta|.\end{aligned}$$

Here $N=\max(|\alpha|,1)$, that is,

$$\begin{aligned}|q_f(\zeta,\eta,\lambda)-q_f(\zeta,\xi,\lambda)|\le\mathrm{vol}(K)C_N(f)(1+|\zeta|)^{-N}|\xi-\eta|\cdot\\ \sup_{t\in[0,1]}(1+|\xi+t(\eta-\xi)|)^{m-1}(1+|\xi+t(\eta-\xi)|+|\lambda|^{\frac1m})^{r-m}.\end{aligned} \tag{4.6}$$

To estimate the norm of $\mathrm{Op}(f)\,\mathrm{Op}(g)-\mathrm{Op}(f\cdot g)$ as an operator from $H^s$ to $H^{s+m-r}$ we need to estimate the norm of the integral operator in $L^2(\mathbb{R}^n)$ whose kernel is given by (see (4.3))

$$\widetilde{k}(\tau,\xi,\lambda)=(1+|\tau|)^{s+m-r}k(\tau,\xi,\lambda)(1+|\xi|)^{-s},$$

where $k(\tau,\xi,\lambda)$ is defined in (4.4). From (4.4), (4.5) and (4.6) we infer

$$\begin{aligned}&|\widetilde{k}(\tau,\xi,\lambda)|\\ &\le C_N(f)\widetilde{C}_N(g)\int(1+|\eta-\tau|)^{-N}|\xi-\eta|\,(1+|\xi-\eta|)^{-N}(1+|\xi|+|\lambda|^{\frac1m})^{-m}(1+|\tau|)^{s+m-r}\\ &\qquad\cdot(1+|\xi|)^{-s}\sup_{t\in[0,1]}(1+|\xi+t(\eta-\xi)|)^{m-1}(1+|\xi+t(\eta-\xi)|+|\lambda|^{\frac1m})^{r-m}đ\eta.\end{aligned} \tag{4.7}$$

Note that we may choose $N$ as large as we please. We now distinguish two cases.
*Case I*: $|\eta-\xi|\le\frac12|\xi|$. Then for $0\le t\le1$, $\frac12|\xi|\le|\xi+t(\eta-\xi)|\le\frac32|\xi|$, and thus the

integrand of the right hand side of (4.7) can be estimated (absorbing another constant into $C_N(f)\widetilde{C}_N(g)$) by

$$\le C_N(f)\widetilde{C}_N(g)(1+|\eta-\tau|)^{-N}(1+|\xi-\eta|)^{1-N}(1+|\tau|)^{s+m-r} (1+|\xi|)^{-s+m-1}(1+|\xi|+|\lambda|^{\frac{1}{m}})^{r-2m}. \quad (4.8)$$

Using PEETRE's Inequality, we have

$$(1+|\tau|)^{s+m-r}(1+|\xi|)^{-s+m-1} \le (1+|\tau-\xi|)^{|s+m-r|}(1+|\xi|)^{2m-r-1}.$$

Then (4.8)

$$\le C_N(f)\widetilde{C}_N(g)(1+|\eta-\tau|)^{-N}(1+|\xi-\eta|)^{1-N}(1+|\tau-\xi|)^{|s+m-r|} (1+|\xi|)^{2m-r-1}(1+|\xi|+|\lambda|^{\frac{1}{m}})^{r-2m}. \quad (4.9)$$

For $0 \le r \le m$,

$$(1+|\xi|)^{2m-r-1}(1+|\xi|+|\lambda|^{\frac{1}{m}})^{r-2m} \le \begin{cases} (1+|\lambda|^{\frac{1}{m}})^{-1}, & 2m-r-1 \le 0, \\ (1+|\lambda|^{\frac{1}{m}})^{-m}, & 2m-r-1 > 0. \end{cases}$$

Thus (4.9)

$$\le C_N(f)\widetilde{C}_N(g)(1+|\eta-\tau|)^{-N}(1+|\xi-\eta|)^{1-N}(1+|\tau-\xi|)^{|s+m-r|} (1+|\lambda|)^{-\min(\frac{1}{m},1)}. \quad (4.10)$$

Again PEETRE's Inequality gives that for $N > n+1$,

$$\begin{aligned}\int_{\mathbb{R}^n}(1+|\eta-\tau|)^{-N}(1+|\xi-\eta|)^{1-N}đ\eta &\le \int_{\mathbb{R}^n}(1+|\eta|)^{-N}(1+|\xi-\eta-\tau|)^{1-N+n}đ\eta \\ &\le \int_{\mathbb{R}^n}(1+|\eta|)^{-n-1}(1+|\xi-\tau|)^{1-N+n}đ\eta.\end{aligned}$$

Taking this into account and integrating the right side of (4.10) over $\eta$ yields

$$\int_{|\eta-\xi|\le\frac{1}{2}|\xi|}\cdots đ\eta \le C_N(f)\widetilde{C}_N(g)\int(1+|\eta|)^{-n-1}đ\eta (1+|\xi-\tau|)^{1+n+|s+m-r|-N}(1+|\lambda|)^{-\min(\frac{1}{m},1)}. \quad (4.11)$$

Here we choose $N$ large enough such that $N > n+1+|s+m-r|$.
*Case II*: $|\eta-\xi| > \frac{1}{2}|\xi|$. Then the integrand of the right hand side of (4.7) is estimated by

$$\le C_N(f)\widetilde{C}_N(g)(1+|\eta-\tau|)^{-N}(1+|\xi-\eta|)^{m-N}(1+|\tau|)^{s+m-r} (1+|\xi|)^{-s+m-1}(1+|\lambda|^{\frac{1}{m}})^{r-2m}.$$

Since $\frac{1}{2}|\xi| < |\eta - \xi|$, we estimate

$$(1+|\xi|)^{-s+m-1} \leq \begin{cases} 1, & -s+m-1 \leq 0, \\ C_{s,m}(1+|\xi-\eta|)^{-s+m-1}, & -s+m-1 > 0. \end{cases}$$

Now we proceed as in Case I.

In sum we have proved that for $N$ large enough,

$$|\widetilde{k}(\tau, \xi, \lambda)| \leq C_N(f)\widetilde{C}_N(g)(1+|\xi-\tau|)^{-n-1}(1+|\lambda|)^{-\min(\frac{1}{m},1)}.$$

The lemma follows from SCHUR's test finally. □

## 5. Key estimates

Before proving the main result, we give some more estimates.

**Lemma 5.1.** *Given $A \in \mathrm{Ell}^m_{\Gamma_+}(M, E)$. Then for $s \in \mathbb{R}$, $0 \leq p \leq m$, and all $\lambda \in \Gamma_+$ we have*

$$\|\mathrm{Op}(\psi(a_m - \lambda)^{-1})\|_{s,s+p} \leq C_s(A)|\lambda|^{-1+\frac{p}{m}}. \tag{5.1}$$

*Furthermore, to $s$ there is $N_s \in \mathbb{N}$ such that $C_s(A)$ is bounded by the $C^{N_s}$-norm of $a_m$ on $S^*M$.*

In other words, to $A$ there is an open neighborhood $U$ of $a_m$ (in the $C^{N_s}$-topology) such that the map $B \mapsto C_s(B)$ is bounded on the open set $\sigma_m^{-1}(U)$.

PROOF. Use the standard method of estimating norms of pseudo-differential operators as in SEELEY [**See67**, Lemma 2]. Of course it also follows from the method presented in the preceding section. □

**Lemma 5.2.** *Given $A \in \mathrm{Ell}^m_{\Gamma_+}(M, E)$. Then for $s \in \mathbb{R}$ and all $\lambda \in \Gamma_+$*

$$\|\mathrm{Op}(\psi(a_m - \lambda)^{-1}) - (A-\lambda)^{-1}\|_{s,s+m} \leq C_s(A)|\lambda|^{-\min(\frac{1}{m},1)}. \tag{5.2}$$

*$C_s(A)$ has the same property as in* Lemma 5.1.

PROOF. Put $A = \mathrm{Op}(a)$ for the complete symbol $a$. Write $a = a_m + a_{m-1}$. Then we have

$$\begin{aligned} &(A-\lambda)(\mathrm{Op}(\psi(a_m-\lambda)^{-1} - (A-\lambda)^{-1})) \\ &= \mathrm{Op}(a_m-\lambda)\mathrm{Op}(\psi(a_m-\lambda)^{-1}) - \mathrm{Op}(\psi) - \mathrm{Op}(1-\psi) + \mathrm{Op}(a_{m-1})\mathrm{Op}(\psi(a_m-\lambda)^{-1}). \end{aligned}$$

Note that $(A-\lambda)(A-\lambda)^{-1} = I = \mathrm{Op}(1)$ and $\mathrm{Op}(1-\psi)$ is a smoothing operator (because $1-\psi$ is compactly supported). Hence

$$\begin{aligned}
& \|\mathrm{Op}(\psi(a_m-\lambda)^{-1}) - (A-\lambda)^{-1}\|_{s,s+m} \\
\leq\ & \|(A-\lambda)^{-1}\|_{s,s+m}\|\mathrm{Op}(a_m-\lambda)\mathrm{Op}(\psi(a_m-\lambda)^{-1}) - \mathrm{Op}(\psi)\|_{s,s} \\
& +\|(A-\lambda)^{-1}\|_{s+m,s+m}\|\mathrm{Op}(1-\psi)\|_{s,s+m} \\
& +\|(A-\lambda)^{-1}\|_{s,s+m}\|\mathrm{Op}(a_{m-1})\|_{s+m-1,s}\|\mathrm{Op}(\psi(a_m-\lambda)^{-1})\|_{s,s+m-1} \\
\leq\ & C_s(A)|\lambda|^{-\min(\frac{1}{m},1)}
\end{aligned}$$

by the Main Technical Lemma 4.2, applied to $f = a_m - \lambda$, $g = \psi(a_m-\lambda)^{-1}$ and Lemma 5.1. The local boundedness claim on $C_s(A)$ also follows from this lemma. □

## 6. Applications

As an application of the preceding estimates, we prove that the sectorial projections are zero order operators and continuously depend on the initial operators in the topology $\mathcal{T}$ to be fixed below.

### 6.1. The operator type of the sectorial projection.

**Proposition 6.1.** *For $s \in \mathbb{R}$ the operator $P_{\Gamma_+}(A)$ is bounded $H^s(M;E) \to H^s(M;E)$. In fact, it differs from a pseudo-differential operator of order 0 by an operator mapping $H^s(M;E)$ continuously into $H^{s+m}(M;E)$.*

**Remark 6.2.** With some more effort (e.g., like in [**BBWo93**, pp. 91-100]) one can show that $P_{\Gamma_+}(A)$ is a pseudo-differential operator of order 0.

Proof. As usually, we argue locally. By Proposition 3.1a, $\Phi_0(a_m) \in \mathrm{CL}^{-m}(U,E)$. Furthermore, by Lemma 5.2

$$\|\mathrm{Op}\big(\psi(a_m-\lambda)^{-1}\big) - (A-\lambda)^{-1}\|_{s,s+m} \leq C_s(A)|\lambda|^{-\min(\frac{1}{m},1)}$$

for $\lambda \in \Gamma_+$, thus

$$\big\|P_{\Gamma_+}(A) - A\Phi_0(a_m)\big\|_{s,s+m} \leq C_s(A)\int_{\Gamma_+} |\lambda|^{-1-\min(\frac{1}{m},1)}|d\lambda|,$$

and the claim follows. □

**6.2. The locally convex topology $\mathcal{T}$ on $\mathrm{L}^m_{\mathrm{sc}}(M,E)$.** Next we define a locally convex topology on $\mathrm{L}^m_{\mathrm{sc}}(M,E)$. Recall the exact sequence (2.9), adjusted due to the presence of the vector bundle $E$,

$$0 \longrightarrow \mathrm{L}^{m-1}(M,E) \hookrightarrow \mathrm{L}^m_{\mathrm{sc}}(M,E) \xrightarrow{\sigma_m} C^\infty(S^*M;\mathrm{End}(\pi^*E)) \longrightarrow 0. \tag{6.1}$$

Here $\pi : S^*M \to M$ denotes the canonical projection map.

We fix a right inverse

$$\mathrm{Op} : C^\infty(S^*M; \mathrm{End}(\pi^*E)) \longrightarrow \mathrm{L}^m_{\mathrm{sc}}(M, E)$$

of $\sigma_m$, obtained by patching together the local Op-maps (2.4) via a partition of unity. Each choice of the splitting Op induces a vector space isomorphism

$$\begin{array}{ccccc} \mathrm{L}^m_{\mathrm{sc}}(M,E) & \longrightarrow & \mathrm{L}^{m-1}(M,E) & \oplus & C^\infty(S^*M;\mathrm{End}(\pi^*E)) \\ T & \mapsto & T - \mathrm{Op}(\sigma_m(T)) & \oplus & \sigma_m(T). \end{array} \tag{6.2}$$

(For easy reading, we suppress the required cut-off functions). We topologize the right hand side of (6.2) as follows:

1. On $\mathrm{L}^{m-1}(M, E)$ we take the countably many semi-norms $\|T\|_{k+m-1,k}$ , $k \in \mathbb{Z}$.
2. The summand $C^\infty(S^*M; \mathrm{End}(\pi^*E))$ is equipped with the $C^\infty$-topology. This is known to be a Fréchet-topology, hence is generated by countably many semi-norms $\big(p_j\big)_{j\in\mathbb{Z}_+}$.

**Definition 6.3.** The locally convex topology on $\mathrm{L}^m_{\mathrm{sc}}(M, E)$ induced by the countably many semi-norms $\|\cdot\|_{k+m-1,k}$ , $k \in \mathbb{Z}$ and $p_j$, $j \in \mathbb{Z}_+$ is denoted by $\mathcal{T}$.

It follows from complex interpolation that for each *real* $s$ the (semi-)norm $\|\cdot\|_{s+m-1,s}$ is continuous with regard to $\mathcal{T}$.

It is straightforward to see that $\mathcal{T}$ is independent of the choice of Op.

It is worth noting that $\mathcal{T}$ is not complete. By construction, the completion of $\mathrm{L}^m_{\mathrm{sc}}(M, E)$ is a Fréchet space which is of the form

$$\mathrm{CZ}^{m-1}(M, E) \oplus C^\infty(S^*M; \mathrm{End}(\pi^*E)).$$

Here $\mathrm{CZ}^{m-1}(M, E)$ is (a variant of) the well-known Calderón-Zygmund graded algebra (cf. [**Pal65**, Chapter 16]).

Finally, we record that a sequence $(T_n)_{n\in\mathbb{N}} \subset \mathrm{L}^m_{\mathrm{sc}}(M, E)$ converges to $T \in \mathrm{L}^m_{\mathrm{sc}}(M, E)$ if and only if

(i) $\sigma_m(T_n) \longrightarrow \sigma_m(T)$ in the $C^\infty$-topology of $C^\infty(S^*M; \mathrm{End}(\pi^*E))$, and
(ii) $T_n - \mathrm{Op}(\sigma_m(T_n)) \longrightarrow T - \mathrm{Op}(\sigma_m(T))$ with regard to $\|\cdot\|_{k+m-1,k}$ for all $k \in \mathbb{Z}$.

For simplicity we have again suppressed the required cut-off functions.

Combining Proposition 3.1b and the preceding (i), we get the following result (the easy part of the perturbation):

**Corollary 6.4.** *The map*

$$\begin{array}{ccc} \mathrm{Ell}^m_{\Gamma_+}(M,E) & \longrightarrow & \mathcal{B}(H^s(M;E), H^{s+m}(M;E)) \\ A & \mapsto & \Phi_0(A) = \mathrm{Op}\Big(\int_{\Gamma_+} \lambda^{-1}\psi(\xi)\big(a_m(x,\xi)-\lambda\big)^{-1}\,d\lambda\Big) \end{array}$$

*is continuous for all* $s \in \mathbb{R}$.

**6.3. Proof of the perturbation result.** Now we shall finish the proof of the continuous dependence on $A$ of $\Phi$.

Proof of Theorem 1.1. From now on we equip $\mathrm{Ell}^m_{\Gamma_+}(M,E)$ with the topology $\mathcal{T}$. Let $A \in \mathrm{Ell}^m_{\Gamma_+}(M,E)$ and $\Delta A$ be in a neighborhood of 0. Since $\Phi(A) = \int_{\Gamma_+} \lambda^{-1}(A-\lambda)^{-1} d\lambda$ and $A \to \Phi_0(A)$ is continuous, it is sufficient to prove an estimate, uniformly on $\Gamma_+$, of the form

$$\begin{aligned}\|(A+\Delta A-\lambda)^{-1} - (A-\lambda)^{-1} - \mathrm{Op}\big(\psi((a_m+\Delta a_m-\lambda)^{-1} - (a_m-\lambda)^{-1})\big)\|_{s,s+m} \\ \le C_s(A,\Delta A)|\lambda|^{-\min(\frac{1}{m},1)}, \quad (6.3)\end{aligned}$$

such that the following holds: given $\epsilon > 0$, there is a neighborhood $U$ of 0 (in the locally convex topology $\mathcal{T}$) such that for all $\Delta A \in U$, $C_s(A,\Delta A) < \epsilon$.

To prove (6.3), we make an elementary algebraic re-ordering of the left side of (6.3) into five summands and invoke the triangle inequality successively:

$$\begin{aligned}(6.3)\text{, left side} &= \|(A-\lambda)^{-1}(-\Delta A)(A+\Delta A-\lambda)^{-1} \\ &\qquad - \mathrm{Op}\Big(\psi(a_m-\lambda)^{-1}\,(-\Delta a_m)(a_m+\Delta a_m-\lambda)^{-1}\Big)\|_{s,s+m} \\ &\le \|(A-\lambda)^{-1} - \mathrm{Op}(\chi(a_m-\lambda)^{-1})\|_{s,s+m} \\ &\qquad \cdot \|\Delta A(A+\Delta A-\lambda)^{-1}\|_{s,s} \\ &\quad + \|\,\mathrm{Op}(\chi(a_m-\lambda)^{-1})\|_{s,s+m} \cdot \\ &\qquad \|\Delta A - \mathrm{Op}(\chi_1 \Delta a_m)\|_{s+m-1,s}\, \|(A+\Delta A-\lambda)^{-1}\|_{s,s+m-1} \\ &\quad + \|\,\mathrm{Op}(\chi(a_m-\lambda)^{-1})\,\mathrm{Op}(\chi_1\Delta a_m)\|_{s+m,s+m} \cdot \\ &\qquad \|A+\Delta A-\lambda)^{-1} - \mathrm{Op}(\chi_2(a_m+\Delta a_m-\lambda)^{-1})\|_{s,s+m} \\ &\quad + \|\Big\{\mathrm{Op}\big(\chi(a_m-\lambda)^{-1}\big)\,\mathrm{Op}(\chi_1\Delta a_m) - \mathrm{Op}\big(\chi(a_m-\lambda)^{-1}\Delta a_m\big)\Big\} \\ &\qquad \cdot \mathrm{Op}\big(\chi_2(a_m+\Delta a_m-\lambda)^{-1}\big)\|_{s,s+m} \\ &\quad + \|\,\mathrm{Op}\big(\chi(a_m-\lambda)^{-1}\Delta a_m\big)\,\mathrm{Op}\big(\chi_2(a_m+\Delta a_m-\lambda)^{-1}\big) \\ &\qquad - \mathrm{Op}\big(\psi(a_m-\lambda)^{-1}\,\Delta a_m\,(a_m+\Delta a_m-\lambda)^{-1}\big)\|_{s,s+m}.\end{aligned}$$

Here we choose $\chi, \chi_1, \chi_2$ with the same properties like $\psi$ such that $\chi = \chi\chi_1$ and $\chi\chi_2 = \psi$.

Now apply the Main Technical Lemma 4.2 to the last two summands and the Lemmata 5.1 and 5.2 to the first three summands, and we are done. ☐

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