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Old Babylonian “Algebra”, and What It Teaches Us about Possible Kinds of Mathematics

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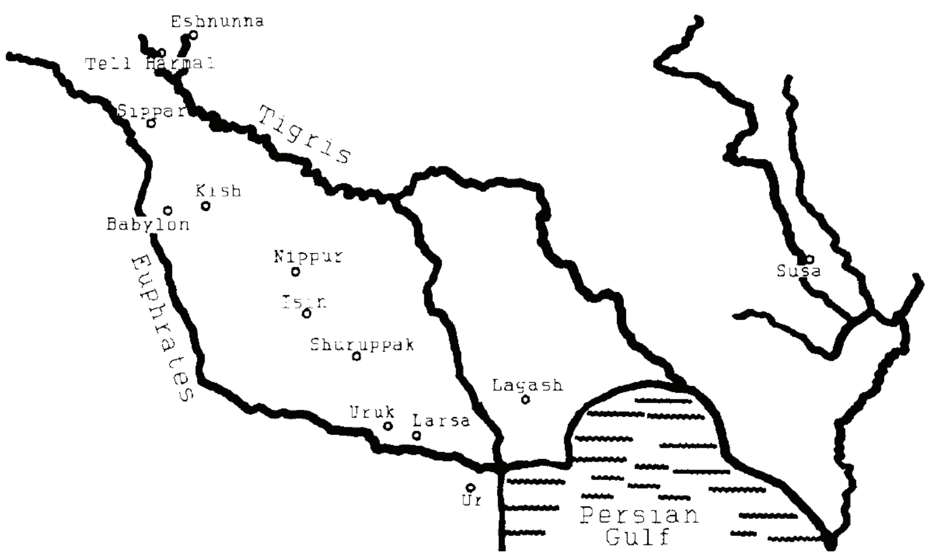
Abstract

Until recently, Old Babylonian "algebra" (mostly identified simply as "Babylonian) either looked very much like recent equation algebra in presentations of the history of mathematics, or it was characterized as "empirical", a collection of rules found by trial and error or other (unidentified) methods not based on reasoning. In the former case, the implicit message was a confirmation of the status of our present type of mathematics as *mathematics itself*. The message inherent in the second portrait is not very different: if mathematics is not of the type we know, and whose roots we customarily trace to the Greeks, it is just a collection of mindless recipes (a type we also know, indeed, from teaching of those social classes that are not supposed to possess or exercise reason) – *tertium non datur!*

More precise analysis of Old Babylonian mathematical texts – primarily the so-called algebraic texts, the only ones extensive enough to allow such analysis – shows that both traditional views are wrong. The prescriptions turn out to be neither renderings of algebraic computations as we know them nor mindless rules (or algorithms) to be followed blindly; they describe a particular type of geometric manipulation, which like modern equation algebra is analytical in character, and which displays the correctness of its procedures without being explicitly demonstrative.

The paper explains this, adding substance, shades and qualifications to the picture, and then takes up the implications for our global understanding of the possible types of mathematics – in particular the question whether the notion of an "algorithmic" type offers relevant insights.



Map of southern and central Mesopotamia,
with ancient coast line and main rivers

Three readers of "Babylonian algebra"

In many general expositions of the history of mathematics one finds a treatment of "Babylonian mathematics", mostly presented without chronological distinction between the "Old Babylonian" epoch (2000 to 1600 BCE according to the "middle chronology") and the Seleucid period (third and second century BCE) – the only periods from which these expositions know about mathematical texts.

As a matter of fact, the texts from the two periods are rather different in character, and in what follows I shall only speak about the mathematics of the former period – omitting except for a brief remark even the nineteenth-century (?) texts from Ur, which constitute a historical dead-end.

The so-called "algebra" is indeed best known from texts written between 1800 and 1600 BCE. Being the earliest example of what can be considered "advanced mathematics", this mathematical genre is treated sometimes briefly, sometimes extensively in many historical presentations. The authors of these, being unable to read the original texts, depend on what they have found in the commentaries in the text editions of Otto Neugebauer, François Thureau-Dangin and (after 1961) Evert M. Bruins, and furthermore read selectively through their own understanding of mathematics. We may look at three examples.

In [1953], J. E. Hofmann published the first of three volumes of an utterly condensed *Geschichte der Mathematik*. Borrowing an observation from Neugebauer (sharpening it unduly and locating it wrongly as can happen when one does not fully understand the background for an observation) he states on p. 11 that

> the particular ideographic writing causes the prescriptions for the solution of practical problems to be almost untranslatable into language, showing thus a certain kinship with the representation through algebraic formulae[1]

after which follows this formula for the truncated pyramid:

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[1] "Infolge der eigenartigen ideographischen Bezeichnungen sind die zur Behandlung praktischer Aufgaben gegebenen Anweisungen sprachlich beinahe unübersetzbar und zeigen daher gewisse Verwandtschaft mit der formelmäßigen algebraischen Wiedergabe". My translation, as all translations in the following when nothing else is stated.

Neugebauer's observation concerns the so-called series texts, which contain no prescriptions but only list problem statements, and they are as far from anything practical as Babylonian texts can be.

$$h[(\tfrac{a+b}{2})^2+\tfrac{1}{3}(\tfrac{a-b}{2})^2]\ .$$

More significant is perhaps what follows on p. 14:

> On other occasions, linear equations with several unknowns are solved diligently, and further equations of the form $ax+by = C$, $xy = D$ and those that can be reduced too it, where the transformation $(x+y)^2 = (x-y)^2+4xy$ is constantly made use of.[2]

Possibly, Hofmann is aware that his "linear equations with several unknowns" are (e.g.) word problems dealing with fields of different area and different rent per area unit – problems whose translation into algebraic equations (Viète's *zetetics*) presupposes choices and therefore is not unambiguous; but he gives his readers no possibility to discover, what he speaks about is exclusively the result of blunt mapping onto the conceptual grid of present-day mathematics.

Carl Boyer's *A History of Mathematics* from [1968] is not very different on this account. He states (p. 33) that

> in an Old Babylonian text we find two simultaneous linear equations in two unknown quantities, called respectively the "first silver ring" and the "second silver ring"

and goes on to explain that "if we call these x and y in our notation, the equations are $x/7+y/11 = 1$ and $6x/7 = 10y/11$"; again, the problem seems to be born as an equation, not as a description of a situation whose translation into an equation already involves choices.

On pp. 34*f* we are told that

> quadratic equations in ancient and Medieval times – and even in the early modern period – were classified under three types:
>
> $$(1)\qquad x^2+px = q$$
> $$(2)\qquad x^2 = px+q$$
> $$(3)\qquad x^2+q = px$$
>
> All three types are found in Old Babylonian texts of some 4000 years ago. The first two types are illustrated by the problems given above; the third appears frequently in problem texts, where it is treated as equivalent to the simultaneous system $x+y = p$, $xy = q$.

Of Boyer's two "problems given above", the first is said (adequately) to call "for the side of a square if the area less the side is 14,30" and then to be equivalent to solving $x^2-x = 870$ (NB, not to $x^2 = x+870$ as type (2) would dictate); the second

[2] "Bei anderer Gelegenheit werden lineare Gleichungen mit mehreren Unbekanntgen geschickt gelöst, außerdem Gleichungen der Form $ax+by = C$, $xy = D$ und darauf zurückführbare, wobei fortwährend die Umformung $(x+y)^2 = (x-y)^2+4xy$ herangezogen wird".

is referred to simply as "the equation $11x^2+7x = 6;15$". What is said about the classification of equation types cannot come from any inspection of sources preceding al-Khwārizmī, even in translation (I wonder whether the inspiration might be an utterly sloppy reading of [Gandz 1937]). No Babylonian text ever adds a number and either a length or an area, which rules out the presence of (2) and (3) as such in Babylonian texts. There is *one* line in *one* text which corresponds to a transformed shape of (3), stating the excess of the square side over the area (in Boyer's symbols, $px–x^2 = q$),[3] but since Bruins does not even comment upon this single line[4] Boyer will not have known it. So this type is not "treated as equivalent to the simultaneous system $x+y = p$, $xy = q$" – what hides behind these words is that the texts contain problems where the sum of the sides and the area of a rectangle are known, which in symbolic translation *coincides with* the "simultaneous system".

In one respect, even Morris Kline's *Mathematical Thought from Ancient to Modern Times* from [1972] is not very different. Kline also speaks of algebra (etc.), and translates Babylonian results into modern symbolic formulae. About a problem which he translates (p. 9)

> I have multiplied Length and Breadth and the Area is 10. I have multiplied the Length by itself and obtained an Area. The excess of Length over Breadth I have multiplied and this result by 9. And this Area is Area obtained by multiplying the Length by itself. What are the Length and Breadth?

he says without explaining why that it is "obvious here that the words *Length, Breadth* and *Area* are merely convenient terms for two unknowns and their product, respectively", and further that today

> we would write this problem as
> $$xy = 10$$
> $$9(x–y)^2 = x^2.$$
> The solution, incidentally, leads to a fourth-degree equation in x, with the x and x^3 terms missing so that it can be and was solved as a quadratic in x^2.

10 in the first line should be interpreted as 10`, i.e., 600.[5] What is meant by the

[3] TMS V, III.15, see [Bruins & Rutten 1961: 47, 48].

[4] Perhaps because he forgot to do so when preparing the edition, but perhaps instead, as he claimed in a letter to me, because he wanted to demonstrate the incompetence of colleagues whom he (justly, as it turned out) expected to overlook the line if there was no commentary.

[5] The mathematical texts write numbers in a sexagesimal floating point place value system. Fixing the sexagesimal point in his translations, Otto Neugebauer would write a number

claim that the problem *was* solved as a quadratic in x^2 is not clear. The actual method, if translated into symbols, finds that $3(x-y) = x$, whence $x = 3z$, $y = 2z$, and in consequence $6z^2 = 600$ (see below, p. 11). But whether anything is really meant is doubtful. In a summary "evaluation of Babylonian mathematics", Kline observes (p. 14) that the Babylonians

> did solve by correct procedures rather complicated equations involving unknowns. However, they gave verbal instructions only on the steps to be made and offered no justification of the steps. Almost surely, the arithmetic and algebraic processes and the geometrical rules were the end result of physical evidence, trial and error, and insight. That the methods worked was sufficient justification to the Babylonians for their continued use.

The final period removes any substance one might believe to be implied by the term "insight". Since

> The concept of proof, the notion of logical structure based on principles warranting acceptance on one ground or another, and the consideration of such questions as under what conditions solutions to problems can exist, are not found in Babylonian mathematics

Babylonian calculators obviously did not base their mathematics on understanding. As seen by Kline, Babylonian mathematics was thus not really mathematics at all – in complete agreement with his claim (p. 3) that

> mathematics as an organized, independent and reasoned discipline did not exist before the classical Greeks of the period from 600 to 300 B.C. entered upon the scene.

So, while Hofmann and Boyer find our kind of mathematics in the Babylonian texts, Kline finds something different – but so different that it does not really count as mathematics. The implied message (probably resulting because it is the implicit starting point for all) is the same: there is only one kind of mathematics: *ours*.

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like $a\cdot60^2+b\cdot60+c+d\cdot60^{-1}+e\cdot60^{-2}$ (where a, b, c, d and e are integers between 0 and 59) as $a,b,c\,;d,e$. This notation is used by Boyer in the above. François Thureau-Dangin, instead, would write *a˵b`c°d´e˝*, an extension of our usual degree-minute-second-system (which indeed descends from the Mesopotamian place value system) which I am going to use. ´ can thus be pronounced "minute", ˝ "second", ‴ "third", etc. For ` I shall suggest g à l , Sumerian for "great" and used exactly about enlargement by a factor 60.

Interpretations

Hofmann, Boyer and Kline all depended on the translations and interpretations made by Neugebauer, Thureau-Dangin and Bruins, and on more popular expositions like Neugebauer's *Vorgriechische Mathematik* [1934] and *Exact Sciences in Antiquity* [1957] and B. L. Van der Waerden's *Erwachende Wissenschaft* [1956]. Neugebauer and Thureau-Dangin made their translations at a moment when the technical terminology had yet to be cracked (the late 1920s and the 1930s), and apart from a few terms whose mathematical meaning could be guessed from their general interpretation, the only way to do so was to start from the numbers. An operation which produces 30 from 5 and 6 was thus supposed to be a multiplication. The first approximation to the meaning of the texts – and hence the early translations – thereby came to presuppose a purely arithmetical reading.

In the early 1980s I started looking more closely at the mathematical terminology, taking inspiration from the study of literature – namely from the principles of close reading and structural analysis; the first thorough presentation of my results is in [Høyrup 1990], while [Høyrup 2002a] can be considered the final outcome of the project. It turned out that two different (not synonymous) operations had been interpreted as addition; similarly, there were two different "subtractions" (only one of which is the inverse of one of the "additions"), two different "halves", and no less than four distinct "multiplications". This did not fit the arithmetical interpretation, within which there is space for only one of each class ("there *is* only one multiplication", as Thureau-Dangin observes somewhere as a reason to consider the various multiplicative terms as mere synonyms).

In my presentation of the results, I have used the principle of "conformal translation": apart from well-established logographic equivalence,[6] different terms

[6] Our texts are written in cuneiform, in the Babylonian dialect of the Akkadian language. Words may either be rendered in phonetic writing, where each sign stands for a syllable, or by means of logograms, signs standing each for a whole word (usually without indication of grammatical form, even though grammatical or phonetic complements may provide this information). The same signs have both functions, and the same sign may have several logographic interpretations (mostly however used in different contexts or periods), and also stand for whole groups of related sound values, where sounds belonging to the same group (e.g., *bal*, *pal*) may occur in the same text while the different groups normally belong to different contexts or periods.

are always translated differently, and the same term (unless the word is unambiguously used in distinct functions) is always translated in the same way. The translations are chosen so as to correspond to the general meaning of the original terms.[7] Moreover, words order is conserved in as far as possible, since it structures the architecture of the argument.

All of this may make reading rather awkward, but one can get accustomed to this conformal translation. I shall use it in the following.

The two additions are:

– *to append* (*waṣābum* / DAḪ), an asymmetric operation where one entity is joined to another one, which conserves its identity – as a capital conserves its identity when interest (in Babylonian "the appended") is added. It is by necessity concrete;
– *to accumulate*[8] (*kamārum* / GAR.GAR). This is a symmetric operation, collecting into one sum two magnitudes *or their measuring numbers*. In the latter case, the operation need not be concrete, and magnitudes of different kinds (lengths and areas – areas and volumes – workers, bricks and working days) can be collected.

The two subtractions are:

– *to tear out* (*nasāḫum* / ZI), with synonym *cutting off* (*ḫarāṣum*) and various near-synonyms used in particular situations;[9] it is the inverse of *appending*, a concrete removal of an entity from another quantity of which it is a part;
– *comparison*, the observation that *one quantity goes so and so much beyond another*

The phonetic and logographic writings of the same word are *logographic equivalents*. However, equivalence within one text group or in one function need not extend to other text groups or functions.

Phonetic writing and phonetic representation of Babylonian words are rendered in *italics*. Logograms are rendered as SMALL CAPS, giving either the supposed pronunciation of the corresponding (mostly Sumerian) word or the "sign name", a *possible* Sumerian value.

[7] I have met the objection that technical terms have to be translated *as* and hence *by* technical terms, but the point is that the technical meaning of the Babylonian terms has to be learned from their use, not imported from a different conceptual structure – who would get the idea to translate chemical texts from the early eighteenth century using terms like oxygen, hydrogen etc.?

[8] I would now have chosen "to heap", but I stick to the choices of [Høyrup 2002a] for the ease of comparison.

[9] [Høyrup 1993] contains a detailed discussion of these.

one (*eli* ... *watārum* / UGU ... DIRIG). Even this operation is concrete, it made no more sense to the Babylonians than to us to claim that a length exceeds or falls short of an area.

The inverse of *accumulating* is no subtraction but a splitting into constituent parts (*bêrum*, *sapāḫum*, apparently synonyms). This operation occurs in only a couple of texts.

The four operations originally interpreted as multiplications are:

– *steps of* (A.RÀ), the term used in tables of multiplication, "5 steps of 6" being 30. It is thus a multiplication of pure number by pure number;
– *to raise* (*našum* / ÍL), originally used in volume calculation, where the base is "raised" from the default height of one cubit to the real height, and then transferred metaphorically to other calculations of concrete magnitudes by multiplication;[10]
– *to make* [two segments] *hold*, namely "hold" a rectangle (*šutakūlum* / $GU_7.GU_7$, with a variety of synonyms). This is thus no genuine multiplication but a construction, mostly implying, however, the determination of the resulting area;
– *to repeat* or *repeat until n* (*eṣēpum* / TAB), a concrete doubling (e.g., of a right triangle into a rectangle) or *n-doubling* (*n* being sufficiently small to be intuitively graspable, $2 \leq n \leq 9$).

The construction of a square can be described as "making the side hold". However, there is also the possibility to *make it confront itself* (*šutamḫurum*). The square configuration itself is called a *confrontation* (*mitḫartum*, more precisely designating a situation characterized by the confrontation of equals). It refers to the square frame rather than to the area it contains – whereas our square *is* (say) 4 m^2 and *has* a side 2 m, a "confrontation" *is* 2 m and *has* an area 4 m^2. However, finding the side *s* of an area *Q* laid out as a square is expressed in a Sumerian phrase: *by* Q, s *is equal* (*Q*.E *s* $ÍB.SI_8$). The ambiguity of "by" corresponds to historical development and to a Sumerian ambiguity. Originally, it meant that *s* was equal (*viz*, to the other sides) *close by* the square area *Q* (thus still in texts from nineteenth-century (?) Ur). Other Old Babylonian scribes, however, understood that *s* was *made equal* (that is, made a square-side) *by Q*.

[10] Always implying some kind of proportionality, even where *we* do not notice it. For instance, the calculation of an area can be understood as an expansion of a "broad line" (see below, p. 24) from the standard width of one length unit to the real width.

We may remember that the Euclidean definition of multiplication (*Elements* VII, def. 15) is based on the same principle.

In certain text groups, ÍB.SI$_8$ is used as a noun, in which case I shall translate it "the equalside". When one side is found, the other side which it meets in a corner may be spoken of as its *counterpart* (*meḫrum* / GABA), the name also used (e.g.) about the exact copy of a tablet.

It is an oft-repeated claim that the Babylonians did not know division. This statement is ambiguous and only partly true. Of course they knew the division *problem* "What shall I raise to *b* which gives me *A*", and many mathematical texts formulate that question. However, division was no *operation* for them. Here, if possible, they used multiplication by the reciprocal of the divisor – more precisely, by its IGI, the reciprocal as listed in a table. Finding the IGI (probably from the table memorized in school) was spoken of as "detaching" it, probably thought of as detaching 1 from a bundle of *n* parts.

In practical calculation, it was always possible to find the IGI: technical constants were chosen to be "regular" numbers, numbers whose reciprocal was a finite sexagesimal number. In the mathematical school texts, on the other hand, division by irregular numbers turns up time and again. Here, the text asks exactly the division question "What shall I ...", and states the answer immediately. Indeed, since the problems where it happens were always constructed backwards from known results, the quotient would always exist – and always be known by the author of the problem.[11]

Halves, as stated, were two. One is the "normal" half, a fraction belonging to the same family as $^1/_3$, $^1/_4$, etc. It can be a number (30´) or the *half of something*, found then via multiplication by 30´. But a half (then only the *half of something*) can also be a "natural" or "necessary" half, as the radius of a circle is the necessary half of the diameter: it has a role quite different from that of, say, $^1/_3$ of the diameter. This natural half I shall designate a *moiety* (*bamtum*); the operation producing it is called *to break* (*ḫepûm* / GAZ).

Old Babylonian "algebra" deals with squares and rectangles and their sides. These were taken in the first interpretation to be mere names (cf. Kline as quoted above; but on this account he is not alone). Actually, this was a mistake, as we

[11] We know from three texts from the third millennium and from a couple of Old Babylonian tables of reciprocals of irregular numbers (probably prepared as exercises, not for practical use) that the Mesopotamian calculators did possess techniques for dividing by irregular divisors. The third-millennium texts suggest that the techniques were created *ad hoc* according to convenience in the single case. The situation may of course have been different in the second millennium.

shall see. In any case, the essential terminology is geometrical, and comprises the following:

– the *length* (UŠ) of a rectangle;
– the *width* (SAG) of a rectangle ("front" would have been a better choice, but once again I stick to my original translation for the ease of comparison);
– the *confrontation*, the square configuration numerically parametrized by the side, cf. above;
– and the *surface* (*eqlum* / A.ŠÀ), the area of a rectangle or square (or any other figure), designated by a word which in general usage means "field" (as did "area" originally, but since we normally do not think of this Latin etymology, "surface" is a more adequate translation).

As already follows from this list of essential terms, the geometry in question is of a particular kind: it is a geometry of measurable segments and the areas they contain, so to say within a square grid.

Some texts

We are now prepared to look at some texts. We may start by reconsidering the problem referred to by Kline (VAT 8390 #1):[12]

VAT 8390 #1[13]

Obv. I

1. [Length and width] I have made hold: 10` the surface.
2. [The length t]o itself I have made hold:
3. [a surface] I have built.
4. [So] much as the length over the width went beyond
5. I have made hold, to 9 I have repeated:
6. as much as that surface which the length by itself
7. was [ma]de hold.
8. The length and the width what?
9. 10` the surface posit,
10. and 9 (to) which he[14] has repeated posit:
11. The equalside of 9 (to) which he has repeated what? 3.
12. 3 to the length posit
13. 3 t[o the w]idth posit.
14. Since "so [much as the length] over the width went beyond
15. I have made hold", he has said
16. 1 from ⌊3 which t]o the width you have posited
17. tea[r out:] 2 you leave.
18. 2 which yo[u have l]eft to the width posit.
19. 3 which to the length you have posited
20. to 2 which ⟨to⟩ the width you have posited raise, 6.
21. Igi 6 detach: 10´.
22. 10´ to 10` the surface raise, 1`40.

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[12] Cuneiform tablets are written in lines, numbered in the editions, mostly on the obverse as well as the reverse of the tablet, and often in columns (indicated by Roman numerals). Mostly, they are identified by their museum numbers (in the present case, Berlin, *Vorderasiatische Texte*, tablet no. 8390).

Damages on the tablet are indicated in square brackets. Because the texts are heavily repetitive, damaged passages can often be reconstructed. What corresponds to parts of words omitted during writing stands in pointed brackets, while explanatory words are added in round brackets.

[13] First published in [Neugebauer 1935: I, 335*f*]. Here following [Høyrup 2002a: 61–63].

[14] This "he" in the reference to the statement shows that the voice which explains the procedure is supposed to differ from the one which states the problem.

23. The equalside of 1`40 what? 10.

Obv. II

1. 10 to 3 wh[ich to the length you have posited]
2. raise, 30 the length.
3. 10 to 2 which to the width you have po[sited]
4. raise, 20 the width.
5. If 30 the length, 20 the width,
6. the surface what?
7. 30 the length to 20 the width raise, 10` the surface.
8. 30 the length together with 30 make hold: 15`.
9. 30 the length over 20 the width what goes beyond? 10 it goes beyond.
10. 10 together with [10 ma]ke hold: 1`40.
11. 1`40 to 9 repeat: 15` the surface.
12. 15` the surface, as much as 15` the surface which the length
13. by itself was made hold.

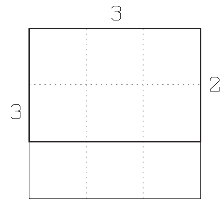

Figure 1. The configuration dealt with in VAT 8390 #1.

As we see, the text starts by constructing a rectangle. The sides are unknown, but the area is given. In the next step, the square on the length is constructed, and since its area is so far unknown, it is only stated that a surface results. Then a square is constructed on the excess of the length over the width (indicated by the standard expression "So much as" (*mala*), functioning as an algebraic parenthesis) and "repeated until 9", which is "as much as" (*kīma*) the square on the length. The length and width are then asked for.

This already shows us what a Babylonian "equation" is: a statement that (the measure) of a (mostly composite) entity is so and so much, or that (the measure of) one entity is "as much as" (the measure of) another entity.[15] This is no different from the equations of any applied algebra; the difference, as we shall see, is that the Babylonians did not *operate* on their equations.

At first (lines I.9–10) the given numbers are "posited", that is, taken note of materially (if only memorized, they would have to be "held in your head"). Then (lines 12–13), since the square on the excess is "repeated until" 9 and then becomes a square, the "equalside" of 9 is found to be 3, a number which is posited *to* length as well as width (see Figure 1). The following step is argued with a quotation from the statement "since ... he has said" (a standard phrase for this): the small square, of which the large square contains 3×3 copies, is the square on the excess of length over width. When removing ("tearing out") 1 of these from the width, 2 are left, which must correspond to the width of the rectangle. Therefore, the number 2 is posited to the width. In total, the rectangle

[15] The excess of one over the other is also a possibility.

therefore contains 2·3 = 6 small squares (found by "raising", the rectangle being already there; line 20).

Lines 21–22 now perform an igi-division of 10` (identified as the rectangle surface) by 6, finding the area of the single small square to be 1`40; line 23 then finds that its side is 10. Finally (lines II.1–4) the sides are found by raising the numbers "posited to" the length and the width to this side. Lines II.5–13 contain a proof, that is, a control of the correctness of the result.

This is what Kline characterized as "a fourth-degree equation in x [that] was solved as a quadratic in x^2". As we see there are two equations, not one. Before they can be transformed into one equation, the passage I.11–20 – the passage where everything tricky is found – has to be worked through. One may suspect that Kline never read Neugebauer's translation but only his mathematical commentary [1935: I, 339*f*], in which the problem is first stated (as *two* equations in x and y) and then reduced. However, Neugebauer has the introductory clause that "one would have to proceed more or less as follows" – not exactly a claim that this is what the Babylonians did! And indeed, if we look at the text without prejudice it is difficult to find anything in it which looks like an algebraic transformation.

One observation should be added: it looks as if the numbers 3 and 2 that are posited to the width are posited to *the same* width. Presumably, the diagram to which the text refers should be understood as nothing but a rough rectangular structure diagram which could serve for the square as well as the rectangle. Since the diagrams serving the solution are never drawn on the tablet (only diagrams supporting the statement are found on these[16]), they were probably made in dust or sand (as were the working diagrams of Ancient Greek geometers), which allowed that lines that had been drawn or numbers that had been written could be erased and replaced by other lines or numbers.

BM 13901 #1[17]

This problem is the first and the simplest from a tablet containing in total 24 "algebraic" problems about one or several squares; translated into modern algebraic symbols, it also becomes the simplest of all mixed second-degree problems, $x^2+x = \alpha$ ($\alpha = {}^3/_4$)).

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[16] Actually, there is one exception to this rule, the tablet YBC 8633, see [Høyrup 2002a: 254]. However, this is definitely not an "algebraic" text in even the most general sense.

[17] First published by Thureau-Dangin [1936: 31], then (translation and transliteration only) in [Neugebauer 1935 : III, 1, 5*f*]. Here following [Høyrup 2002a: 50–52].

Obv. I

1. The surfa[ce] and my confrontation I have accu[mulated]: 45´ is it. 1, the projection,
2. you posit. The moiety of 1 you break, [3]0´ and 30´ you make hold.
3. 15´ to 45´ you append: ⌊by] 1, 1 is equal. 30´ which you have made hold
4. in the inside of 1 you tear out: 30´ the confrontation.

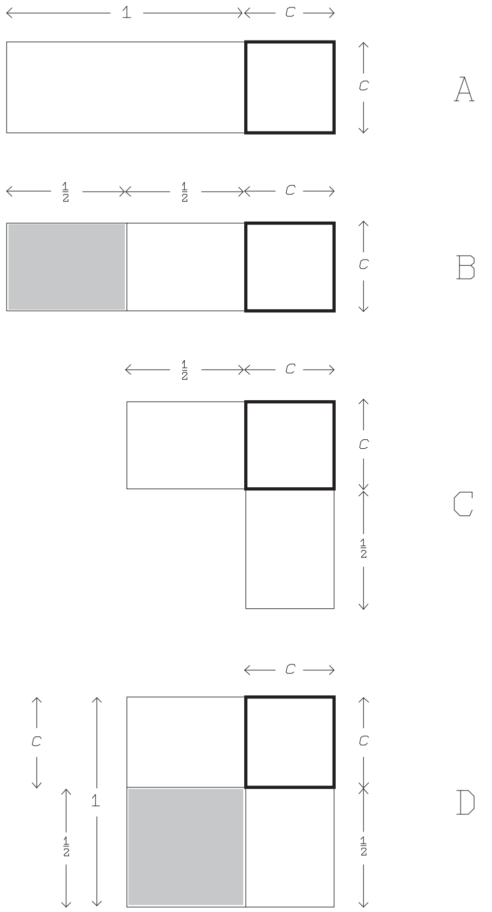

Figure 2. The procedure of BM 13901 #1, in slightly distorted proportions.

Initially, we are told that the sum of (the measures of) a square area and the side (the "confrontation") is 45´(= $^3/_4$). In order to make this sum concretely meaningful, the side is provided with a "projection" – in Babylonian *waṣītum*, meaning something which sticks out or projects, e.g. (in architecture) from a building. Thereby the side is transformed into a rectangle, which can meaningfully be glued onto the area – see Figure 2. This "projection" is bisected and the outer part (together with the appurtenant part of the rectangle) is moved so as to form a gnomon. The two halves are now caused to "hold" a supplementary square, which is "appended" to the gnomon. The surface of the resulting square is 1, close by which "1 is equal" – that is, its side is 1. Removing ("tearing out") from "the inside" of this side[18] that half of the projection which was moved, we are left with the original side, which must hence be 1–30´ = 30´.

We may compare with the procedure by which we solve the corresponding modern equation (disregarding negative numbers, which the Babylonians did not have):

$$
\begin{aligned}
x^2 + 1\cdot x = {}^3/_4 \quad &\Leftrightarrow \quad x^2 + 1\cdot x + ({}^1/_2)^2 = {}^3/_4 + ({}^1/_2)^2 \\
&\Leftrightarrow \quad x^2 + 1\cdot x + ({}^1/_2)^2 = {}^3/_4 + {}^1/_4 = 1 \\
&\Leftrightarrow \quad (x + {}^1/_2)^2 = 1
\end{aligned}
$$

[18] Literally "from the heart" or "from the bowels". This word was always omitted from the early translations, as it made no sense in the arithmetical interpretation.

$$\Leftrightarrow \quad x + {}^1\!/_2 = \sqrt{1} = 1$$
$$\Leftrightarrow \quad x = 1 - {}^1\!/_2 = {}^1\!/_2 \ .$$

We observe that the sequence of numbers occurring in the Babylonian text coincides with what we find here. Moreover, the Babylonian method is *analytical* in the same (classical) sense as the solution by equation: we suppose that the object sought after (the "confrontation" respectively the number represented by *x*) exists; we take note of what we know about it, treat it as a normal entity, thus transforming what we know about it until we have disentangled it.

Finally, both methods are "naive": we do not argue explicitly for their correctness, but we "see" immediately that they are correct. We *could* undertake a Kantian critique, investigating *in which sense and to which extent* our procedure is justified[19] – but even we mostly do not feel the need for that.

There are thus much better reasons to consider this an algebraic text than the mere possibility to translate its presumed "mathematical substance" into equation transformations. Whether the reasons are sufficient is a different matter.

TMS XVI #1[20]

Our next examples shows that the Babylonians *did* argue in some way for the correctness of their procedures – not deductively, as did Euclid and as does modern theoretical mathematics, but by imparting conceptual understanding of what goes on. At first we shall look at the transformation of a first-degree equation.

1. [The 4th of the width, from] the length and the width to tear out, 45´. You, 45´
2. [to 4 raise, 3 you] see. 3, what is that? 4 and 1 posit,
3. [50´ and] 5´, to tear out, ⌈posit⌉. 5´ to 4 raise, 1 width. 20´ to 4 raise,
4. 1°20´ you ⟨see⟩, 4 widths. 30´ to 4 raise, 2 you ⟨see⟩, 4 lengths. 20´, 1 width, to tear out,
5. from 1°20´, 4 widths, tear out, 1 you see. 2, the lengths, and 1, 3 widths,

---

[19] In the case of the equations by using the Euclidean axioms (or an updated version): "when equals are added to equals, the wholes are equal", etc. In the Babylonian case, for instance by emulating the proof of *Elements* II.6.

In a sense, analysis is always "naive", assuming the existence and properties of the objects it deals with without having proved either; if we end up with a contradiction, as is possible, we have an indirect proof that the initial naive assumption is impossible. Synthesis is the corresponding critique. However, the two solutions under discussion are naive also in other respects.

[20] First published in [Bruins & Rutten 1961: 91*f*], where the commentary is unfortunately almost as mistaken as can be. I follow the presentation in [Høyrup 2002a: 85–89].

accumulate, 3 you see.

6. Igi 4 de[ta]ch, 15´ you see. 15´ to 2, lengths, raise, [3]0´ you ⟨see⟩, 30´ the length.

7. 15´ to 1 raise, [1]5´ the contribution of the width. 30´ and 15´ hold.

8. Since "The 4th of the width, to tear out", it is said to you, from 4, 1 tear out, 3 you see.

9. Igi 4 de⟨tach⟩, 15´ you see, 15´ to 3 raise, 45´ you ⟨see⟩, 45´ as much as (there is) of [widths].

10. 1 as much as (there is) of lengths posit. 20, the true width take, 20 to 1´ raise, 20´ you see.

11. 20´ to 45´ raise, 15´ you see. 15´ from $^{30}{}_{15}$´ [tear out],

12. 30´ you see, 30´ the length.

As we see, the equation deals with the length and width of a rectangle; however, already in line 1 these are added with a mere "and", an ellipsis for "accumulation". If one had been "appended" to the other, we would still be bound to the rectangle; as things actually are the rectangle is left behind from the very beginning – its only role is to place its sides at disposal. Apart from its conflation of the two additions, a translation into a symbolic equation seems adequate (in particular if we notice that all multiplications occurring in the transformation are "raisings"):

$$(l+w)-\tfrac{1}{4}\,w = 45' \ .$$

In any case we notice that the subtraction here is a "tearing-out". One fourth of the width is indeed part of the accumulation of length and width.

The text starts by raising 45´ – the right-hand side of the equation – to 4, from which results 3, and then asks for an explanation of this number. This explanation is given in lines 2–5. In line 3 we see that the two "unknowns" are already known; the transformation is thus explained on the basis of a figure with known dimensions. Each of the contributions 5´, 20´and 30´ is "raised" to 4, and the resulting numbers 20´, 1°20´ and 2 are identified, respectively, with w, $4w$ and 4ℓ. Tearing out 20´ = w from 1°20´ = $4w$ yields 1 = $3w$ – in total thus 3, as was to be explained:

$$4\ell+3w = 3 \ .$$

Next, the texts goes back, dividing this new equation by 4, that is, raising to IGI 4 = 15´. This gives the *contributions* of length and width in the original equation, 30´ respectively 15´ ("held" in memory in line 7), and "how much there is" of each, that is, the *coefficients*, respectively 1 and 45´. A final control shows that multiplication of length and width with these coefficients and removal of 45´w from the sum (already written in a way that corresponds to the numbers memorized in line 7) leaves the (contribution of the) length.

The occurrence of a “true width” which is apparently indistinguishable from the width without epithet (line 10) is unique in the corpus (the notion of a “true” entity is not, cf. below, p. 26). Probably, the “true” length and width were 30 NINDAN and 20 NINDAN, the NINDAN or “rod” (the standard unit of horizontal distance) being c. 6 m. Evidently, a rectangle of 120 m × 180 m would not fit in the school yard; so, the standard “school extensions” were reduced by an order of magnitude, resulting in a rectangle 2 m × 3 m, which fits nicely within the yards of Babylonian houses.

In line 8 we observe once more a quotation from the statement used as argument for at particular step.

This is certainly no deductive proof of anything. It is a nice didactical explanations of concepts relevant to the understanding of the equations and of what goes on in their transformation. Already Neugebauer argued that many Babylonian problems were so complex that solutions could never have been found without good understanding, and supposed that such explanations were imparted orally. The present text is from Susa, a peripheral area (in a valley in the Zagros), which may explain that an oral tradition of this kind had to be put into writing. However, once we recognize the style, we can find traces elsewhere in the corpus of similar though more rudimentary expositions, combined with problem solutions.

TMS IX[21]

The next text we shall look at, also from Susa, contains precisely this combination of (certainly not rudimentary) didactical explanation and problem solution.

1. The surface and 1 length accumulated, 4[0´. ¿30, the length,? 20´ the width.]
2. As 1 length to 10´ ⸢the surface, has been appended,]
3. or 1 (as) base to 20´, [the width, has been appended,]
4. or 1°20´ [¿is posited?] to the width which 40´ together ⸢with the length ¿holds?]
5. or 1°20´ toge⟨ther⟩ with 30´ the length hol[ds], 40´ (is) [its] name.
6. Since so, to 20´ the width, which is said to you,
7. 1 is appended: 1°20´ you see. Out from here
8. you ask. 40´ the surface, 1°20´ the width, the length what?
9. [30´ the length. T]hus the procedure.

[21] First published in [Bruins & Rutten 1961: 63*f*], once again with a misleading commentary (which has been largely accepted in the literature). Here after [Høyrup 2002a: 89–95]. The tablet is damaged and the text sometimes without parallel, for which reasons the wording of some restitutions (marked ¿…?) is uncertain.

#2

10. [Surface, length, and width accu]mulated, 1. By the Akkadian (method).
11. [1 to the length append.] 1 to the width append. Since 1 to the length is appended,
12. [1 to the width is app]ended, 1 and 1 make hold, 1 you see.
13. [1 to the accumulation of length,] width and surface append, 2 you see.
14. [To 20´ the width, 1 appe]nd, 1°20´. To 30´ the length, 1 append, 1°30´.
15. [¿Since? a surf]ace, that of 1°20´ the width, that of 1°30´ the length,
16. [¿the length together with? the wi]dth, are made hold, what is its name?
17. 2 the surface.
18. Thus the Akkadian (method).

#3

19. Surface, length, and width accumulated, 1 the surface. 3 lengths, 4 widths accumulated,
20. its [17]th to the width appended, 30´.
21. [You], 30´ to 17 go: 8°30´ [yo]u see.
22. [To 17 widths] 4 widths append, 21 you see.
23. [21 as] much as of widths posit. 3, of three lengths,
24. [3, as] much as lengths posit. 8°30´, what is its name?
25. [3] lengths and 2[1 wid]ths accumulated.
26. 8°30´ you see
28. [Since 1 to] the length is appended [and 1 t]o the width is appended, make hold:
29. 1 to the accumulation of surface, length, and width append, 2 you see,
30. [2 the sur]face. Since the length and the width of 2 the surface,
31. [1°30´, the length, toge]ther with 1°20´, the width, are made hold,
32. [1 the appe]nded of the length and 1 the appended of the width,
33. [make hold ¿1 you see? 1 and 1,] the various (things), accumulate, 2 you see.
34. [3 ..., 21 ..., and 8°30´ accumulate,] 32°30´ you see;
35. [s]o you ask.
36. [...] of widths, to 21, that accumulation:
37. [...] to 3, lengths, raise,
38. [1`3 you see. 1`3 t]o 2, the surface, raise:
39. [2`6 you see, ¿2`6 the surface?.] 32°30´ the accumulation break, 16°15´ you ⟨see⟩.
40. {...}. 16°15´ the counterpart posit, make hold,
41. 4`24°3´45˝ you see. 2`6 [¿erasure?]
42. from 4`[2]4°3´45˝ tear out, 2`18°3´45˝ you see.
43. What is equal? 11°45´ is equal, 11°45´ to 16°15´ append,
44. 28 you see. From the 2nd tear out, 4°30´ you see.
45. Igi 3, of the lengths, detach, 20´ you see. 20´ to 4°[30´]
46. {...} raise: 1°30´ you see,
47. 1°30´ the length of 2 the sur[face. What] to 21, the widths, [may I posit]
48. which 28 give[s me? 1°20´ po]sit, 1°20´ the width
49. of 2 the surface. Turn back. 1 from 1°[30´ tear out,]
50. 30´ you see. 1 from 1°20´ tear [out,]
51. 20´ you see.

The text consists of three sections, #1 and #2 of which are didactical explanations, while #3 is an application of the methods explained in TMS XVI #1 and in #2 of the present text.

All three sections are based on the usual rectangle $\sqsubset\!\sqsupset(20',30')$. #1 has the task to explain the geometric interpretation of the equation that the accumulation of the area and 1 length is 40´. It explains that *appending* 1 length to the area is equivalent to appending a “base” of 1 to the width[22] – cf. Figure 3. Together with the length, this extended width holds a rectangle $\sqsubset\!\sqsupset(1°20',30')$ with surface 40´. In the end it is shown how one may find the length from the surface and the extended width.

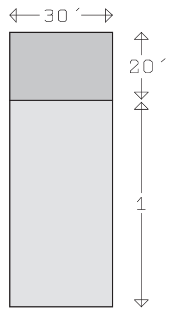

Figure 3. The configuration described in TMS IX #1.

As we observe, the “base” of the present text is equivalent to the “projection” of BM 13901 #1.

#2 accumulates both sides and the surface. It applies the same stratagem in order to make this addition concretely meaningful, but in the first instance this only brings about a quasi-gnomon, a rectangle from which a square □(1) is lacking in a corner – cf. Figure 4. In order to get a useful configuration this square has to be appended, which gives a total surface 1+1 = 2 (line 13). And indeed, the rectangle $\sqsubset\!\sqsupset(1°20',1°30')$ has the surface 2 (line 17).

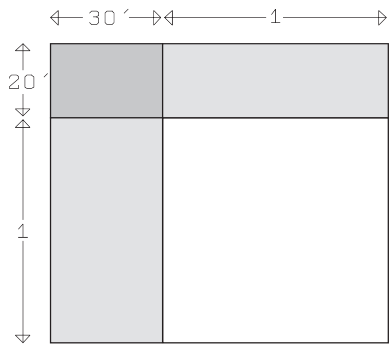

Figure 4. The configuration of TMS IX #2.

The trick to be used is announced in line 10 to be “the Akkadian [method]”, and referred to as such again in line 18. Since the only innovation with respect to #1 is the quadratic completion (though an aberrant variant), this trick must then be what is known as “Akkadian”.

#3 combines the equation that was examined in #2 with an equation of the type that was dealt with in TMS XVI #1 – in short:

$$\sqsubset\!\sqsupset(\ell,w)+\ell+w = 1 \ , \quad \tfrac{1}{17}\,(3\ell+4w)+w = 30' \ .$$

At first, the linear equation is multiplied by the denominator 17; the operation is described as “going”, conceptually related to “steps of” but in Akkadian syllabic writing and only found in some Susa texts, either indicating multiplication or repeated “appending” (both indeed iterative procedures). Thereby this equation becomes

[22] The logogram KI.GUB.GUB is not known from elsewhere, but seems to indicate something which stands erected permanently on the ground. “Base” seems an adequate translation.

$$3\ell+(17+4)w = 8°30´ .$$

We observe that the term for a coefficient turns up again.

Next, the trick from #2 is repeated (lines 28–33). The length and width “of 2 the surface” are introduced – we may call them $\lambda = \ell+1$ and $\omega = w+1$, where thus $⊏⊐(\lambda,\omega) = 2$. Moreover (the damages prevent us from knowing the exact formulation) it is found that

$$3\lambda+21\omega = 32°30´ .$$

If Λ designates 3λ and Ω stands for 21ω (no particular name for these entities occur in the text, we should notice!), we thus have (lines 34–39) that[23]

$$\Lambda+\Omega = 32°30 ,\quad ⊏⊐(\Lambda,\Omega) = (3\cdot21)\cdot2 = 2`6 .$$

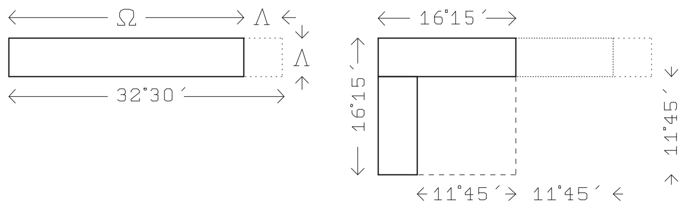

Figure 5. The transformed system of TMS IX #3.

This is a standard problem, the companion piece to BM 13901 #1 reformulated as a rectangle problem (a rectangle of which the area and the difference between the sides is known – Figure 2A shows the equivalence), and it is solved by a different but kindred cut-and-paste procedure – see Figure 5:

That linear extension which is known – here $\Lambda+\Omega = 32°30$ – is “broken”. The resulting half 16°15´ is “made hold” together with its “counterpart”. This produces a square $\square(16°15) = 4`24°3´45´´$, and inside it the parts of the rectangle corresponding to the broken line form a gnomon with area equal to $⊏⊐(\Lambda,\Omega) = 2`6$. The area of the completing square must therefore be $4`24°3´45´´–2`6 = 2`18°3´45´´$, and its “equalside” 11°45´. Addition and subtraction then gives us Λ and Ω, and in two further steps ℓ and w.

We observe that the “name” of an entity (*šumum*) may as well be the explanation of its composition (line 25) as its numerical value (lines 5, 16). As also reflected in the notions of length / width “of 2 the surface”, numerical values may indeed be used to identify entities. In some texts such numerical names are even used for magnitudes whose numerical value is so far unknown. However, the calculators knew perfectly well to distinguish between such values as were *given* and such as were *merely known*. What looks like overdetermination simply reflects the need to give names to entities if the procedure is to be

[23] A simpler variant of the same trick is used to solve non-normalized problems about squares and sides, transforming a problem $\alpha\square(c)+\beta c = \delta$ into $\square(\alpha c)+\beta(\alpha c) = \alpha\delta$. Cf. below, p. 31.

explained; the value of the merely known quantities are never used in the calculation.

A text like this one should bury any belief that Old Babylonian mathematics was merely "empirical" and based on trial-and-error.

YBC 6967[24]

In a way – but only in a way – this simple text presents us with the rectangle version of BM 13901 #1:

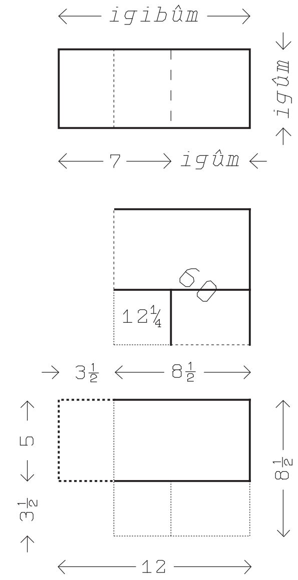

Figure 6. The procedure of YBC 6967.

Obv.

1. [The *igib*]*ûm* over the *igûm*, 7 it goes beyond
2. [*igûm*] and *igibûm* what?
3. Yo[u], 7 which the *igibûm*
4. over the *igûm* goes beyond
5. to two break: 3°30´;
6. 3°30´ together with 3°30´
7. make hold: 12°15´.
8. To 12°15´ which comes up for you
9. [1` the surf]ace append: 1`12°15´.
10. [The equalside of 1`]12°15´ what? 8°30´.
11. [8°30´ and] 8°30´, its counterpart, lay down.

Rev.

1. 3°30´, the made-hold,
2. from one tear out,
3. to one append.
4. The first is 12, the second is 5.
5. 12 is the *igibûm*, 5 is the *igûm*.

Indeed, its topic is not geometrical at all but a couple of numbers from the IGI-table, *igûm* and *igibûm*, Akkadianized versions of IGI and IGI.BI, "the IGI" and "its IGI". However, the reference to the product of the two as a "surface" in obv. 9 shows that these numbers are *represented* by the sides of a rectangle (just as we *represent* geometrical magnitudes by pure numbers when treating geometrical objects and relations algebraically).

The product is not 1°, as we might expect, but 1` – perhaps a reflection of the historical origin of the sexagesimal IGI, which at first were apparently meant as fractions of 60, not of 1 [Steinkeller 1979: 187]. In any case, the value is possible because of the floating-point nature of the place-value system.

[24] First published in [Neugebauer & Sachs 1945: 129]. Here following [Høyrup 2002a: 55–58].

With this in mind, we can easily follow the text and see that it describes the transformations of Figure 6. (The "made-hold" / *takīltum* of line is a verbal noun, equivalent to the relative clause "which you have made hold" of BM 13901, line 3.) We observe that this time it is the gnomon which is "appended" to the completing square, not vice versa. Since both remain in place, this is possible; in contrast, no text ever "appends" en entity which stays in place in this geometric reading to something which is moved around.

Lines rev. 2–3 are interesting. Normally, the Babylonians would let addition precede subtraction, just as we do. Here, however, the subtractive process comes first. The explanation is that it is *the same* piece which is "torn out" and "appended", and before it can be appended it must of course be at disposal. In TMS IX, lines 43–44, in contrast, "appending" comes first. Here, indeed, it is not the same piece that is involved.

This "norm of concreteness", first observed by Aisik A. Vajman [1961: 100], is not respected in all texts. In the texts IM 53965, IM 54559, and Db_2-146, all three quite early (initial eighteenth century BCE), we just find the prescription "to 1 append, from 1 tear out". The norm appears to be a secondary development, introduced by school teachers insisting that operations should be meaningful.[25]

AO 8862[26]

This text is written on a square clay prism. For palaeographic reasons, both Thureau-Dangin and Neugebauer considered it one of the oldest mathematical texts known by then (which should mean 1750 or before). In the meantime, a number of texts have been found in Eshnunna in the north-eastern periphery which date from around 1775. However, the present text may still be one of the

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[25] I shall not elaborate on the origin of Old Babylonian "algebra", only state that there are good reasons to assume that its starting point was a collection of geometrical riddles circulating among Akkadian-speaking, non-scribal practical surveyors – but see [Høyrup 2002a: 362–387].

The name given to the quadratic completion, "the Akkadian method" (above, p. 18), of course fits into the picture.

[26] First partially published by Thureau-Dangin and Neugebauer in several publications, then completely in [Thureau-Dangin 1932] and [Neugebauer 1935: I, 108–111; II, pl. 36; III, 53]. Here following [Høyrup 2002a: 162–174].

oldest from the core area; much in its terminology and certain procedures support this dating.[27]

In total, the text contains 7 problems. I give the translation of the first four, dealing with rectangular fields and their sides:

I

edge Nisaba

#1

1. Length, width. Length and width I have made hold:
2. A surface I have built.
3. I turned around (it). As much as length over width
4. went beyond,
5. to inside the surface I have appended:
6. 3`3. I turned back. Length and width
7. I have accumulated: 27. Length, width, and surface w[h]at?
8. You, by your proceeding,
9. 27, the things accumulated, length and width,
10. to inside [3`3] append:
11. 3`30. 2 to 27 append:
12. 29. Its moiety, that of 29, you break:
13. 14°30´ steps of 14°30´, 3`30°15´.
14. From inside 3`30°15´
15. 3`30 you tear out:
16. 15´ the remainder. By 15´, 30´ is equ[al.]
17. 30´ to one 14°30´
18. append: 15 the length.
19. 30´ [fr]om the second 14°30´
20. you cut off: 14 the width.
21. 2 which to 27 you have appended,
22. from 14, the width, you tear out:
23. 12 the true width.
24. 15, the length, and 12, the width, I have made hold:
25. 15 steps of 12, 3` the surface.
26. 15, the length, over 12, the width,
27. what goes beyond?
28. 3 it goes beyond. 3 to inside 3` the surface append,

[27] I have a hypothesis, so far (and perhaps forever) supported only by indirect evidence but to my knowledge contradicted by neither direct nor indirect evidence: that the characteristic complex of Old Babylonian mathematics emerged in Eshnunna and was then brought to the south after Hammurapi's conquest of the Eshnunna state in 1761. There is indeed nothing in the mathematical texts from nineteenth-century (?) Ur (published in [Friberg 2000], cf. [Høyrup 2002a: 352–354] which points forwards to what develops in the south in the eighteenth century but some affinities with the Eshnunna texts. Similarly, the texts from pre-conquest Mari (i.e., pre-1758, maybe decades older) on the north-western periphery do not point forward to what was to come, nor to contemporary developments in Eshnunna.

29. 3`3 the surface.

#2

30. Length, width. Length and width
31. I have made hold: A surface I have built.
32. I turned around (it). The half of the length
33. and the third of the width
34. to the inside of my surface
35. [I have] appended: 15.
36. [I tu]rned back. Length and width
37. [I have ac]cumulated: 7.

II

1. Length and width what?
2. You, by your proceeding,
3. [2 (as) in]scr[i]ption of the half
4. [and] 3 (as) inscription
5. of the [th]ird you ins[cr]ibe:
6. Igi 2, 30´, you detach:
7. 30´ steps of 7, 3°30´; to 7,
8. the things accumulated, length and width,
9. I bring:
10. 3°30´ from 15, my things accu[mu]lated,
11. cut off:
12. 11°30´ the remainder.
13. Do n[ot] go beyond. 2 and 3 make hold:
14. 3 steps of 2, 6.
15. Igi 6, 10´ it gives you.
16. 10´ from 7, your things accumulated,
17. length and width, I tear out:
18. 6°50´ the remainder.
19. It[s] moiety, that of 6°50´, I break:
20. 3°25´ it gives you.
21. 3°25´ until twice
22. you inscribe; 3°25´ steps of 3°25´,
23. 11°40´[25˝]; from the inside
24. 11°30´ I tear out:
25. 10´25˝ the remainder. 〈By 10´25˝, 25´ is equal〉.
26. To the first 3°25´
27. 25´ you append: 3°50´,
28. and (that) which from the things accumulated of
29. length and width I ha[ve] torn out
30. to 3°50´ you append:
31. 4 the length. From the second 3°25´
32. 25´ I tear out: 3 the width.

32a. 7 the things accumulated.

32b. 4, the length
3, the width   12, the surface

#3

33. Length, width. Length and width
34. I have made hold:
35. A surface I have built.

III

1. I turned around (it). So much as length over width
2. goes beyon[d], together with the things accumulated,
3. my length and [width], I have made hold:
4. To the in[si]de of my surface
5. I have app[end]ed:
6. 1``13`20. I [tu]rned back. Length and width
7. I have accumul[ated:] 1`40.

1`40	1``13`20 the things accumulated
1` the length 40 the width	40` the surface

8. You, by your proceeding,
9. 1,40 the things accumulated, length and width,
10. 1`40 steps of 1`40, 2``46`40.
11. From 2`46`40, 1``13`20 the surface
12. you tear out: 1``33`20.
13. Do not go beyond. The moiety, that of 1`40,
14. you break: 50 steps of 50,
15. 41`40 to 1``33`20 you append.
16. By 2``15`, 1`30 is equal.
17. 1`40 over 1`30 what goes beyond?
18. 10 it goes beyond. 10 to 50 append,
19. 1` the length. 10 from 50 cut off,
20. 40 the width.
40 sag̃

#4

21. Length, width. Length and width
22. {...} I have made hold: A surface I have built.
23. I turned back. Length and width I have accumulated:
24. together with the surface, they confront each other (as equals).
25. Length, width, and surface I have accumulated:
26. 9. Length, width, and surface what?

Nisaba is the goddess of justice and of the scribal arts. The initial invocation is the closest any Old Babylonian mathematical text comes to religion or esoteric matters. Old Babylonian mathematics was no priestly endeavour.

There are two striking deviations from normal usage. One is that linear magnitudes are repeatedly "appended" directly to a surface, without being provided with a "projection" or a "base". The very existence of several names for the same thing suggests that their introduction was a secondary development, once more introduced by school teachers insisting that operations should be meaningful: like Plato (*Laws* 819D–820B) they may have disliked the widespread habit of practical geometers to make use of "broad lines" – i.e., lines carrying a default breadth of one length unit, which allows one to measure areas and lengths in the same units.[28] The present text, however, has no qualms – which is one of the substantial reasons to consider it early.

The other deviation is that the operation of "breaking" is not followed by a construction (the two moieties being "made hold") but by the numerical calculation. Whereas other texts specify the construction and leave the calculation implicit, this text does the opposite (I.12–13; II.19–22;[29] III.13–15), However, when there is no preceding "breaking" operation (II.13–14), that of "making hold" is explicit. So it is when the surface is "built" in the beginning of each statement.

The methods used are interesting not only in themselves but also as alternatives to possibilities which the calculator knew about.

Terminologically we may observe that "turning around" and "turning back", in later texts used just as demarcations within the text of the beginning of new sections, seem here really to speak of the surveyor walking around while laying out a field. Even the dimensions of the fields are not the "school dimensions" discussed above (p. 16).

The text has a tendency to "tear out" from surfaces but to "cut off" from lines, thus choosing among synonyms according to general-language connotations. "Cutting off" possessing no Sumerographic equivalent, it is likely to have been borrowed from a "lay" (i.e., non-scribal) surveyor's environment.

#1

In this first problem, it is given that

$$\sqsubset\!\sqsupset(\ell,w)+(\ell-w) = 3`3\ ,\ \ell+w = 27\ .$$

"Appending" $\ell+w$ *to* $\sqsubset\!\sqsupset(\ell,w)+(\ell-w)$, we get

$$\sqsubset\!\sqsupset(\ell,w)+2\ell = 3`30\ ,$$

which is the situation analyzed in TMS IX #1. The present text follows the same procedure (see Figure 7), thus transforming the system into

$$\sqsubset\!\sqsupset(\ell,w+2) = 3`30,\ \ell+(w+2) = 29$$

[28] For this notion of the "broad line" and its widespread appearance, see [Høyrup 1995].

One may observe that the "broad lines" were never quite suppressed. When asking for the elimination of a side from a square, the texts request it to be "torn out"; no subtractive equivalent of "accumulation" being at hand (the addition which might concern measuring numbers), there was no alternative.

[29] There is a slight doubt in this case. The double "inscription" could refer to the drawing of lines with these lengths. It is more likely, however, that what is spoken of is the customary inscription of the number twice on a tablet for rough work – see, e.g., specimens in [Robson 2000: 24*f*].

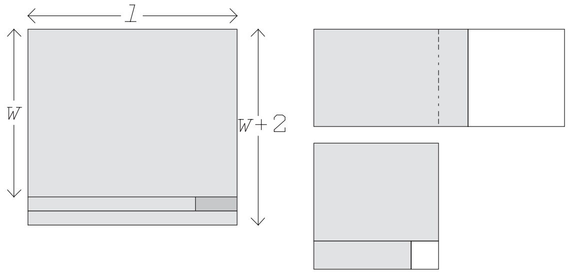

Figure 7. The situation and procedure of AO 8862 #1.

– the standard problem type we encountered in TMS IX #3, and solved by the same procedure.

We notice the distinction between "the width" (w +2) and the "true width" (w).

#2

In #2, half of the length and one third of the width are "appended to the inside" of the surface of a rectangle, with outcome 15. Moreover, "accumulation" of length and width gives 7:

$$\sqsubset\!\sqsupset(\ell,w)+{}^1\!/_2\,\ell+{}^1\!/_3\,w = 15\ , \quad \ell+w = 7\ .$$

This *could* have been solved by the trick of TMS IX #3, by introducing $\lambda = \ell+{}^1\!/_3$, $\omega = w+{}^1\!/_2$ and adding a supplementary rectangle $\sqsubset\!\sqsupset({}^1\!/_2,{}^1\!/_3)$. However, the text chooses a reduction to a situation similar to that of TMS IX #1 (if only ${}^1\!/_3(\ell+w)$ had been torn out, the analogy with TMS #1 and #1 of the present text had been perfect). It halves $\ell+w$ and then "brings" ${}^1\!/_2\,\ell+{}^1\!/_3\,w$ to the place where it can be "torn out" from $\sqsubset\!\sqsupset(\ell,w)+{}^1\!/_2\,\ell+{}^1\!/_3\,w$ – see Figure 8. This eliminates ${}^1\!/_2\,\ell$ but more than ${}^1\!/_3\,w$.

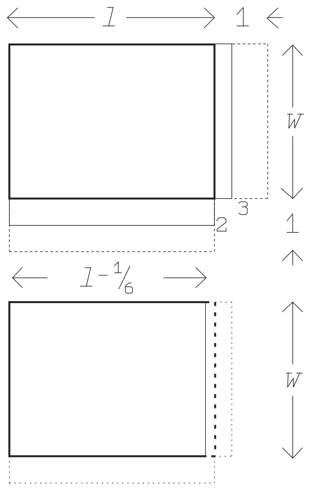

Figure 8. The configuration of AO 8862 #2.

How much more is found by an only halfway explicit geometric argument to be ${}^1\!/_6$ = 10´ (lines II.13–15, see Figure 9). That the difference is not just found as 30´–20´ seems astonishing; *maybe* the method reflects what was done in the non-scribal environment inspiring the problem. In any case, we now know that $\sqsubset\!\sqsupset(\ell$–10´,$w)$ = 15–3°30´ = 11°30´; we further find that (ℓ–10´)+w = 7–10´ = 6°50´. Then everything goes as in #1, with the only difference that ℓ–10´ is not spoken of as a "length", for which reason there is no reason to speak of ℓ itself as the "true" length.

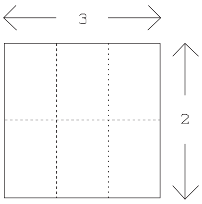

Figure 9. The geometrical determination of ${}^1\!/_2$–${}^1\!/_3$.

#3

In symbolic writing, #3 has as givens that

$$\sqsubset\!\sqsupset(\ell,w)+\sqsubset\!\sqsupset(\ell+w,\ell-w) = 1``13`20,\ \ell+w = 1`40\ .$$

This might once more have been reduced to a form where the TMS IX #1 would serve. Indeed, using the second equation we may rewrite the first equation as

$$\sqsubset\!\sqsupset(\ell,w)+1`40\cdot(\ell-w) = 1``13`20 \,.$$

Adding $1`40\cdot(\ell+w)$ to the left and $140\cdot1`40 = 2``46`40$ to the right (the trick used in #1), we would know that $\sqsubset\!\sqsupset(\ell,w)+3`20\cdot\ell = \sqsubset\!\sqsupset(\ell,w+3`20)$.

However, the text proceeds differently – see Figure 10. It finds $\square(\ell+w) = 2``46`40$, and tears out $\sqsubset\!\sqsupset(\ell,w)+\sqsubset\!\sqsupset(\ell+w,\ell-w) = 1``13`20$. What is left can be identified as $\square(w)+1`40w$, which equals $2``46`40-1``13`20 = 1``33`20$. The equation

$$\square(w)+1`40w = 1``33`20$$

can be solved according to the standard model of BM 13901 #1, and does indeed follow its pattern until the point where $w+\frac{\ell+w}{2}$ is seen to be 1`30. Then, however, something happens which shows how fundamental the *average* ($a$) and *deviation* ($d$) between two magnitudes was in Old Babylonian mathematical thought (here, $\ell = a+d$, $w = a-d$). Since $1`30 = w+\frac{\ell+w}{2} = w+a$ while $1`40 = 2a$, $d$ must be $1`40-1`30 = 10$. This is appended to and cut off from $a = 50$, which gives us ℓ and w. As we see, the “norm of concreteness” is not observed, which fits an early date for the text.

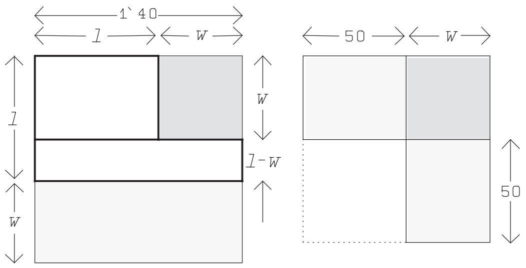

Figure 10. The configuration and procedure of AO 8862 #3.

#4

In the last rectangle problem, the area of the rectangle is stated to equal the sum of the sides, and the sum of all three to be 9 :

$$\sqsubset\!\sqsupset(\ell,w) = \ell+w \,, \quad \sqsubset\!\sqsupset(\ell,w)+\ell+w = 9 \,.$$

The procedure is not indicated, but an easy way would be to change the system into

$$\sqsubset\!\sqsupset(\ell,w) = 4°30' \,, \quad \ell+w = 4°30' \,.$$

#5–7

Problems #5–7 of the text (not translated above) are brickwork calculations, and refer to a technical constant determining a work norm. #5–6 are first-degree problems about proportional sharing between workers. #7 is of the second degree, adding men, days, and bricks produced (the number of which is proportional to the number of man-days), and thus of a kind related in mathematical structure to #1–4 and to TMS IX, #2–3. It thereby presents us with a second instance of a phenomenon we have already encountered in YBC 6967 : algebraic *representation*, the use of the geometry of measurable segments and areas in a square grid –

the *fundamental representation* – to represent entities of other kinds ; such representation is also the very foundation of modern applied algebra (though with an arithmetical fundamental representation). In the present case, workers and working days will have to be represented by line segments, the bricks they produce by an area.

BM 13901 #12[30]

Representation of a different but no less sophisticated kind in involved in this problem, coming from the catalogue of problems about one or more squares which was already mentioned above (p. 12).[31]

Obv. II
27. The surfaces of my two confrontations I have accumulated: 21´40″.
28. My confrontations I have made hold: 10´.
29. The moiety of 21´40″ you break: 10´50″ and 10´50″ you make hold,
31. inside 1´57″21{+25}‴40⁗ you tear out: by 17″21{+25}‴40⁗, 4´10″ is equal.
32. 4´10″ to one 10´50″ you append: by 15´, 30´ is equal.
33. 30´ the first confrontation.
34. 4´10″ inside the second 10´50″ you tear out: by 6´40″, 20´ is equal.
35. 20´ the second confrontation.

As we see, the problem deals with two squares $\square(c_1)$ and $\square(c_2)$, about which we know the sum of their areas and the area of the rectangle contained by their sides,

$$\square(c_1)+\square(c_2) = 21'40'' \;,\quad \sqsubset\!\sqsupset(c_1,c_2) = 10' \;.$$

From this it could be easily derived, on the basis of a diagram used elsewhere on the same tablet, that

$$\square(c_1+c_2) = \square(c_1)+\square(c_2)+2\sqsubset\!\sqsupset(c_1,c_2) = 41'40'' \;,$$

whence

$$c_1+c_2 = 50' \;,$$

which together with the second initial condition would be a standard problem. The text, however, chooses a different path, transforming the problem into a problem concerning a different rectangle about which we know the area and

[30] First published by Thureau-Dangin [1936: 38], then (translation and transliteration only) in [Neugebauer 1935: III, 3, 7]. Here following [Høyrup 2002a: 71–73].

[31] There is a miscalculation in line 30, due to a repeated insertion of an intermediate product on the calculation board [Høyrup 2002b: 2–3]. It is carried over to the following lines, but disappears when the “equalside” is found – obviously because this was not calculated but taken from the known end result.

the sum of the sides – the problem type to which, for instance, TMS IX #3 was reduced.

The sides of this rectangle are the *areas* of the two squares. Their sum, we remember, has to be bisected, and the two moieties caused to contain a square. This happens in lines 29–30. Then the area of the rectangle has to be "torn out" – but this presupposes that it be known, and this magnitude is indeed calculated in line 30. In the end, the sides of the rectangle are found together with their "equalsides", which are the sides of the original squares.

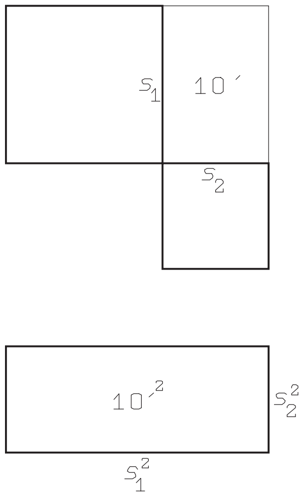

Figure 11. The procedure of BM 13901 #12.

The calculator was thus fully aware that a rectangle contained by two squares equals the square on the rectangle,

$$\sqsubset\!\sqsupset(\square(c_1),\square(c_2)) = \square(\sqsubset\!\sqsupset(c_1,c_2)) \ .$$

We may be tempted to see in this a proof that the geometry of Old Babylonian "algebra" was merely a disguise for underlying arithmetical thinking – but the fallacious reason is that *our* thinking, and our fundamental representation for such relations, is arithmetical. But the present procedure illustrates that our very categories of "arithmetic" and "geometry" are inadequate: the "geometry" of Old Babylonian "algebra" was not only a geometry of measurable segments in a square grid, it also made possible extensions like the present one.[32] Disregarding the difference of mathematical level, nineteenth-century experiments with imaginary geometry come to mind.

BM 13901 #14[33]

Our last example comes from the same tablet as the previous one. Its mathematics is more intricate but its ontology more straightforward – the only "representation" it offers having the character of a "change of variable". But it is interesting for a different reason.

[32] A much more sophisticated example of such representation is found in the problem TMS XIX #2 [Høyrup 2002a: 197–200], where the area of a rectangle $\sqsubset\!\sqsupset(\ell,w)$ is given together with the area of the rectangle $\sqsubset\!\sqsupset(d,\square(\ell))$ contained by the diagonal and the cube on the length. This leads to a bi-biquadratic equation, which is solved correctly – once again apart from an error made on the calculation board..

[33] First published by Thureau-Dangin [1936: 40], then (translation and transliteration only) in [Neugebauer 1935 : III, 3, 8]. Here following [Høyrup 2002a: 73–77].

The present part of the tablet is heavily damaged, but the parallel in #24 of the same tablet (dealing with three squares, and thus more complicated) allows safe reconstruction.

Obv. II

44. The surfaces of my two confrontations I have accumulated: ⌈25´]25˝.
45. The confrontation, two-thirds of the confrontation [and 5´, nind]an.
46. 1 and 40´ and 5´ [over-going 4]0´ you inscribe.
47. 5´ and 5´ [you make hold, 25˝ inside 25´25˝ you tear out:]

Rev. I

1. [25´ you inscribe. 1 and 1 you make hold, 1. 40´ and 40´ you make hold,]
2. [26´40˝ to 1 you append: 1°26´40˝ to 25´ you raise:]
3. [36´6˝40‴ you inscribe. 5´ to 4]0´ yo[u raise: 3´20˝]
4. [and 3´20˝ you make hold, 11˝6‴40⁗] to [36´]6˝40‴ [you append:]
5. [by 36´17˝46‴40⁗, 46´40˝ is equal. 3´]20˝ which you have made hold
6. [inside 46´40˝ you tear out]: 43´20˝ you inscrib[e].
7. [igi 1°26´40˝ is not det]ached. What to 1°2[6´4]0˝
8. [may I posit which 43´20˝ g]ives me? 30´ its *bandûm*.[34]
9. [30´ to 1 you raise: 30´] the first confrontation.
10. [30´ to 40´ you raise: 20´], and 5´ you append:
11. [25´] the second [confrontat]ion

The problem, as we see, deals with two "confrontations" c_1 and c_2. The sum of their surfaces is $\square(c_1)+\square(c_2)$ = 25´25˝, and $c_2 = {}^2\!/_3 c_1$+5´. The "inscriptions" in line II.46 corresponds to the expression of c_1 and c_2 in terms of a new "confrontation" c,

$$c_1 = 1\cdot c\ ,\ \ c_2 = 40'\cdot c+5'\ .$$

That a *new* entity is meant is confirmed by lines III.9–11, where c_1 and c_2 are found in parallel from c. $\square(c_2)$ can then be developed as $p\cdot\square(c)+2q\cdot c+\square(5')$, where p and q are to be determined later. At first, line II.47 determines $\square(5')$ = 25˝ and "tears it out" from $\square(c_1)+\square(c_2)$, leaving 25´.

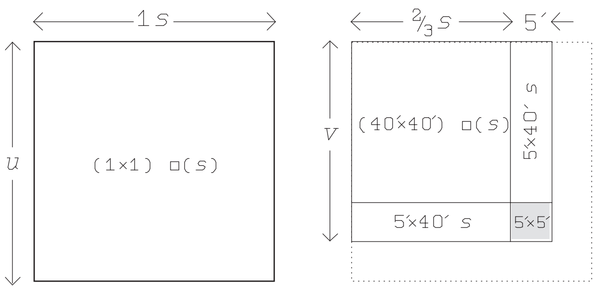

Figure 12. The squares of BM 13901 #14.

Next it is found that $\square(c_1) = \square(1)\cdot\square(c)$[35], and through a similar calculation that

[34] A Sumerian loanword. Its mathematical function is obvious, the number which (in the present case) should be "raised" to 1°16´40˝ in order to give 43´20˝ (functionally thus simply a quotient). The etymological meaning of the term is unclear, but it *might* be "that which is to be given together with". Though rare in the extant text material, the term must have been quite important: in the early part of the first millennium BCE, a calculator was termed a "scribe of the *bandûm*".

[35] We may notice that the principle of this transformation is close to the one used in the previous problem, $\sqsubset\!\sqsupset(\square(c_1),\square(c_2)) = \square(\sqsubset\!\sqsupset(c_1),c_2))$; we only need to replace $1\cdot c$ by $\sqsubset\!\sqsupset(1,c)$.

p is $\square(40')$ = 26´40˝. In total, the number of squares $\square(c)$ contained in $\square(c_1)+\square(c_2)$ is thus 1°26´40˝. In consequence we have that

$$1°26'40''\square(c)+2q\cdot c = 25' \ .$$

The trick routinely used to normalize such non-normalized equations was mentioned in note 23: the equation is "raised" to 1°26´40˝, thus becoming

$$\square(1°26'40''c)+2q\cdot(1°26'40''c) = 1°26'40''\cdot 25' = 36'6''40''' \ ,$$

a problem of the same type as BM 13901 #1, just with the coefficient 1 replaced by $2q$ and a different right-hand side, and grosso modo solved in the same way. One difference is noteworthy, however: line 3 does not find $2q$ but only q, because $2q$ would anyhow have to be "broken" in the next step. We may think of the joke about the difference between the mathematician and the physicist:

> A physicist and a mathematician are put in front of a cooker with two gas-rings, a match-box and an empty kettle standing on the left gas-ring. Asked how to cook water for tea they both tell that you fill the kettle with water and put it back; you turn on the gas, and then you use a match to light the gas. Asked what is to be done if the kettle is to the right, the physicist says "Act correspondingly". The mathematician has a different solution: You move the kettle to the left, reducing thus the situation to the previous case.

The thought of the mathematician of the joke is *algorithmic:* once a subroutine has been constructed, it can be embedded within larger prescriptions and need no further thought. That of our Babylonian "mathematician" is not: he sees the procedure as a whole, and if a step from the embedding part is reversed by a step within the embedded subroutine he eliminates both.

A general characterization

The examples discussed so far do not show all aspects of Old Babylonian "algebra", and even less all aspects of Old Babylonian mathematics. However, they already allow us to make some generalizations.

It is obvious that anything like deductive chains is absent from all texts. Moreover, the absence of all traces even from such didactic texts as TMS XVI and TMS IX and the sometimes rather circular character of their exposition – a feature which goes against the very gist of deductivity – allows us to conclude that their authors had no familiarity with the possibility of deductive presentation. In so far Kline is right that

> The concept of proof, the notion of logical structure based on principles warranting acceptance on one ground or another [...] are not found in Babylonian mathematics.

He is mistaken, however, when he concludes from this that Babylonian mathematics was not reasoned and thus not really mathematics – unless he would also exile the whole history of analysis before Cauchy (or before Weierstraß?) to the pre-history of mathematics.

Old Babylonian "algebra" was certainly "naive" – but so was infinitesimal analysis during the epoch where faith was promised by d'Alembert to come from practising. So was even European *algebra* until the early nineteenth century – the attempts to make it rigorous by basing it on Euclidean geometry failed as soon as theoretical developments went beyond the second degree (and already when it operated with negative numbers).

This naive character of essential branches of Early and not so early Modern mathematics did not prevent the appearance of criticism – criticism is a project and hardly ever a totally stabilized outcome. But even Old Babylonian "algebra" presents us with such attempts at criticism. The "norm of concreteness" can be considered such an attempt, and the abolition of the "broad lines" as another one. Both were apparently introduced as the algebraic discipline became the object of institutionalized teaching in or in the vicinity of the scribe school.[36]

[36] It should perhaps be stressed that we have no external, independent evidence for the existence of such an institutionalized teaching of sophisticated mathematics. However, the standardized format of the texts leaves no doubt that institutionalization had taken place. What we cannot know for certain is the precise relation of this kind of teaching to the normal scribe school (similarly known from the evidence presented by a highly standardized syllabus, not from independent external testimonials).

Both did indeed give a guarantee of consistency and possible existence by linking the "algebraic" technique to a domain of which one could feel sure, eliminating thus the risk that the entities dealt with had "no existence, if not that on paper", in Georg Cantor's vicious words [1895: 501] about Veronese's infinite numbers. Much in the same vein, modern metamathematics ascertains the consistency of a mathematical domain by linking it to the integers, about whose consistency we do feel confident.

Induction based on only a couple of examples is a daring leap. None the less, the Old Babylonian example taken together with "our" mathematics and its Greek, Arabic and Early Modern European ancestor types suggests that mathematics going beyond a very elementary level cannot avoid being reasoned.[37]

An algebra?

Much ink, and much ire, have been disbursed in order to disprove that various pre-Cartesian mathematical theories and techniques can justly be considered "algebraic". Taking these arguments to their full consequence we easily end up in the position once formulated by my friend and colleague Bernhelm Booß-Bavnbek: *there was no algebra before Emmy Noether!* Leo Corry [2004: 397], finding the question about the essence of algebra "ill-posed", suggests instead to "ask 'What is the algebra of Fermat, Descartes and Viète?' or 'What is van der Waerden's algebra?', or even, 'What was the algebra of the Greeks?'" and *then* discuss whether "the Greeks were, or were not, doing algebra like it was later done in the seventeenth century, or like it is done in the twentieth century". Similarly, let us resume the characteristics of Old Babylonian "algebra":

Firstly, what we know about was a *technique*, no mathematical theory. Insights of a quasi-theoretical kind may have been necessary in order to see, e.g., that a problem like TMS XIX #2 *could be solved* (see note 32), but they have left no written trace, and we can only guess at their nature.

Its "fundamental representation" was a geometry of measurable segments and squares and rectangles in a square grid. However, it could be applied to entities of other sorts whose mutual relations were analogous to those of the

[37] Other mathematical traditions could be certainly included in the induction, strengthening its validity; I abstain from mentioning them, not having worked enough on them myself.

fundamental representation – numbers, workers and their working days, areas instead of segments, etc.[38]

It was formulated in words, in a very standardized but not always unambiguous language.[39] However, in contrast to rhetorical algebra of al-Khwārizmī's type, its operations were not made *within* language. Even though it seems justified to speak of the verbal statements as "equations", there is thus the fundamental difference that the Babylonian calculators did not *operate* on their equations, as did al-Khwārizmi and as does modern symbolic equation algebra. The Babylonian prescriptions *describe* what is done in the geometric representation, just as we may describe in words what we are doing to the equation – "then we halve the coefficient of *x* and square it and add it to both sides of the equation", etc., in the example on p. 13.

As already discussed, Old Babylonian "algebra" was "naive", though with attempts at criticism. So, as a last characteristic we shall recall that it was *analytical*, as is modern equation algebra: it presupposed the existence and the properties of the objects it was looking for.

Whether all of this is sufficient to include the Old Babylonian technique in an extended family of "algebras" is a matter of taste and epistemological convenience. However, whether we include it or we exclude it we should remain aware of the precise criteria used to delimit this family and which Old Babylonian

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[38] It is noteworthy, but says more about the kind of real-world problems encountered by Babylonian calculators than about their mathematical technique that all problems of the second or higher degrees which we find in the texts are artificial. Not a single one of them corresponds to a task a scribe might encounter in his working practice – unless his work was to teach mathematics!

[39] Pretending that a language one only understands approximately is ambiguous is always risky. In the present case, however, there is no doubt. In BM 13901, for instance, we find exactly the same phrase *mi-it-ḫa-ra-ti-ia uš-ta-ki-il*$_5$*-ma*, "my confrontations I have made hold", with two different meanings. In obv. II.28 (#12, cf. above), however, it is followed by a single number, showing that the two "confrontations" are meant to be sides of a single rectangle; in rev. I.50 (#19), it is followed by the statement that "the surface I have accumulated", showing that (at least) two figures are involved, and thus that each "confrontation" holds its own square.

The very compact "series text" (containing only statements and sometimes results but no description of the procedure) are much worse in this respect. There, ambiguity often is not eliminated by immediate context as here but only (if at all) by the sequence of surrounding problems – see, e.g, the copious examples in YBC 4714 [Høyrup 2002a: 112–137]. Their heavy use of logographic abbreviations and their lack of interest in the rules of natural grammar do not imply that they are approaching symbolism (cf. note 1 and preceding text): they are *stenographic*.

“algebra” fulfils or fails to fulfil. To assert that it *was* an algebra merely because its procedures can be described in modern equations, or to declare that it *was not* because it did not itself write such equations is perhaps a bit superficial.

An algorithmic type of mathematics?

In recent decades, the concept of "algorithms" has been widely used in examinations of various branches of early mathematics. Often, reference is made to a paper by Donald Knuth from [1972] on "Ancient Babylonian Algorithms". In contrast to many others inspired by him, Knuth did not see "algorithms" as an alternative to the algebraic interpretation but as a specification. He states indeed (p. 622) that

> Babylonian mathematicians [...] were adept at solving many types of algebraic equations. But they did not have an algebraic notation that is quite as transparent as ours; they represented each formula by a step-by-step list of rules for its evaluation, i.e. by an algorithm for computing that formula. In effect, they worked with a "machine language" representation of formulas instead of a symbolic language.

The reference to a "machine language" was built on the translations that were current at the time, which saw the texts as dealing with pure numbers. The identifications within the texts (e.g., "30´ which you have made hold" in BM 13901) could then be seen (although Knuth does not explain that) as "comment fields", external to the algorithms.

On this level, Knuth can thus be claimed to be mistaken, misled by the translations he used. However, the passage contains another point, namely the definition of an "algorithm" as "a step-by-step list of rules". This remains adequate, and seems to correspond well to the words which open the prescription in many Old Babylonian problem texts: "You, by your proceeding" (others close by the phrase "the procedure"). However, it corresponds just as well to, say, the prescription of how to construct an equilateral triangle in *Elements* I.1. Here, the ensuing proof takes the place of the "comment field".

It must be noticed that Knuth only finds "straight-line calculations, without any branching or decision-making involved. In order to construct algorithms that are really non-trivial from a computer-scientist's point of view, we need to have some operations that affect the flow of control". The constructional algorithm of *Elements* I.1 has the same "trivial" character. We may legitimately ask if we are not breaking a butterfly on the wheel when applying the modern algorithm concept, created to describe non-trivial procedures, to the ancient linear prescriptions. But we should be aware that no definitive answer can be given unless we specify the question.

One aspect of the question is whether an algorithmic *notation* can be a useful analytic tool. Here, the answer appears to be affirmative. As an example I shall take one of my own translations of a Late Babylonian procedure [Høyrup 2002a: 393] into a formula

$$\frac{½\cdot([d+\ell]^2-w^2)}{d+\ell}\;.$$

This translation is only unambigous because $d+\ell$ is a *given* number. If it had been calculated, the formula, even when read strictly, would not tell whether this magnitude was calculated twice, or once and then saved and retrieved. An algorithmic formalism able to grasp the structure and details of complicated calculations for analysis was proposed by Jim Ritter [2004][40] and used in adapted shape by Annette Imhausen first in [2002] and next in her dissertation [2003].

Another aspect is how well the notion of "algorithm" corresponds to the thought of the ancients themselves. This question is of course too broad to be answered, "the ancients" being as badly defined as a category as (say) "non-Western" or (my favourite polemical example) "non-yellow colours". Mathematical cultures where it was current to formulate first an abstract rule and to illustrate it afterwards by one or several paradigmatic examples (India, China) may be adequately described by the (trivial) algorithms concept, but they are not my topic.[41]

The terms "proceeding" (*epēšum*) and "procedure" (*nēpešum*) appear *not* to be used about any procedure of more general validity than the single paradigmatic example (except in one Eshnunna text, Tell Haddad 104 [al-Rawi & Roaf 1984], where the latter term may precede several variants and thus several different "algorithms"). Hence none of them corresponds to a notion of an algorithm or rule of general validity (as, say, the "rule of three"); they must be regarded as irrelevant for our question.

Also a possibility to be considered, and also irrelevant at closer inspection, are the few methods which were given a particular *name*. One is the "Akkadian

[40] The paper circulated for long before this date of publication. I read it myself in 1997; a preprint appeared in [1998].

[41] There are a couple Old Babylonian texts containing such rules. One is an oblique quotation of the "Pythagorean rule" in the proof in Db_2-146, the other a very opaque rule formulated in AO 6770 #1 – see [Høyrup 2002a: 257–261, 179–181]. Both texts are early, and the absence of traces of similar rules from all later texts shows that such rules formulated in abstract terms, though originally known, were deliberately discarded by the school. Abstractly formulated rules reappear in texts from the Late Babylonian period (now in less opaque formulation), as mathematics was once more adopted by the scholar-scribes from an environment of lay calculators [Høyrup 2002a: 389].

(method)" of TMS IX #2. As we saw, it referred there to a very atypical quadratic completion, but it almost certainly also referred to the standard completion of a square gnomon (there would be no reason to have a name for the atypical variant only). The other one is "bundling" (*makṣarum*), which may refer both to the division of a surface (in the actual case, a triangle) and of a volume (in the actual case, a cube) into a "bundle" of smaller surfaces or volumes of the same shape [Høyrup 2002a: 66, 254]. Once again, no specific algorithm but an idea that may be varied.

Jim Ritter [2004] points out that the technique to solve a normalized second-degree problem may be seen as a subroutine, embedded towards the end of the solution of more complex problems, and thus finds a not quite trivial algorithmic structure. We have indeed seen this embedding several times above. Ritter uses as his example the text Str. 368,[42] which however presents us with exactly the phenomenon that was discussed above in connection with BM 13901 #14: the "breaking" is eliminated from the subroutine together with a doubling in the embedding text. The algorithm concept is still a valid tool *for us* when we want to explain what goes on, and to compare this to the standard procedure; but evidently the author of the text was able to understand from a higher vantage point the structure of the calculation. *His* thinking was not algorithmic.[43] That training (beyond the training of basic skills and elementary routines) was not thought of in terms we may call algorithmic is also illustrated for example by AO 8862. Instead of solving #3 as #1, with the minimal variation that 100 times ℓ–w is added to the area instead of ℓ–w, a wholly different road is chosen (and

[42] Transliteration and translation in [Neugebauer 1935: I, 311*f*].

[43] Jim Ritter is obviously not blind to this. His notion of an algorithm is much broader than Knuth's "step-by-step list of rules". He introduces "another, more general level of the algorithm, more general than that of the calculational techniques or that of the arithmetical operations, the level of method of solution, the choice of strategy of solution". As an example he gives the method of a single false position, which can be seen to underlie several of his examples. In this way, the carriers' higher-level understanding is integrated into the "algorithm". The disadvantage is that the algorithm concept is dissolved by the inclusion of a level which is not linked to the steps of the algorithm (as are the "comments", Babylonian as well as Euclidean) and which is furthermore common to many algorithms consisting of different steps. As I have observed on another occasion [Høyrup 2008: 268],

> Instead of seeing the algorithms used in weather prediction as encompassing the physical theories and differential equations on which they are based, it seems to me to leave more room for analysis to separate the physical and mathematical theories from their implementation in computer algorithms.

instead of a reduction of #2 to the situation of #1, a slight variation is selected). The message seems to be that many methods are available, and that one should choose intelligently and flexibly.

All in all, Old Babylonian mathematics seems to me to be no more "algorithmic" in essence than any other mathematics generating and applying methods of more or less general validity.

Moreover, I should add, I see a danger that those who are located outside the field of professional history of mathematics (or, like Kline, only know about Greek and post-Renaissance European mathematics) will merely learn once more from the description of Babylonian, Egyptian, Indian and Chinese mathematics as specifically "algorithmic" that "Western" mathematics is reasoned while the "non-Western" types of mathematics are based on rules without proof nor reason.

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