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Carsten Lunde Petersen and Tan Lei: Branner-Hubbard motions and attracting dynamics

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Abstract:
Branner-Hubbard motion is a systematic way of deforming an attracting holomorphic dynamical system f into a family $(f_s)_{s\in\mathbb{L}}$, via a holomorphic motion which is also a group action. We establish the analytic dependence of f_s on s (a result first stated by Lyubich) and the injectivity of f_s on f. We prove that the stabilizer of f (in terms of s) is either the full group $\mathbb{L}$ (rigidity), or a discrete subgroup (injectivity). The first case means that f_s is Möbius conjugate to f for all $s \in \mathbb{L}$, and it happens for instance at the center of a hyperbolic component. In the second case the map $s \mapsto f_s$ is locally injective. We show that BH-motion induces a periodic holomorphic motion on the parameter space of cubic polynomials, and that the corresponding quotient motion has a natural extension to its isolated singularity. We give another application in the setting of Lavaurs enriched dynamical systems within a parabolic basin.

Branner-Hubbard motions and attracting dynamics

C.L. Petersen and Tan Lei, 23rd February 2005

1 Introduction

This paper describes systematic perturbations of holomorphic dynamical systems via structured holomorphic motions that are also group actions. The technique, commonly known as the Branner Hubbard motion or in short BH-motion, was introduced by Branner and Hubbard in [BH1] to study parameter spaces of monic polynomials. It was later used by, for example, Branner's Ph.D. student Willumsen [Wi] (see also [Ta] in this volume).

In order to better exhibit the general properties of BH-motions we introduce the notion of attracting dynamics (see Definition 2.3), on which the BH-motion naturally acts. Given an attracting dynamics f, a BH-motion provides a parametrized family of attracting dynamics $(f_s)_{s\in\mathbb{L}}$, with the parameter space $\mathbb{L}$ equal to the right half complex plane, equipped with its usual complex structure and with a specific real Lie group structure. The map $s \mapsto f_s$ is naturally a group action.

We give a thorough description of the construction of the BH-motion together with basic properties. We prove then the holomorphic dependence of f_s on s (Theorem 2.5.(2)), in a holomorphic motion context more general than BH-motions (Theorem 2.7). This result was first stated without proof by Michael Lyubich. We proceed to prove the injectivity of $f \mapsto f_s$ (Theorem 2.5.(6)). These two results will be our main tool, while performing BH-motions on a full slice of cubic polynomial attracting dynamics, to promote a holomorphic motion of the dynamical planes to a holomorphic motion of the parameter space of such polynomials.

We then study the mapping properties of $s \mapsto f_s$. We show that the stabilizer (see Definition 3.1) exhibits the following dichotomy: it is

• either the full group $\mathbb{L}$, in which case f behaves like the center of a hyperbolic component, in other words all critical points attracted by the attracting cycle actually lay on the cycle or its preimages;

• or a discrete subgroup of $\mathbb{L}$ contained in the vertical line $1 + i\mathbb{R}$, in which case f is necessarily a non-center, in other words at least one attracted critical point has an infinite orbit (Theorem 3.3).

There are many possible applications of BH-motions. We have chosen here two of them which we find illustrative for the diversity of applications.

The first one concerns a family (P_a) of cubic polynomials, such that 0 is an attracting fixed point of multiplier independent of a, and attracts exactly one simple critical point. We will perform a BH-motion on each P_a, thus obtain a double indexed family $P_{a,s}$ of cubic polynomials. As mentioned above, we prove that these dynamically defined BH-motions promote to a holomorphic motion of the a-slice within the space of cubic polynomials, which turns out to be $2\pi i$-periodic on s (Theorem 4.1). This induces naturally a quotient motion over $\mathbb{D}^* \approx \mathbb{L}/2\pi i\mathbb{Z}$, which is the most natural way to change the multiplier of P_a at 0, but keeping the remaining part of the dynamics fixed. We then make one more effort to extend this motion over $\mathbb{D}$, and thus succeed in deforming systematically the attracting fixed point into a superattracting fixed point (Theorem 4.2).

The second one concerns the BH-motion of the basin of ∞ of the quadratic cauliflower $z \mapsto z^2 + \frac{1}{4}$, enriched by a Lavaurs map g. We give a detailed study of the effect of the BH-motion on the enrichment of the dynamics (Theorem 4.6).

For other illustrations beyond the paper of Willumsen and the original paper of Branner and Hubbard, the reader may want to consult the beautiful master thesis of Uhre [U], the paper [D] in this volume and the paper [P-T] which explores further the notion of attracting dynamics.

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2 Definition and basic properties of BH-motions

In this paper we shall use the notion of holomorphic motions in a slightly more general sense than the usual definition:

Definition 2.1. *(Holomorphic motion) Let $(\mathcal{X}, \Lambda, p)$ be a triple with $\mathcal{X}, \Lambda$ two complex analytic manifolds and $p : \mathcal{X} \to \Lambda$ an analytic surjective mapping. Denote by $\mathcal{X}_\lambda$ the fiber $p^{-1}(\lambda)$. Choose $\lambda_0 \in \Lambda$ a* **base point** *and $E \subset \mathcal{X}_{\lambda_0}$. A* **holomorphic motion** *of E over Λ into $\mathcal{X}$ is a mapping $H : \Lambda \times E \to \mathcal{X}$, $(\lambda, z) \mapsto H(\lambda, z)$ satisfying:*

1. *For any fixed λ, $z \mapsto H(\lambda, z)$ is injective on E and maps E into $\mathcal{X}_\lambda$.*
2. *For any fixed $z \in E$, $\lambda \mapsto H(\cdot, z)$ is analytic.*
3. *$H(\lambda_0, \cdot)$ is the identity map on E.*

In practice, we often have $\mathcal{X} = \Lambda \times X$, in which case we suppress the first coordinate of H and write $H : \Lambda \times E \longrightarrow X$.

2.1 The model BH-motion

The notation below is taken from [Wi]. Further calculations can be found there.

Define $\mathbb{L} = \{u + iv, u > 0\}$ and for any $s = s_x + is_y \in \mathbb{L}$, define an $\mathbb{R}$-linear diffeomorphism $\widetilde{l}_s : \mathbb{C} \to \mathbb{C}$ by:

$$\widetilde{l}_s(z) = (s-1)z_x + z = sz_x + iz_y = \frac{s+1}{2}z + \frac{s-1}{2}\bar{z} = \begin{pmatrix} s_x & 0 \\ s_y & 1 \end{pmatrix}\begin{pmatrix} z_x \\ z_y \end{pmatrix} = \begin{pmatrix} s_x z_x \\ s_y z_x + z_y \end{pmatrix};$$

where $z = z_x + iz_y$. Moreover define a homeomorphism $l_s : \mathbb{C} \to \mathbb{C}$ by

$$l_s(z) = l_s(re^{2\pi i\theta}) = r^s e^{2\pi i\theta} = z \cdot r^{s-1} = z \cdot e^{(s-1)\log r} \ ,$$

so that $\exp \circ \widetilde{l}_s = l_s \circ \exp$. Then the almost complex structure $\widetilde{\sigma}_s = \widetilde{l}_s^*(\sigma_0)$ obtained by pulling back the standard almost complex structure σ_0, is given by the 'constant' Beltrami form $t_s \frac{d\bar{z}}{dz}$ where the constant $t_s = \frac{s-1}{s+1}$ depends only on s, but not the position z.

Let $\star : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ denote the group structure for which the map $s \mapsto \widetilde{l}_s$, is a group isomorphism onto its image, i.e. $s' \star s \mapsto \widetilde{l}_{s'} \circ \widetilde{l}_s$ and in algebra

$$s' \star s = \frac{s'(s+\bar{s}) + (s-\bar{s})}{2} = s'_x s_x + i(s'_y s_x + s_y) \ , \text{ and } s^{-1} = \frac{1}{s_x} - i\frac{s_y}{s_x} = \frac{2 - s + \bar{s}}{s+\bar{s}} \ .$$

Note that $s = (s_x + i0) \star (1 + is_y)$. The group $(\mathbb{L}, \star)$ is therefore a real Lie group and is generated by the two Abelian but non mutually commuting subgroups $(\mathbb{W}, \star)$ and $(\mathbb{S}, \star)$ called wring and stretch respectively. Where $\mathbb{W} = 1 + i\mathbb{R}$ with $\star$ is given by addition of imaginary parts and where $\mathbb{W}$ acts on the group $(\mathbb{L}, \star)$ from the right by addition of the imaginary part. And $\mathbb{S} = \mathbb{R}_+$ with $\star$ given by multiplication and where $\mathbb{S}$ acts on the group $(\mathbb{L}, \star)$ from the right by multiplication. The (collection of) maps $\widetilde{l}_s$ and l_s have many useful properties. We state them below as:

Define $\mathbb{H}_\pm = \{z \in \mathbb{C} | \pm \Re(z) > 0\}$, the right (left) half plane.

Lemma 2.2. *(basic properties of $\widetilde{l}_s$ and l_s)*

1. *The map $\widetilde{l}_s$ is the unique linear map mapping the ordered triple $(i, 0, 1)$ to the ordered triple $(i, 0, s)$. The maps*

   $$(s, z) \mapsto \widetilde{l}_s(z), \quad \mathbb{L} \times \mathbb{C} \to \mathbb{C}, \quad \mathbb{L} \times \mathbb{H}_\pm \to \mathbb{H}_\pm$$

   $$(s, z) \mapsto l_s(z), \quad \mathbb{L} \times \mathbb{C} \to \mathbb{C}$$

   satisfy simultaneously the following properties:

   - *they are* **holomorphic motions** *over $\mathbb{L}$, with base point $s_0 = 1$*
   - *they are* **group actions** *of $(\mathbb{L}, \star)$, acting on the left.*
   - *they are* **dynamical conjugacies***, more precisely $\widetilde{l}_s$ conjugates $z \mapsto kz$ $(k \in \mathbb{R})$ to itself and conjugates $z \mapsto z + L$ to $z \mapsto z + L_s$, where $L_s := \widetilde{l}_s(L)$; and l_s conjugates $z \mapsto z^k$ $(k \in \mathbb{N})$ to itself and conjugates $z \mapsto \lambda z$ to $z \mapsto \lambda_s z$, where $\lambda_s := l_s(\lambda) = \lambda \cdot |\lambda|^{s-1}$.*

2. *$\widetilde{l}_s|_{i\mathbb{R}} = Id$, it maps $\mathbb{R}$ to the oblique line passing through 0 and s, and any other horizontal line $iy + \mathbb{R}$ to the line parallel to s, passing through iy. It maps any vertical line to a vertical line.*

3. *For any $z \in \mathbb{H}_\pm$: $\mathrm{d}_{\mathbb{H}_\pm}(z, \widetilde{l}_s(z)) = \mathrm{d}_{\mathbb{H}_+}(1, s) = C_s$, where $\mathrm{d}_{\mathbb{H}_\pm}$ denotes the hyperbolic distance.*

4. *For* $s_1 = 1 + 2\pi i$, $\widetilde{l}_{s_1}$ *maps* $m + i\cdot\mathbb{R}$ ($m \in \mathbb{Z}$) *onto itself, mapping* m *to* $m + 2\pi i m$.

5. $l_s|_{\{|z|=1\}} = id$, l_s *maps the circle* $|z| = r$ *to the circle* $\{|z| = r^u\}$ (*where* $u = \Re(s)$), *and the ray* $e^{2\pi i\theta}\cdot\mathbb{R}$ *to an oblique (logarithmic) ray.*

6. *The Beltrami coefficient of* $\widetilde{l}_s(z)$ *is a constant depending only on* s, *i.e. is translationally invariant and that of* l_s *is invariant under linear maps* $z \mapsto az$, $a \neq 0$:

$$\frac{\overline{\partial}\widetilde{l_s}}{\partial\widetilde{l_s}}(z) \equiv \frac{s-1}{s+1} =: t_s \in \mathbb{D}\ , \qquad \frac{\overline{\partial}l_s}{\partial l_s}(z) \equiv \frac{z}{\overline{z}}\frac{s-1}{s+1} =: t_s \in \mathbb{D}\ .$$

Moreover the dilatations $K(l_s) = K(\widetilde{l}_s) = \frac{1+|t|}{1-|t|}$ *are also constants.*

7. *When* s *varies from* 1 *to* $1 + 2\pi i$, *the circle* $\{|z| = e^m\}$ *makes* m*-turns, relative to the unit circle, for all* $m \in \mathbb{Z}$, *under the action of* l_s.

Proof. We will only prove 3 for $\mathbb{H}_+$, the rest being straight forward. Fix $z_0 \in \mathbb{H}_+$. The map $w \mapsto w - i\cdot\Im(z_0)$ is a hyperbolic isometry of $\mathbb{H}_+$, mapping z_0 to $\Re(z_0)$ and $\widetilde{l}_s(z_0)$ to $\widetilde{l}_s(\Re(z_0))$ (by the conjugacy property). Now $w \mapsto w/\Re(z_0)$ is again an isometry, mapping $\Re(z_0)$ to 1 and $\widetilde{l}_s(\Re(z_0))$ to $\widetilde{l}_s(1) = s$ (by the conjugacy property). $\square$

2.2 BH-motion of an attracting dynamics (f, W, α)

Definition 2.3. *We say that* (f, W, α) *or in short* (f, α) *or* f, *is an* **attracting dynamics**, *if: i)* $W \subset \overline{\mathbb{C}}$ *is open, ii)* $f : W \longrightarrow \overline{\mathbb{C}}$ *is holomorphic and iii)* $\alpha \in W$ *is an attracting or superattracting periodic point for* f.

Any attracting dynamics (f, W, α) *comes with a* **long string** *of informations*

$$(f, W, \alpha, k(f), \lambda(f), \widetilde{B}(\alpha), B(\alpha), \phi, U)$$

defined as follows: $k = k(f) \in \mathbb{N}$ *is the exact period of* α, *and* $\lambda(f) \in \mathbb{D}$ *denotes the multiplier* $(f^k)'(\alpha)$. *The set*

$$\widetilde{B}(\alpha) := \{z \in W|\ \forall n\ f^n(z) \in W\ \&\ f^{nk+l}(z) \underset{n\to+\infty}{\longrightarrow} \alpha \text{ for some } l \in \mathbb{N}\}$$

denotes the entire attracted basin of the orbit of α, *and* $B(\alpha)$ *denotes the immediate basin of* α, *i.e. the connected component of* $\widetilde{B}(\alpha)$ *containing* α. *The map* $\phi : U \longrightarrow \mathbb{C}$ *is a choice of a linearizing (possibly a Böttcher) coordinate defined and univalent on some neighborhood* U *of* α.

Note that different choices of ϕ on a given U differ by a multiplicative constant. In what follows we shall for any subset $W \subseteq \overline{\mathbb{C}}$ denote by W^c the complement $\overline{\mathbb{C}}\setminus W$.

Definition 2.4. *Define* **a BH-motion** *of* (f, W, α) *to be a map:*

$$s \mapsto (\sigma_s, h_s,\ (f_s, W_s, \alpha_s),\ \phi_s, U_s)\ ,\ s \in \mathbb{L} \qquad \text{or in short} \qquad s \mapsto h_s\ ,\ s \in \mathbb{L}$$

as follows (see the diagram (1) below):

- σ_s *is the measurable and bounded Beltrami form defined by*

$$\sigma_s = \begin{cases} (l_s \circ \phi)^*(\sigma_0) & \text{on } U \\ (f^n)^*\sigma_s & \text{on } f^{-n}(U)\ ,\ n \in \mathbb{N} \\ \sigma_0 & \text{on } \widetilde{B}(\alpha)^c \end{cases} .$$

That is σ_s is given by the above formulas on U and $\widetilde{B}(\alpha)^c$ and extended to $\widetilde{B}(\alpha)$ by iterated pull-backs of f. Note that for every $z_0 \in U$ the assignment $s \mapsto \sigma_s(z_0)$ is a complex analytic function on $\mathbb{L}$. In fact if we write $\sigma_s(z) = \mu_s(z)\frac{d\overline{z}}{dz}$ in some local coordinate z on $W' \subseteq W$, then for every fixed $z_0 \in U$ the map $s \mapsto \mu_s(z_0) : \mathbb{L} \longrightarrow \mathbb{D}$ is a Möbius transformation. On $\mathbb{C}$ which has a natural preferred chart the identity, we shall abuse the notation and simply write μ for the Beltrami form $\mu\frac{d\overline{z}}{dz}$.

- $h_s = h_{s,f} : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ *is a family of* **integrating maps** *for σ_s normalized so as to depend complex analytically on s, as supplied by the measurable Riemann mapping theorem with parameters.*
- $(f_s, W_{f_s}, \alpha_{f_s}) = (f_s, W_s, \alpha_s) = (h_s \circ f \circ h_s^{-1},\ h_s(W),\ h_s(\alpha))$.
- $\phi_s = \phi_{f_s} = l_s \circ \phi \circ h_s^{-1}$ *and* $U_s = U_{f_s} = h_s(U)$.

Note that another s-analytic normalization $\widehat{h}_s$ of h_s differs by an s-analytic family of Möbius transformations. In other words $\widehat{h}_s = M_s \circ h_s$ with M_s Möbius and analytic in s.

Example 1. $(f, W, \alpha) = (e^{-1}z + z^2, \overline{\mathbb{C}}, 0)$ is an attracting dynamics with $k(f) = 1$ and $\lambda(f) = e^{-1}$. In a BH-motion of it, we may normalize the integrating maps h_s so that they fix 0 and ∞ and they are tangent to the identity at ∞. Then one checks easily that $f_s(z) = e^{-s}z + z^2$.

Example 2. Let P be a monic centered polynomial. We do a BH-motion for $(f, W, \alpha) = (P, \overline{\mathbb{C}}, \infty)$. We normalize h_s so that each f_s is again monic centered. In case that P has a connected Julia set, a theorem of Branner-Hubbard ([BH], Prop. 8.3) shows that $f_s \equiv P$. We will reprove this result below in a more general setting.

The basic properties of a BH-motion are:

Theorem 2.5. *(BH-motion of dynamics) Let (f, W, α) be an attracting dynamics with $k(f) = k$ and $\lambda(f) = \lambda$ and let $s \mapsto (\sigma_s, h_s,\ (f_s, W_s, \alpha_s),\ \phi_s, U_s)$ be a BH-motion of (f, W, α). Then:*

1. *For any $z \in \overline{\mathbb{C}}$, the assignment $s \mapsto \sigma_s(z)$ is independent of the choices of (ϕ, U) in the long string information.*
2. *The two maps of two complex variables $(s, z) \mapsto f_s(z)$ and $(s, z) \mapsto \phi_s(z)$ are complex analytic in $\{(s, z), s \in \mathbb{L}, z \in W_s\}$ and $\{(s, z), s \in \mathbb{L}, z \in U_s\}$ respectively.*

3. *For any $s \in \mathbb{L}$ the triple (f_s, W_s, α_s) is again an attracting dynamics, whose long string of information takes the form:*

$$(f_s, W_s, \alpha_s, k, \lambda|\lambda|^{s-1}, h_s(\widetilde{B}(\alpha)), h_s(B(\alpha)), \phi_s, U_s) \ .$$

   If $\lambda \in \mathbb{D}^*$ then $s \mapsto \lambda_s$ is holomorphic and depends on λ only.

   If $\lambda = 0$ then $\lambda_s \equiv 0$ and ϕ_s is a Böttcher coordinate for (f_s, α_s), defined and univalent on U_s. Moreover write $f^k(z) - \alpha = a(z-\alpha)^d +$higher order terms, with $a \neq 0$ and $d > 1$ the local degree at α, then $f_s^k(z) - \alpha_s = a(s)(z-\alpha_s)^d +$higher order terms, with $a(s)$ non vanishing and holomorphic in s.

4. *If ϕ extends holomorphically to some domain $U' \subset \widetilde{B}(\alpha)$ then $l_s \circ \phi \circ h_s^{-1}$ is a holomorphic extension of ϕ_s to $U'_s = h_s(U') \subset \widetilde{B}(\alpha_s)$.*

5. *The maps $(s, z) \mapsto h_s(z)$*
   - *form a* **holomorphic motion** *of $\overline{\mathbb{C}}$ over $\mathbb{L}$ with base point $s = 1$;*
   - *are* **dynamical conjugacies** *as indicated in the following diagram:*

$$\begin{array}{ccccc}
(f) & & (\lambda z, z^m) & & (z+L, mz) \\
W \supset \widetilde{B}(\alpha) \supset U & \xrightarrow{\phi} & \phi(U) \subset \mathbb{C} & \xleftarrow{\exp} & \mathbb{C},\ (i,0,1) \\
\downarrow h_s & & \downarrow l_s & & \downarrow \widetilde{l}_s \ \textit{linear} \\
W_s \supset \widetilde{B}(\alpha_s) \supset U_s & \underset{\phi_s := l_s \phi h_s^{-1}}{\longrightarrow} & \phi_s(U_s) \subset \mathbb{C} & \underset{\exp}{\longleftarrow} & \mathbb{C},\ (i,0,s) \\
(f_s) & & (\lambda_s z, z^m) & & (z+L_s, mz)
\end{array} \qquad (1)$$

   - *are* **group actions***. More precisely, $(f_s)_{s'} = f_{s' \star s}$, $h_{s'\star s, f} = h_{s', f_s} \circ h_{s,f}$ and $\phi_{s' \star s, f} = (\phi_{s,f})_{s', f_s}$, (subject to suitable normalizations). And for any fixed $s \in \mathbb{L}$, the map*

$$s' \mapsto (\sigma_{s'\star s}, h_{s'\star s}, (f_{s'\star s}, W_{s'\star s}, \alpha_{s'\star s}), \phi_{s'\star s}, U_{s'\star s})$$

     is a BH-motion of f_s.

6. *(***injectivity***) If $(f_{s_0}, W_{f_{s_0}}, \alpha_{f_{s_0}})$ and $(g_{s_0}, W_{g_{s_0}}, \alpha_{g_{s_0}})$ are Möbius conjugate by M (see Definition 2.6 below) for some $s_0 \in \mathbb{L}$ and some pair of attracting dynamics (f, W_f, α_f) and (g, W_g, α_g). Then $(f_s, W_{f_s}, \alpha_{f_s})$ and $(g_s, W_{g_s}, \alpha_{g_s})$ are Möbius conjugate for all s by a holomorphically varying family M_s of Möbius transformations with $M_{s_0} = M$.*

The proof of this theorem is postponed to the next subsection.

Definition 2.6. *Two attracting dynamics (f, W_f, α_f) and (g, W_g, α_g) are* **Möbius conjugate***, if there is a Möbius transformation M, with $M(\alpha_f) = \alpha_g$, $M(W_f) = W_g$ and $M \circ f = g \circ M$.*

Remark. The map $s \mapsto f_s$ may not extend continuously to the boundary $i\mathbb{R}$ of $\mathbb{L}$. See [BH1], [Wi], [KN] or [Ta] for details.

2.3 Proof of Theorem 2.5

The non trivial part of Theorem 2.5 is the analytic dependence on s of $f_s(z)$ and $\phi_s(z)$. It is a consequence of a theorem first stated by Lyubich, which we restate and prove below. It requires however a little setup.

Let $U, V \subset \overline{\mathbb{C}}$ be open subsets and $f : U \longrightarrow V$ be a holomorphic map. Let Λ be a complex analytic manifold and suppose $\sigma : \Lambda \times V \longrightarrow \mathrm{Bel}(V)$ is an analytically varying family of bounded measurable Beltrami forms supported on V. Let $\Psi : \Lambda \times \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ be a corresponding analytically varying family of integrating quasi-conformal homeomorphisms as supplied by the measurable Riemann mapping theorem with parameters. That is for each fixed $z \in \overline{\mathbb{C}}$: $\lambda \mapsto \Psi_\lambda(z)$ is holomorphic and for each λ the map $\Psi_\lambda = \Psi(\lambda, \cdot)$ is a quasi-conformal homeomorphism with $\Psi_\lambda^*(\sigma_0) = \sigma_\lambda$ on V and $\Psi_\lambda^*(\sigma_0) = \sigma_0$ on V^c. Let similarly $\Phi : \Lambda \times \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ be an analytically varying family of integrating quasi-conformal homeomorphisms $\Phi_\lambda = \Phi(\lambda, \cdot)$ for the pulled-back structures $\widehat{\sigma}_\lambda = f^*(\sigma_\lambda)$, with $\Phi_\lambda^*(\sigma_0) = \widehat{\sigma}_\lambda$ on U and $\Phi_\lambda^*(\sigma_0) = \sigma_0$ on U^c. See diagram (2) .

$$\begin{array}{ccc} (U, \widehat{\sigma}_\lambda) & \stackrel{\Phi_\lambda}{\longrightarrow} & (U_\lambda, \sigma_0) \\ f\downarrow & & \downarrow f_\lambda \\ (V, \sigma_\lambda) & \stackrel{\Psi_\lambda}{\longrightarrow} & (V_\lambda, \sigma_0) \end{array} \tag{2}$$

Define for each $\lambda \in \Lambda$: $U_\lambda = \Phi_\lambda(U)$ and $V_\lambda = \Psi_\lambda(V)$ and open subsets $\mathcal{U}, \mathcal{V} \subseteq \Lambda \times \overline{\mathbb{C}}$ by $\mathcal{U} = \{(\lambda, z) | z \in U_\lambda\}$ and $\mathcal{V} = \{(\lambda, z) | z \in V_\lambda\}$. Finally define a continuous map (homeomorphism if f is bi holomorphic) $F : \mathcal{U} \longrightarrow \mathcal{V}$ by

$$F(\lambda, z) = (\lambda, f_\lambda(z)) := (\lambda, \Psi_\lambda \circ f \circ \Phi_\lambda^{-1}(z)) \ .$$

Note that although $\Psi_\lambda^{-1}(z)$ is still quasi-conformal in z and continuous in (λ, z), it is in general no more analytic in λ. However, we have, as a miracle,

Theorem 2.7. *(Lyubich) The above map F is complex analytic or equivalently $(\lambda, z) \mapsto f_\lambda(z)$ is complex analytic.*

Proof. The map $(\lambda, z) \mapsto f_\lambda(z) : \mathcal{U} \longrightarrow \overline{\mathbb{C}}$ is continuous, because the two maps

$$(\lambda, z) \mapsto (\lambda, \Phi_\lambda(z)) : \Lambda \times U \longrightarrow \mathcal{U} \quad \text{and} \quad (\lambda, z) \mapsto (\lambda, \Psi_\lambda(z)) : \Lambda \times V \longrightarrow \mathcal{V}$$

are homeomorphisms. Moreover f_λ is holomorphic for each fixed λ_0 as f_λ pulls back the standard Beltrami form σ_0 to itself,

$$f_\lambda^*(\sigma_0) = (\Psi_\lambda \circ f \circ \Phi_\lambda^{-1})^*(\sigma_0) = (\Phi_\lambda^{-1})^* \circ f^* \circ \Psi_\lambda^*(\sigma_0) = (\Phi_\lambda^{-1})^*(f^*(\sigma_\lambda)) = \sigma_0.$$

Thus we need only check that for each fixed z the map $\lambda \mapsto f_\lambda(z)$ is holomorphic in each of the coordinate functions of a complex analytic local chart on Λ. Equivalently we need to prove that $f_\lambda(z)$ has a complex partial derivative at each point $(\lambda_0, z_0) \in \mathcal{U}$ with respect to each such coordinate function. Hence the theorem is an immediate consequence of the following one variable version. $\square$

Proposition 2.8. *In the notation above suppose $\Lambda, U, V \subset \mathbb{C}$ are open subsets and $\mathcal{U}, \mathcal{V} \subseteq \Lambda \times \mathbb{C}$. For $(\lambda_0, z_0) \in \mathcal{U}$ write $w_0 = \Phi_{\lambda_0}^{-1}(z_0) \in U$ then*

$$\frac{\partial f_\lambda}{\partial \lambda}(z_0)\Big|_{\lambda=\lambda_0} := \lim_{\lambda \to \lambda_0} \frac{f_\lambda(z_0) - f_{\lambda_0}(z_0)}{\lambda - \lambda_0} = \frac{\partial \Psi_\lambda}{\partial \lambda}(f(w_0))\Big|_{\lambda=\lambda_0} - f'_{\lambda_0}(z_0) \cdot \frac{\partial \Phi_\lambda}{\partial \lambda}(w_0)\Big|_{\lambda=\lambda_0}.$$

The same proof shows that if λ is a real parameter and $\Phi_\lambda(w_0)$ and $\Psi_\lambda(f(w_0))$ are real partially differentiable as functions of λ at λ_0, then $f_\lambda(z_0)$ is partially real differentiable at λ_0 with the same formula for the partial derivative. See the following diagram:

$$\begin{array}{ccc} w_0 & \xrightarrow{\Phi_{\lambda_0}} & z_0 \\ f \downarrow & & \downarrow f_{\lambda_0} \\ f(w_0) & \xrightarrow{\Psi_{\lambda_0}} & f_{\lambda_0}(z_0) \end{array}$$

Proof. At first $(\lambda, z) \mapsto f_\lambda(z)$ is continuous on (λ, z) and analytic on z. By the Cauchy integral formula, $f'_\lambda(z)$ and $f''_\lambda(z)$ depend continuously on (λ, z). In particular, for

$$u(\lambda, z) := f_\lambda(z) - f_\lambda(z_0) - f'_\lambda(z_0)(z - z_0),$$

there is some $\kappa > 0$ such that $|u(\lambda, z)| \le \kappa |z - z_0|^2$ for (λ, z) in a neighborhood of (λ_0, z_0).

As the maps $\lambda \mapsto \Phi_\lambda(z_0)$ and $\lambda \mapsto \Psi_\lambda(f(w_0))$ are $\mathbb{C}$-differentiable at λ_0, we can write $\frac{\partial \Phi_\lambda}{\partial \lambda}(w_0)|_{\lambda_0} = A$, $\frac{\partial \Psi_\lambda}{\partial \lambda}(f(w_0))|_{\lambda_0} = B$ and $\Phi_\lambda(w_0) - z_0 = A(\lambda - \lambda_0) + \mathfrak{o}(|\lambda - \lambda_0|)$. From the relation $f_\lambda \circ \Phi_\lambda = \Psi_\lambda \circ f$ we obtain

$$\begin{aligned} B \underset{\lambda \to \lambda_0}{\longleftarrow} \frac{\Psi_\lambda(f(w_0)) - \Psi_{\lambda_0}(f(w_0))}{\lambda - \lambda_0} &= \frac{f_\lambda(\Phi_\lambda(w_0)) - f_{\lambda_0}(\Phi_{\lambda_0}(w_0))}{\lambda - \lambda_0} = \frac{f_\lambda(\Phi_\lambda(w_0)) - f_{\lambda_0}(z_0)}{\lambda - \lambda_0} \\ &= \frac{f_\lambda(\Phi_\lambda(w_0)) - f_\lambda(z_0)}{\lambda - \lambda_0} + \frac{f_\lambda(z_0) - f_{\lambda_0}(z_0)}{\lambda - \lambda_0} \\ &= \frac{f'_\lambda(z_0)(\Phi_\lambda(w_0) - z_0) + u(\lambda, \Phi_\lambda(w_0))}{\lambda - \lambda_0} + \frac{f_\lambda(z_0) - f_{\lambda_0}(z_0)}{\lambda - \lambda_0} \\ &= \frac{f'_\lambda(z_0)A(\lambda - \lambda_0) + \mathfrak{o}(|\lambda - \lambda_0|)}{\lambda - \lambda_0} + \frac{f_\lambda(z_0) - f_{\lambda_0}(z_0)}{\lambda - \lambda_0}. \end{aligned}$$

It follows that

$$\lim_{\lambda \to 0} \frac{f_\lambda(z_0) - f_{\lambda_0}(z_0)}{\lambda - \lambda_0} = B - f'_{\lambda_0}(z_0) \cdot A.$$

□

Proof of Theorem 2.5.

1. Let (ϕ_i, U_i), $i = 1, 2$ be two choices of the linearizer (or Böttcher coordinate). Then on a neighborhood $U \subset U_1 \cap U_2$ of α, there is a constant $a \neq 0$ such that $\phi_1 = a \cdot \phi_2$. But $(a\cdot)^*(l_s^* \sigma_0) = l_s^* \sigma_0$, by *6.* of Lemma 2.2. Thus on U, $(l_s \circ \phi_1)^* \sigma_0 = (l_s \circ \phi_2)^* \sigma_0$. It follows from the f-invariance of σ_s that $\sigma_s(z)$ is independent of the choice of (ϕ, U).

2. Complex analyticity of the maps $(s,z) \mapsto f_s(z)$ and $(s,z) \mapsto \phi_s(z)$ follow immediately from Theorem 2.7 applied to the following two commutative diagrams:

$$\begin{array}{ccc} & \xrightarrow{h_s} & \\ f \downarrow & & \downarrow f_s \\ & \xrightarrow{h_s} & \end{array} \quad , \quad \begin{array}{ccc} & \xrightarrow{h_s} & \\ \phi \downarrow & & \downarrow \phi_s \\ & \xrightarrow{l_s} & \end{array} \; .$$

3. As h_s is a homeomorphism and $f_s(z) = h_s \circ f \circ h_s^{-1}$ it follows that $\alpha_s = h_s(\alpha)$ is a k-periodic point. Moreover

$$\phi_s \circ f_s^k \circ \phi_s^{-1} = l_s \circ \phi \circ h_s^{-1} \circ h_s \circ f^k \circ h_s^{-1} \circ h_s \circ \phi^{-1} \circ l_s^{-1} = l_s \circ \phi \circ f^k \circ \phi^{-1} \circ l_s^{-1}.$$

If $\lambda \neq 0$ then $\phi \circ f^k \circ \phi^{-1}(z) = \lambda z$ and

$$\phi_s \circ f_s^k \circ \phi_s^{-1}(z) = l_s \circ \lambda \cdot \circ l_s^{-1}(z) = \lambda_s z = \lambda |\lambda|^{s-1} z \; .$$

If $\lambda = 0$, we have $f^k(z) - \alpha = a(z-\alpha)^d +$ higher order terms for some $a \neq 0$ and $\phi \circ f^k \circ \phi^{-1}(d) = z^d$ then

$$\phi_s \circ f_s^k \circ \phi_s^{-1}(z) = l_s \circ z^d \circ l_s^{-1}(z) = z^d \; .$$

Moreover $f_s^k(z) - \alpha_s = a(s)(z-\alpha_s)^d +$ higher order terms with $a(s) = (\phi'(\alpha_s))^{d-1}$ depending holomorphically on s.

The facts that $\widetilde{B}(\alpha_s) = h_s(\widetilde{B}(\alpha))$ and $B(\alpha_s) = h_s(B(\alpha))$ are immediate from the definition.

4. The extension of ϕ_s is immediate from the definitions.

5. The map $(s,z) \mapsto h_s$ is a holomorphic motion by the measurable Riemann mapping theorem with parameters. Each h_s is automatically a dynamical conjugacy by the commutative diagram. To prove the group action properties, just look at the following diagram:

$$\begin{array}{ccc} f,\ W \supset \widetilde{B}(\alpha_f) \supset U & \overset{\phi = \phi_f}{\longrightarrow} & \phi_f(U) \subset \mathbb{C}\,, \\ \downarrow h_{s,f} & & \downarrow l_s \\ (s,f) = f_s,\ W_s \supset \widetilde{B}(\alpha_s) \supset U_s & \overset{\phi_s := l_s \phi h_s^{-1}}{\longrightarrow} & \phi_s(U_s) \subset \mathbb{C} \\ \downarrow h_{s',f_s} & & \downarrow l_{s'} \\ g = (s', f_s),\ (W_s)_{s'} \supset \widetilde{B}(\alpha_g) \supset (U_s)_{s'} & \overset{\phi_g := l_{s'} \phi_s h_{s',f_s}^{-1}}{\longrightarrow} & \phi_g((U_s)_{s'}) \subset \mathbb{C} \end{array} \; .$$

As $l_{s'} \circ l_s = l_{s' \star s}$, the map $h_{s',f_s} \circ h_{s,f}$ integrates the complex structure pulled back by $l_{s'\star s} \circ \phi_f$. So, up to normalization, $h_{s',f_s} \circ h_{s,f} = h_{s'\star s,f}$ and $(f_s)_{s'} = f_{s'\star s}$. Similarly

$$\phi_g = l_{s'} \circ \phi_s \circ h_{s',f_s}^{-1} = l_{s'} \circ l_s \circ \phi \circ h_s^{-1} \circ h_{s',f_s}^{-1} = l_{s'\star s} \circ \phi \circ h_{s'\star s}^{-1}.$$

6. **Injectivity**. If $s_0 = 1$ so that $g = M \circ f \circ M^{-1}$. Then $M^*(\sigma_{s,g}) = \sigma_{s,f}$ so that $M_s = h_{s,g} \circ M \circ h_{s,f}^{-1}$ is a Möbius transformation, depends holomorphically on $s \in \mathbb{L}$ and conjugates $(f_s, W_{f_s}, \alpha_{f_s})$ and $(g_s, W_{g_s}, \alpha_{g_s})$. For the general case note that by the group action property for any $s \in \mathbb{L}$ the inverse map $h_{s,f}^{-1} : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ integrates the f_s invariant almost complex structure σ_{s^{-1},f_s}, i.e. $(h_{s,f}^{-1})^*(\sigma_0) = \sigma_{s^{-1},f_s}$. And similarly for g. If $(f_{s_0}, W_{f_{s_0}}, \alpha_{f_{s_0}})$ and $(g_{s_0}, W_{g_{s_0}}, \alpha_{g_{s_0}})$ are Möbius conjugate by some M. Then $M^*(\sigma_{s_0^{-1},g_{s_0}}) = \sigma_{s_0^{-1},f_{s_0}}$ and hence $M_1 = h_{s_0,g}^{-1} \circ M \circ h_{s_0,f}$ is a Möbius conjugacy between (f, W_f, α_f) and (g, W_g, α_g). So the first part applies and yields $M_{s_0} = M$.

$$
\begin{array}{ccc}
\underset{f\,\circlearrowleft}{\mathbb{C}}\ (\sigma_{s,f}, \sigma_0) & \overset{h_{s,f}}{\longrightarrow} & \underset{f_s\,\circlearrowleft}{\mathbb{C}}\ (\sigma_0, \sigma_{s^{-1},f_s}) \\
\downarrow M_1 & & \downarrow M_s \\
\underset{g\,\circlearrowleft}{\mathbb{C}}\ \sigma_{s,g}, \sigma_0 & \overset{h_{s,g}}{\longrightarrow} & \underset{g_s\,\circlearrowleft}{\mathbb{C}}\ \sigma_0, \sigma_{s^{-1},g_s} \\
\downarrow \phi_g & & \downarrow \phi_{g_s} \\
\mathbb{C}\ l_s^*(\sigma_0), \sigma_0 & & \mathbb{C}\ \sigma_0, l_{s^{-1}}^*(\sigma_0)
\end{array}
.
$$

□

3 Centers, stabilizers and proper attracting dynamics

In this section we study the mapping properties of $s \mapsto f_s$ in a BH-motion. We will show that it is either locally injective (as in Example 1, page 5) or constant (rigid, as in Example 2, page 5). For this we will need some notations:

Definition 3.1. *In a BH-motion of an attracting dynamics (f, W, α), we denote by* $\mathrm{Stab}(f) \subset \mathbb{L}$, *the* **stabilizer**, *to be the set of s for which (f_s, W_s, α_s) is Möbius conjugate to (f, W, α).*

Definition 3.2. *We say that an attracting dynamics (f, W, α) is* **proper**, *if on every connected component Ω of $\widetilde{B}(\alpha)$ the restriction $f : \Omega \longrightarrow f(\Omega)$ is a proper map (for example a rational map with an attracting cycle and with the choice $W = \overline{\mathbb{C}}$ is always proper).*

We shall use the term **central orbit** *synonymously with the grand orbit of α : $G.O.(\alpha) = \{z | \exists n \in \mathbb{N} : f^n(z) = \alpha\}$.*

A proper attracting dynamics (f, α) is a **center** *if all critical points in $\widetilde{B}(\alpha)$ are central, i.e. belongs to the central orbit, in particular α is f-superattracting. In other words, f is not a center if either $\lambda(f) \neq 0$ or $\lambda(f) = 0$, but at least one critical point in $\widetilde{B}(\alpha)$ is not a preimage of α.*

Theorem 3.3. *(Injective or rigid) Let (f, W, α) be a proper attracting dynamics with $k(f) = k$, $\lambda(f) = \lambda$.*

- *Assume that (f, α) is not a center. Then* $\mathrm{Stab}(f)$ *is a discrete subgroup of* $\mathbb{W} := (1 + i\mathbb{R}, \star)$ *and the map $s \mapsto f_s$ is injective on the semi strips $\{\Re(s) > 0, |\Im(s - s_0)| < \delta\}$ for some $\delta > 0$ and any $s_0 \in \mathbb{L}$. Moreover* $\mathrm{Stab}(f_s) = s^{-1} \star \mathrm{Stab}(f) \star s$.

- *Assume now (f, α) IS a center. Then* $\mathrm{Stab}(f) \equiv \mathrm{Stab}(f_s) = \mathbb{L}$ *and $f_s \equiv f$ for all $s \in \mathbb{L}$ (after suitable normalization of h_s).*

Remark that Branner and Hubbard's original result ([BH]'s Prop. 8.3, see also [Wi]'s Prop.5.5) corresponds to the case that f is a polynomial and (f, ∞) is a center, which means in this case the absence of escaping critical points, or equivalently the connectedness of the Julia set.

The first step in our proof is the following:

Proposition 3.4. *For any attracting dynamics (f, α) (with $k(f) = k$, $\lambda(f) = \lambda$), the stabilizer* $\mathrm{Stab}(f)$ *is a subgroup of* $(\mathbb{L}, \star)$ *and is independent of the normalizations of h_s. If $\lambda \in \mathbb{D}^*$ then* $\mathrm{Stab}(f)$ *is a discrete subgroup of* $\mathbb{W}$*. If $\lambda = 0$ and if* $\mathrm{Stab}(f)$ *is not discrete, then there is a sequence $s'_n \xrightarrow{\neq} 1$ such that $f_{s'_n} \equiv f$, after suitable normalization of h_s. Consequently $f_s \equiv f$ under this normalization.*

Proof. Assume $s_1, s_2 \in \mathrm{Stab}(f)$. We shall prove that $s_1^{-1} \star s_2 \in \mathrm{Stab}(f)$, which implies that $(\mathrm{Stab}(f), \star)$ is a group.

By Theorem 2.5.(5), the maps $s \mapsto f_{s\star s_1}$, $s \mapsto f_{s \star s_2}$ are BH-motions of f_{s_1} and f_{s_2} respectively.

For $i = 1, 2$ let M_i be Möbius transformations with $M_i(\alpha) = \alpha_{s_i}$ and $f = M_i^{-1} \circ f_{s_i} \circ M_i$. Then $f_{s_1} = N^{-1} \circ f_{s_2} \circ N$, where $N = M_2 \circ M_1^{-1}$. By Theorem 2.5.(6) the dynamics $f_{s \star s_1}$ and $f_{s\star s_2}$ are Möbius conjugate for all $s \in \mathbb{L}$. Setting $s = s_1^{-1}$ we get that f and $f_{s_1^{-1} \star s_2}$ are Möbius conjugate, i.e. $s_1^{-1} \star s_2 \in \mathrm{Stab}(f)$.

Therefore $\mathrm{Stab}(f)$ is a subgroup. Now different normalizations of h_s lead to Möbius conjugated f_s, and therefore the same $\mathrm{Stab}(f)$.

Assume now $\lambda \in \mathbb{D}^*$. A necessary condition for $s \in \mathrm{Stab}(f)$ is that $\lambda_s = \lambda$, i.e. $s \in \{1 + i\frac{2\pi n}{\log|\lambda|} | n \in \mathbb{Z}\}$ which is a discrete subgroup of $\mathbb{W}$.

Assume now $\lambda = 0$. We will make a sequence of Möbius conjugations to reduce f to a suitable normal form.

We start by remarking that there is a Möbius transformation G with $G(\alpha) = 0$ such that in a neighborhood of the origin

$$G^{-1} \circ f^k \circ G(z) = z^d(1 + \mathfrak{O}(z)) = 1 \cdot z^d + p \cdot z^{d+1} + \mathfrak{O}(z^{d+2}) ,$$

where d is the local degree of f^k at α.

Next we choose a further Möbius transformation M fixing 0 so that $(G \circ M)^{-1} \circ f^k \circ G \circ M(z)$ has a local expansion in the following normal form

$$z^d(1 + \mathfrak{O}(z^2)) = 1 \cdot z^d + 0 \cdot z^{d+1} + \mathfrak{O}(z^{d+2}) \ . \tag{3}$$

By looking at the local expansions one can check easily that such M exists, and there are precisely $d-1$ of them, in the form

$$M(z) = N(\rho z) \ , \ \rho^{d-1} = 1 \ , \ N(z) = \frac{z}{\frac{p}{d}z + 1} \ . \tag{4}$$

It follows easily that there are exactly $d-1$ choices of the composed Möbius map $G \circ M$ to reduce f to its normal form.

Now for each s we may and shall post-compose h_s by Möbius maps $G_s \circ M_s$ to reduce f_s to its normal form. Again there are exactly $d-1$ choices for each given s. By Theorem 2.5.(2), the map $s \mapsto f_s$ is analytic. So $G_s \circ M_s$ can be chosen to be s-analytic.

Therefore there are s-analytic normalizations of h_s so that all f_s have the above normal form. We may and shall thus suppose that all f_s have already the above normal form.

Assume that $\mathrm{Stab}(f)$ is not discrete. Then there is a sequence $s_n \in \mathrm{Stab}(f)$ with $s_n \xrightarrow{\neq} s' \in \mathbb{L}$. Then there is a sequence of Möbius transformations M_n with $M_n(0) = 0$ and $M_n^{-1} \circ f \circ M_n = f_{s_n}$. As both f and f_{s_n} have the normal form (3), by (4) we conclude that $M_n(z) = \rho_n z$ with $\rho_n^{d-1} = 1$.

If there is a subsequence (n_p) for which $\rho_{n_p} = 1$, then $f_1 = f_{s_{n_p}}$. But the right hand side converges to $f_{s'}$. So $f_{s_{n_p}} = f_1 = f_{s'}$.

Otherwise there is a subsequence with $\rho_{n_p} = \rho$ for some fixed ρ with $\rho^{d-1} = 1$. Thus $M_{n_p}(z) = M(z) = \rho z$ and $M^{-1} \circ f \circ M = f_{s_{n_p}}$. Again the right hand side converges to $f_{s'}$. So $f_{s_{n_p}} = M^{-1} \circ f \circ M = f_{s'}$.

In both cases we get $f = (f_{s'})_{(s')^{-1}} = f_{(s')^{-1} \star s_{n_p}}$. Setting $s'_n = (s')^{-1} \star s_{n_p}$, we get the proposition, except the final consequence.

But due to Theorem 2.5.(2) the map $s \mapsto f_s(z)$ is analytic for each fixed z. By the isolated zero theorem we conclude that $f_s(z) \equiv f(z)$. $\square$

Using Riemann-Hurwitz formula, it is quite easy to prove that (f, α) is a center if and only if any connected component Λ of $\widetilde{B}(\alpha)$ is simply connected and has a unique point in the central orbit.

Proof of Theorem 3.3, non-center part. If $\lambda \in \mathbb{D}^*$ we know already that $\mathrm{Stab}(f) \subset \{1 + i\frac{2\pi n}{\log|\lambda|} |\ n \in \mathbb{Z}\}$ which is a discrete subgroup of $\mathbb{W}$.

For $\lambda = 0$ we have to work a little harder. Let $\phi : U \longrightarrow V$ be a Böttcher coordinate for f^k near α.

Case 1. There is at least one non central critical point in the immediate basin $B(\alpha)$. There is a maximal radius $0 < r < 1$ and an open subset $U^0 \subset B(\alpha)$ such that ∂U^0

contains at least one at most finitely many critical points c^j and ϕ extends as a biholomorphic map $\phi : U^0 \longrightarrow \mathbb{D}(r)$. The radius r is a conformal invariant. For the Böttcher coordinate ϕ_s of f_s the maximal radius $r(s) = r^{\Re(s)}$. Hence if $\Re(s) \neq 1$ then f_1 and f_s can not be Möbius conjugate, yielding $\mathrm{Stab}(f) \subset \mathbb{W} := \{s, \Re(s) = 1\}$.

Assume that $\mathrm{Stab}(f)$ is not discrete. Then by Proposition 3.4 we have $f_s \equiv f$ after suitable normalization of h_s. For s close to 1 the map h_s is close to the identity and must map c^j to some $c^{j'}$. So $h_s(c^j) \equiv c^j$ for all s close to 1. But as $s \mapsto h_s(c^j)$ is continuous (even holomorphic), we have $h_s(c^j) \equiv c^j$. Similarly $\phi_s(h_s(c^j)) = constant$. This is not possible as $\phi_s(h_s(c^j)) = l_s(\phi(c^j)) \neq constant$. We conclude that $\mathrm{Stab}(f)$ is a discrete subgroup of $\mathbb{W}$.

Case 2. The point α is the sole critical point of f^k in $B(\alpha)$. Then ϕ extends to a biholomorphic map $\phi : B(\alpha) \longrightarrow \mathbb{D}$ and by assumption there is at least one connected component Ω of $\widetilde{B}(\alpha)$ containing a critical point not in the central orbit. Let n be the minimal iterate for which $f^n(\Omega) = B(\alpha)$ and let r be the maximal modulus of the critical values of $\phi \circ f^n$ on Ω. Then again r is a conformal invariant, and the corresponding value for f_s is $r(s) = r^{\Re(s)}$. Reasoning as above we may conclude that $\mathrm{Stab}(f)$ is again a discrete subgroup of $\mathbb{W}$.

Finally we show that $s \mapsto f_s$ is injective on the semi strips. Let $\delta > 0$ be minimal so that $1 + \delta i \in \mathrm{Stab}(f)$. Then $\mathrm{Stab}(f) = \{1 + in\delta, n \in \mathbb{Z}\}$ and

$$f_{s_1} = f_{s_2} \Longrightarrow s_2^{-1} \star s_1 \in \mathrm{Stab}(f) \Longrightarrow \exists n \in \mathbb{Z},\ s_1 = s_2 \star (1 + in\delta) \Longrightarrow \Im(s_1 - s_2) \in \delta\mathbb{Z}\ .$$

□

Rather than proving now the center part, we prove at first a slightly more general result, unrelated to BH-motions:

Definition 3.5. *We say that two attracting dynamics (f, W, α) and (f_0, W_0, α_0) are* **hybridly equivalent***, if there is a q.c. homeomorphism $h : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$, $h(\alpha) = \alpha_0$, $h(W) = W_0$ which is conformal a.e. on $\widetilde{B}(\alpha)^c$, and which is a conjugacy $h \circ f = f_0 \circ h$ on a neighborhood of $\widetilde{B}(\alpha)^c$. We also call h a* **hybrid conjugacy***.*

Proposition 3.6. *(rigidity) Let (f_1, W_1, α_1) and (f_2, W_2, α_2) be two proper attracting dynamics which are centers and which are hybridly equivalent to each other by a quasi-conformal map h (in particular $k(f_1) = k(f_2) := k$). Then they are Möbius conjugate (see Definition 2.6), by a Möbius transformation M, which coincides with h on $\widetilde{B}(\alpha)^c$.*

Proof. Let $\rho_i : B(\alpha_i) \longrightarrow \mathbb{D}$ denote Riemann maps (Böttcher coordinates) such that $\rho_i \circ f_i^k = (\rho_i)^{p+1}$, where $p + 1 = \deg(f^k : B(\alpha) \to B(\alpha))$. The maps are unique modulo multiplication by a pth-root of unity. Let $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hybrid conjugacy. Then the quasi-conformal homeomorphism $\eta := \rho_2 \circ h \circ \rho_1^{-1} : \mathbb{D} \longrightarrow \mathbb{D}$ extends by reflection to a global quasi-conformal homeomorphism and conjugates $z \mapsto z^{p+1}$ to itself on a neighborhood of $\mathbb{S}^1$. It follows that its restriction to $\mathbb{S}^1$ equals to a rigid rotation of order p, that is ωz with $\omega^p = 1$ (see Lemma 3.8 below and its trailing remark). Hence given a choice of ρ_1 we can choose ρ_2 such that the restriction of η to $\mathbb{S}^1$ is the identity. Define

$\phi = \rho_2^{-1} \circ \rho_1 : B(\alpha_1) \longrightarrow B(\alpha_2)$. We shall also express this fact that $\eta(z) = z$ on $\mathbb{S}^1$ by saying that ϕ and h are identical on the ideal boundary or $h^{-1} \circ \phi$ is the identity on the ideal boundary. See the following diagram.

$$\begin{array}{ccccc} \Omega & \xrightarrow{f_1^n} & B(\alpha_1) & \xrightarrow{\rho_1} & \mathbb{D} \\ h \downarrow\downarrow \phi_\Omega & & h \downarrow\downarrow \phi & & \eta \downarrow\downarrow id \\ h(\Omega) & \xrightarrow[f_2^n]{} & B(\alpha_2) & \xrightarrow[\rho_2]{} & \mathbb{D} \end{array} \tag{5}$$

Since η is quasi-conformal and equal to the identity on $\mathbb{S}^1$, there exists a constant $C > 0$ depending only on the maximal dilatation of η (and thus implicitly on h) such that $\forall z \in \mathbb{D}$: $\mathrm{d}_{\mathbb{D}}(z, \eta(z)) \leq C$ or equivalently $\forall z \in B(\alpha_1)$: $\mathrm{d}_{B(\alpha_2)}(h(z), \phi(z)) \leq C$. This is a classic compactness result for q.c. mappings. For completeness we reprove it in Lemma 3.9 below.

For Ω any connected component of $\widetilde{B}(\alpha_i)$ let $n = n(\Omega) = \min\{m | f_i^m(\Omega) = B(\alpha_i)\}$ and $p(\Omega) = \deg(f_i^n : \Omega \to B(\alpha_i))$. Then $p(\Omega) = p(h(\Omega))$ and $n(\Omega) = n(h(\Omega))$ and $\phi \circ f_1^n$ lifts by f_2^n to an isomorphism $\phi_\Omega : \Omega \longrightarrow h(\Omega)$. This lift is uniquely determined up to post composition by a deck-transformation for f_2^n. Also h and ϕ_Ω differs on the ideal boundary by a deck-transformation for the action of f_2^n on the ideal boundary. We let ϕ_Ω be the unique choice of lift for which h and ϕ_Ω are identical on the ideal boundary of Ω. Then by the same argument as above we have $\forall z \in \Omega$: $\mathrm{d}_{h(\Omega)}(h(z), \phi_\Omega(z)) \leq C$, with the same C, since $h \circ \phi_\Omega^{-1}$ is K-qc with the same K as above.

Define $H : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ by $H = h$ on $\widetilde{B}(\alpha_1)^c$ and by $H = \phi_\Omega$ on each connected component Ω of $\widetilde{B}(\alpha_1)$.

In order to prove that H is continuous, choose two distinct points β, γ outside $\widetilde{B}(\alpha_1)$. Let $w_n \in \Omega_n$, where the Ω_n are (not necessarily distinct) connected components of $\widetilde{B}(\alpha_1)$ and $w_n \to w \in \partial\widetilde{B}(\alpha_1)$. (The point w may be one of β, γ, but is never $\alpha_1 \in \widetilde{B}(\alpha_1)$). Then

$$\mathrm{d}_{\overline{\mathbb{C}} \setminus h(\{\alpha_1, \beta, \gamma, w\})}(h(w_n), H(w_n)) \leq d_{h(\Omega_n)}(h(w_n), \phi_{\Omega_n}(w_n)) \leq C < \infty \ .$$

By a classical inequality (see for example Milnor [Mi]) we have $H(w_n) \to h(w)$ as $n \to \infty$.

Thus H is a homeomorphism, which coincides with h outside $\widetilde{B}(\alpha_1)$.

But then by Rickman's lemma ([Ri], see also [DH2], Lemma 2) H is also quasi-conformal, because h is globally quasi-conformal and the patches ϕ_Ω are also quasi-conformal, in fact conformal. Moreover H is 1-quasi-conformal, because the maps ϕ_Ω are conformal and h is conformal a.e. on $\widetilde{B}(\alpha_1)^c$. Finally H is conformal and thus a Möbius transformation by Weyl's lemma. □

Remark. Note that the key point here is the existence of bi-holomorphic conjugacies ϕ, ϕ_Λ equal to h on the ideal boundaries, as indicated in the diagram (5) . We may thus replace the assumption of being centers by this requirement and obtain a more general rigidity result. We will need this fact twice in Section 4.

Definition 3.7. *A degree $d \geq 2$ orientation preserving covering map $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is called weakly expanding iff $\forall x, y \in \mathbb{S}^1$ and for each of the two complementary subarcs $I_1 = [x, y]$ and $I_2 = [y, x]$ of $\{x, y\}$ in $\mathbb{S}^1$ there exists an $n \in \mathbb{N}$ such that $f^n(I_i) = \mathbb{S}^1$ or equivalently f^n is not injective on any of the two arcs.*

The following Lemma is a classical result included for completeness.

Lemma 3.8. *For any pair of degree $d \geq 2$ weakly expanding covering maps $f_i : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ and any choice of fixed points $\alpha_i \in \mathbb{S}^1$, $f_i(\alpha_i) = \alpha_i$ for $i = 1, 2$. There exists a unique orientation preserving homeomorphism $h : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ with $h(\alpha_1) = \alpha_2$ and $h \circ f_1 = f_2 \circ h$.*

Proof. Existence: Let $h_0 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be any orientation preserving homeomorphism with $h_0(\alpha_1) = \alpha_2$, e.g. $h_0(z) = \frac{\alpha_1}{\alpha_2} z$. Define recursively $h_n : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ to be the unique lift of $h_{n-1} \circ f_1$ to f_2, with $h_n(\alpha_1) = \alpha_2$. (Equivalently define h_n to be the unique lift of $h_0 \circ f_1^n$ to f_2^n.) Then each f_n is order preserving and for every $m \geq n : \ h_m(f_1^{-n}(\alpha_1)) = f_2^{-n}(\alpha_2)$. As each f_i is weakly expanding, both families $\{h_n\}_n$ and $\{h_n^{-1}\}_n$ are equicontinuous and hence pre-compact. Let $h : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be any limit map. Then h is a homeomorphism and $h \circ f_1 = f_2 \circ h$ on the subset, $\cup_n f_1^{-n}(\alpha_1)$, which is dense because f_1 is weakly expanding. Hence $h \circ f_1 = f_2 \circ h$ on $\mathbb{S}^1$ as desired.

Uniqueness: If $\widehat{h} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is any orientation preserving conjugacy with $\widehat{h}(\alpha_1) = \alpha_2$, then $\widehat{h} = h$ on the dense subset $\cup_n f_1^{-n}(\alpha_1)$ and hence everywhere in $\mathbb{S}^1$. □

As an immediate consequence of this Lemma the automorphism group of $z \mapsto z^d$ for $d \geq 2$ (i.e. the set of orientation preserving homeomorphisms which commutes with z^d) equals the set of rigid rotations $\{z \mapsto \rho z | \rho^{d-1} = 1\}$, since $\{\rho | \rho^{d-1} = 1\}$ equals the set of fixed points in $\mathbb{S}^1$ of z^d.

Lemma 3.9. *There exists $C = C(K) > 0$ such that for any K-qc homeomorphism, $h : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ with $h = \mathrm{id}$ on $\mathbb{S}^1$: $\forall z \in \mathbb{D}$: $\qquad \mathrm{d}_{\mathbb{D}}(z, h(z)) \leq C$.*

Proof. Define

$$\mathcal{K}_K = \{h : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}} | h \text{ is } K\text{-qc and } h = \mathrm{id} \text{ on } \mathbb{S}^1\}.$$

Then $\mathcal{K}_K$ is compact, because any $h \in \mathcal{K}_K$ extends by Schwarz-reflection in $\mathbb{S}^1$ to a global K-qc map, which fixes three distinct points say $1, i, -1$. Since the map $h \mapsto \mathrm{d}_{\mathbb{D}}(0, h(0))$ is continuous on the compact set $\mathcal{K}_K$ we can define $C = C(K)$ as its maximal value.

Let $h \in \mathcal{K}_K$ and $z_0 \in \mathbb{D}$ be arbitrary and let $M(z) = \frac{z+z_0}{1+\overline{z}_0 z}$, so that $M(0) = z_0$. Then $M^{-1} \circ h \circ M \in \mathcal{K}_K$ and since M is a hyperbolic isometry we have

$$\mathrm{d}_{\mathbb{D}}(z_0, h(z_0)) = \mathrm{d}_{\mathbb{D}}(0, M^{-1} \circ h \circ M(0)) \leq C.$$

□

End of the proof of Theorem 3.3, the center part. This can be deduced easily as follows: We normalize h_s so that it fixes α and two points of $\widetilde{B}(\alpha)^c$. For any s the maps f_1, f_s

and h_s satisfy the hypothesis of Proposition 3.6 and the Möbius conjugacy M_s fixes three points on the sphere. So $f_s \equiv f$ and $\mathrm{Stab}(f) = \mathbb{L}$. $\square$

Remark. The proof of Proposition 3.6 can also be done explicitly using the formula of $\widetilde{l}_s$.

4 Applications

4.1 Cubic slices

The first of our two examples is the two parameter family of cubic polynomials

$$P_{\lambda,a}(z) := \lambda z + \sqrt{a} z^2 + z^3 \ , \qquad \lambda, a \in \mathbb{C} \ . \tag{6}$$

Here the two different determinations of $\sqrt{a}$ yields maps which are conjugate by the map $z \mapsto -z$ and hence are holomorphically equivalent. Moreover any cubic polynomial admitting 0 as a fixed point is linearly conjugate to $P_{\lambda,a}$ for some unique parameters λ, a. Let

$$\widetilde{\mathcal{H}} = \{(\lambda, a) \in \mathbb{D} \times \mathbb{C} \mid \text{ both critical points belong to } \widetilde{B}_{\lambda,a}(0)\}$$

$$\widetilde{\mathcal{H}}^c = \{(\lambda, a) \in \mathbb{D} \times \mathbb{C} \mid \text{only one simple critical point belongs to } \widetilde{B}_{\lambda,a}(0)\}$$

$$\mathcal{P}_\lambda = \{(\lambda', a) | \lambda' = \lambda\} \simeq \mathbb{C} \ , \qquad \widetilde{\mathcal{H}}_\lambda = \widetilde{\mathcal{H}} \cap \mathcal{P}_\lambda \ , \qquad \text{in particular}$$

$$\widetilde{\mathcal{H}}^c_{e^{-1}} = \{a \in \mathbb{C} \mid \text{only one simple critical point belongs to } \widetilde{B}_{e^{-1},a}(0)\} \ .$$

We study the effect of the BH-motion on both the parameter space and the dynamical plane, and determine completely the stabilizers.

Theorem 4.1. *There exists a holomorphic motion $H : (s, a, z) \mapsto (v(s, a), h(s, a, z))$, $\mathbb{L} \times \mathbb{C}^2 \to \mathbb{C}^2$ over $\mathbb{L}$ based at $s_0 = 1$ such that for $\lambda_0 = \mathrm{e}^{-1} = \mathrm{e}^{-s_0}$:*

1. *For each fixed a, the map $s \mapsto h(s, a, \cdot)$ is a BH-motion of the attracting dynamics $(P_{e^{-1},a}, \overline{\mathbb{C}}, 0)$, in particular $h_{s,a} \circ P_{e^{-1},a} = P_{e^{-s},v(s,a)} \circ h_{s,a}$.*
2. *For each fixed (s, a) the quasi-conformal conjugacy $z \mapsto h(s, a, z)$ is conformal on the exterior of $\widetilde{B}_{e^{-1},a}(0)$.*
3. *The holomorphic motion $(s, a) \mapsto v(s, a)$ restricted to $\mathbb{L} \times \widetilde{\mathcal{H}}^c_{e^{-1}}$ is $2\pi i$-periodic in the s variable. In particular $\mathrm{Stab}(P_{e^{-1},a}) = 1 + 2\pi i\mathbb{Z}$. Furthermore the holomorphic motion H restricted to $\{(s, a, z), s \in \mathbb{L}, a \in \widetilde{\mathcal{H}}^c_{e^{-1}}, z \in \widetilde{B}_{e^{-1},a}(0)^c\}$ is also $2\pi i$-periodic in the s variable (but has a more complicated monodromy structure elsewhere).*

By the λ-lemma for holomorphic motions, the maps $(s, z) \mapsto h(s, a, z)$ for $a \in \mathbb{C}$ fixed and $(s, a) \mapsto v(s, a)$ are continuous as functions of two variables. However in general the map $h(s, a, z)$ is discontinuous with respect to a.

The periodicity leads naturally to the following operations:

Define $\widehat{v} : \mathbb{D}^* \times \widetilde{\mathcal{H}}^c_{e^{-1}} \longrightarrow \mathbb{C}$ by $\widehat{v}(e^{-s}, a) = v(s,a)$; define $\widehat{h} : \mathbb{D}^* \times \widetilde{\mathcal{H}}^c_{e^{-1}} \times \widetilde{B}_{e^{-1},a}(0)^c \longrightarrow \mathbb{C}$ by $\widehat{h}(e^{-s}, a, z) = h(s,a,z)$, and finally define

$$\widehat{H} : \{(\lambda, a, z), \lambda \in \mathbb{D}^*, a \in \widetilde{\mathcal{H}}^c_{e^{-1}}, z \in \widetilde{B}_{e^{-1},a}(0)^c\} \longrightarrow \overline{\mathbb{C}}^2$$

by $\widehat{H}(\lambda, a, z) = (\widehat{v}(\lambda, a), \widehat{h}(\lambda, a, z))$. Then both $\widehat{v}(\cdot,\cdot)$ and $\widehat{H}(\cdot, a, \cdot)$ are holomorphic motions over $\mathbb{D}^*$ with base point $\lambda_0 = \mathrm{e}^{-1}$. We call them the quotient motions.

With a little extra work we obtain:

Theorem 4.2. *The quotient holomorphic motions $\widehat{v}(\cdot,\cdot)$ and $\widehat{H}(\cdot, a, \cdot)$ both extend to holomorphic motions over $\mathbb{D}$ with base point* 0.

Proof of Theorem 4.1. For $a \in \mathbb{C}$ fixed and a fixed choice of $\sqrt{a}$, we consider a BH-motion for the attracting dynamics $(f, W, \alpha) := (P_{e^{-1},a}, \overline{\mathbb{C}}, 0)$. We normalize the integrating maps $h_s = h_{s,a}$ so that $h_{s,a}(0) = 0$, $h_{s,a}(\infty) = \infty$ and $h_{s,a}$ is tangent to the identity at ∞. This implies that the new maps f_s are again a cubic polynomial in the form (6) . Further, by Theorem 2.5.(3), the new multipliers $\lambda(f_s)$ equal to $e^{-1}|e^{-1}|^{s-1} = e^{-s}$. Therefore $f_s = P_{e^{-s},v(s,a)}$ for some $v(s,a) \in \mathbb{C}$. Set $h(s,a,z) := h_{s,a}(z)$ and

$$H(s,a,z) := (v(s,a), h(s,a,z)) \ .$$

We check at first that $H : \mathbb{L} \times \mathbb{C}^2 \to \mathbb{C}^2$ is a holomorphic motion:

• Injectivity on (a,z): For any fixed $s \in \mathbb{L}$, assume $H(s,a,z) = H(s,a',z')$. In particular $v(s,a) = v(s,a')$ and so $P_{e^{-s},v(s,a)} = P_{e^{-s},v(s,a')}$. By Theorem 2.5.(6), we conclude that $P_{e^{-1},a}$ and $P_{e^{-1},a'}$ are linearly conjugate and hence $a = a'$. Consequently $h(s,a,z) = h(s,a,z')$. But $h(s,a,\cdot) = h_{s,a}(\cdot)$ is a homeomorphism of $\mathbb{C}$, so we conclude that $z = z'$.

• Analyticity in s: The map $h(s,a,z)$ is analytic in s for any fixed $a, z \in \mathbb{C}$, by the measurable Riemann mapping theorem with parameters. Due to Theorem 2.5.(2), for $a \in \mathbb{C}$ fixed, the map

$$s \mapsto P_{e^{-s},v(s,a)}(z) = \lambda_s z + \widetilde{v}(s,a) z^2 + z^3 = h_{s,a} \circ (z \mapsto e^{-1}z + \sqrt{a}z^2 + z^3) \circ h^{-1}_{s,a}(z) \ ,$$

where $v(s,a) = \widetilde{v}(s,a)^2$ is analytic in s for every fixed z. It follows $\widetilde{v}(s,a)$ and hence $v(s,a)$ depends complex analytically on s.

• Identity at the base point $s = 1$. In this case $h_{1,a}(z) \equiv z$ and consequently $P_{\mathrm{e}^{-1},v(1,a)} := h_{1,a} \circ P_{e^{-1},a} \circ h^{-1}_{1,a} = P_{e^{-1},a}$. So $v(1,a) = a$ and $H(1,a,z) = (a,z)$.

This proves that H is indeed a holomorphic motion over $\mathbb{L}$. We proceed to prove the remaining part of Theorem 4.1.

(1) By construction.

(2) is obvious.

(3). Fix now any $a \in \widetilde{\mathcal{H}}^c_{e^{-1}}$, and set $B(0) = B_{e^{-1},a}(0)$ and $\widetilde{B}(0) = \widetilde{B}_{e^{-1},a}(0)$. Recall that by definition of $\widetilde{\mathcal{H}}^c_{e^{-1}}$, the entire attracted basin $\widetilde{B}(0)$ of 0 for $P_{e^{-1},a}$ contains a unique critical point $c_0 = c_0(a)$. This critical point is in the immediate basin $B(0)$ and has local degree 2.

Let $s \in \mathrm{Stab}(P_{e^{-1},a})$. Thus $P_{e^{-s},v(a,s)}$ is Möbius conjugate to $P_{e^{-1},a}$. A necessary condition on s is that $e^{-s} = e^{-1}$. In other words $\mathrm{Stab}(P_{e^{-1},a}) \subset 1 + 2\pi i\mathbb{Z}$.

Claim. We have $v(1+2\pi i, a) = v(1,a) \equiv a$, i.e. $f_{1+2\pi i} = P_{e^{-1},v(1+2\pi i,a)}$ equals to $f = f_1 := P_{e^{-1},a}$. Furthermore $h_{1+2\pi i,a}(z) = h_{1,a}(z) \equiv z$ for all $z \in \widetilde{B}(0)^c$. Consequently $\mathrm{Stab}(P_{e^{-1},a}) = 1 + 2\pi i\mathbb{Z}$.

Proof of Claim.

Let $\phi : B(0) \longrightarrow \mathbb{C}$ be the linearizer with $\phi(c_0) = 1$ and let $\psi : \mathbb{D} \longrightarrow U$ be the local inverse carrying 0 to 0. Define $\widehat{h} := \psi \circ l_{1+2\pi i} \circ \phi : U \longrightarrow U$ so that $\widehat{h}(f(c_0)) = f(c_0)$, and extend $\widehat{h}$ by iterated lifting to a quasi conformal homeomorphism $\widehat{h} : B(0) \longrightarrow B(0)$, which integrates $\sigma_{1+2\pi i}$ and which conjugates f to itself. By further iterated lifting we can uniquely extend $\widehat{h}$ to a q.c. homemorphism $\widehat{h} : \widetilde{B}(0) \longrightarrow \widetilde{B}(0)$ which preserves each connected component of $\widetilde{B}(0)$.

We now prove that $\widehat{h}$ equals to the identity on the ideal boundary of $B(0)$. Let $\eta : B(0) \longrightarrow \mathbb{D}$ denote a Riemann map fixing 0 and define $R = \eta \circ f \circ \eta^{-1}$. Then R is a quadratic Blaschke product fixing the origin. Define $\widetilde{h} = \eta \circ \widehat{h} \circ \eta^{-1}$ and extend $\widetilde{h}$ to a global q.c. homeomorphism $\widetilde{h} : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ by reflection in the unit circle. Then $\widetilde{h} \circ R = R \circ \widetilde{h}$ and in particular this holds on the unit circle which is invariant by both R and $\widetilde{h}$. However since the degree of R is 2 there is a unique self homeomorphism of $\mathbb{S}^1$, which commutes with R. It is the identity. Thus $\widetilde{h}$ equals to the identity on $\mathbb{S}^1$. It follows that $\widehat{h}$ equals to the identity on the ideal boundary of $B(0)$.

Similarly $\widehat{h}$ equals to the identity on the ideal boundary of every connected component of $\widetilde{B}(0)$. As in the proof of Proposition 3.6 (see also the Remark following its proof), we conclude that the extension

$$\widehat{h} = \begin{cases} \widehat{h} & \text{on } \widetilde{B}(0) \\ id & \text{on } \widetilde{B}(0)^c \end{cases}$$

is a global quasi-conformal homeomorphism, conjugating f to itself. As $\widehat{h}$ integrates the almost complex structure $\sigma_{1+2\pi i}$, fixes the origin and is tangent to the identity at ∞, it follows that $\widehat{h} = h_{1+2\pi i}$, $f = f_{1+2\pi i}$ and $h_{1+2\pi i}$ equals to the identity on $\widetilde{B}(0)^c$. This ends the proof of the Claim.

Now we may prove that $s \mapsto P_{e^{-s},v(s,a)}$ is $2\pi i$-periodic, or equivalently $s \mapsto v(s,a)$ is $2\pi i$-periodic. We have

$$v(s+2\pi i, a) = v(s \star (1+2\pi i), a) = v(s, v(1+2\pi i, a)) = v(s,a) \tag{7}$$

where the first equality is due to the (simple) equality $s + 2\pi i = s \star (1+2\pi i)$, the second is due to the group action property $(f_{s'})_s = f_{s\star s'}$ in Theorem 2.5.(5), and the third is due to the claim above.

Assume now $z \in \widetilde{B}(0)^c$, we want to prove that $s \mapsto h(s,a,z)$ is also $2\pi i$-periodic. In particular the maps $h_{1+n\cdot 2\pi i,a}(z)$, $n \in \mathbb{Z}$ are the identity on $\widetilde{B}(0)^c$. Again

$$h(s+2\pi i,a,z) = h(s \star (1+2\pi i),a,z) = h(s,v(1+2\pi i,a),h(1+2\pi i,a,z)) = h(s,a,z)\ ,$$

where the second equality is due to the group action property $h_{s\star s',f} = h_{s,f_{s'}} \circ h_{s',f}$ in Theorem 2.5.(5), and the third is due to the claim above. $\square$

In the proof of Theorem 4.2 the hard work is really to prove that $\widehat{H}$ extends to a holomorphic motion to $\lambda = 0$. Because then one can change the base point from e^{-1} to 0 as follows: for any holomorphic motion $K : \Lambda \times E \longrightarrow \mathcal{X}$ with base point $\lambda_0 \in \Lambda$ define $E_{\lambda_0} := E$ and more generally $E_\lambda := K_\lambda(E)$ for $\lambda \in \Lambda$. Then $K' : \Lambda \times E_{\lambda_1} \longrightarrow \mathcal{X}$ given by $K'(\lambda,z) = K(\lambda, K_{\lambda_1}^{-1}(z))$ is a holomorphic motion with the same fibers and the same set of graphs $\{K(\Lambda,z)|z \in E_{\lambda_0}\} = \{K'(\Lambda,z)|z \in E_{\lambda_1}\}$, but with base point λ_1.

We start by proving

Lemma 4.3. *Both maps $\widehat{v} : \mathbb{D}^* \times \widetilde{\mathcal{H}}^c_{e^{-1}} \longrightarrow \mathbb{C}$ and $\widehat{h} : \mathbb{D}^* \times \widetilde{\mathcal{H}}^c_{e^{-1}} \times \widetilde{B}_{e^{-1},a}(0)^c \longrightarrow \mathbb{C}$ have unique extensions to $\lambda = 0$ such that $\lambda \mapsto \widehat{v}(\lambda,a)$ and $\lambda \mapsto \widehat{h}(\lambda,a,z)$ are holomorphic for every fixed $a \in \widetilde{\mathcal{H}}^c_{e^{-1}}$ and every fixed $z \in \widetilde{B}_{e^{-1},a}(0)^c$.*

For P a monic polynomial of degree d denote by $g_P : \mathbb{C} \longrightarrow [0,\infty[$ the Böttcher potential at infinity. That is $g = g_P$ is the unique sub harmonic function, which satisfies i) $g(P(z)) = d \cdot g(z)$, ii) $g(z) \equiv 0$ on $\mathbb{C} \setminus B_P(\infty)$ and iii) $g(z) - \log|z| = \mathfrak{o}(1)$ at ∞. Denote by $g_{\lambda,a}$ the map $g_{P_{\lambda,a}}$.

Proof of Lemma 4.3. The conjugacy $h_{s,a}$ is conformal on the attracted basin of infinity $B_{e^{-1},a}(\infty)$ for $f = P_{e^{-1},a}$ and hence preserves the Böttcher potential, i.e. $g_{e^{-s},v(s,a)}(h(s,a,z)) = g_{e^{-1},a}(z)$. Hence for any $z \in \widetilde{B}_{e^{-1},a}(0)^c$ the map $\lambda \mapsto \widehat{h}_a(\lambda,z)$ is bounded. To see this note that the Böttcher coordinate at ∞ for $P_{\lambda,a'}$ is tangent to the identity at ∞ for any $\lambda, a' \in \mathbb{C}$, and apply the compactness of normalized univalent maps. Similarly $\widehat{v}(\lambda,a)$ is bounded because $\widehat{v}(\lambda,a) \simeq 3c_1(\lambda,a)/2$ for small λ, where $c_1(\lambda,a) = \widehat{h}_a(\lambda,c_1)$ denotes the critical point of $P_{\lambda,\widehat{v}(\lambda,a)}$ not in $\widehat{h}_{\lambda,a}(\widetilde{B}_{e^{-1},a}(0))$ and c_1 denotes the critical point of $P_{e^{-1},a}$ not in $\widetilde{B}_{e^{-1},a}(0)$. By the theorem of removable singularities both $s \mapsto \widehat{v}(\cdot,a)$ and $s \mapsto \widehat{h}(\cdot,a,z)$ extends holomorphically to $\lambda = 0$. $\square$

To complete the proof of Theorem 4.2 we need to check that the extended maps $a \mapsto \widehat{v}(0,a)$ and $z \mapsto \widehat{h}_a(0,z)$ are injective on $\widetilde{\mathcal{H}}^c_{e^{-1}}$, respectively on $\widetilde{B}_{e^{-1},a}(0)^c$ for $a \in \widetilde{\mathcal{H}}^c_{e^{-1}}$. For this we prove at first the following

Theorem 4.4. *For any $a \in \widetilde{\mathcal{H}}^c_{e^{-1}}$ and any $\lambda \in \mathbb{D}$ the map $P_{\lambda,\widehat{v}(\lambda,a)}$ is hybridly equivalent to $P_{0,\widehat{v}(0,a)}$. More precisely there exist hybrid conjugacies $H_{\lambda,a} : \mathbb{C} \longrightarrow \mathbb{C}$ between $P_{\lambda,\widehat{v}(\lambda,a)}$ and $P_{0,\widehat{v}(0,a)}$, which are asymptotic to the identity at ∞, such that, for each fixed a, and as $\lambda \to 0$, the dilatations $||\overline{\partial} H_{\lambda,a}/\partial H_{\lambda,a}||_\infty$ converges to zero uniformly , and $H_{\lambda,a}$ converges locally uniformly to the identity.*

Proof. In the following we fix an arbitrary $a \in \widetilde{\mathcal{H}}^c_{e^{-1}}$ and suppress a in the rest of the proof, e.g write P_λ for $P_{\lambda,\widehat{v}(\lambda,a)}$ etc. It is easy to see (see below) that there is a center attracting dynamics (see Definition 3.2) which is hybridly equivalent to $P_{e^{-1}}$. Once properly normalized, this center is a cubic polynomial of the form $P_{0,b}$, with $b \in \mathbb{C} \setminus \{0\}$ as it is a cubic polynomial with a superattracting fixed point of order 2. Moreover as $\widehat{h}_\lambda$ is a hybrid conjugacy between $P_{e^{-1}}$ and P_λ, the map $P_{0,b}$ is a common center for all P_λ, $\lambda \in \mathbb{D}$, by uniqueness of centers (Proposition 3.6). Thus to prove that $\widehat{v}(0,a) = b$ we need only to show that, as $\lambda \to 0$, the coefficients of P_λ converge to those of $P_{0,b}$, or equivalently, the non-captured P_λ-critical point $c_1(\lambda) \in \widetilde{B}_{\lambda,v(\lambda,a)}(0)^c$ converges to the $P_{0,b}$-critical point $c_1(b) \in \widetilde{B}_{0,b}(0)^c$ (as the captured P_λ-critical point $c_0(\lambda) \in \widetilde{B}_{\lambda,v(\lambda,a)}(0)$ converges to 0 and the critical points determine P_0).

To this end let $\phi_\lambda : B_\lambda(0) \longrightarrow \mathbb{D}$ be the Riemann map with $\phi_\lambda(0) = 0$ and $R_\lambda := \phi_\lambda^{-1} \circ P_\lambda \circ \phi_\lambda = z\frac{z+\lambda}{1+\overline{\lambda}z}$. Note that $\overline{\mathbb{D}(|\lambda|)}$ contains the critical point of R_λ in $\mathbb{D}$ as well as both preimages of 0. Define $V'_\lambda := \phi_\lambda^{-1}(\mathbb{D}(\sqrt{|\lambda|}))$ and $V_\lambda := P_\lambda^{-1}(V'_\lambda) \cap B_\lambda(0)$ so that $V'_\lambda \subset\subset V_\lambda$ and define a new map $\widehat{P_\lambda} : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$\widehat{P_\lambda} = \begin{cases} P_\lambda & \text{on } \mathbb{C} \setminus V_\lambda \\ \phi_\lambda^{-1} \circ (z \mapsto z^2) \circ \phi_\lambda & \text{on } \overline{V'_\lambda} \\ \text{degree 2 quasi-regular interpolation} & \text{on } A := V_\lambda \setminus \overline{V'_\lambda}\,. \end{cases}$$

Note that on the boundary of the annulus A the map $\widehat{P_\lambda}$ is a priori defined as real analytic covering maps of degree 2. It easily follows that there exists a, say C^1 extension also denoted $\widehat{P_\lambda}$, $\widehat{P_\lambda} : A \longrightarrow A' := \phi_\lambda^{-1}(\{z \mid |\lambda| \le |z| \le \sqrt{|\lambda|}\})$, which is also a degree 2 covering. We need a little more, namely we need that this extension can be chosen so that its complex dilatation converges to 0 as $|\lambda| \to 0$. Using the Riemann map ϕ_λ we can transport the problem to $\mathbb{D}$, solve it and transport the solution back. That it can be solved (for R_λ in $\mathbb{D}$) is a consequence of Lemma 4.5 below. The map $\widehat{P_\lambda}$ is evidently quasi regular. And any point of z passes at most once through the zone A, where $\widehat{P_\lambda}$ is not conformal. With the quasi regular extension on A given by Lemma 4.5 through ϕ_λ let μ denote the measurable $\widehat{P_\lambda}$-invariant Beltrami form, which equals the standard 0 Beltrami form on V'_λ and on $\widetilde{B}_{e^{-1},a}(0)^c$. Let also $H_\lambda : \mathbb{C} \longrightarrow \mathbb{C}$ denote the integrating map for μ, given by the measurable Riemann mapping theorem, and normalized by $H_\lambda(0) = 0$ and H_λ being tangent to the identity at ∞. Then $P_{0,b} = H_\lambda \circ \widehat{P_\lambda} \circ H_\lambda^{-1}$ and the dilatation of H_λ is bounded by that of μ on A, which is bounded by $16 \cdot 2 \cdot \sqrt{|\lambda|}/\log\sqrt{|\lambda|}$, as shown in Lemma 4.5. This bound of distortion tends to 0 as $\lambda \to 0$. With the chosen normalization it follows that H_λ converges uniformly to the identity on compact sets of $\mathbb{C}$. In particular the P_λ-critical point $c_1(\lambda) \in \widetilde{B}_{\lambda,v(\lambda,a)}(0)^c$ converges to the $P_{0,b}$-critical point $c_1(b) \in \widetilde{B}_{0,b}(0)^c$. We conclude then $P_{0,b} = P_0$. □

Lemma 4.5. *There exists $0 < r_0 < 1$ with the following property: Let $R : \mathbb{D} \longrightarrow \mathbb{D}$ be any degree $d > 1$ Blaschke product fixing 0 and 1 for which the critical values and the zeros are contained in $\mathbb{D}(r)$ for some $r < r_0$. Define $\widehat{V}' = \mathbb{D}(\sqrt{r})$ and $\widehat{V} = R^{-1}(\widehat{V}')$ (we have $\widehat{V} \supset \widehat{V}'$ by Schwarz Lemma). Then there exists a degree d quasi regular branched covering $F : \mathbb{D} \longrightarrow \mathbb{D}$ with $F(z) = z^d$ on $\overline{\widehat{V}'}$ and $F(z) = R(z)$ on $\mathbb{D} \setminus \widehat{V}$ such that*

$$\left|\frac{\overline{\partial} F}{\partial F}\right| \le \frac{16d\sqrt{r}}{|\log\sqrt{r}|}\,. \tag{8}$$

Proof. Let $A' = \{z | r < |z| < 1\}$, $A = R^{-1}(A')$ and $A'' = \{z | r^{1/d} < |z| < 1\}$. Let $\Gamma : A'' \longrightarrow A$ denote the lift of z^d to R which fixes 1, i.e $R(\Gamma(z)) = z^d$. Set $C = \{z | r^{1/2d} < |z| < 1\}$. Then it suffices to construct a quasi-conformal map G, which is the identity on $\widehat{V}'$ and equals Γ on C, because then

$$F(z) := (G^{-1}(z))^d$$

would be the required map. Write $r = \exp(dm)$ (with $m < 0$) and let $\widetilde{A}, \widetilde{A}'', \widetilde{C}, \widetilde{V'}$ be the preimages by $\exp(z)$ of the corresponding un-tilded (and hatted for V) sets in $\mathbb{D}^*$. In particular

$$\widetilde{A}'' = \{x + iy | x \in]m, 0[\}, \quad \widetilde{C} = \{x + iy | x \in]\frac{m}{2}, 0[\}, \quad \widetilde{V'} = \{x + iy | x < \frac{dm}{2}\} \ .$$

Denote by $\widetilde{\Gamma} : \widetilde{A}'' \longrightarrow \widetilde{A}$ the lift of $\Gamma \circ \exp(z)$ to $\exp(z)$ which fixes 0. Then $\widetilde{\Gamma}(z + 2\pi) = \widetilde{\Gamma}(z) + 2\pi$ and we shall construct a q.c. homeomorphism $\widetilde{G} : \mathbb{H}_- \longrightarrow \mathbb{H}_- = \{x + iy | x < 0\}$ which equals $\widetilde{\Gamma}$ on $\widetilde{C}$ and Id on $\widetilde{V'}$, which satisfies $\widetilde{G}(z + 2\pi) = \widetilde{G}(z) + 2\pi$ as well as (8). Then

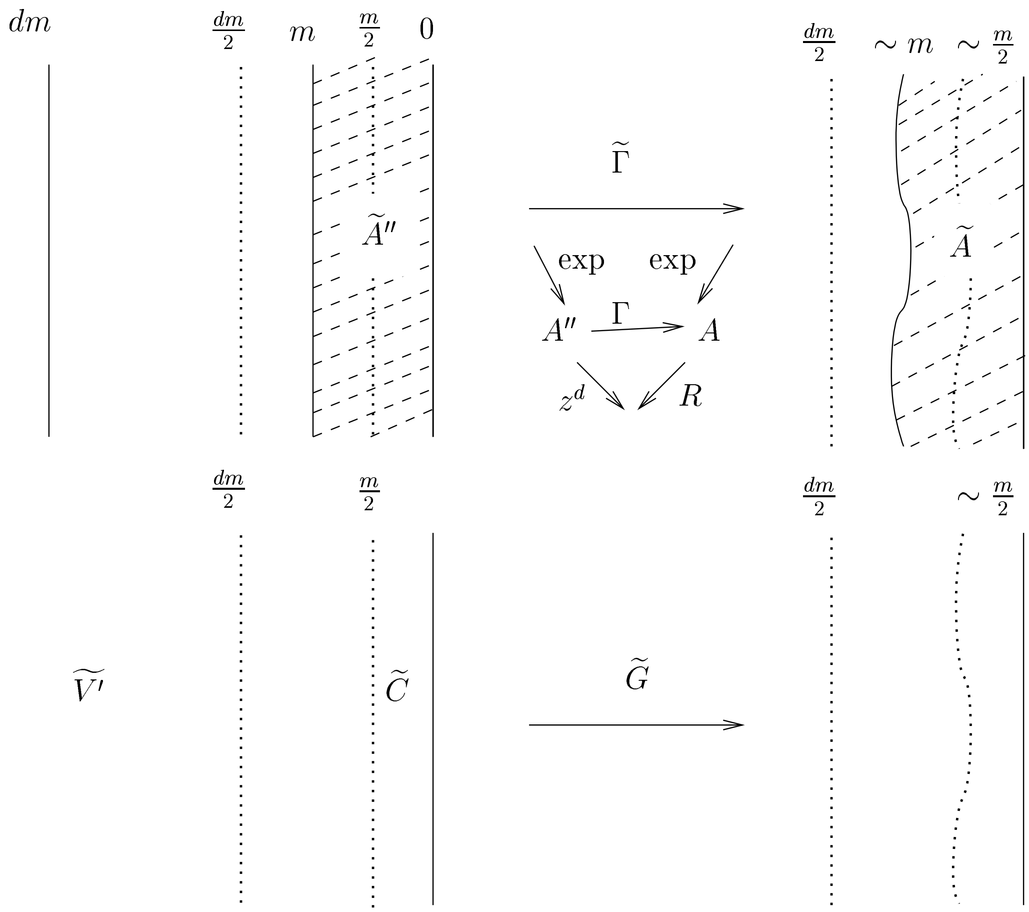
$$F(z) := \exp(d \cdot \widetilde{G}^{-1}(\log z)) \tag{9}$$

is the required map. To construct $\widetilde{G}$ we need only construct its values on the vertical strip $\{x + iy | x \in [\frac{dm}{2}, \frac{m}{2}]\}$ so that it verifies (8) and such that $\widetilde{G}$ is continuous. The extension to this strip is the standard affine extension, i.e. it maps each horizontal segment affinely to the segment whose endpoints are determined by the values of $\widetilde{G}$ already defined. More precisely, set $\widetilde{\Gamma}(u) = u + \omega(u)$. For $x \in [a, b] := [\frac{dm}{2}, \frac{m}{2}]$ and $y \in \mathbb{R}$, we have

$$x + iy = \frac{b - x}{b - a}(a + iy) + \frac{x - a}{b - a}(b + iy) \ .$$

Set

$$\widetilde{G}(x + iy) = \frac{b - x}{b - a}(a + iy) + \frac{x - a}{b - a}\widetilde{\Gamma}(b + iy) = x + iy + \frac{x - a}{b - a}\omega(b + iy) \ .$$



The reader may check easily that $\widetilde{G}$ is a homeomorphism provided $|\omega'(b+iy)| < \frac{1}{4}$ for every $y \in \mathbb{R}$.

To give more precise estimates we write $\Gamma(z) = z(1+g(z))$ and $\Gamma^{-1}(z) = z(1+h(z))$. Then

$$z = \Gamma^{-1}(\Gamma(z)) = z(1+g(z))(1+h(\Gamma(z)), \quad |\log(1+g(z))| = |\log(1+h(\Gamma(z))| \ . \tag{10}$$

Assume now $a_1, \cdots, a_d$ are the zeros of the Blaschke product R, we have

$$z^d(1+h(z))^d = R(z) = \prod_{j=1}^{d} \frac{1-\bar{a}_j}{1-a_j}\frac{z-a_j}{1-\bar{a}_j z} \ , \quad \text{so} \quad (1+h(z))^d = \prod_{j=1}^{d} \frac{1-\bar{a}_j}{1-a_j}\frac{1-a_j/z}{1-\bar{a}_j z} \ .$$

and

$$|\log(1+h(z))| \leq \frac{1}{d}\sum_{j=1}^{d} \Big(|\log(1-\frac{a_j}{z})| + |\log(1-\bar{a}_j z)| + |\log(1-\bar{a}_j)| + |\log(1-a_j)| \Big) \ .$$

As $|a_i| \leq r$ and $|z| \geq \sqrt{r}$, and $|\log(1+v)| \leq 2|v|$ for $|v| \leq \frac{1}{2}$, there is $r_0 > 0$ so that if $r < r_0$,

$$|\log(1+h(z))| \leq 2(\sqrt{r} + 3r) \leq 8\sqrt{r} \ .$$

Therefore

$$\forall u \in \widetilde{A}'', \quad |\omega(u)| = |\log(1+g(e^u))| \overset{(10)}{=} |\log(1+h(\Gamma(e^u)))| \le 8\sqrt{r}\ . \tag{11}$$

Now we use the Cauchy integral to estimate $\omega'(b+iy)$. Set $\rho = \frac{-m}{2} = -\frac{\log r}{2d}$. Note that $D(b+iy,\rho) \subset \widetilde{A}''$. So

$$|\omega'(b+iy)| \le \frac{8\sqrt{r}}{\rho} \longrightarrow_{r\to 0} 0\ . \tag{12}$$

We may thus adjust r_0 so that for $r < r_0$, $|\omega'(b+iy)| \le \frac{1}{4}$ and therefore $\widetilde{G}$ is a homeomorphism. We can readily estimate the Beltrami coefficient of $\widetilde{G}$: for $x \in]a,b[$,

$$\frac{\partial \widetilde{G}}{\partial x}(x+iy) = 1 + \frac{\omega(b+iy)}{b-a}, \qquad \frac{1}{i}\frac{\partial \widetilde{G}}{\partial y}(x+iy) = 1 + \frac{x-a}{b-a}\omega'(b+iy)\ .$$

$$\frac{\partial \widetilde{G}}{\partial z}(x+iy) = \frac{1}{2}\left(\frac{\partial \widetilde{G}}{\partial x}(x+iy) + \frac{1}{i}\frac{\partial \widetilde{G}}{\partial y}(x+iy)\right) = 1 + \frac{\omega(b+iy)+(x-a)\omega'(b+iy)}{2(b-a)}$$

$$\frac{\partial \widetilde{G}}{\partial \bar{z}}(x+iy) = \frac{1}{2}\left(\frac{\partial \widetilde{G}}{\partial x}(x+iy) - \frac{1}{i}\frac{\partial \widetilde{G}}{\partial y}(x+iy)\right) = \frac{\omega(b+iy)-(x-a)\omega'(b+iy)}{2(b-a)}\ .$$

We may therefore adjust again r_0 so that if $r < r_0$, $|\frac{\partial \widetilde{G}}{\partial z}(x+iy)| \ge 1/2$ for all $x \in]a,b[$. Note that $b-a = (d-1)\rho$. So

$$\left|\frac{\overline{\partial} F}{\partial F}\right| \overset{(9)}{\le} \sup_{x\in]a,b[}\left|\frac{\overline{\partial}\widetilde{G}}{\partial \widetilde{G}}\right| \le 2\left|\frac{\partial \widetilde{G}}{\partial \bar{z}}\right| \le \frac{|\omega(b+iy)|}{(d-1)\rho} + |\omega'(b+iy)| \le \frac{8\sqrt{r}}{\rho} + \frac{8\sqrt{r}}{\rho} = \frac{16d\sqrt{r}}{|\log(\sqrt{r})|}\ .$$

See [Sh] for a similar estimate. □

Proof of Theorem 4.2.

Let us first show that the map $a \mapsto \widehat{v}(0,a)$ is injective on $\widetilde{\mathcal{H}}^c_{e^{-1}}$. It follows from Theorem 4.4 that if $\widehat{v}(0,a) = \widehat{v}(0,a')$, for some a, a', then $P_{e^{-1},a}$ and $P_{e^{-1},a'}$ are hybridly conjugate. The dynamics of $P_{e^{-1},a}$ and $P_{e^{-1},a'}$ are conformally conjugate on the immediate attracted basins of 0, by a unique biholomorphic map fixing the origin, because both basins are quadratic and the multipliers at the origin are identical. Hence essentially repeating the proof of Proposition 3.6 (see also the Remark following its proof) one proves that this hybrid equivalence coincides with a Möbius conjugacy on $\widetilde{B}_{e^{-1},a}(0)^c$. Details are left to the reader. It follows that $a = a'$.

To prove that each $\widehat{h}_{0,a}$ is injective, we prove that it has a quasi conformal extension to all of $\mathbb{C}$. As in the proof of Theorem 4.4 fix an arbitrary $a \in \widetilde{\mathcal{H}}^c_{e^{-1}}$ and let H_λ, $\lambda \in \mathbb{D}$ be the hybrid conjugacies whose existence is assured by Theorem 4.4. Fix $\lambda \in \mathbb{D}$ and define $h = \widehat{h}_\lambda^{-1} \circ H_\lambda^{-1} \circ H_{e^{-1}}$, so that each h is a hybrid equivalence from $P_{e^{-1}}$ to itself, which is tangent to the identity at ∞. Repeating once more the proof of Proposition 3.6 we find that h coincides with the identity on $\widetilde{B}_{e^{-1}}(0)^c$. Hence the two maps $\widehat{h}_\lambda$ and $H_\lambda^{-1} \circ H_{e^{-1}}$

coincides on $\widetilde{B}_{e^{-1}}(0)^c$. Since H_λ^{-1} converges locally uniformly to the identity, it follows that $\widehat{h}_\lambda$ converges locally uniformly to the quasi conformal homeomorphism $H_{e^{-1}} : \mathbb{C} \longrightarrow \mathbb{C}$ on $\widetilde{B}_{e^{-1}}(0)^c$, from which the injectivity of $\widehat{h}_0 = H_{e^{-1}}$ on $\widetilde{B}_{e^{-1}}(0)^c$ follows.

Finally,

$$\begin{aligned} \widehat{H}(0,a,z) = \widehat{H}(0,a',z') &\Longrightarrow \widehat{v}(0,a) = \widehat{v}(0,a') \;\&\; \widehat{h}(0,a,z) = \widehat{h}(0,a',z') \\ &\Longrightarrow a = a' \;\&\; \widehat{h}(0,a,z) = \widehat{h}(0,a',z') \\ &\Longrightarrow a = a' \;\&\; \widehat{h}(0,a,z) = \widehat{h}(0,a,z') \\ &\Longrightarrow a = a' \;\&\; z = z' \ . \end{aligned}$$

So $\widehat{H}$ is injective. □

4.2 Lavaurs motion

The following example is different from the first and most other applications of the BH-motion. It is a motion of a two generator dynamical system (P, g_σ) consisting of a center attracting dynamics $(P, \overline{\mathbb{C}}, \infty)$ with $P(z) = z^2 + \frac{1}{4}$ (which is invariant under the BH-motion, by Proposition 3.6) and a so called parabolic enrichment or Lavaurs map g_σ coming from the complementary parabolic basin $B(0)$. Let $\phi : B(\infty) \longrightarrow \overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$ denote the Böttcher coordinate at ∞. Let $\Phi : B(0) \longrightarrow \mathbb{C}$ denote an attracting Fatou coordinate and $\Psi : \mathbb{C} \longrightarrow \mathbb{C}$ denote a repelling Fatou parameter, i.e. $\Phi \circ P(z) = 1 + \Phi(z)$ and $P \circ \Psi(z) = \Psi(z+1)$ where defined. We shall normalize Φ and Ψ by $\Phi(0) = 0$ and $\Psi(0) = \phi^{-1}(e)$. Define the Lavaurs map $g_\sigma : B(0) \longrightarrow \mathbb{C}$ of phase $\sigma \in \mathbb{C}$ by $g_\sigma(z) = \Psi \circ T_\sigma \circ \Phi$, where $T_\sigma(z) = z+\sigma$ and let $\Sigma = \{\sigma \in \mathbb{C} \mid g_\sigma(0) \in B(\infty)\} = \Psi^{-1}(B(\infty)) \ni 0$. The Julia set $J(P, g_\sigma)$ of the Lavaurs enriched dynamical system is the closure of $\cup_{n,m\geq 0} P^{-n}(g_\sigma^{-m}(J_P))$.

Theorem 4.6. *There exists a holomorphic motion $h : \Sigma \times \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$, such that $h_\sigma \circ P = P \circ h_\sigma$ and $h_\sigma \circ g_0 = g_\sigma \circ h_\sigma$.*

The map h_σ fixes J_P point-wise, but moves the points of the enrichment. However the the enriched Julia set is 1 periodic as a compact set, because $g_{\sigma+1} = P \circ g_\sigma = g_\sigma \circ P$ and hence the two enriched dynamical systems (P, g_σ) and $(P, g_{\sigma+1})$ have the same enriched Julia set.

Proof. Denote by $\psi : \overline{\mathbb{C}}\setminus\overline{\mathbb{D}} \longrightarrow B(\infty)$ the inverse of ϕ, the Böttcher parameter at ∞. We consider a BH-motion of the enriched dynamical system (P, g_0) based at the super attracting fixed point ∞. More precisely let μ_s denote the unique Beltrami form which equals $(l_s \circ \phi)^*(\mu_0)$ on $B(\infty)$, where $\mu_0 = 0$ is the zero or standard Beltrami form and which is invariant under the enriched dynamical system (P, g_0), i.e. $P^*(\mu_s) = \mu_s$ and $g_0^*(\mu_s) = \mu_s$. Note that μ_s is also supported in $B(1/2)$, because g_0 maps part of $B(1/2)$ into $B(\infty)$.

Let $h_s : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ denote the solution of the Beltrami equation $\overline{\partial}h = \mu_s \partial h$ normalized by $h_s(0) = 0$, $h_s(1/2) = 1/2$ and $h_s(\infty) = \infty$. Then $f = h_s \circ P \circ h_s^{-1}$ is a centered quadratic polynomial with a parabolic fixed point of multiplier 1 at 1/2. There is only one such polynomial, it is P. So h_s conjugates P to itself.

Define $\widehat{\mu}_s = \Psi^*(\mu_s)$ on $\mathbb{C}$, then $T_1^*(\widehat{\mu}_s) = \widehat{\mu}_s$. Let $\eta : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ denote the solution of the Beltrami equation $\overline{\partial} f = \widehat{\mu}_s \partial f$ normalized by $\eta_s(0) = 0$, $\eta_s(1) = 1$ and $\eta_s(\infty) = \infty$. We have $\eta_s(z+1) = \eta_s(z) + 1$, because $\widehat{\mu}_s$ is 1-periodic.

For each s the map $\eta_s \circ \Phi \circ h_s^{-1}$ is holomorphic, conjugates P to translation by 1 and fixes 0. Hence it equals Φ by uniqueness of normalized Fatou coordinates. Arguing similarly we find that $h_s \circ \Psi \circ \eta_s^{-1} = \Psi \circ T_{\sigma(s)}$ for some complex number $\sigma(s) \in \Sigma$. We have $h_s \circ \Psi \circ \eta_s^{-1}(0) = h_s(\psi(e)) = \psi(e^s)$ by the normalization of Ψ. Thus $\sigma(s)$ can be chosen to depend continuously on s. Furthermore $h_s \circ g_0 = g_\sigma \circ h_s$. See the following diagram.

$$
\begin{tikzcd}
\mathbb{C}, 0 & \mathbb{C}, 0 \arrow[l, "\eta_s"'] \arrow[r, "T_0"] & \mathbb{C}, 0 \arrow[r, "\eta_s"] \arrow[d, "\Psi"'] & \mathbb{C}, 0 \arrow[r, "T_{\sigma(s)}"] & \mathbb{C}, * \arrow[d, "\Psi"] \\
B(0), 0 \arrow[u, "\Phi"] & B(0), 0 \arrow[l, "h_s"] \arrow[u, "\Phi"] \arrow[r, "g_0"'] & \mathbb{C}, \psi(e) \arrow[rr, "h_s"'] & & \mathbb{C} \supset B(\infty), \psi(e^s).
\end{tikzcd}
$$

The restriction $\Psi : \Sigma \longrightarrow B(\infty)$ is a universal covering, (see e.g. [P]) with $\Psi(0) = \psi(\mathrm{e})$. And $\psi \circ \exp : \mathbb{H} \longrightarrow B(\infty)$ is also a universal covering, but with $\psi \circ \exp(1) = \psi(e)$. Hence there exists a unique lift $\widehat{\sigma} : \mathbb{H} \longrightarrow \Sigma$ of $\psi \circ \exp$ to Ψ with $\widehat{\sigma}(1) = 0$. This lift is an isomorphism, since both coverings are universal, and it satisfies $\widehat{\sigma}(2s) = \widehat{\sigma}(s) + 1$. However since $\Psi \circ T_{\widehat{\sigma}(s)} = \Psi \circ T_{\sigma(s)}$ and both functions σ and $\widehat{\sigma}$ are continuous, we have $\sigma = \widehat{\sigma}$. As σ is an isomorphism from $\mathbb{H}$ to Σ we may use σ as a parameter for the holomorphic motion, replacing the parameter space $\mathbb{H}$ by Σ and the base point 1 by 0. $\square$

References

[BH1] B. Branner & J.H. Hubbard, The iteration of cubic polynomials. Part I: the global topology of parameter space, Acta Mathematica 160 (1988), 143-206.

[D] A. Douady, in this Volume

[DH1] A. Douady & J.H. Hubbard, Etude dynamique des polynômes complexes, I, publications mathématiques d'Orsay, 84-02, 1984.

[DH2] A. Douady & J.H. Hubbard, On the dynamics of polynomial-like mappings, Ann. Scient. Ec. Norm. Sup., t.18, p. 287-343, 1985.

[KN] Komori, Y. & Nakane, Sh., Landing property of stretching rays for real cubic polynomials, Conformal geom. and dyn. 8 (2004), 87-114.

[Mi] J. Milnor, Dynamics in One Complex Variable, Introductory Lectures, vieweg, 1999.

[P] C. L. Petersen, *On the Pommerenke-Levin-Yoccoz inequality*, Ergod. Th. & Dynam. Sys. , **13** (1993) 785–806.

[P-T] C. L. Petersen and Tan Lei, *BH-deformations of attracting dynamics*, manuscript in preparation.

[Sh] M. Shishikura, *Bifurcation of parabolic fixed points*, in "The Mandelbrot set, Theme and Variations", Ed. Tan Lei, LMS Lect. Note Ser. 274, Cambridge Univ. Press, 2000, p.325-363.

[Ta] Tan Lei, Stretching rays and their accumulations, following Pia Willumsen, in this volume.

[Ri] S. Rickman, Removability theorems for quasi-conformal mappings, Ann. Ac. Scient. Fenn. 449, p. 1-8, 1969.

[U] E. Uhre, Construction of a Holomorphic Motion in Part of the Parameter Space for a Family of Quadratic Rational Maps. Master Thesis, IMFUFA, RUC, Denmark 2004.

[Wi] P. Willumsen, Holomorphic dynamics, On accumulation of stretching rays, Ph.D. thesis, Danmarks Tekniske Universitet, 1997.

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